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Magnetic moment for a localized magnetostatic current

Motivation. I was once again reading my Jackson [2]. This time I found that his presentation of magnetic moment didn't really make sense to me. Here's my own pass through it, filling in a number of details. As I did last time, I'll also translate into SI units as I go.

Vector potential. The Biot-Savart expression for the magnetic field can be factored into a curl expression using the usual tricks

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|^3} d^3 x' \\ &= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \boldsymbol{\nabla} \frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} d^3 x' \\ &= \frac{\mu_0}{4\pi} \boldsymbol{\nabla} \times \int \frac{\mathbf{J}(\mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|} d^3 x',\end{aligned} \tag{1.1}$$

so the vector potential, through its curl, defines the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{J(\mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|} d^3 x'. \tag{1.2}$$

If the current source is localized (zero outside of some finite region), then there will always be a region for which $\left|\mathbf{x}\right| \gg \left|\mathbf{x}'\right|$, so the denominator yields to Taylor expansion

$$\begin{aligned}\frac{1}{\left|\mathbf{x} - \mathbf{x}'\right|} &= \frac{1}{\left|\mathbf{x}\right|} \left(1 + \frac{\left|\mathbf{x}'\right|^2}{\left|\mathbf{x}\right|^2} - 2 \frac{\mathbf{x} \cdot \mathbf{x}'}{\left|\mathbf{x}\right|^2} \right)^{-1/2} \\ &\approx \frac{1}{\left|\mathbf{x}\right|} \left(1 + \frac{\mathbf{x} \cdot \mathbf{x}'}{\left|\mathbf{x}\right|^2} \right) \\ &= \frac{1}{\left|\mathbf{x}\right|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{\left|\mathbf{x}\right|^3}.\end{aligned} \tag{1.3}$$

so the vector potential, far enough away from the current source is

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{J(\mathbf{x}')}{\left|\mathbf{x}\right|} d^3 x' + \frac{\mu_0}{4\pi} \int \frac{(\mathbf{x} \cdot \mathbf{x}') J(\mathbf{x}')}{\left|\mathbf{x}\right|^3} d^3 x'. \tag{1.4}$$

Jackson uses a sneaky trick to show that the first integral is killed for a localized source. That trick appears to be based on evaluating the following divergence

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\mathbf{J}(\mathbf{x}) x_i) &= (\boldsymbol{\nabla} \cdot \mathbf{J}) x_i + (\boldsymbol{\nabla} x_i) \cdot \mathbf{J} \\
&= (\mathbf{e}_k \partial_k x_i) \cdot \mathbf{J} \\
&= \delta_{ki} J_k \\
&= J_i.
\end{aligned}
\tag{1.5}
$$

Note that this made use of the fact that $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ for magnetostatics. This provides a way to rewrite the current density as a divergence

$$
\begin{aligned}
\int \frac{J(\mathbf{x}')}{|\mathbf{x}|} d^3 x' &= \mathbf{e}_i \int \frac{\boldsymbol{\nabla}' \cdot (x_i' \mathbf{J}(\mathbf{x}'))}{|\mathbf{x}|} d^3 x' \\
&= \frac{\mathbf{e}_i}{|\mathbf{x}|} \int \boldsymbol{\nabla}' \cdot (x_i' \mathbf{J}(\mathbf{x}')) d^3 x' \\
&= \frac{1}{|\mathbf{x}|} \oint \mathbf{x}' (d\mathbf{a} \cdot \mathbf{J}(\mathbf{x}')).
\end{aligned}
\tag{1.6}
$$

When $\mathbf{J}$ is localized, this is zero provided we pick the integration surface for the volume outside of that localization region.

It is now desired to rewrite $\int \mathbf{x} \cdot \mathbf{x}' \mathbf{J}$ as a triple cross product since the dot product of such a triple cross product has exactly this term in it

$$
\begin{aligned}
-\mathbf{x} \times \int \mathbf{x}' \times \mathbf{J} &= \int (\mathbf{x} \cdot \mathbf{x}') \mathbf{J} - \int (\mathbf{x} \cdot \mathbf{J}) \mathbf{x}' \\
&= \int (\mathbf{x} \cdot \mathbf{x}') \mathbf{J} - \mathbf{e}_k x_i \int J_i x_k',
\end{aligned}
\tag{1.7}
$$

so

$$
\int (\mathbf{x} \cdot \mathbf{x}') \mathbf{J} = -\mathbf{x} \times \int \mathbf{x}' \times \mathbf{J} + \mathbf{e}_k x_i \int J_i x_k'.
\tag{1.8}
$$

To get of this second term, the next sneaky trick is to consider the following divergence

$$
\begin{aligned}
\oint d\mathbf{a}' \cdot (\mathbf{J}(\mathbf{x}') x_i' x_j') &= \int dV' \boldsymbol{\nabla}' \cdot (\mathbf{J}(\mathbf{x}') x_i' x_j') \\
&= \int dV' (\boldsymbol{\nabla}' \cdot \mathbf{J}) + \int dV' \mathbf{J} \cdot \boldsymbol{\nabla}' (x_i' x_j') \\
&= \int dV' J_k \cdot \left(x_i' \partial_k x_j' + x_j' \partial_k x_i' \right) \\
&= \int dV' J_k x_i' \delta_{kj} + J_k x_j' \delta_{ki} \\
&= \int dV' J_j x_i' + J_i x_j'.
\end{aligned}
\tag{1.9}
$$

The surface integral is once again zero, which means that we have an antisymmetric relationship in integrals of the form

$$\int J_j x'_i = -\int J_i x'_j. \tag{1.10}$$

Now we can use the tensor algebra trick of writing $y = (y + y)/2$,

$$\begin{aligned}
\int (\mathbf{x}\cdot\mathbf{x}')\mathbf{J} &= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} + \mathbf{e}_k x_i \int J_i x'_k \\
&= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} + \frac{1}{2}\mathbf{e}_k x_i \int \left(J_i x'_k + J_i x'_k\right) \\
&= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} + \frac{1}{2}\mathbf{e}_k x_i \int \left(J_i x'_k - J_k x'_i\right) \\
&= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} + \frac{1}{2}\mathbf{e}_k x_i \int (\mathbf{J}\times\mathbf{x}')_j \epsilon_{ikj} \\
&= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} - \frac{1}{2}\epsilon_{kij}\mathbf{e}_k x_i \int (\mathbf{J}\times\mathbf{x}')_j \\
&= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} - \frac{1}{2}\mathbf{x}\times\int \mathbf{J}\times\mathbf{x}' \\
&= -\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} + \frac{1}{2}\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J} \\
&= -\frac{1}{2}\mathbf{x}\times\int \mathbf{x}'\times\mathbf{J},
\end{aligned} \tag{1.11}$$

so

$$\mathbf{A}(\mathbf{x}) \approx \frac{\mu_0}{4\pi|\mathbf{x}|^3}\left(-\frac{\mathbf{x}}{2}\right)\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x'. \tag{1.12}$$

Letting

$$\boxed{\mathbf{m} = \frac{1}{2}\int \mathbf{x}'\times\mathbf{J}(\mathbf{x}')d^3x',} \tag{1.13}$$

the far field approximation of the vector potential is

$$\boxed{\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}.} \tag{1.14}$$

Note that when the current is restricted to an infintisimally thin loop, the magnetic moment reduces to

$$\mathbf{m}(\mathbf{x}) = \frac{I}{2}\int \mathbf{x}\times d\mathbf{l}'. \tag{1.15}$$

Refering to [1] (pr. 1.60), this can be seen to be I times the "vector-area" integral.

Bibliography

[1] David Jeffrey Griffiths and Reed College. *Introduction to electrodynamics*. Prentice hall Upper Saddle River, NJ, 3rd edition, 1999. 1

[2] JD Jackson. *Classical Electrodynamics*. John Wiley and Sons, 2nd edition, 1975. 1