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Average power for circuit elements

In [2] §2.2 is a comparison of field energy expressions with their circuit equivalents. It's clearly been too long since I've worked with circuits, because I'd forgotten all the circuit energy expressions:

$$\begin{aligned}
W_{\mathrm{R}} &= \frac{R}{2} \left\lvert I \right\rvert^2 \\
W_{\mathrm{C}} &= \frac{C}{4} \left\lvert V \right\rvert^2 \\
W_{\mathrm{L}} &= \frac{L}{4} \left\lvert I \right\rvert^2 \\
W_{\mathrm{G}} &= \frac{G}{2} \left\lvert V \right\rvert^2
\end{aligned} \tag{1.1}$$

Here's a recap of where these come from

Energy lost to resistance Given

$$v(t) = R i(t) \tag{1.2}$$

the average power lost to a resistor is

$$\begin{aligned}
p_{\mathrm{R}} &= \frac{1}{T} \int_0^T v(t) i(t) dt \\
&= \frac{1}{T} \int_0^T \mathrm{Re}\left(V e^{j\omega t} \right) \mathrm{Re}\left(I e^{j\omega t} \right) dt \\
&= \frac{1}{4T} \int_0^T \left(V e^{j\omega t} + V^\ast e^{-j\omega t} \right) \left(I e^{j\omega t} + I^\ast e^{-j\omega t} \right) dt \\
&= \frac{1}{4T} \int_0^T \left(V I e^{2 j\omega t} + V^\ast I^\ast e^{-2 j\omega t} + V I^\ast + V^\ast I \right) dt \\
&= \frac{1}{2} \mathrm{Re}\left(V I^\ast \right) \\
&= \frac{1}{2} \mathrm{Re}\left(I R I^\ast \right) \\
&= \frac{R}{2} \left\lvert I \right\rvert^2.
\end{aligned} \tag{1.3}$$

Here it is assumed that the averaging is done over some integer multiple of the period, which kills off all the exponentials.

Energy stored in a capacitor I tried the same sort of analysis for a capacitor in phasor form, but everything cancelled out. Referring to [1], the approach used to figure this out is to operate first strictly in the time domain. Specifically, for the capacitor where $i = C dv/dt$ the power supplied up to a time t is

$$\begin{aligned} p_{\mathrm{C}}(t) &= \int_{-\infty}^{t} C \frac{dv}{dt} v(t) dt \\ &= \frac{1}{2} C v^2(t). \end{aligned} \tag{1.4}$$

The $v^2(t)$ term can now be expanded in terms of phasors and averaged for

$$\begin{aligned} \bar{p}_{\mathrm{C}} &= \frac{C}{2T} \int_0^T \frac{1}{4} \left(V e^{j\omega t} + V^* e^{-j\omega t} \right) \left(V e^{j\omega t} + V^* e^{-j\omega t} \right) dt \\ &= \frac{C}{2T} \int_0^T \frac{1}{4} 2 \left| V \right|^2 dt \\ &= \frac{C}{4} \left| V \right|^2. \end{aligned} \tag{1.5}$$

Energy stored in an inductor The inductor energy is found the same way, with

$$\begin{aligned} p_{\mathrm{L}}(t) &= \int_{-\infty}^{t} L \frac{di}{dt} i(t) dt \\ &= \frac{1}{2} L i^2(t), \end{aligned} \tag{1.6}$$

which leads to

$$\bar{p}_{\mathrm{L}} = \frac{L}{4} \left| I \right|^2. \tag{1.7}$$

Energy lost due to conductance Finally, we have conductance. In phasor space that is defined by

$$G = \frac{I}{V} = \frac{1}{R}, \tag{1.8}$$

so power lost due to conductance follows from power lost due to resistance. In the average we have

$$\begin{aligned} p_{\mathrm{G}} &= \frac{1}{2G} \left| I \right|^2 \\ &= \frac{1}{2G} \left| V G \right|^2 \\ &= \frac{G}{2} \left| V \right|^2 \end{aligned} \tag{1.9}$$

Bibliography

[1] J.D. Irwin. *Basic Engineering Circuit Analysis*. MacMillian, 1993. 1

[2] David M Pozar. *Microwave engineering*. John Wiley & Sons, 2009. 1