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ECE1236H Microwave and Millimeter-Wave Techniques: Transmission lines. Taught by Prof. G.V. Eleftheriades

Disclaimer Peeter's lecture notes from class. These may be incoherent and rough.

These are notes for the UofT course ECE1236H, Microwave and Millimeter-Wave Techniques, taught by Prof. G.V. Eleftheriades, covering ch. 2 [1] content.

1.1 Requirements

A transmission line requires two conductors as sketched in fig. 1.1, which shows a 2-wire line such a telephone line, a coaxial cable as found in cable TV distribution, and a microstrip line as found in cell phone RF interconnects.

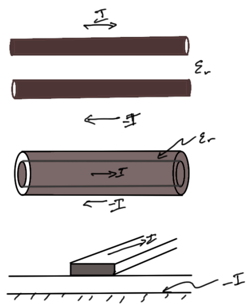

Figure 1.1: Transmission line examples.

A two-wire line becomes a transmission line when the wavelength of operation becomes comparable to the size of the line (or higher spectral component for pulses). In general a transmission line much support (TEM) transverse electromagnetic modes.

1.2 Time harmonic solutions on transmission lines

In fig. 1.2, an electronic representation of a transmission line circuit is sketched.

In this circuit all the elements have per-unit length units. With $I = C dV/dt \sim j\omega CV$, $v = IR$, and $V = L dI/dt \sim j\omega LI$, the KVL equation is

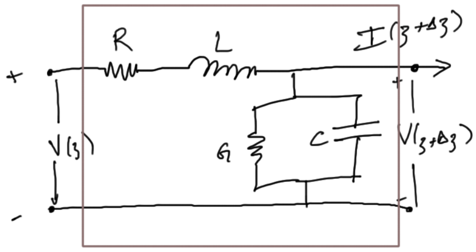

Figure 1.2: Transmission line equivalent circuit.

$$V(z) - V(z+\Delta z) = I(z)\Delta z\left(R + j\omega L\right), \tag{1.1}$$

or in the $\Delta z \to 0$ limit

$$\frac{\partial V}{\partial z} = -I(z)\left(R + j\omega L\right). \tag{1.2}$$

The KCL equation at the interior node is

$$-I(z) + I(z+\Delta z) + \left(j\omega C + G\right)V(z+\Delta z) = 0, \tag{1.3}$$

or

$$\frac{\partial I}{\partial z} = -V(z)\left(j\omega C + G\right). \tag{1.4}$$

This pair of equations is known as the telegrapher's equations

$$\boxed{\begin{aligned}\frac{\partial V}{\partial z} &= -I(z)\left(R + j\omega L\right)\\ \frac{\partial I}{\partial z} &= -V(z)\left(j\omega C + G\right).\end{aligned}} \tag{1.5}$$

The second derivatives are

$$\begin{aligned}\frac{\partial^2 V}{\partial z^2} &= -\frac{\partial I}{\partial z}\left(R + j\omega L\right)\\ \frac{\partial^2 I}{\partial z^2} &= -\frac{\partial V}{\partial z}\left(j\omega C + G\right),\end{aligned} \tag{1.6}$$

which allow the V, I to be decoupled

$$\boxed{\begin{aligned}\frac{\partial^2 V}{\partial z^2} &= V(z)\left(j\omega C + G\right)\left(R + j\omega L\right)\\ \frac{\partial^2 I}{\partial z^2} &= I(z)\left(R + j\omega L\right)\left(j\omega C + G\right),\end{aligned}} \tag{1.7}$$

With a complex propagation constant

$$\begin{aligned} \gamma &= \alpha + j\beta \\ &= \sqrt{\left(j\omega C + G\right)\left(R + j\omega L\right)} \\ &= \sqrt{RG - \omega^2 LC + j\omega(LG + RC)}, \end{aligned} \tag{1.8}$$

the decouple equations have the structure of a wave equation for a lossy line in the frequency domain

$$\boxed{\begin{aligned} \frac{\partial^2 V}{\partial z^2} - \gamma^2 V = 0 \\ \frac{\partial^2 I}{\partial z^2} - \gamma^2 I = 0. \end{aligned}} \tag{1.9}$$

We write the solutions to these equations as

$$\begin{aligned} V(z) &= V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \\ I(z) &= I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z} \end{aligned} \tag{1.10}$$

Only one of V or I is required since they are dependent through eq. (1.5), as can be seen by taking derivatives

$$\begin{aligned} \frac{\partial V}{\partial z} &= \gamma\left(-V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}\right) \\ &= -I(z)\left(R + j\omega L\right), \end{aligned} \tag{1.11}$$

so

$$I(z) = \frac{\gamma}{R + j\omega L}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right). \tag{1.12}$$

Introducing the characteristic impedance Z_0 of the line

$$\begin{aligned} Z_0 &= \frac{R + j\omega L}{\gamma} \\ &= \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \end{aligned} \tag{1.13}$$

we have

$$\begin{aligned} I(z) &= \frac{1}{Z_0}\left(V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right) \\ &= I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z}, \end{aligned} \tag{1.14}$$

where

$$\begin{aligned} I_0^+ &= \frac{V_0^+}{Z_0} \\ I_0^- &= \frac{V_0^-}{Z_0}. \end{aligned} \tag{1.15}$$

1.3 Mapping TL geometry to per unit length C and L elements

Example 1.1: Coaxial cable.

From electrostatics and magnetostatics the per unit length induction and capacitance constants for a co-axial cable can be calculated. For the cylindrical configuration sketched in fig. 1.3

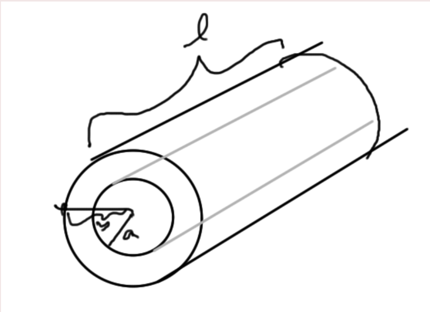

Figure 1.3: Coaxial cable.

From Gauss' law the total charge can be calculated assuming that the ends of the cable can be neglected

$$\begin{aligned} Q &= \int \boldsymbol{\nabla} \cdot \mathbf{D} dV \\ &= \oint \mathbf{D} \cdot d\mathbf{A} \\ &= \epsilon_0 \epsilon_r E (2\pi r) l, \end{aligned} \tag{1.16}$$

This provides the radial electric field magnitude, in terms of the total charge

$$E = \frac{Q/l}{\epsilon_0 \epsilon_r (2\pi r)}, \tag{1.17}$$

which must be a radial field as sketched in fig. 1.4.

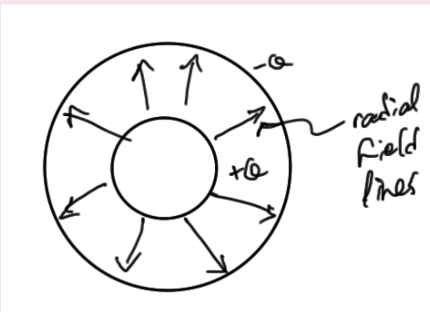

Figure 1.4: Radial electric field for coaxial cable.

The potential difference from the inner transmission surface to the outer is

$$\begin{aligned} V &= \int_a^b E dr \\ &= \frac{Q/l}{2\pi\epsilon_0\epsilon_r}\int_a^b \frac{dr}{r} \\ &= \frac{Q/l}{2\pi\epsilon_0\epsilon_r}\ln\frac{b}{a}. \end{aligned} \tag{1.18}$$

Therefore the capacitance per unit length is

$$C = \frac{Q/l}{V} = \frac{2\pi\epsilon_0\epsilon_r}{\ln\frac{b}{a}}. \tag{1.19}$$

The inductance per unit length can be calculated form Ampere's law

$$\begin{aligned} \int (\boldsymbol{\nabla}\times\mathbf{H})\cdot d\mathbf{S} &= \int \mathbf{J}\cdot d\mathbf{S} + \frac{\partial}{\partial t}\int \cancel{\mathbf{D}\cdot d\mathbf{l}} \\ &= I \\ &= \oint \mathbf{H}\cdot d\mathbf{l} \\ &= H(2\pi r) \\ &= \frac{B}{\mu_0}(2\pi r) \end{aligned} \tag{1.20}$$

The flux is

$$\begin{aligned} \Phi &= \int \mathbf{B}\cdot d\mathbf{A} \\ &= \frac{\mu_0 I}{2\pi}\int_A \frac{1}{r} d dr \\ &= \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{r} l d dr \\ &= \frac{\mu_0 I l}{2\pi}\ln\frac{b}{a}. \end{aligned} \tag{1.21}$$

The inductance per unit length is

$$L = \frac{\Phi/l}{I} = \frac{\mu_0}{2\pi}\ln\frac{b}{a}. \tag{1.22}$$

For a lossless line where $R = G = 0$, we have $\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC}$, so the phase velocity for a (lossless) coaxial cable is

$$
\begin{aligned}
v_\phi &= \frac{\omega}{\beta} \\
&= \frac{\omega}{\mathrm{Im}(\gamma)} \\
&= \frac{\omega}{\omega\sqrt{LC})} \\
&= \frac{1}{\sqrt{LC})}.
\end{aligned}
\tag{1.23}
$$

This gives

$$
\begin{aligned}
v_\phi^2 &= \frac{1}{L}\frac{1}{C} \\
&= \frac{2\pi}{\mu_0 \ln \frac{b}{a}} \frac{\ln \frac{b}{a}}{2\pi\epsilon_0\epsilon_r} \\
&= \frac{1}{\mu_0\epsilon_0\epsilon_r} \\
&= \frac{1}{\mu_0\epsilon}.
\end{aligned}
\tag{1.24}
$$

So

$$
v_\phi = \frac{1}{\sqrt{\epsilon\mu_0}},
\tag{1.25}
$$

which is the speed of light in the medium (ϵ_r) that fills the co-axial cable.

This is not a coincidence. In any two-wire homogeneously filled transmission line, the phase velocity is equal to the speed of light in the unbounded medium that fills the line.

The characteristic impedance (again assuming the lossless $R = G = 0$ case) is

$$
\begin{aligned}
Z_0 &= \sqrt{\frac{\cancel{R} + j\omega L}{\cancel{G} + j\omega C}} \\
&= \sqrt{\frac{L}{C}} \\
&= \sqrt{\frac{\mu_0}{2\pi} \ln \frac{b}{a} \frac{\ln \frac{b}{a}}{2\pi\epsilon_0\epsilon_r}} \\
&= \sqrt{\frac{\mu_0}{\epsilon}} \frac{\ln \frac{b}{a}}{2\pi}.
\end{aligned}
\tag{1.26}
$$

Note that $\eta = \sqrt{\mu_0/\epsilon_0} = 120\pi\Omega$ is the intrinsic impedance of free space. The values a, b in eq. (1.26) can be used to tune the characteristic impedance of the transmission line.

1.4 Lossless line.

The lossless lossless case where $R = G = 0$ was considered above. The results were

$$\gamma = j\omega\sqrt{LC}, \tag{1.27}$$

so $\alpha = 0$ and $\beta = \omega\sqrt{LC}$, and the phase velocity was

$$v_\phi = \frac{1}{\sqrt{LC}}, \tag{1.28}$$

the characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}}, \tag{1.29}$$

and the signals are

$$\begin{aligned} V(z) &= V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ I(z) &= \frac{1}{Z_0}\left(V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z}\right) \end{aligned} \tag{1.30}$$

In the time domain for an infinite line, we have

$$\begin{aligned} v(z,t) &= \text{Re}\left(V(z)e^{j\omega t}\right) \\ &= V_0^+ \,\text{Re}\left(e^{-j\beta z}e^{j\omega t}\right) \\ &= V_0^+ \cos(\omega t - \beta z). \end{aligned} \tag{1.31}$$

In this case the shape and amplitude of the waveform are preserved as sketched in fig. 1.5.

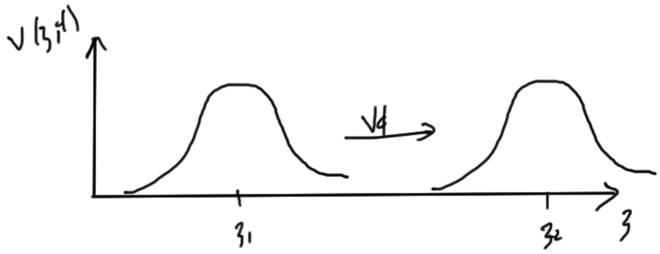

Figure 1.5: Lossless line signal preservation.

1.5 Low loss line.

Assume $R \ll \omega L$ and $G \ll \omega C$. In this case we have

$$\begin{aligned}
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
&= j\omega\sqrt{LC}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\
&\approx j\omega\sqrt{LC}\left(1 + \frac{R}{2j\omega L}\right)\left(1 + \frac{G}{2j\omega C}\right) \\
&\approx j\omega\sqrt{LC}\left(1 + \frac{R}{2j\omega L} + \frac{G}{2j\omega C}\right) \\
&= j\omega\sqrt{LC} + j\omega\frac{R\sqrt{C/L}}{2j\omega} + j\omega\frac{G\sqrt{L/C}}{2j\omega} \\
&= j\omega\sqrt{LC} + \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right),
\end{aligned} \tag{1.32}$$

so

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) \\
\beta &= \omega\sqrt{LC}.
\end{aligned} \tag{1.33}$$

Observe that this value for β is the same as the lossless case to first order. We also have

$$\begin{aligned}
Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\
&\approx \sqrt{\frac{L}{C}},
\end{aligned} \tag{1.34}$$

also the same as the lossless case. We must also have $v_\phi = 1/\sqrt{LC}$. To consider a time domain signal note that

$$\begin{aligned}
V(z) &= V_0^+ e^{-\gamma z} \\
&= V_0^+ e^{-\alpha z} e^{-j\beta z},
\end{aligned} \tag{1.35}$$

so

$$\begin{aligned}
v(z,t) &= \text{Re}\left(V(z)e^{j\omega t}\right) \\
&= \text{Re}\left(V_0^+ e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\right) \\
&= V_0^+ e^{-\alpha z}\cos(\omega t - \beta z).
\end{aligned} \tag{1.36}$$

The phase factor can be written

$$\omega t - \beta z = \omega \left(t - \frac{\beta}{\omega} z \right) \omega \left(t - z/v_\phi \right), \tag{1.37}$$

so the signal still moves with the phase velocity $v_\phi = 1/\sqrt{LC}$, but in a diminishing envelope as sketched in fig. 1.6.

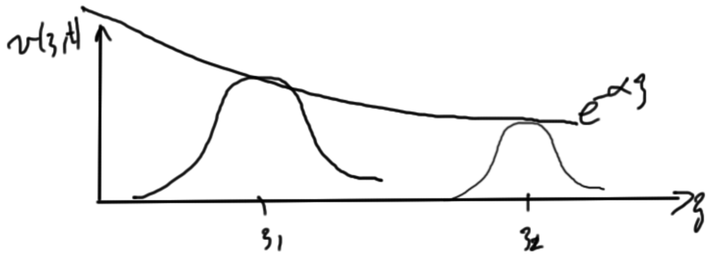

Figure 1.6: Time domain envelope for loss loss line.

Notes

- The shape is preserved but the amplitude has an exponential attenuation along the line.
- In this case, since $\beta(\omega)$ is a linear function to first order, we have no dispersion. All of the Fourier components of a pulse travel with the same phase velocity since $v_\phi = \omega/\beta$ is constant. i.e. $v(z,t) = e^{-\alpha z} f(t - z/v_\phi)$. We should expect dispersion when the $R/\omega L$ and $G/\omega C$ start becoming more significant.

1.6 Distortionless line.

Motivated by the early telegraphy days, when low loss materials were not available. Therefore lines with a constant attenuation and constant phase velocity (i.e. no dispersion) were required in order to eliminate distortion of the signals. This can be achieved by setting

$$\frac{R}{L} = \frac{G}{C}. \tag{1.38}$$

When that is done we have

$$
\begin{aligned}
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
&= j\omega\sqrt{LC}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\
&= j\omega\sqrt{LC}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{R}{j\omega L}\right)} \\
&= j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right) \\
&= R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC} \\
&= \sqrt{RG} + j\omega\sqrt{LC}.
\end{aligned}
\tag{1.39}
$$

We have

$$
\begin{aligned}
\alpha &= \sqrt{RG} \\
\beta &= \omega\sqrt{LC}.
\end{aligned}
\tag{1.40}
$$

The phase velocity is the same as that of the lossless and low-loss lines

$$
v_\phi = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}.
\tag{1.41}
$$

1.7 Terminated lossless line.

Consider the load configuration sketched in fig. 1.7.

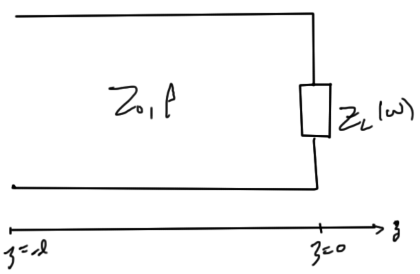

Figure 1.7: Terminated line.

Recall that

$$
\begin{aligned}
V(z) &= V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z} \\
I(z) &= \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z}
\end{aligned}
\tag{1.42}
$$

At the load ($z = 0$), we have

$$
\begin{aligned}
V(0) &= V_0^+ + V_0^- \\
I(0) &= \frac{1}{Z_0}\left(V_0^+ - V_0^-\right)
\end{aligned}
\tag{1.43}
$$

So

$$
\begin{aligned}
Z_L &= \frac{V(0)}{I(0)} \\
&= Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \\
&= Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L},
\end{aligned}
\tag{1.44}
$$

where

$$
\Gamma_L \equiv \frac{V_0^-}{V_0^+},
\tag{1.45}
$$

is the reflection coefficient at the load.
The phasors for the signals take the form

$$
\begin{aligned}
V(z) &= V_0^+ \left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right) \\
I(z) &= \frac{V_0^+}{Z_0} \left(e^{-j\beta z} - \Gamma_L e^{+j\beta z}\right).
\end{aligned}
\tag{1.46}
$$

Observe that we can rearranging for Γ_L in terms of the impedances

$$
(1 - \Gamma_L) Z_L = Z_0 \frac{1 + \Gamma_L}{},
\tag{1.47}
$$

or

$$
\Gamma_L (Z_0 + Z_L) = Z_L - Z_0,
\tag{1.48}
$$

or

$$
\Gamma_L = \frac{Z_L - Z_0}{Z_0 + Z_L}.
\tag{1.49}
$$

Power The average (time) power on the line is

$$
\begin{aligned}
P_{av} &= \frac{1}{2} \mathrm{Re}\left(V(Z) I^*(z)\right) \\
&= \frac{1}{2} \mathrm{Re}\left(V_0^+ \left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right) \left(\frac{V_0^+}{Z_0}\right)^* \left(e^{j\beta z} - \Gamma_L^* e^{-j\beta z}\right)\right) \\
&= \frac{|V_0^+|^2}{2Z_0} \mathrm{Re}\left(1 + \Gamma_L e^{2j\beta z} - \Gamma_L^* e^{-2j\beta z} - |\Gamma_L|^2\right) \\
&= \frac{|V_0^+|^2}{2Z_0} \left(1 - |\Gamma_L|^2\right).
\end{aligned}
\tag{1.50}
$$

where we've made use of the fact that $Z_0 = \sqrt{L/C}$ is real for the lossless line, and the fact that a conjugate difference $A - A^* = 2j\,\mathrm{Im}(A)$ is purely imaginary.

This can be written as

$$P_{\mathrm{av}} = P^+ - P^-, \tag{1.51}$$

where

$$\begin{aligned} P^+ &= \frac{|V_0^+|^2}{2Z_0} \\ P^+ &= \frac{|V_0^+|^2}{2Z_0} |\Gamma_{\mathrm{L}}|^2. \end{aligned} \tag{1.52}$$

This difference is the power delivered to the load. This is not z-dependent because we are considering the lossless case. Maximum power is delivered to the load when $\Gamma_{\mathrm{L}} = 0$, which occurs when the impedances are matched.

1.8 Return loss and insertion loss. Defined.

Return loss (dB) is defined as

$$\begin{aligned} \mathrm{RL} &= 10 \log_{10} \frac{P_{\mathrm{inc}}}{P_{\mathrm{refl}}} \\ &= 10 \log_{10} \frac{1}{|\Gamma|^2} \\ &= -20 \log_{10} |\Gamma|. \end{aligned} \tag{1.53}$$

Insertion loss (dB) is defined as

$$\begin{aligned} \mathrm{IL} &= 10 \log_{10} \frac{P_{\mathrm{inc}}}{P_{\mathrm{trans}}} \\ &= 10 \log_{10} \frac{P^+}{P^+ - P^-} \\ &= 10 \log_{10} \frac{1}{1 - |\Gamma|^2} \\ &= -10 \log_{10} \left(1 - |\Gamma|^2\right). \end{aligned} \tag{1.54}$$

1.9 Standing wave ratio

Consider again the lossless loaded configuration of fig. 1.7. Now let $z = -l$, where l is the distance from the load. The phasors at this point on the line are

$$V(-l) = V_0^+ \left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)$$
$$I(-l) = \frac{V_0^+}{Z_0} \left(e^{j\beta l} - \Gamma_L e^{-j\beta l}\right) \tag{1.55}$$

The absolute voltage at this point is

$$\begin{aligned} |V(-l)| &= |V_0^+| \left| e^{j\beta l} + \Gamma_L e^{-j\beta l} \right| \\ &= |V_0^+| \left| 1 + \Gamma_L e^{-2j\beta l} \right| \\ &= |V_0^+| \left| 1 + |\Gamma_L| e^{j\Theta_L} e^{-2j\beta l} \right|, \end{aligned} \tag{1.56}$$

where the complex valued Γ_L is given by $\Gamma_L = |\Gamma_L| e^{j\Theta_L}$.
This gives

$$|V(-l)| = |V_0^+| \left| 1 + |\Gamma_L| e^{j(\Theta_L - 2\beta l)} \right|. \tag{1.57}$$

The voltage magnitude oscillates as one moves along the line. The maximum occurs when $e^{j(\Theta_L - 2\beta l)} = 1$

$$V_{\max} = |V_0^+| |1 + |\Gamma_L||. \tag{1.58}$$

This occurs when $\Theta_L - 2\beta l = 2k\pi$ for $k = 0, 1, 2, \cdots$. The minimum occurs when $e^{j(\Theta_L - 2\beta l)} = -1$

$$V_{\min} = |V_0^+| |1 - |\Gamma_L||, \tag{1.59}$$

which occurs when $\Theta_L - 2\beta l = (2k - 1)\pi$ for $k = 1, 2, \cdots$. The standing wave ratio is defined as

$$\text{SWR} = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}. \tag{1.60}$$

This is a measure of the mismatch of a line. This is sketched in fig. 1.8.

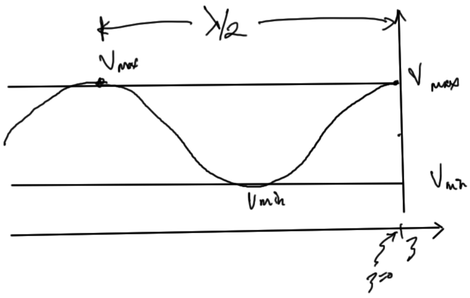

Figure 1.8: SWR extremes.

Notes:

- Since $0 \leq |\Gamma_L| \leq 1$, we have $1 \leq \text{SWR} \leq \infty$. The lower bound is for a matched line, and open, short, or purely reactive termination leads to the infinities.
- The distance between two successive maxima (or minima) can be determined by setting $\Theta_L - 2\beta l = 2k\pi$ for two consecutive values of k. For $k = 0$, suppose that V_{max} occurs at d_1

$$\Theta_L - 2\beta d_1 = 2(0)\pi, \tag{1.61}$$

or

$$d_1 = \frac{\Theta_L}{2\beta}. \tag{1.62}$$

For $k = 1$, let the max occur at d_2

$$\Theta_L - 2\beta d_2 = 2(1)\pi, \tag{1.63}$$

or

$$d_2 = \frac{\Theta_L - 2\pi}{2\beta}. \tag{1.64}$$

The difference is

$$\begin{aligned} d_1 - d_2 &= \frac{\Theta_L}{2\beta} - \frac{\Theta_L - 2\pi}{2\beta} \\ &= \frac{\pi}{\beta} \\ &= \frac{\pi}{2\pi/\lambda} \\ &= \frac{\lambda}{2}. \end{aligned} \tag{1.65}$$

The distance between two consecutive maxima (or minima) of the SWR is $\lambda/2$.

1.10 Impedance Transformation.

Referring to fig. 1.9, let's solve for the impedance at the load where $z = 0$ and at $z = -l$.
At any point on the line we have

$$V(z) = V_0^+ e^{-j\beta z}\left(1 + \Gamma_L e^{2j\beta z}\right), \tag{1.66}$$

so at the load and input we have

$$\begin{aligned} V_L &= V_0^+ (1 + \Gamma_L) \\ V(-l) &= V^+ (1 + \Gamma_L(-1)), \end{aligned} \tag{1.67}$$

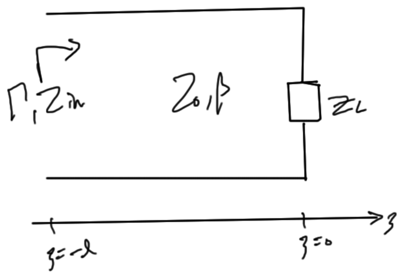

Figure 1.9: Configuration for impedance transformation.

where

$$
\begin{aligned}
V^+ &= V_0^+ e^{j\beta l} \\
\Gamma_L(-1) &= \Gamma_L e^{-2j\beta l}
\end{aligned}
\tag{1.68}
$$

Similarly

$$
I(-l) = \frac{V^+}{Z_0} \left(1 - \Gamma_L(-1)\right). \tag{1.69}
$$

Define an input impedance as

$$
\begin{aligned}
Z_{\text{in}} &= \frac{V(-l)}{I(-l)} \\
&= Z_0 \frac{1 + \Gamma_L(-1)}{1 - \Gamma_L(-1)}
\end{aligned}
\tag{1.70}
$$

This is analogous to

$$
Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{1.71}
$$

From eq. (1.49), we have

$$
\begin{aligned}
Z_{\text{in}} &= Z_0 \frac{Z_0 + Z_L + (Z_L - Z_0)\, e^{-2j\beta l}}{Z_0 + Z_L - (Z_L - Z_0)\, e^{-2j\beta l}} \\
&= Z_0 \frac{(Z_0 + Z_L)\, e^{j\beta l} + (Z_L - Z_0)\, e^{-j\beta l}}{(Z_0 + Z_L)\, e^{j\beta l} - (Z_L - Z_0)\, e^{-j\beta l}} \\
&= Z_0 \frac{Z_L \cos(\beta l) + j Z_0 \sin(\beta l)}{Z_0 \cos(\beta l) + j Z_L \sin(\beta l)},
\end{aligned}
\tag{1.72}
$$

or

$$
\boxed{Z_{\text{in}} = \frac{Z_L + j Z_0 \tan(\beta l)}{Z_0 + j Z_L \tan(\beta l)}.} \tag{1.73}
$$

This can be thought of as providing a reflection coefficient function along the line to the load at any point as sketched in fig. 1.10.

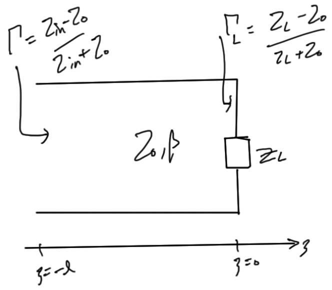

Figure 1.10: Impedance transformation reflection on the line.

Bibliography

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