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Dipole field from spherical harmonics

As indicated in Jackson [1], the components of the electric field can be obtained directly from the multipole moments

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum \frac{4\pi}{(2l+1)r^{l+1}} q_{lm} Y_{lm}, \tag{1.1}$$

so for the l, m contribution to this sum the components of the electric field are

$$E_r = \frac{1}{\epsilon_0} \sum \frac{l+1}{(2l+1)r^{l+2}} q_{lm} Y_{lm}, \tag{1.2}$$

$$E_\theta = -\frac{1}{\epsilon_0} \sum \frac{1}{(2l+1)r^{l+2}} q_{lm} \partial_\theta Y_{lm} \tag{1.3}$$

$$\begin{aligned} E_\phi &= -\frac{1}{\epsilon_0} \sum \frac{1}{(2l+1)r^{l+2}\sin\theta} q_{lm} \partial_\phi Y_{lm} \\ &= -\frac{1}{\epsilon_0} \sum \frac{jm}{(2l+1)r^{l+2}\sin\theta} q_{lm} Y_{lm}. \end{aligned} \tag{1.4}$$

Here I've translated from CGS to SI. Let's calculate the $l = 1$ electric field components directly from these expressions and check against the previously calculated results.

$$\begin{aligned} E_r &= \frac{1}{\epsilon_0} \frac{2}{3r^3} \left(2 \left(-\sqrt{\frac{3}{8\pi}} \right)^2 \mathrm{Re} \left((p_x - jp_y) \sin\theta e^{j\phi} \right) + \left(\sqrt{\frac{3}{4\pi}} \right)^2 p_z \cos\theta \right) \\ &= \frac{2}{4\pi\epsilon_0 r^3} \left(p_x \sin\theta\cos\phi + p_y \sin\theta\sin\phi + p_z\cos\theta \right) \\ &= \frac{1}{4\pi\epsilon_0 r^3} 2\mathbf{p}\cdot\hat{\mathbf{r}}. \end{aligned} \tag{1.5}$$

Note that

$$\partial_\theta Y_{11} = -\sqrt{\frac{3}{8\pi}} \cos\theta e^{j\phi}, \tag{1.6}$$

and

$$\partial_\theta Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \cos\theta e^{-j\phi}, \tag{1.7}$$

so

$$\begin{aligned}
E_\theta &= -\frac{1}{\epsilon_0}\frac{1}{3r^3}\left(2\left(-\sqrt{\frac{3}{8\pi}}\right)^2 \operatorname{Re}\left((p_x - jp_y)\cos\theta e^{j\phi}\right) - \left(\sqrt{\frac{3}{4\pi}}\right)^2 p_z \sin\theta\right) \\
&= -\frac{1}{4\pi\epsilon_0 r^3}\left(p_x \cos\theta\cos\phi + p_y\cos\theta\sin\phi - p_z\sin\theta\right) \\
&= -\frac{1}{4\pi\epsilon_0 r^3}\mathbf{p}\cdot\hat{\boldsymbol{\theta}}.
\end{aligned} \tag{1.8}$$

For the $\hat{\boldsymbol{\phi}}$ component, the $m = 0$ term is killed. This leaves

$$\begin{aligned}
E_\phi &= -\frac{1}{\epsilon_0}\frac{1}{3r^3\sin\theta}\left(jq_{11}Y_{11} - jq_{1,-1}Y_{1,-1}\right) \\
&= -\frac{1}{3\epsilon_0 r^3\sin\theta}\left(jq_{11}Y_{11} - j(-1)^{2m}q_{11}^* Y_{11}^*\right) \\
&= \frac{2}{\epsilon_0}\frac{1}{3r^3\sin\theta}\operatorname{Im} q_{11}Y_{11} \\
&= \frac{2}{3\epsilon_0 r^3\sin\theta}\operatorname{Im}\left(\left(-\sqrt{\frac{3}{8\pi}}\right)^2 (p_x - jp_y)\sin\theta e^{j\phi}\right) \\
&= \frac{1}{4\pi\epsilon_0 r^3}\operatorname{Im}\left((p_x - jp_y)e^{j\phi}\right) \\
&= \frac{1}{4\pi\epsilon_0 r^3}\left(p_x\sin\phi - p_y\cos\phi\right) \\
&= -\frac{\mathbf{p}\cdot\hat{\boldsymbol{\phi}}}{4\pi\epsilon_0 r^3}.
\end{aligned} \tag{1.9}$$

That is

$$\boxed{\begin{aligned}
E_r &= \frac{2}{4\pi\epsilon_0 r^3}\mathbf{p}\cdot\hat{\mathbf{r}} \\
E_\theta &= -\frac{1}{4\pi\epsilon_0 r^3}\mathbf{p}\cdot\hat{\boldsymbol{\phi}} \\
E_\phi &= -\frac{1}{4\pi\epsilon_0 r^3}\mathbf{p}\cdot\hat{\boldsymbol{\phi}}.
\end{aligned}} \tag{1.10}$$

These are consistent with equations (4.12) from the text for when $\mathbf{p}$ is aligned with the z-axis.

Observe that we can sum each of the projections of $\mathbf{E}$ to construct the total electric field due to this $l = 1$ term of the multipole moment sum

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0 r^3}\left(2\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \hat{\boldsymbol{\phi}}(\mathbf{p}\cdot\hat{\boldsymbol{\phi}}) - \hat{\boldsymbol{\theta}}(\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\right) \\
&= \frac{1}{4\pi\epsilon_0 r^3}\left(3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}\right),
\end{aligned}
\tag{1.11}
$$

which recovers the expected dipole moment approximation.

Bibliography

[1] JD Jackson. *Classical Electrodynamics*. John Wiley and Sons, 2nd edition, 1975. 1