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Tschebyscheff polynomials

In ancient times (i.e. 2nd year undergrad) I recall being very impressed with Tschebyscheff polynomials for designing lowpass filters. I'd used Tschebyscheff filters for the hardware we used for a speech recognition system our group built in the design lab. One of the benefits of these polynomials is that the oscillation in the $|x| < 1$ interval is strictly bounded. This same property, as well as the unbounded nature outside of the $[-1, 1]$ interval turns out to have applications to antenna array design.

The Tschebyscheff polynomials are defined by

$$T_m(x) = \cos\left(m \cos^{-1} x\right), \quad |x| < 1 \tag{1.1a}$$

$$T_m(x) = \cosh\left(m \cosh^{-1} x\right), \quad |x| > 1. \tag{1.1b}$$

Range restrictions and hyperbolic form. Prof. Eleftheriades's notes made a point to point out the definition in the $|x| > 1$ interval, but that can also be viewed as a consequence instead of a definition if the range restriction is removed. For example, suppose $x = 7$, and let

$$\cos^{-1} 7 = \theta, \tag{1.2}$$

so

$$\begin{aligned} 7 &= \cos\theta \\ &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= \cosh(i\theta), \end{aligned} \tag{1.3}$$

or

$$-i \cosh^{-1} 7 = \theta. \tag{1.4}$$

$$\begin{aligned} T_m(7) &= \cos(-mi \cosh^{-1} 7) \\ &= \cosh(m \cosh^{-1} 7). \end{aligned} \tag{1.5}$$

The same argument clearly applies to any other value outside of the $|x| < 1$ range, so without any restrictions, these polynomials can be defined as just

$$\boxed{T_m(x) = \cos\left(m \cos^{-1} x\right).} \tag{1.6}$$

Polynomial nature. Equation (1.6) does not obviously look like a polynomial. Let's proceed to verify the polynomial nature for the first couple values of m.

- $m = 0$.

$$\begin{aligned} T_0(x) &= \cos(0 \cos^{-1} x) \\ &= \cos(0) \\ &= 1. \end{aligned} \tag{1.7}$$

- $m = 1$.

$$\begin{aligned} T_1(x) &= \cos(1 \cos^{-1} x) \\ &= x. \end{aligned} \tag{1.8}$$

- $m = 2$.

$$\begin{aligned} T_2(x) &= \cos(2 \cos^{-1} x) \\ &= 2\cos^2 \cos^{-1}(x) - 1 \\ &= 2x^2 - 1. \end{aligned} \tag{1.9}$$

To examine the general case

$$\begin{aligned} T_m(x) &= \cos(m \cos^{-1} x) \\ &= \operatorname{Re} e^{jm \cos^{-1} x} \\ &= \operatorname{Re} \left(e^{j \cos^{-1} x}\right)^m \\ &= \operatorname{Re} \left(\cos\cos^{-1} x + j \sin\cos^{-1} x\right)^m \\ &= \operatorname{Re} \left(x + j\sqrt{1 - x^2}\right)^m \\ &= \operatorname{Re} \left(x^m + \binom{m}{1} jx^{m-1} \left(1 - x^2\right)^{1/2} \right. \\ &\quad \left. - \binom{m}{2} x^{m-2} \left(1 - x^2\right)^{2/2} - \binom{m}{3} jx^{m-3} \left(1 - x^2\right)^{3/2} + \binom{m}{4} x^{m-4} \left(1 - x^2\right)^{4/2} + \cdots\right) \\ &= x^m - \binom{m}{2} x^{m-2} \left(1 - x^2\right) + \binom{m}{4} x^{m-4} \left(1 - x^2\right)^2 - \cdots \end{aligned} \tag{1.10}$$

This expansion was a bit cavaliar with the signs of the $\sin\cos^{-1} x = \sqrt{1-x^2}$ terms, since the negative sign should be picked for the root when $x \in [-1, 0]$. However, that doesn't matter in the end since the real part operation selects only powers of two of this root.

The final result of the expansion above can be written

$$T_m(x) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k} (-1)^k x^{m-2k} \left(1 - x^2\right)^k. \tag{1.11}$$

This clearly shows the polynomial nature of these functions, and is also perfectly well defined for any value of x. The even and odd alternation with m is also clear in this explicit expansion.

Some plots The first couple polynomials are plotted in fig. 1.1.

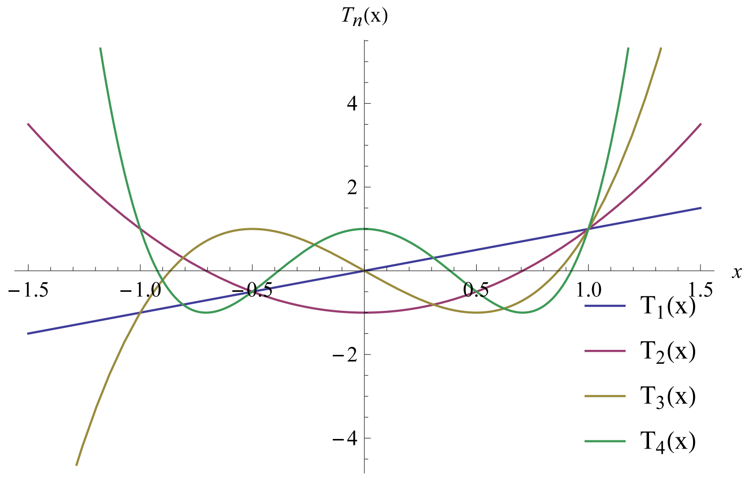

Figure 1.1: A couple Chebychev plots.

Properties In [1] a few properties can be found for these polynomials

$$T_m(x) = 2xT_{m-1} - T_{m-2} \tag{1.12a}$$

$$0 = \left(1 - x^2\right) \frac{dT_m(x)}{dx} + mxT_m(x) - mT_{m-1}(x) \tag{1.12b}$$

$$0 = \left(1 - x^2\right) \frac{d^2T_m(x)}{dx^2} - x\frac{dT_m(x)}{dx} + m^2T_m(x) \tag{1.12c}$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x)T_n(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n = 0 \\ \pi/2 & \text{if } m = n, m \neq 0 \end{cases} \tag{1.12d}$$

Exercise 1.1 Recurrance relation.

Prove eq. (1.12a).

Answer for Exercise 1.1

To show this, let

$$x = \cos\theta. \tag{1.13}$$

$$2xT_{m-1} - T_{m-2} = 2\cos\theta\cos((m-1)\theta) - \cos((m-2)\theta). \tag{1.14}$$

Recall the cosine addition formulas

$$\begin{aligned}
\cos(a+b) &= \operatorname{Re} e^{j(a+b)} \\
&= \operatorname{Re} e^{ja} e^{jb} \\
&= \operatorname{Re} \left(\cos a + j\sin a\right)\left(\cos b + j\sin b\right) \\
&= \cos a\cos b - \sin a\sin b.
\end{aligned} \tag{1.15}$$

Applying this gives

$$\begin{aligned}
2xT_{m-1} - T_{m-2} &= 2\cos\theta\left(\cos(m\theta)\cos\theta + \sin(m\theta)\sin\theta\right) - \left(\cos(m\theta)\cos(2\theta) + \sin(m\theta)\sin(2\theta)\right) \\
&= 2\cos\theta\left(\cos(m\theta)\cos\theta + \sin(m\theta)\sin\theta)\right) \\
&\quad - \left(\cos(m\theta)(\cos^2\theta - \sin^2\theta) + 2\sin(m\theta)\sin\theta\cos\theta\right) \\
&= \cos(m\theta)\left(\cos^2\theta + \sin^2\theta\right) \\
&= T_m(x). \qquad \square
\end{aligned} \tag{1.16}$$

Exercise 1.2 First order LDE relation.

Prove eq. (1.12b).

Answer for Exercise 1.2

To show this, again, let

$$x = \cos\theta. \tag{1.17}$$

Observe that

$$1 = -\sin\theta\frac{d\theta}{dx}, \tag{1.18}$$

so

$$
\begin{aligned}
\frac{d}{dx} &= \frac{d\theta}{dx}\frac{d}{d\theta} \\
&= -\frac{1}{\sin\theta}\frac{d}{d\theta}.
\end{aligned} \tag{1.19}
$$

Plugging this in gives

$$
\begin{aligned}
&(1-x^2)\frac{d}{dx}T_m(x) + mxT_m(x) - mT_{m-1}(x) \\
&= \sin^2\theta\left(-\frac{1}{\sin\theta}\frac{d}{d\theta}\right)\cos(m\theta) + m\cos\theta\cos(m\theta) - m\cos((m-1)\theta) \\
&= -\sin\theta(-m\sin(m\theta)) + m\cos\theta\cos(m\theta) - m\cos((m-1)\theta).
\end{aligned} \tag{1.20}
$$

Applying the cosine addition formula eq. (1.15) gives

$$
m\left(\sin\theta\sin(m\theta) + \cos\theta\cos(m\theta)\right) - m\left(\cos(m\theta)\cos\theta + \sin(m\theta)\sin\theta\right) = 0. \qquad \square \tag{1.21}
$$

Exercise 1.3 Second order LDE relation.

Prove eq. (1.12c).

Answer for Exercise 1.3

This follows the same way. The first derivative was

$$
\begin{aligned}
\frac{dT_m(x)}{dx} &= -\frac{1}{\sin\theta}\frac{d}{d\theta}\cos(m\theta) \\
&= -\frac{1}{\sin\theta}(-m)\sin(m\theta) \\
&= m\frac{1}{\sin\theta}\sin(m\theta),
\end{aligned} \tag{1.22}
$$

so the second derivative is

$$
\begin{aligned}
\frac{d^2T_m(x)}{dx^2} &= -m\frac{1}{\sin\theta}\frac{d}{d\theta}\frac{1}{\sin\theta}\sin(m\theta) \\
&= -m\frac{1}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\sin(m\theta) + \frac{1}{\sin\theta}m\cos(m\theta)\right).
\end{aligned} \tag{1.23}
$$

Putting all the pieces together gives

$$
\begin{aligned}
&(1-x^2)\frac{d^2T_m(x)}{dx^2} - x\frac{dT_m(x)}{dx} + m^2T_m(x) \\
&= m\left(\frac{\cos\theta}{\sin\theta}\sin(m\theta) - m\cos(m\theta)\right) - \cos\theta\, m\frac{1}{\sin\theta}\sin(m\theta) + m^2\cos(m\theta) \\
&= 0. \qquad \square
\end{aligned} \tag{1.24}
$$

Exercise 1.4 Orthogonality relation

Prove eq. (1.12d).

Answer for Exercise 1.4

First consider the 0,0 inner product, making an $x = \cos\theta$, so that $dx = -\sin\theta d\theta$

$$\begin{aligned}\langle T_0, T_0 \rangle &= \int_{-1}^{1} \frac{1}{\left(1-x^2\right)^{1/2}} dx \\ &= \int_{-\pi}^{0} \left(-\frac{1}{\sin\theta}\right) - \sin\theta d\theta \\ &= 0 - (-\pi) \\ &= \pi.\end{aligned} \tag{1.25}$$

Note that since the $[-\pi, 0]$ interval was chosen, the negative root of $\sin^2\theta = 1 - x^2$ was chosen, since $\sin\theta$ is negative in that interval.

The m,m inner product with $m \neq 0$ is

$$\begin{aligned}\langle T_m, T_m \rangle &= \int_{-1}^{1} \frac{1}{\left(1-x^2\right)^{1/2}} \left(T_m(x)\right)^2 dx \\ &= \int_{-\pi}^{0} \left(-\frac{1}{\sin\theta}\right) \cos^2(m\theta) - \sin\theta d\theta \\ &= \int_{-\pi}^{0} \cos^2(m\theta) d\theta \\ &= \frac{1}{2}\int_{-\pi}^{0} \left(\cos(2m\theta) + 1\right) d\theta \\ &= \frac{\pi}{2}.\end{aligned} \tag{1.26}$$

So far so good. For $m \neq n$ the inner product is

$$\begin{aligned}\langle T_m, T_m \rangle &= \int_{-\pi}^{0} \cos(m\theta)\cos(n\theta) d\theta \\ &= \frac{1}{4}\int_{-\pi}^{0} \left(e^{jm\theta} + e^{-jm\theta}\right)\left(e^{jn\theta} + e^{-jn\theta}\right) d\theta \\ &= \frac{1}{4}\int_{-\pi}^{0} \left(e^{j(m+n)\theta} + e^{-j(m+n)\theta} + e^{j(m-n)\theta} + e^{j(-m+n)\theta}\right) d\theta \\ &= \frac{1}{2}\int_{-\pi}^{0} \left(\cos((m+n)\theta) + \cos((m-n)\theta)\right) d\theta \\ &= \frac{1}{2}\left(\frac{\sin((m+n)\theta)}{m+n} + \frac{\sin((m-n)\theta)}{m-n}\right)\Bigg|_{-\pi}^{0} \\ &= 0. \qquad \square\end{aligned} \tag{1.27}$$

Bibliography

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. Dover publications, 1964. 1