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Green's function inversion of magnetostatic equation

A previous example of inverting a gradient equation was the electrostatics equation. We can do the same for the magnetostatics equation, which has the following Geometric Algebra form in linear media

$$\boldsymbol{\nabla} I \mathbf{B} = -\mu \mathbf{J}. \tag{1.1}$$

The Green's inversion of this is

$$\begin{aligned} I\mathbf{B}(\mathbf{x}) &= \int_V dV' G(\mathbf{x}, \mathbf{x}') \boldsymbol{\nabla}' I \mathbf{B}(\mathbf{x}') \\ &= \int_V dV' G(\mathbf{x}, \mathbf{x}') (-\mu \mathbf{J}(\mathbf{x}')) \\ &= \frac{1}{4\pi} \int_V dV' \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3} (-\mu \mathbf{J}(\mathbf{x}')). \end{aligned} \tag{1.2}$$

We expect the LHS to be a bivector, so the scalar component of this should be zero. That can be demonstrated with some of the usual trickery

$$\begin{aligned} -\frac{\mu}{4\pi} \int_V dV' \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3} \cdot \mathbf{J}(\mathbf{x}') &= \frac{\mu}{4\pi} \int_V dV' \left(\boldsymbol{\nabla} \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} \right) \cdot \mathbf{J}(\mathbf{x}') \\ &= -\frac{\mu}{4\pi} \int_V dV' \left(\boldsymbol{\nabla}' \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} \right) \cdot \mathbf{J}(\mathbf{x}') \\ &= -\frac{\mu}{4\pi} \int_V dV' \left(\boldsymbol{\nabla}' \cdot \frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|} - \frac{\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|} \right). \end{aligned} \tag{1.3}$$

The current $\mathbf{J}$ is not unconstrained. This can be seen by premultiplying eq. (1.1) by the gradient

$$\boldsymbol{\nabla}^2 I \mathbf{B} = -\mu \boldsymbol{\nabla} \mathbf{J}. \tag{1.4}$$

On the LHS we have a bivector so must have $\boldsymbol{\nabla} \mathbf{J} = \boldsymbol{\nabla} \wedge \mathbf{J}$, or $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$. This kills the $\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{x}')$ integrand numerator in eq. (1.3), leaving

$$\begin{aligned} -\frac{\mu}{4\pi} \int_V dV' \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3} \cdot \mathbf{J}(\mathbf{x}') &= -\frac{\mu}{4\pi} \int_V dV' \boldsymbol{\nabla}' \cdot \frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|} \\ &= -\frac{\mu}{4\pi} \int_{\partial V} dA' \hat{\mathbf{n}} \cdot \frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|}. \end{aligned} \tag{1.5}$$

This shows that the scalar part of the equation is zero, provided the normal component of $\mathbf{J}/|\mathbf{x}-\mathbf{x}'|$ vanishes on the boundary of the infinite sphere. This leaves the Biot-Savart law as a bivector equation

$$I\mathbf{B}(\mathbf{x}) = \frac{\mu}{4\pi}\int_V dV'\mathbf{J}(\mathbf{x}') \wedge \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{1.6}$$

Observe that the traditional vector form of the Biot-Savart law can be obtained by premultiplying both sides with $-I$, leaving

$$\mathbf{B}(\mathbf{x}) = \frac{\mu}{4\pi}\int_V dV'\mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{1.7}$$

This checks against a trusted source such as [1] (eq. 5.39).

Bibliography

[1] David Jeffrey Griffiths and Reed College. *Introduction to electrodynamics*. Prentice hall Upper Saddle River, NJ, 3rd edition, 1999. 1