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Hamiltonian for a scalar field

Exercise 1.1

In [1] it is left as an exersize to expand the scalar field Hamiltonian in terms of the raising and lowering operators. Let's do that.

Answer for Exercise 1.1

The field operator expanded in terms of the raising and lowering operators is

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right). \tag{1.1}$$

Note that x and k here are both four-vectors, so this field is dependent on a spacetime point, but the integration is over a spatial volume.

The Hamiltonian in terms of the fields was

$$H = \frac{1}{2}\int d^3 x \left(\Pi^2 + \left(\boldsymbol{\nabla}\phi\right)^2 + \mu^2\phi^2 \right). \tag{1.2}$$

The field derivatives are

$$\begin{aligned}
\Pi &= \partial_0 \phi \\
&= \partial_0 \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \right) \\
&= i \int \frac{d^3 k}{(2\pi)^{3/2}\frac{\omega_k}{2\omega_k}} \left(-a_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \right),
\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
\partial_n \phi &= \partial_n \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \right) \\
&= i \int \frac{d^3 k k^n}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} - a_k^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}} \right).
\end{aligned} \tag{1.4}$$

Introducing a second set of momentum variables with $j = |\mathbf{j}|$, the momentum portion of the Hamiltonian is

$$\begin{aligned}
\frac{1}{2}\int d^3x\Pi^2 &= -\frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x\int d^3jd^3k\frac{1}{\sqrt{4\omega_j\omega_k}}\omega_j\omega_k\left(-a_je^{-i\omega_jt+i\mathbf{j}\cdot\mathbf{x}}+a_j^\dagger e^{i\omega_jt-i\mathbf{j}\cdot\mathbf{x}}\right)\\
&\qquad\left(-a_ke^{-i\omega_kt+i\mathbf{k}\cdot\mathbf{x}}+a_k^\dagger e^{i\omega_kt-i\mathbf{k}\cdot\mathbf{x}}\right)\\
&= -\frac{1}{4}\frac{1}{(2\pi)^3}\int d^3x\int d^3jd^3k\sqrt{\omega_j\omega_k}\Big(\\
&\qquad a_j^\dagger a_k^\dagger e^{i(\omega_k+\omega_j)t-i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}}+a_ja_ke^{-i(\omega_j+\omega_k)t+i(\mathbf{j}+\mathbf{k})\cdot\mathbf{x}}\\
&\qquad - a_j^\dagger a_ke^{-i(\omega_k-\omega_j)t-i(\mathbf{j}-\mathbf{k})\cdot\mathbf{x}} - a_ja_k^\dagger e^{-i(\omega_j-\omega_k)t-i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}}\Big)\\
&= -\frac{1}{4}\int d^3jd^3k\sqrt{\omega_j\omega_k}\Big(\\
&\qquad a_j^\dagger a_k^\dagger e^{i(\omega_k+\omega_j)t}\delta^3(\mathbf{k}+\mathbf{j})+a_ja_ke^{-i(\omega_j+\omega_k)t}\delta^3(-\mathbf{j}-\mathbf{k})\\
&\qquad - a_j^\dagger a_ke^{-i(\omega_k-\omega_j)t}\delta^3(\mathbf{j}-\mathbf{k}) - a_ja_k^\dagger e^{-i(\omega_j-\omega_k)t}\delta^3(\mathbf{k}-\mathbf{j})\Big)\\
&= -\frac{1}{4}\int d^3k\omega_k\left(a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt}+a_{-k}a_ke^{-2i\omega_kt}-a_k^\dagger a_k-a_ka_k^\dagger\right).
\end{aligned} \tag{1.5}$$

For the gradient portion of the Hamiltonian we have

$$\begin{aligned}
\frac{1}{2}\int d^3x\left(\boldsymbol{\nabla}\phi\right)^2 &= -\frac{1}{2}\frac{1}{(2\pi)^3}\int d^3x\int d^3jd^3k\frac{1}{\sqrt{4\omega_j\omega_k}}\left(\sum_{n=1}^{3}j^nk^n\right)\left(a_je^{-i\omega_jt+i\mathbf{j}\cdot\mathbf{x}}-a_j^\dagger e^{i\omega_jt-i\mathbf{j}\cdot\mathbf{x}}\right)\\
&\qquad\left(a_ke^{-i\omega_kt+i\mathbf{k}\cdot\mathbf{x}}-a_k^\dagger e^{i\omega_kt-i\mathbf{k}\cdot\mathbf{x}}\right)\\
&= -\frac{1}{4}\int d^3jd^3k\frac{\mathbf{j}\cdot\mathbf{k}}{\sqrt{\omega_j\omega_k}}\Big(\\
&\qquad a_j^\dagger a_k^\dagger e^{i(\omega_k+\omega_j)t}\delta^3(\mathbf{k}+\mathbf{j})+a_ja_ke^{-i(\omega_j+\omega_k)t}\delta^3(-\mathbf{j}-\mathbf{k})\\
&\qquad - a_j^\dagger a_ke^{-i(\omega_k-\omega_j)t}\delta^3(\mathbf{j}-\mathbf{k}) - a_ja_k^\dagger e^{-i(\omega_j-\omega_k)t}\delta^3(\mathbf{k}-\mathbf{j})\Big)\\
&= -\frac{1}{4}\int d^3k\frac{\mathbf{k}^2}{\omega_k}\Big(\\
&\qquad - a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt} - a_{-k}a_ke^{-2i\omega_kt}\\
&\qquad - a_k^\dagger a_k - a_ka_k^\dagger\Big).
\end{aligned} \tag{1.6}$$

Finally, for the mass term, we have

$$
\begin{aligned}
\frac{1}{2}\int d^3x\mu^2\phi^2 = &\frac{\mu^2}{2}\frac{1}{(2\pi)^3}\int d^3x\int d^3jd^3k\frac{1}{\sqrt{4\omega_j\omega_k}}\left(a_je^{-i\omega_jt+i\mathbf{j}\cdot\mathbf{x}}+a_j^\dagger e^{i\omega_jt-i\mathbf{j}\cdot\mathbf{x}}\right)\\
&\left(a_ke^{-i\omega_kt+i\mathbf{k}\cdot\mathbf{x}}+a_k^\dagger e^{i\omega_kt-i\mathbf{k}\cdot\mathbf{x}}\right)\\
=&\frac{\mu^2}{2}\frac{1}{(2\pi)^3}\int d^3x\int d^3jd^3k\frac{1}{\sqrt{4\omega_j\omega_k}}\Big(\\
&a_ja_ke^{-i(\omega_k+\omega_j)t+i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}}+a_j^\dagger a_k^\dagger e^{i(\omega_j+\omega_k)t-i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}}\\
&+a_ja_k^\dagger e^{i(\omega_k-\omega_j)t-i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}}+a_j^\dagger a_ke^{-i(\omega_k+\omega_j)t-i(\mathbf{j}-\mathbf{k})\cdot\mathbf{x}}\Big)\\
=&\frac{\mu^2}{2}\int d^3jd^3k\frac{1}{\sqrt{4\omega_j\omega_k}}\Big(\\
&a_ja_ke^{-i(\omega_k+\omega_j)t}\delta^3(-\mathbf{k}-\mathbf{j})+a_j^\dagger a_k^\dagger e^{i(\omega_j+\omega_k)t}\delta^3(\mathbf{k}+\mathbf{j})\\
&+a_ja_k^\dagger e^{i(\omega_k-\omega_j)t}\delta^3(\mathbf{k}-\mathbf{j})+a_j^\dagger a_ke^{-i(\omega_k+\omega_j)t}\delta^3(\mathbf{j}-\mathbf{k})\Big)\\
=&\frac{\mu^2}{4}\int d^3k\frac{1}{\omega_k}\left(a_{-k}a_ke^{-2i\omega_kt}+a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt}+a_ka_k^\dagger+a_k^\dagger a_k\right).
\end{aligned}
\tag{1.7}
$$

Now all the pieces can be put back together again

$$
\begin{aligned}
H = &\frac{1}{4}\int d^3k\frac{1}{\omega_k}\Big(\\
&-\omega_k^2\left(a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt}+a_{-k}a_ke^{-2i\omega_kt}-a_k^\dagger a_k-a_ka_k^\dagger\right)\\
&+\mathbf{k}^2\left(a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt}+a_{-k}a_ke^{-2i\omega_kt}+a_k^\dagger a_k+a_ka_k^\dagger\right)\\
&+\mu^2\left(a_{-k}a_ke^{-2i\omega_kt}+a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt}+a_ka_k^\dagger+a_k^\dagger a_k\right)\Big)\\
=&\frac{1}{4}\int d^3k\frac{1}{\omega_k}\Big(a_{-k}^\dagger a_k^\dagger e^{2i\omega_kt}\left(-\omega_k^2+\mathbf{k}^2+\mu^2\right)\\
&+a_{-k}a_ke^{-2i\omega_kt}\left(-\omega_k^2+\mathbf{k}^2+\mu^2\right)\\
&+a_ka_k^\dagger\left(\omega_k^2+\mathbf{k}^2+\mu^2\right)\\
&+a_k^\dagger a_k\left(\omega_k^2+\mathbf{k}^2+\mu^2\right)\Big).
\end{aligned}
\tag{1.8}
$$

With $\omega_k^2=\mathbf{k}^2+\mu^2$, the time dependent terms are killed leaving

$$
H=\frac{1}{2}\int d^3k\omega_k\left(a_ka_k^\dagger+a_k^\dagger a_k\right). \tag{1.9}
$$

Bibliography

[1] Michael Luke. *PHY2403F Lecture Notes: Quantum Field Theory*, 2015. URL https://s3.amazonaws.com/piazza-resources/i87nj8g7yie7nh/ihdwuk7wva13qq/lecturenotes.pdf?AWSAccessKeyId=AKIAIEDNRLJ4AZKBW6HA&Expires=1451803428&Signature=IF6qOjlKqOYL01FwqT%2FGV6BSDb8%3D. [Online; accessed 02-Jan-2016]. 1.1