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Poynting relationship

Exercise 1.1 Poynting theorem.

Given

$$\boldsymbol{\nabla} \times \mathbf{E} = -\mathbf{M}_i - \frac{\partial \mathbf{B}}{\partial t}, \tag{1.1}$$

and

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_i + \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}, \tag{1.2}$$

expand the divergence of $\mathbf{E} \times \mathbf{H}$ to find the form of the Poynting theorem.

Answer for Exercise 1.1

First we need the chain rule for of this sort of divergence. Using primes to indicate the scope of the gradient operation

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) &= \boldsymbol{\nabla}' \cdot \left(\mathbf{E}' \times \mathbf{H}\right) - \boldsymbol{\nabla}' \cdot \left(\mathbf{H}' \times \mathbf{E}\right) \\
&= \mathbf{H} \cdot \left(\boldsymbol{\nabla}' \times \mathbf{E}'\right) - \mathbf{H} \cdot \left(\boldsymbol{\nabla}' \times \mathbf{H}'\right) \\
&= \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}).
\end{aligned} \tag{1.3}$$

In the second step, cyclic permutation of the triple product was used. This checks against the inside front cover of Jackson [1]. Now we can plug in the Maxwell equation cross products.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot \left(-\mathbf{M}_i - \frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \cdot \left(\mathbf{J}_i + \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t}\right) \\
&= -\mathbf{H} \cdot \mathbf{M}_i - \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \mathbf{E} \cdot \mathbf{J}_i - \mathbf{E} \cdot \mathbf{J}_c - \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t},
\end{aligned} \tag{1.4}$$

or

$$\boxed{0 = \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) + \frac{\epsilon}{2} \frac{\partial}{\partial t} |\mathbf{E}|^2 + \frac{\mu}{2} \frac{\partial}{\partial t} |\mathbf{H}|^2 + \mathbf{H} \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i + \sigma |\mathbf{E}|^2.} \tag{1.5}$$

Bibliography

[1] JD Jackson. *Classical Electrodynamics*. John Wiley and Sons, 2nd edition, 1975. 1