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Vector Area

One of the results of this problem is required for a later one on magnetic moments that I'd like to do.

Exercise 1.1 **Vector Area.** (*[1] pr. 1.61*)

The integral

$$\mathbf{a} = \int_S d\mathbf{a}, \tag{1.1}$$

is sometimes called the vector area of the surface S.

1. Find the vector area of a hemispherical bowl of radius R.
2. Show that $\mathbf{a} = 0$ for any closed surface.
3. Show that $\mathbf{a}$ is the same for all surfaces sharing the same boundary.
4. Show that
$$\mathbf{a} = \frac{1}{2} \oint \mathbf{r} \times d\mathbf{l}, \tag{1.2}$$
where the integral is around the boundary line.
5. Show that
$$\oint (\mathbf{c} \cdot \mathbf{r})\, d\mathbf{l} = \mathbf{a} \times \mathbf{c}. \tag{1.3}$$

Answer for Exercise 1.1

Part 1.

$$
\begin{aligned}
\mathbf{a} &= \int_0^{\pi/2} R^2 \sin\theta d\theta \int_0^{2\pi} d\phi \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right) \\
&= R^2 \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \left(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta\right) \\
&= 2\pi R^2 \int_0^{\pi/2} d\theta \mathbf{e}_3 \sin\theta\cos\theta \\
&= \pi R^2 \mathbf{e}_3 \int_0^{\pi/2} d\theta \sin(2\theta) \\
&= \pi R^2 \mathbf{e}_3 \left. \left(\frac{-\cos(2\theta)}{2} \right) \right|_0^{\pi/2} \\
&= \pi R^2 \mathbf{e}_3 \left(1 - (-1)\right)/2 \\
&= \pi R^2 \mathbf{e}_3 .
\end{aligned}
\tag{1.4}
$$

Part 2. As hinted in the original problem description, this follows from

$$
\int dV \boldsymbol{\nabla} T = \oint T d\mathbf{a}, \tag{1.5}
$$

simply by setting $T = 1$.

Part 3. Suppose that two surfaces sharing a boundary are parameterized by vectors $\mathbf{x}(u, v), \mathbf{x}(a, b)$ respectively. The area integral with the first parameterization is

$$
\begin{aligned}
\mathbf{a} &= \int \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} du dv \\
&= \epsilon_{ijk} \mathbf{e}_i \int \frac{\partial x_j}{\partial u} \frac{\partial x_k}{\partial v} du dv \\
&= \epsilon_{ijk} \mathbf{e}_i \int \left(\frac{\partial x_j}{\partial a} \frac{\partial a}{\partial u} + \frac{\partial x_j}{\partial b} \frac{\partial b}{\partial u} \right) \left(\frac{\partial x_k}{\partial a} \frac{\partial a}{\partial v} + \frac{\partial x_k}{\partial b} \frac{\partial b}{\partial v} \right) du dv \\
&= \epsilon_{ijk} \mathbf{e}_i \int du dv \left(\frac{\partial x_j}{\partial a} \frac{\partial a}{\partial u} \frac{\partial x_k}{\partial a} \frac{\partial a}{\partial v} + \frac{\partial x_j}{\partial b} \frac{\partial b}{\partial u} \frac{\partial x_k}{\partial b} \frac{\partial b}{\partial v} + \frac{\partial x_j}{\partial b} \frac{\partial b}{\partial u} \frac{\partial x_k}{\partial a} \frac{\partial a}{\partial v} + \frac{\partial x_j}{\partial a} \frac{\partial a}{\partial u} \frac{\partial x_k}{\partial b} \frac{\partial b}{\partial v} \right) \\
&= \epsilon_{ijk} \mathbf{e}_i \int du dv \left(\frac{\partial x_j}{\partial a} \frac{\partial x_k}{\partial a} \frac{\partial a}{\partial u} \frac{\partial a}{\partial v} + \frac{\partial x_j}{\partial b} \frac{\partial x_k}{\partial b} \frac{\partial b}{\partial u} \frac{\partial b}{\partial v} \right) + \epsilon_{ijk} \mathbf{e}_i \int du dv \left(\frac{\partial x_j}{\partial b} \frac{\partial x_k}{\partial a} \frac{\partial b}{\partial u} \frac{\partial a}{\partial v} - \frac{\partial x_k}{\partial a} \frac{\partial x_j}{\partial b} \frac{\partial a}{\partial u} \frac{\partial b}{\partial v} \right).
\end{aligned}
\tag{1.6}
$$

In the last step a j, k index swap was performed for the last term of the second integral. The first integral is zero, since the integrand is symmetric in j, k. This leaves

$$
\begin{aligned}
\mathbf{a} &= \epsilon_{ijk}\mathbf{e}_i \int dudv \left(\frac{\partial x_j}{\partial b}\frac{\partial x_k}{\partial a}\frac{\partial b}{\partial u}\frac{\partial a}{\partial v} - \frac{\partial x_k}{\partial a}\frac{\partial x_j}{\partial b}\frac{\partial a}{\partial u}\frac{\partial b}{\partial v} \right) \\
&= \epsilon_{ijk}\mathbf{e}_i \int \frac{\partial x_j}{\partial b}\frac{\partial x_k}{\partial a} \left(\frac{\partial b}{\partial u}\frac{\partial a}{\partial v} - \frac{\partial a}{\partial u}\frac{\partial b}{\partial v} \right) dudv \\
&= \epsilon_{ijk}\mathbf{e}_i \int \frac{\partial x_j}{\partial b}\frac{\partial x_k}{\partial a}\frac{\partial(b,a)}{\partial(u,v)} dudv \\
&= -\int \frac{\partial \mathbf{x}}{\partial b} \times \frac{\partial \mathbf{x}}{\partial a} dadb \\
&= \int \frac{\partial \mathbf{x}}{\partial a} \times \frac{\partial \mathbf{x}}{\partial b} dadb.
\end{aligned}
\tag{1.7}
$$

However, this is the area integral with the second parameterization, proving that the area-integral for any given boundary is independant of the surface.

Part 4. Having proven that the area-integral for a given boundary is independent of the surface that it is evaluated on, the result follows by illustration as hinted in the full problem description. Draw a "cone", tracing a vector $\mathbf{x}'$ from the origin to the position line element, and divide that cone up into infinitesimal slices as sketched in fig. 1.1.

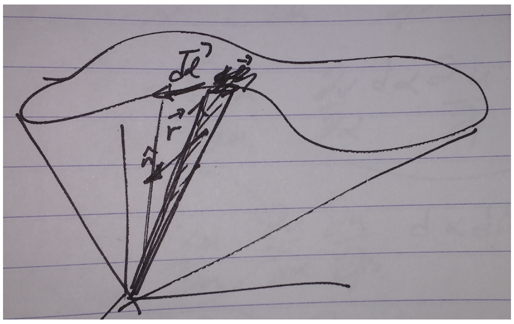

Figure 1.1: Cone configuration.

The area of each of these triangular slices is

$$\frac{1}{2}\mathbf{x}' \times d\mathbf{l}'. \tag{1.8}$$

Summing those triangles proves the result.

Part 5. As hinted in the problem, this follows from

$$\int \boldsymbol{\nabla} T \times d\mathbf{a} = -\oint T d\mathbf{l}. \tag{1.9}$$

Set $T = \mathbf{c} \cdot \mathbf{r}$, for which

$$
\begin{aligned}
\boldsymbol{\nabla} T &= \mathbf{e}_k \partial_k c_m x_m \\
&= \mathbf{e}_k c_m \delta_{km} \\
&= \mathbf{e}_k c_k \\
&= \mathbf{c},
\end{aligned} \tag{1.10}
$$

so

$$
\begin{aligned}
(\boldsymbol{\nabla} T) \times d\mathbf{a} &= \int \mathbf{c} \times d\mathbf{a} \\
&= \mathbf{c} \times \int d\mathbf{a} \\
&= \mathbf{c} \times \mathbf{a}.
\end{aligned} \tag{1.11}
$$

so

$$
\mathbf{c} \times \mathbf{a} = -\oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l}, \tag{1.12}
$$

or

$$
\oint (\mathbf{c} \cdot \mathbf{r}) d\mathbf{l} = \mathbf{a} \times \mathbf{c}. \qquad \square \tag{1.13}
$$

Bibliography

[1] David Jeffrey Griffiths and Reed College. *Introduction to electrodynamics*. Prentice hall Upper Saddle River, NJ, 3rd edition, 1999. 1.1