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Transverse gauge

Jackson [1] has an interesting presentation of the transverse gauge. I'd like to walk through the details of this, but first want to translate the preliminaries to SI units (if I had the 3rd edition I'd not have to do this translation step).

Gauge freedom The starting point is noting that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ the magnetic field can be expressed as a curl

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{1.1}$$

Faraday's law now takes the form

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \\ &= \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial}{\partial t} (\boldsymbol{\nabla} \times \mathbf{A}) \\ &= \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right). \end{aligned} \tag{1.2}$$

Because this curl is zero, the interior sum can be expressed as a gradient

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \equiv -\boldsymbol{\nabla} \Phi. \tag{1.3}$$

This can now be substituted into the remaining two Maxwell's equations.

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{D} &= \rho_v \\ \boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned} \tag{1.4}$$

For Gauss's law, in simple media, we have

$$\begin{aligned} \rho_v &= \epsilon \boldsymbol{\nabla} \cdot \mathbf{E} \\ &= \epsilon \boldsymbol{\nabla} \cdot \left(-\boldsymbol{\nabla} \Phi - \frac{\partial \mathbf{A}}{\partial t} \right) \end{aligned} \tag{1.5}$$

For simple media again, the Ampere-Maxwell equation is

$$\frac{1}{\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{J} + \epsilon\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t}\right). \tag{1.6}$$

Expanding $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = -\boldsymbol{\nabla}^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})$ gives

$$-\boldsymbol{\nabla}^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu\mathbf{J} - \epsilon\mu\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t}. \tag{1.7}$$

Maxwell's equations are now reduced to

$$\boxed{\begin{aligned} \boldsymbol{\nabla}^2\mathbf{A} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \epsilon\mu\frac{\partial\Phi}{\partial t}\right) - \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu\mathbf{J} \\ \boldsymbol{\nabla}^2\Phi + \frac{\partial\boldsymbol{\nabla}\cdot\mathbf{A}}{\partial t} &= -\frac{\rho_v}{\epsilon}. \end{aligned}} \tag{1.8}$$

There are two obvious constraints that we can impose

$$\boldsymbol{\nabla}\cdot\mathbf{A} - \epsilon\mu\frac{\partial\Phi}{\partial t} = 0, \tag{1.9}$$

or

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{1.10}$$

The first constraint is the Lorentz gauge, which I've played with previously. It happens to be really nice in a relativistic context since, in vacuum with a four-vector potential $A = (\Phi/c, \mathbf{A})$, that is a requirement that the four-divergence of the four-potential vanishes ($\partial_\mu A^\mu = 0$).

Transverse gauge Jackson identifies the latter constraint as the transverse gauge, which I'm less familiar with. With this gauge selection, we have

$$\boldsymbol{\nabla}^2\mathbf{A} - \epsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} + \epsilon\mu\boldsymbol{\nabla}\frac{\partial\Phi}{\partial t} \tag{1.11a}$$

$$\boldsymbol{\nabla}^2\Phi = -\frac{\rho_v}{\epsilon}. \tag{1.11b}$$

What's not obvious is the fact that the irrotational (zero curl) contribution due to Φ in eq. (1.11a) cancels the corresponding irrotational term from the current. Jackson uses a transverse and longitudinal decomposition of the current, related to the Helmholtz theorem to allude to this.

That decomposition follows from expanding $\boldsymbol{\nabla}^2 J/R$ in two ways using the delta function $-4\pi\delta(\mathbf{x}-\mathbf{x}') = \boldsymbol{\nabla}^2 1/R$ representation, as well as directly

$$\begin{aligned} -4\pi\mathbf{J}(\mathbf{x}) &= \int \boldsymbol{\nabla}^2\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= \boldsymbol{\nabla}\int\boldsymbol{\nabla}\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \boldsymbol{\nabla}\cdot\int\boldsymbol{\nabla}\wedge\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \\ &= -\boldsymbol{\nabla}\int\mathbf{J}(\mathbf{x}')\cdot\boldsymbol{\nabla}'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\wedge\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right) \\ &= -\boldsymbol{\nabla}\int\boldsymbol{\nabla}'\cdot\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \boldsymbol{\nabla}\int\frac{\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'\right) \end{aligned} \tag{1.12}$$

The first term can be converted to a surface integral

$$-\boldsymbol{\nabla} \int \boldsymbol{\nabla}' \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' = -\boldsymbol{\nabla} \int d\mathbf{A}' \cdot \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \tag{1.13}$$

so provided the currents are either localized or $|\mathbf{J}|/R \rightarrow 0$ on an infinite sphere, we can make the identification

$$\mathbf{J}(\mathbf{x}) = \boldsymbol{\nabla} \frac{1}{4\pi} \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \frac{1}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \equiv \mathbf{J}_l + \mathbf{J}_t, \tag{1.14}$$

where $\boldsymbol{\nabla} \times \mathbf{J}_l = 0$ (irrotational, or longitudinal), whereas $\boldsymbol{\nabla} \cdot \mathbf{J}_t = 0$ (solenoidal or transverse). The irrotational property is clear from inspection, and the transverse property can be verified readily

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{X})) &= -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{X})) \\
&= -\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{X} - \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{X}) \right) \\
&= -\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{X} \right) + \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \cdot \mathbf{X}) \\
&= 0.
\end{aligned} \tag{1.15}$$

Since

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon} \int \frac{\rho_v(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{1.16}$$

we have

$$\begin{aligned}
\boldsymbol{\nabla} \frac{\partial \Phi}{\partial t} &= \frac{1}{4\pi\epsilon} \boldsymbol{\nabla} \int \frac{\partial_t \rho_v(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\
&= \frac{1}{4\pi\epsilon} \boldsymbol{\nabla} \int \frac{-\boldsymbol{\nabla}' \cdot \mathbf{J}}{|\mathbf{x} - \mathbf{x}'|} d^3x' \\
&= \frac{\mathbf{J}_l}{\epsilon}.
\end{aligned} \tag{1.17}$$

This means that the Ampere-Maxwell equation takes the form

$$\boldsymbol{\nabla}^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \mu \mathbf{J}_l = -\mu \mathbf{J}_t. \tag{1.18}$$

This justifies the "transverse" in the label transverse gauge.

Bibliography

[1] JD Jackson. *Classical Electrodynamics*. John Wiley and Sons, 2nd edition, 1975. 1