Peeter Joot
peeter.joot@gmail.com

SHO translation operator expectation

Exercise 1.1 SHO translation operator expectation ([1] pr. 2.12)

Using the Heisenberg picture evaluate the expectation of the position operator $\langle x \rangle$ with respect to the initial time state

$$\left|\alpha, 0\right\rangle = e^{-ip_0 a/\hbar} \left|0\right\rangle, \tag{1.1}$$

where p_0 is the initial time position operator, and a is a constant with dimensions of position.

Answer for Exercise 1.1

Recall that the Heisenberg picture position operator expands to

$$\begin{aligned} x^{\mathrm{H}}(t) &= U^\dagger x U \\ &= x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t), \end{aligned} \tag{1.2}$$

so the expectation of the position operator is

$$\begin{aligned} \langle x \rangle &= \left\langle 0\right| e^{ip_0 a/\hbar} \left(x_0 \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) \right) e^{-ip_0 a/\hbar} \left|0\right\rangle \\ &= \left\langle 0\right| \left(e^{ip_0 a/\hbar} x_0 \cos(\omega t) e^{-ip_0 a/\hbar} \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) \right) \left|0\right\rangle . \end{aligned} \tag{1.3}$$

The exponential sandwich above can be expanded using the Baker-Campbell-Hausdorff [2] formula

$$\begin{aligned} e^{ip_0 a/\hbar} x_0 e^{-ip_0 a/\hbar} &= x_0 + \frac{ia}{\hbar} [p_0, x_0] + \frac{1}{2!} \left(\frac{ia}{\hbar}\right)^2 [p_0, [p_0, x_0]] + \cdots \\ &= x_0 + \frac{ia}{\hbar} (-i\hbar) + \frac{1}{2!} \left(\frac{ia}{\hbar}\right)^2 [p_0, -i\hbar] + \cdots \\ &= x_0 + a. \end{aligned} \tag{1.4}$$

The position expectation with respect to this translated state is

$$\begin{aligned} \langle x \rangle &= \left\langle 0\right| \left((x_0 + a) \cos(\omega t) + \frac{p_0}{m\omega} \sin(\omega t) \right) \left|0\right\rangle \\ &= a \cos(\omega t). \end{aligned} \tag{1.5}$$

The final simplification above follows from $\left\langle n\right| x \left|n\right\rangle = \left\langle n\right| p \left|n\right\rangle = 0$.

Bibliography

[1] Jun John Sakurai and Jim J Napolitano. *Modern quantum mechanics*. Pearson Higher Ed, 2014. 1.1

[2] Wikipedia. Baker-campbell-hausdorff formula — wikipedia, the free encyclopedia, 2015. URL https://en.wikipedia.org/w/index.php?title=Baker%E2%80%93Campbell%E2%80%93Hausdorff_formula&oldid=665123858. [Online; accessed 16-August-2015]. 1