Peeter Joot
peeterjoot@protonmail.com

Electric and magnetic fields at an interface

As pointed out in [1] the fields at an interface that is not a perfect conductor on either side are related by

$$\begin{aligned} \hat{\mathbf{n}} \cdot \left(\mathbf{D}_2 - \mathbf{D}_1 \right) &= \rho_{es} \\ \hat{\mathbf{n}} \times \left(\mathbf{E}_2 - \mathbf{E}_1 \right) &= -\mathbf{M}_s \\ \hat{\mathbf{n}} \cdot \left(\mathbf{B}_2 - \mathbf{B}_1 \right) &= \rho_{ms} \\ \hat{\mathbf{n}} \times \left(\mathbf{H}_2 - \mathbf{H}_1 \right) &= \mathbf{J}_s. \end{aligned} \tag{1.1}$$

Given the fields in medium 1, assuming that boths sets of media are linear, we can use these relationships to determine the fields in the other medium.

$$\begin{aligned} \hat{\mathbf{n}} \cdot \mathbf{E}_2 &= \frac{1}{\epsilon_2} \left(\epsilon_1 \hat{\mathbf{n}} \cdot \mathbf{E}_1 + \rho_{es} \right) \\ \hat{\mathbf{n}} \wedge \mathbf{E}_2 &= \hat{\mathbf{n}} \wedge \mathbf{E}_1 - I \mathbf{M}_s \\ \hat{\mathbf{n}} \cdot \mathbf{B}_2 &= \hat{\mathbf{n}} \cdot \mathbf{B}_1 + \rho_{ms} \\ \hat{\mathbf{n}} \wedge \mathbf{B}_2 &= \mu_2 \left(\frac{1}{\mu_1} \hat{\mathbf{n}} \wedge \mathbf{B}_1 + I \mathbf{J}_s \right). \end{aligned} \tag{1.2}$$

Now the fields in interface 2 can be obtained by adding the normal and tangential projections. For the electric field

$$\begin{aligned} \mathbf{E}_2 &= \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \mathbf{E}_2) + \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \wedge \mathbf{E}_2) \\ &= \frac{1}{\epsilon_2} \hat{\mathbf{n}} \left(\epsilon_1 \hat{\mathbf{n}} \cdot \mathbf{E}_1 + \rho_{es} \right) + \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \wedge \mathbf{E}_1 - I \mathbf{M}_s). \end{aligned} \tag{1.3}$$

Expanding $\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \wedge \mathbf{E}_1) = \mathbf{E}_1 - \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \mathbf{E}_1)$, and $\hat{\mathbf{n}} \cdot (I \mathbf{M}_s) = -\hat{\mathbf{n}} \times \mathbf{M}_s$, that is

$$\boxed{\mathbf{E}_2 = \mathbf{E}_1 + \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \mathbf{E}_1) \left(\frac{\epsilon_1}{\epsilon_2} - 1 \right) + \frac{\rho_{es}}{\epsilon_2} + \hat{\mathbf{n}} \times \mathbf{M}_s.} \tag{1.4}$$

For the magnetic field

$$\begin{aligned} \mathbf{B}_2 &= \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \mathbf{B}_2) + \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \wedge \mathbf{B}_2) \\ &= \hat{\mathbf{n}} \left(\hat{\mathbf{n}} \cdot \mathbf{B}_1 + \rho_{ms} \right) + \mu_2 \hat{\mathbf{n}} \cdot \left(\left(\frac{1}{\mu_1} \hat{\mathbf{n}} \wedge \mathbf{B}_1 + I \mathbf{J}_s \right) \right), \end{aligned} \tag{1.5}$$

which is

$$\boxed{\mathbf{B}_2 = \frac{\mu_2}{\mu_1}\mathbf{B}_1 + \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{B}_1)\left(1 - \frac{\mu_2}{\mu_1}\right) + \hat{\mathbf{n}}\rho_{ms} - \hat{\mathbf{n}} \times \mathbf{J}_s.} \tag{1.6}$$

These are kind of pretty results, having none of the explicit angle dependence that we see in the Fresnel relationships. In this analysis, it is assumed there is only a transmitted component of the ray in question, and no reflected component. Can we do a purely vectoral treatment of the Fresnel equations along these same lines?

Bibliography

[1] Constantine A Balanis. *Advanced engineering electromagnetics*. Wiley New York, 1989. 1