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ECE1228H Electromagnetic Theory. Lecture 10: Fresnel relations. Taught by Prof. M. Mojahedi

Motivation In class, an overview of the Fresnel relations for a TE mode electric field were presented. Here's a fleshing out of the details is presented, as well as the equivalent for the TM mode.

Single interface TE mode. The Fresnel reflection geometry for an electric field **E** parallel to the interface (TE mode) is sketched in fig. 1.1.

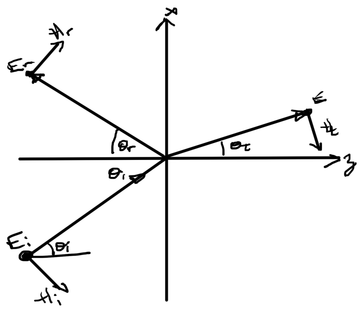

Figure 1.1: Electric field TE mode Fresnel geometry.

$$\boldsymbol{\mathcal{E}}_i = \mathbf{e}_2 E_i e^{j\omega t - j\mathbf{k}_i \cdot \mathbf{x}}, \tag{1.1}$$

with an assumption that this field maintains it's polarization in both its reflected and transmitted components, so that

$$\boldsymbol{\mathcal{E}}_r = \mathbf{e}_2 r E_i e^{j\omega t - j\mathbf{k}_r \cdot \mathbf{x}}, \tag{1.2}$$

and

$$\boldsymbol{\mathcal{E}}_t = \mathbf{e}_2 t E_i e^{j\omega t - j\mathbf{k}_t \cdot \mathbf{x}}, \tag{1.3}$$

Measuring the angles $\theta_i, \theta_r, \theta_t$ from the normal, with $i = \mathbf{e}_3\mathbf{e}_1$ the wave vectors are

$$\begin{aligned}
\mathbf{k}_i &= \mathbf{e}_3 k_1 e^{i\theta_i} = k_1 \left(\mathbf{e}_3 \cos\theta_i + \mathbf{e}_1 \sin\theta_i \right) \\
\mathbf{k}_r &= -\mathbf{e}_3 k_1 e^{-i\theta_r} = k_1 \left(-\mathbf{e}_3 \cos\theta_r + \mathbf{e}_1 \sin\theta_r \right) \\
\mathbf{k}_t &= \mathbf{e}_3 k_2 e^{i\theta_t} = k_2 \left(\mathbf{e}_3 \cos\theta_t + \mathbf{e}_1 \sin\theta_t \right)
\end{aligned} \tag{1.4}$$

So the time harmonic electric fields are

$$
\begin{aligned}
\mathbf{E}_i &= \mathbf{e}_2 E_i \exp\left(-jk_1\left(z\cos\theta_i + x\sin\theta_i\right)\right) \\
\mathbf{E}_r &= \mathbf{e}_2 r E_i \exp\left(-jk_1\left(-z\cos\theta_r + x\sin\theta_r\right)\right) \\
\mathbf{E}_t &= \mathbf{e}_2 t E_i \exp\left(-jk_2\left(z\cos\theta_t + x\sin\theta_t\right)\right).
\end{aligned} \tag{1.5}
$$

The magnetic fields follow from Faraday's law

$$
\begin{aligned}
\mathbf{H} &= \frac{1}{-j\omega\mu}\boldsymbol{\nabla}\times\mathbf{E} \\
&= \frac{1}{-j\omega\mu}\boldsymbol{\nabla}\times\mathbf{e}_2 e^{-j\mathbf{k}\cdot\mathbf{x}} \\
&= \frac{1}{j\omega\mu}\mathbf{e}_2\times\boldsymbol{\nabla} e^{-j\mathbf{k}\cdot\mathbf{x}} \\
&= -\frac{1}{\omega\mu}\mathbf{e}_2\times\mathbf{k} e^{-j\mathbf{k}\cdot\mathbf{x}} \\
&= \frac{1}{\omega\mu}\mathbf{k}\times\mathbf{E}
\end{aligned} \tag{1.6}
$$

We have

$$
\begin{aligned}
\hat{\mathbf{k}}_i\times\mathbf{e}_2 &= -\mathbf{e}_1\cos\theta_i + \mathbf{e}_3\sin\theta_i \\
\hat{\mathbf{k}}_r\times\mathbf{e}_2 &= \mathbf{e}_1\cos\theta_r + \mathbf{e}_3\sin\theta_r \\
\hat{\mathbf{k}}_t\times\mathbf{e}_2 &= -\mathbf{e}_1\cos\theta_t + \mathbf{e}_3\sin\theta_t,
\end{aligned} \tag{1.7}
$$

Note that

$$
\begin{aligned}
\frac{k}{\omega\mu} &= \frac{k}{kv\mu} \\
&= \frac{\sqrt{\mu\epsilon}}{\mu} \\
&= \sqrt{\frac{\epsilon}{\mu}} \\
&= \frac{1}{\eta}.
\end{aligned} \tag{1.8}
$$

so

$$
\begin{aligned}
\mathbf{H}_i &= \frac{E_i}{\eta_1}\left(-\mathbf{e}_1\cos\theta_i + \mathbf{e}_3\sin\theta_i\right)\exp\left(-jk_1\left(z\cos\theta_i + x\sin\theta_i\right)\right) \\
\mathbf{H}_r &= \frac{rE_i}{\eta_1}\left(\mathbf{e}_1\cos\theta_r + \mathbf{e}_3\sin\theta_r\right)\exp\left(-jk_1\left(-z\cos\theta_r + x\sin\theta_r\right)\right) \\
\mathbf{H}_t &= \frac{tE_i}{\eta_2}\left(-\mathbf{e}_1\cos\theta_t + \mathbf{e}_3\sin\theta_t\right)\exp\left(-jk_2\left(z\cos\theta_t + x\sin\theta_t\right)\right).
\end{aligned} \tag{1.9}
$$

The boundary conditions at $z = 0$ with $\hat{\mathbf{n}} = \mathbf{e}_3$ are

$$
\begin{aligned}
\hat{\mathbf{n}} \times \mathbf{H}_1 &= \hat{\mathbf{n}} \times \mathbf{H}_2 \\
\hat{\mathbf{n}} \cdot \mathbf{B}_1 &= \hat{\mathbf{n}} \cdot \mathbf{B}_2 \\
\hat{\mathbf{n}} \times \mathbf{E}_1 &= \hat{\mathbf{n}} \times \mathbf{E}_2 \\
\hat{\mathbf{n}} \cdot \mathbf{D}_1 &= \hat{\mathbf{n}} \cdot \mathbf{D}_2,
\end{aligned}
\tag{1.10}
$$

At $x = 0$, this is

$$
\begin{aligned}
-\frac{1}{\eta_1} \cos\theta_i + \frac{r}{\eta_1} \cos\theta_r &= -\frac{t}{\eta_2} \cos\theta_t \\
k_1 \sin\theta_i + k_1 r \sin\theta_r &= k_2 t \sin\theta_t \\
1 + r &= t
\end{aligned}
\tag{1.11}
$$

When $t = 0$ the latter two equations give Shell's first law

$$
\boxed{\sin\theta_i = \sin\theta_r.}
\tag{1.12}
$$

Assuming this holds for all r, t we have

$$
k_1 \sin\theta_i (1 + r) = k_2 t \sin\theta_t,
\tag{1.13}
$$

which is Snell's second law in disguise

$$
k_1 \sin\theta_i = k_2 \sin\theta_t.
\tag{1.14}
$$

With

$$
\begin{aligned}
k &= \frac{\omega}{v} \\
&= \frac{\omega}{c} \frac{c}{v} \\
&= \frac{\omega}{c} n,
\end{aligned}
\tag{1.15}
$$

so eq. (1.14) takes the form

$$
\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t.}
\tag{1.16}
$$

With

$$
\begin{aligned}
k_{1z} &= k_1 \cos\theta_i \\
k_{2z} &= k_2 \cos\theta_t,
\end{aligned}
\tag{1.17}
$$

we can solve for r, t by inverting

$$
\begin{bmatrix} \mu_2 k_{1z} & \mu_1 k_{2z} \\ -1 & 1 \end{bmatrix}
\begin{bmatrix} r \\ t \end{bmatrix}
=
\begin{bmatrix} \mu_2 k_{1z} \\ 1 \end{bmatrix},
\tag{1.18}
$$

which gives

$$
\begin{bmatrix} r \\ t \end{bmatrix}
=
\begin{bmatrix} 1 & -\mu_1 k_{2z} \\ 1 & \mu_2 k_{1z} \end{bmatrix}
\begin{bmatrix} \mu_2 k_{1z} \\ 1 \end{bmatrix},
\tag{1.19}
$$

or

$$\boxed{\begin{aligned} r &= \frac{\mu_2 k_{1z} - \mu_1 k_{2z}}{\mu_2 k_{1z} + \mu_1 k_{2z}} \\ t &= \frac{2\mu_2 k_{1z}}{\mu_2 k_{1z} + \mu_1 k_{2z}} \end{aligned}} \tag{1.20}$$

There are many ways that this can be written. Dividing both the numerator and denominator by $\mu_1 \mu_2 \omega/c$, and noting that $k = \omega n/c$, we have

$$\begin{aligned} r &= \frac{\frac{n_1}{\mu_1}\cos\theta_i - \frac{n_2}{\mu_2}\cos\theta_t}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t} \\ t &= \frac{2\frac{n_1}{\mu_1}\cos\theta_i}{\frac{n_1}{\mu_1}\cos\theta_i + \frac{n_2}{\mu_2}\cos\theta_t}, \end{aligned} \tag{1.21}$$

which checks against (4.32,4.33) in [1].

Single interface TM mode. For completeness, now consider the TM mode.
Faraday's law also can provide the electric field from the magnetic

$$\begin{aligned} \hat{\mathbf{k}} \times \mathbf{H} &= \eta \hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \mathbf{E} \right) \\ &= -\eta \hat{\mathbf{k}} \cdot \left(\hat{\mathbf{k}} \wedge \mathbf{E} \right) \\ &= -\eta \left(\mathbf{E} - \hat{\mathbf{k}} \left(\hat{\mathbf{k}} \cdot \mathbf{E} \right) \right) \\ &= -\eta \mathbf{E}. \end{aligned} \tag{1.22}$$

so

$$\mathbf{E} = \eta \mathbf{H} \times \hat{\mathbf{k}}. \tag{1.23}$$

So the magnetic and electric fields are

$$\begin{aligned} \mathbf{H}_i &= \mathbf{e}_2 \frac{E_i}{\eta_1} \exp\left(-jk_1 \left(z\cos\theta_i + x\sin\theta_i\right)\right) \\ \mathbf{H}_r &= \mathbf{e}_2 r \frac{E_i}{\eta_1} \exp\left(-jk_1 \left(-z\cos\theta_r + x\sin\theta_r\right)\right) \\ \mathbf{H}_t &= \mathbf{e}_2 t \frac{E_i}{\eta_2} \exp\left(-jk_2 \left(z\cos\theta_t + x\sin\theta_t\right)\right) \end{aligned} \tag{1.24a}$$

$$\begin{aligned} \mathbf{E}_i &= -E_i \left(-\mathbf{e}_1 \cos\theta_i + \mathbf{e}_3 \sin\theta_i\right) \exp\left(-jk_1 \left(z\cos\theta_i + x\sin\theta_i\right)\right) \\ \mathbf{E}_r &= -rE_i \left(\mathbf{e}_1 \cos\theta_r + \mathbf{e}_3 \sin\theta_r\right) \exp\left(-jk_1 \left(-z\cos\theta_r + x\sin\theta_r\right)\right) \\ \mathbf{E}_t &= -tE_i \left(-\mathbf{e}_1 \cos\theta_t + \mathbf{e}_3 \sin\theta_t\right) \exp\left(-jk_2 \left(z\cos\theta_t + x\sin\theta_t\right)\right). \end{aligned} \tag{1.24b}$$

Imposing the constraints eq. (1.10), at $x = z = 0$ we have

$$
\begin{aligned}
\frac{1}{\eta_1}(1+r) &= \frac{t}{\eta_2} \\
\cos\theta_i - r\cos\theta_r &= t\cos\theta_t \\
\epsilon_1\left(\sin\theta_i + r\sin\theta_r\right) &= t\epsilon_2\sin\theta_t
\end{aligned}
. \tag{1.25}
$$

At $t = 0$, the first and third of these give $\theta_i = \theta_r$. Assuming this incident and reflection angle equality holds for all values of t, we have

$$
\begin{aligned}
\sin\theta_i(1+r) &= t\frac{\epsilon_2}{\epsilon_1}\sin\theta_t \\
\sin\theta_i\frac{\eta_1}{\eta_2}t &=
\end{aligned}
\tag{1.26}
$$

or

$$
\epsilon_1\eta_1\sin\theta_i = \epsilon_2\eta_2\sin\theta_t. \tag{1.27}
$$

This is also Snell's second law eq. (1.16) in disguise, which can be seen by

$$
\begin{aligned}
\epsilon_1\eta_1 &= \epsilon_1\sqrt{\frac{\mu_1}{\epsilon_1}} \\
&= \sqrt{\epsilon_1\mu_1} \\
&= \frac{1}{v} \\
&= \frac{n}{c}.
\end{aligned}
\tag{1.28}
$$

The remaining equations in matrix form are

$$
\begin{bmatrix} \cos\theta_i & \cos\theta_t \\ -1 & \frac{\eta_1}{\eta_2} \end{bmatrix}
\begin{bmatrix} r \\ t \end{bmatrix}
=
\begin{bmatrix} \cos\theta_i \\ 1 \end{bmatrix},
\tag{1.29}
$$

the inverse of which is

$$
\begin{aligned}
\begin{bmatrix} r \\ t \end{bmatrix}
&= \frac{1}{\frac{\eta_1}{\eta_2}\cos\theta_i + \cos\theta_t}
\begin{bmatrix} \frac{\eta_1}{\eta_2} & -\cos\theta_t \\ 1 & \cos\theta_i \end{bmatrix}
\begin{bmatrix} \cos\theta_i \\ 1 \end{bmatrix} \\
&= \frac{1}{\frac{\eta_1}{\eta_2}\cos\theta_i + \cos\theta_t}
\begin{bmatrix} \frac{\eta_1}{\eta_2}\cos\theta_i - \cos\theta_t \\ 2\cos\theta_i \end{bmatrix},
\end{aligned}
\tag{1.30}
$$

or

$$
\boxed{
\begin{aligned}
r &= \frac{\eta_1\cos\theta_i - \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} \\
t &= \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}.
\end{aligned}
}
\tag{1.31}
$$

Multiplication of the numerator and denominator by $c/\eta_1\eta_2$, noting that $c/\eta = n/\mu$ gives

$$
\begin{aligned}
r &= \frac{\frac{n_2}{\mu_2}\cos\theta_i - \frac{n_1}{\mu_1}\cos\theta_t}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t} \\
t &= \frac{2\frac{n_1}{\mu_1}\cos\theta_i}{\frac{n_2}{\mu_2}\cos\theta_i + \frac{n_1}{\mu_1}\cos\theta_t}
\end{aligned}
\tag{1.32}
$$

which checks against (4.38,4.39) in [1].

Bibliography

[1] E. Hecht. *Optics*. 1998. 1, 1