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Fresnel angular sum and difference formulas

Exercise 1.1 Fresnel sum and difference formulas. ([1] *pr. 4.39*)

Given a $\mu_1 = \mu_2$ constraint, show that the Fresnel equations have the form

$$r^{\text{TE}} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \tag{1.1a}$$

$$r^{\text{TM}} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \tag{1.1b}$$

$$t^{\text{TE}} = \frac{2 \sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)} \tag{1.1c}$$

$$t^{\text{TM}} = \sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t). \tag{1.1d}$$

Answer for Exercise 1.1

We need a couple trig identities to start with.

$$\begin{aligned}
\sin(a+b) &= \operatorname{Im}\left(e^{j(a+b)}\right) \\
&= \operatorname{Im}\left(e^{ja}e^{+jb}\right) \\
&= \operatorname{Im}\left((\cos a + j\sin a)(\cos b + j\sin b)\right) \\
&= \sin a\cos b + \cos a\sin b.
\end{aligned} \tag{1.2}$$

Allowing for both signs we have

$$\begin{aligned}
\sin(a+b) &= \sin a\cos b + \cos a\sin b \\
\sin(a-b) &= \sin a\cos b - \cos a\sin b.
\end{aligned} \tag{1.3}$$

The mixed sine and cosine product can be expressed as a sum of sines

$$2\sin a\cos b = \sin(a+b) + \sin(a-b). \tag{1.4}$$

With $2x = a + b, 2y = a - b$, or $a = x + y, b = x - y$, we find

$$\begin{aligned} 2\sin(x+y)\cos(x-y) &= \sin(2x) + \sin(2y) \\ 2\sin(x-y)\cos(x+y) &= \sin(2x) - \sin(2y). \end{aligned} \tag{1.5}$$

Returning to the problem. When $\mu_1 = \mu_2$ the Fresnel equations were found to be

$$\begin{aligned} r^{\mathrm{TE}} &= \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t} \\ r^{\mathrm{TM}} &= \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t} \\ t^{\mathrm{TE}} &= \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} \\ t^{\mathrm{TM}} &= \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}. \end{aligned} \tag{1.6}$$

Using Snell's law, one of n_1, n_2 can be eliminated, for example

$$n_1 = n_2 \frac{\sin\theta_t}{\sin\theta_i}. \tag{1.7}$$

Inserting this and proceeding with the application of the trig identities above, we have

$$\begin{aligned} r^{\mathrm{TE}} &= \frac{n_2\frac{\sin\theta_t}{\sin\theta_i}\cos\theta_i - n_2\cos\theta_t}{n_2\frac{\sin\theta_t}{\sin\theta_i}\cos\theta_i + n_2\cos\theta_t} \\ &= \frac{\sin\theta_t\cos\theta_i - \cos\theta_t\sin\theta_i}{\sin\theta_t\cos\theta_i + \cos\theta_t\sin\theta_i} \\ &= \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \end{aligned} \tag{1.8a}$$

$$\begin{aligned} r^{\mathrm{TM}} &= \frac{n_2\cos\theta_i - n_2\frac{\sin\theta_t}{\sin\theta_i}\cos\theta_t}{n_2\cos\theta_i + n_2\frac{\sin\theta_t}{\sin\theta_i}\cos\theta_t} \\ &= \frac{\sin\theta_i\cos\theta_i - \sin\theta_t\cos\theta_t}{\sin\theta_i\cos\theta_i + \sin\theta_t\cos\theta_t} \\ &= \frac{\frac{1}{2}\sin(2\theta_i) - \frac{1}{2}\sin(2\theta_t)}{\frac{1}{2}\sin(2\theta_i) + \frac{1}{2}\sin(2\theta_t)} \\ &= \frac{\sin(\theta_i - \theta_t)\cos(\theta_i + \theta_t)}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \\ &= \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \end{aligned} \tag{1.8b}$$

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\begin{aligned}
t^{\mathrm{TE}} &= \frac{2n_2 \frac{\sin\theta_t}{\sin\theta_i} \cos\theta_i}{n_2 \frac{\sin\theta_t}{\sin\theta_i} \cos\theta_i + n_2 \cos\theta_t} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\sin\theta_t \cos\theta_i + \cos\theta_t \sin\theta_i} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)}
\end{aligned} \tag{1.8c}
$$

$$
\begin{aligned}
t^{\mathrm{TM}} &= \frac{2n_2 \frac{\sin\theta_t}{\sin\theta_i} \cos\theta_i}{n_2 \cos\theta_i + n_2 \frac{\sin\theta_t}{\sin\theta_i} \cos\theta_t} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\sin\theta_i \cos\theta_i + \sin\theta_t \cos\theta_t} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\frac{1}{2}\sin(2\theta_i) + \frac{1}{2}\sin(2\theta_t)} \\
&= \frac{2\sin\theta_t \cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}
\end{aligned} \tag{1.8d}
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Bibliography

[1] E. Hecht. *Optics*. 1998. 1.1