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Helmholtz theorem

This is a problem from ece1228. I attempted solutions in a number of ways. One using Geometric Algebra, one devoid of that algebra, and then this method, which combined aspects of both. Of the three methods I tried to obtain this result, this is the most compact and elegant. It does however, require a fair bit of Geometric Algebra knowledge, including the Fundamental Theorem of Geometric Calculus, as detailed in [1], [3] and [2].

Exercise 1.1 Helmholtz theorem

Prove the first Helmholtz's theorem, i.e. if vector **M** is defined by its divergence

$$\boldsymbol{\nabla} \cdot \mathbf{M} = s \tag{1.1}$$

and its curl

$$\boldsymbol{\nabla} \times \mathbf{M} = \mathbf{C} \tag{1.2}$$

within a region and its normal component $\mathbf{M}_{\mathrm{n}}$ over the boundary, then **M** is uniquely specified.

Answer for Exercise 1.1

The gradient of the vector **M** can be written as a single even grade multivector

$$\boldsymbol{\nabla} \mathbf{M} = \boldsymbol{\nabla} \cdot \mathbf{M} + I \boldsymbol{\nabla} \times \mathbf{M} = s + I \mathbf{C}. \tag{1.3}$$

We will use this to attempt to discover the relation between the vector **M** and its divergence and curl. We can express **M** at the point of interest as a convolution with the delta function at all other points in space

$$\mathbf{M}(\mathbf{x}) = \int_V dV' \delta(\mathbf{x} - \mathbf{x}') \mathbf{M}(\mathbf{x}'). \tag{1.4}$$

The Laplacian representation of the delta function in $\mathbb{R}^3$ is

$$\delta(\mathbf{x} - \mathbf{x}') = -\frac{1}{4\pi} \boldsymbol{\nabla}^2 \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|}, \tag{1.5}$$

so **M** can be represented as the following convolution

$$\mathbf{M}(\mathbf{x}) = -\frac{1}{4\pi} \int_V dV' \boldsymbol{\nabla}^2 \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} \mathbf{M}(\mathbf{x}'). \tag{1.6}$$

Using this relation and proceeding with a few applications of the chain rule, plus the fact that $\boldsymbol{\nabla} 1/|\mathbf{x}-\mathbf{x}'| = -\boldsymbol{\nabla}' 1/|\mathbf{x}-\mathbf{x}'|$, we find

$$
\begin{aligned}
-4\pi\mathbf{M}(\mathbf{x}) &= \int_V dV' \boldsymbol{\nabla}^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{M}(\mathbf{x}') \\
&= \left\langle \int_V dV' \boldsymbol{\nabla}^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{M}(\mathbf{x}') \right\rangle_1 \\
&= -\left\langle \int_V dV' \boldsymbol{\nabla} \left(\boldsymbol{\nabla}' \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) \mathbf{M}(\mathbf{x}') \right\rangle_1 \\
&= -\left\langle \boldsymbol{\nabla} \int_V dV' \left(\boldsymbol{\nabla}' \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{\boldsymbol{\nabla}'\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right) \right\rangle_1 \\
&= -\left\langle \boldsymbol{\nabla} \int_{\partial V} dA' \hat{\mathbf{n}} \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right\rangle_1 + \left\langle \boldsymbol{\nabla} \int_V dV' \frac{s(\mathbf{x}') + I\mathbf{C}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right\rangle_1 \\
&= -\left\langle \boldsymbol{\nabla} \int_{\partial V} dA' \hat{\mathbf{n}} \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right\rangle_1 + \boldsymbol{\nabla} \int_V dV' \frac{s(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \boldsymbol{\nabla} \cdot \int_V dV' \frac{I\mathbf{C}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.
\end{aligned}
\tag{1.7}
$$

By inserting a no-op grade selection operation in the second step, the trivector terms that would show up in subsequent steps are automatically filtered out. This leaves us with a boundary term dependent on the surface and the normal and tangential components of $\mathbf{M}$. Added to that is a pair of volume integrals that provide the unique dependence of $\mathbf{M}$ on its divergence and curl. When the surface is taken to infinity, which requires $|\mathbf{M}|/|\mathbf{x}-\mathbf{x}'| \rightarrow 0$, then the dependence of $\mathbf{M}$ on its divergence and curl is unique.

In order to express final result in traditional vector algebra form, a couple transformations are required. The first is that

$$
\begin{aligned}
\left\langle \mathbf{a} I \mathbf{b} \right\rangle_1 &= I^2 \mathbf{a} \times \mathbf{b} \\
&= -\mathbf{a} \times \mathbf{b}.
\end{aligned}
\tag{1.8}
$$

For the grade selection in the boundary integral, note that

$$
\begin{aligned}
\left\langle \boldsymbol{\nabla} \hat{\mathbf{n}} \mathbf{X} \right\rangle_1 &= \left\langle \boldsymbol{\nabla} (\hat{\mathbf{n}} \cdot \mathbf{X}) \right\rangle_1 + \left\langle \boldsymbol{\nabla} (\hat{\mathbf{n}} \wedge \mathbf{X}) \right\rangle_1 \\
&= \boldsymbol{\nabla} (\hat{\mathbf{n}} \cdot \mathbf{X}) + \left\langle \boldsymbol{\nabla} I (\hat{\mathbf{n}} \times \mathbf{X}) \right\rangle_1 \\
&= \boldsymbol{\nabla} (\hat{\mathbf{n}} \cdot \mathbf{X}) - \boldsymbol{\nabla} \times (\hat{\mathbf{n}} \times \mathbf{X}).
\end{aligned}
\tag{1.9}
$$

These give

$$
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\begin{aligned}
\mathbf{M}(\mathbf{x}) = \boldsymbol{\nabla} \frac{1}{4\pi} \int_{\partial V} dA' \hat{\mathbf{n}} \cdot \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \boldsymbol{\nabla} \times \frac{1}{4\pi} \int_{\partial V} dA' \hat{\mathbf{n}} \times \frac{\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\
- \boldsymbol{\nabla} \frac{1}{4\pi} \int_V dV' \frac{s(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \boldsymbol{\nabla} \times \frac{1}{4\pi} \int_V dV' \frac{\mathbf{C}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.
\end{aligned}
}
\tag{1.10}
$$

Bibliography

[1] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, Cambridge, UK, 1st edition, 2003. 1

[2] A. Macdonald. *Vector and Geometric Calculus*. CreateSpace Independent Publishing Platform, 2012. 1

[3] Garret Sobczyk and Omar León Sánchez. Fundamental theorem of calculus. *Advances in Applied Clifford Algebras*, 21(1):221–231, 2011. URL http://arxiv.org/abs/0809.4526. 1