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Continuity equation and Ampere's law

Exercise 1.1 Displacement current and Ampere's law.

Show that without the displacement current $\partial \mathbf{D}/\partial t$, Maxwell's equations will not satisfy conservation relations.

Answer for Exercise 1.1

Without the displacement current, Maxwell's equations are

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}, t) \\
\boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r}, t) &= \mathbf{J} \\
\boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{r}, t) &= \rho_{\mathrm{v}}(\mathbf{r}, t) \\
\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{r}, t) &= 0.
\end{aligned} \tag{1.1}$$

Assuming that the continuity equation must hold, we have

$$\begin{aligned}
0 &= \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho_{\mathrm{v}}}{\partial t} \\
&= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{D}) \\
&= \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{D}) \\
&\ne 0.
\end{aligned} \tag{1.2}$$

This shows that the current in Ampere's law must be transformed to

$$\mathbf{J} \rightarrow \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{1.3}$$

should we wish the continuity equation to be satisfied. With such an addition we have

$$\begin{aligned}
0 &= \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho_{\mathrm{v}}}{\partial t} \\
&= \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right) + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{D}) \\
&= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) - \boldsymbol{\nabla} \cdot \frac{\partial \mathbf{D}}{\partial t} + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{D}).
\end{aligned} \tag{1.4}$$

The first term is zero (assuming sufficient continity of **H**) and the second two terms cancel when the space and time derivatives of one are commuted.

Bibliography