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Vector wave equation in spherical coordinates

For a vector $\mathbf{A}$ in spherical coordinates, let's compute the Laplacian

$$\boldsymbol{\nabla}^2 \mathbf{A}, \tag{1.1}$$

to see the form of the wave equation. The spherical vector representation has a curvilinear basis

$$\mathbf{A} = \hat{\mathbf{r}} A_r + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\phi}} A_\phi, \tag{1.2}$$

and the spherical Laplacian has been found to have the representation

$$\boldsymbol{\nabla}^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}. \tag{1.3}$$

Evaluating the Laplacian will require the following curvilinear basis derivatives

$$\begin{aligned}
\partial_\theta \hat{\mathbf{r}} &= \hat{\boldsymbol{\theta}} \\
\partial_\theta \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{r}} \\
\partial_\theta \hat{\boldsymbol{\phi}} &= 0 \\
\partial_\phi \hat{\mathbf{r}} &= S_\theta \hat{\boldsymbol{\phi}} \\
\partial_\phi \hat{\boldsymbol{\theta}} &= C_\theta \hat{\boldsymbol{\phi}} \\
\partial_\phi \hat{\boldsymbol{\phi}} &= -\hat{\mathbf{r}} S_\theta - \hat{\boldsymbol{\theta}} C_\theta.
\end{aligned} \tag{1.4}$$

We'll need to evaluate a number of derivatives. Starting with the $\hat{\mathbf{r}}$ components

$$\partial_r \left(r^2 \partial_r \left(\hat{\mathbf{r}} \psi \right) \right) = \hat{\mathbf{r}} \partial_r \left(r^2 \partial_r \psi \right) \tag{1.5a}$$

$$\begin{aligned}
\partial_\theta \left(S_\theta \partial_\theta \left(\hat{\mathbf{r}} \psi \right) \right) &= \partial_\theta \left(S_\theta (\hat{\boldsymbol{\theta}} \psi + \hat{\mathbf{r}} \partial_\theta \psi) \right) \\
&= C_\theta (\hat{\boldsymbol{\theta}} \psi + \hat{\mathbf{r}} \partial_\theta \psi) + S_\theta \partial_\theta (\hat{\boldsymbol{\theta}} \psi + \hat{\mathbf{r}} \partial_\theta \psi) \\
&= C_\theta (\hat{\boldsymbol{\theta}} \psi + \hat{\mathbf{r}} \partial_\theta \psi) + S_\theta \partial_\theta ((\partial_\theta \hat{\boldsymbol{\theta}}) \psi + (\partial_\theta \hat{\mathbf{r}}) \partial_\theta \psi) + S_\theta \partial_\theta (\hat{\boldsymbol{\theta}} \partial_\theta \psi + \hat{\mathbf{r}} \partial_{\theta\theta} \psi) \\
&= C_\theta (\hat{\boldsymbol{\theta}} \psi + \hat{\mathbf{r}} \partial_\theta \psi) + S_\theta ((-\hat{\mathbf{r}}) \psi + (\hat{\boldsymbol{\theta}}) \partial_\theta \psi) + S_\theta (\hat{\boldsymbol{\theta}} \partial_\theta \psi + \hat{\mathbf{r}} \partial_{\theta\theta} \psi) \\
&= \hat{\mathbf{r}} \left(C_\theta \partial_\theta \psi - S_\theta \psi + S_\theta \partial_{\theta\theta} \psi \right) + \hat{\boldsymbol{\theta}} \left(C_\theta \psi + 2 S_\theta \partial_\theta \psi \right)
\end{aligned} \tag{1.5b}$$

$$
\begin{aligned}
\partial_{\phi\phi}\left(\hat{\mathbf{r}}\psi\right) &= \partial_\phi\left((\partial_\phi\hat{\mathbf{r}})\psi + \hat{\mathbf{r}}\partial_\phi\psi\right) \\
&= \partial_\phi\left((S_\theta\hat{\boldsymbol{\phi}})\psi + \hat{\mathbf{r}}\partial_\phi\psi\right) \\
&= S_\theta\partial_\phi(\hat{\boldsymbol{\phi}}\psi) + \partial_\phi\left(\hat{\mathbf{r}}\partial_\phi\psi\right) \\
&= S_\theta(\partial_\phi\hat{\boldsymbol{\phi}})\psi + S_\theta\hat{\boldsymbol{\phi}}\partial_\phi\psi + (\partial_\phi\hat{\mathbf{r}})\partial_\phi\psi + \hat{\mathbf{r}}\partial_{\phi\phi}\psi \\
&= S_\theta(-S_\theta\hat{\mathbf{r}} - C_\theta\hat{\boldsymbol{\theta}})\psi + S_\theta\hat{\boldsymbol{\phi}}\partial_\phi\psi + (S_\theta\hat{\boldsymbol{\phi}})\partial_\phi\psi + \hat{\mathbf{r}}\partial_{\phi\phi}\psi \\
&= \hat{\mathbf{r}}\left(-S_\theta^2\psi + \partial_{\phi\phi}\psi\right) + \hat{\boldsymbol{\theta}}\left(-S_\theta C_\theta\psi\right) + \hat{\boldsymbol{\phi}}\left(2S_\theta\hat{\boldsymbol{\phi}}\partial_\phi\psi\right)
\end{aligned} \tag{1.5c}
$$

This gives

$$
\begin{aligned}
\boldsymbol{\nabla}^2(\hat{\mathbf{r}}A_r) = \hat{\mathbf{r}}&\left(\frac{1}{r^2}\partial_r\left(r^2\partial_r A_r\right) + \frac{1}{r^2 S_\theta}\left(C_\theta\partial_\theta A_r - S_\theta A_r + S_\theta\partial_{\theta\theta}A_r\right) + \frac{1}{r^2 S_\theta^2}\left(-S_\theta^2 A_r + \partial_{\phi\phi}A_r\right)\right) \\
&+ \hat{\boldsymbol{\theta}}\left(\frac{1}{r^2 S_\theta}\left(C_\theta A_r + 2S_\theta\partial_\theta A_r\right) - \frac{1}{r^2 S_\theta}S_\theta C_\theta A_r\right) + \hat{\boldsymbol{\phi}}\left(\frac{1}{r^2 S_\theta^2}2S_\theta\partial_\phi A_r\right) \\
&= \hat{\mathbf{r}}\left(\boldsymbol{\nabla}^2 A_r - \frac{2}{r^2}A_r\right) + \frac{\hat{\boldsymbol{\theta}}}{r^2}\left(\frac{C_\theta}{S_\theta}A_r + 2\partial_\theta A_r - C_\theta A_r\right) + \hat{\boldsymbol{\phi}}\frac{2}{r^2 S_\theta}\partial_\phi A_r.
\end{aligned} \tag{1.6}
$$

Next, let's compute the derivatives of the $\hat{\boldsymbol{\theta}}$ projection.

$$
\partial_r\left(r^2\partial_r\left(\hat{\boldsymbol{\theta}}\psi\right)\right) = \hat{\boldsymbol{\theta}}\partial_r\left(r^2\partial_r\psi\right) \tag{1.7a}
$$

$$
\begin{aligned}
\partial_\theta\left(S_\theta\partial_\theta\left(\hat{\boldsymbol{\theta}}\psi\right)\right) &= \partial_\theta\left(S_\theta\left((\partial_\theta\hat{\boldsymbol{\theta}})\psi + \hat{\boldsymbol{\theta}}\partial_\theta\psi\right)\right) \\
&= \partial_\theta\left(S_\theta\left((-\hat{\mathbf{r}})\psi + \hat{\boldsymbol{\theta}}\partial_\theta\psi\right)\right) \\
&= C_\theta\left(-\hat{\mathbf{r}}\psi + \hat{\boldsymbol{\theta}}\partial_\theta\psi\right) + S_\theta\left(-(\partial_\theta\hat{\mathbf{r}})\psi - \hat{\mathbf{r}}\partial_\theta\psi + (\partial_\theta\hat{\boldsymbol{\theta}})\partial_\theta\psi + \hat{\boldsymbol{\theta}}\partial_{\theta\theta}\psi\right) \\
&= C_\theta\left(-\hat{\mathbf{r}}\psi + \hat{\boldsymbol{\theta}}\partial_\theta\psi\right) + S_\theta\left(-(\hat{\boldsymbol{\theta}})\psi - \hat{\mathbf{r}}\partial_\theta\psi + (-\hat{\mathbf{r}})\partial_\theta\psi + \hat{\boldsymbol{\theta}}\partial_{\theta\theta}\psi\right) \\
&= \hat{\mathbf{r}}\left(-C_\theta\psi - 2S_\theta\partial_\theta\psi\right) + \hat{\boldsymbol{\theta}}\left(+C_\theta\partial_\theta\psi - S_\theta\psi + S_\theta\partial_{\theta\theta}\psi\right) \\
&= \hat{\mathbf{r}}\left(-C_\theta\psi - 2S_\theta\partial_\theta\psi\right) + \hat{\boldsymbol{\theta}}\left(+\partial_\theta(S_\theta\partial_\theta\psi) - S_\theta\psi\right)
\end{aligned} \tag{1.7b}
$$

$$
\begin{aligned}
\partial_{\phi\phi}\left(\hat{\boldsymbol{\theta}}\psi\right) &= \partial_\phi\left((\partial_\phi\hat{\boldsymbol{\theta}})\psi + \hat{\boldsymbol{\theta}}\partial_\phi\psi\right) \\
&= \partial_\phi\left((C_\theta\hat{\boldsymbol{\phi}})\psi + \hat{\boldsymbol{\theta}}\partial_\phi\psi\right) \\
&= C_\theta\partial_\phi(\hat{\boldsymbol{\phi}}\psi) + \partial_\phi(\hat{\boldsymbol{\theta}}\partial_\phi\psi) \\
&= C_\theta(\partial_\phi\hat{\boldsymbol{\phi}})\psi + C_\theta\hat{\boldsymbol{\phi}}\partial_\phi\psi + (\partial_\phi\hat{\boldsymbol{\theta}})\partial_\phi\psi + \hat{\boldsymbol{\theta}}\partial_{\phi\phi}\psi \\
&= C_\theta(-\hat{\mathbf{r}}S_\theta - \hat{\boldsymbol{\theta}}C_\theta)\psi + C_\theta\hat{\boldsymbol{\phi}}\partial_\phi\psi + (C_\theta\hat{\boldsymbol{\phi}})\partial_\phi\psi + \hat{\boldsymbol{\theta}}\partial_{\phi\phi}\psi \\
&= -\hat{\mathbf{r}}C_\theta S_\theta\psi + \hat{\boldsymbol{\theta}}\left(-C_\theta C_\theta\psi + \partial_{\phi\phi}\psi\right) + 2\hat{\boldsymbol{\phi}}C_\theta\partial_\phi\psi,
\end{aligned} \tag{1.7c}
$$

which gives

$$
\begin{aligned}
\boldsymbol{\nabla}^2(\hat{\boldsymbol{\theta}}A_\theta) &= \hat{\mathbf{r}}\left(\frac{1}{r^2 S_\theta}\left(-C_\theta A_\theta - 2S_\theta \partial_\theta A_\theta\right) - \frac{1}{r^2 S_\theta^2} C_\theta S_\theta A_\theta\right) \\
&\quad + \hat{\boldsymbol{\theta}}\left(\frac{1}{r^2}\partial_r\left(r^2\partial_r A_\theta\right) + \frac{1}{r^2 S_\theta}\left(+\partial_\theta(S_\theta\partial_\theta A_\theta) - S_\theta A_\theta\right) + \frac{1}{r^2 S_\theta^2}\left(-C_\theta C_\theta A_\theta + \partial_{\phi\phi}A_\theta\right)\right) \\
&\quad + \hat{\boldsymbol{\phi}}\left(\frac{1}{r^2 S_\theta^2} 2C_\theta \partial_\phi A_\theta\right) \\
&= -2\hat{\mathbf{r}}\frac{1}{r^2 S_\theta}\partial_\theta(S_\theta A_\theta) + \hat{\boldsymbol{\theta}}\left(\boldsymbol{\nabla}^2 A_\theta - \frac{1}{r^2}A_\theta - \frac{1}{r^2 S_\theta^2}C_\theta^2 A_\theta\right) + 2\hat{\boldsymbol{\phi}}\left(\frac{1}{r^2 S_\theta^2}C_\theta \partial_\phi A_\theta\right).
\end{aligned}
\tag{1.8}
$$

Finally, we can compute the derivatives of the $\hat{\boldsymbol{\phi}}$ projection.

$$
\partial_r\left(r^2\partial_r\left(\hat{\boldsymbol{\phi}}\psi\right)\right) = \hat{\boldsymbol{\phi}}\partial_r\left(r^2\partial_r\psi\right) \tag{1.9a}
$$

$$
\partial_\theta\left(S_\theta\partial_\theta\left(\hat{\boldsymbol{\phi}}\psi\right)\right) = \hat{\boldsymbol{\phi}}\partial_\theta\left(S_\theta\partial_\theta\psi\right) \tag{1.9b}
$$

$$
\begin{aligned}
\partial_{\phi\phi}\left(\hat{\boldsymbol{\phi}}\psi\right) &= \partial_\phi\left((\partial_\phi\hat{\boldsymbol{\phi}})\psi + \hat{\boldsymbol{\phi}}\partial_\phi\psi\right) \\
&= \partial_\phi\left((-\hat{\mathbf{r}}S_\theta - \hat{\boldsymbol{\theta}}C_\theta)\psi + \hat{\boldsymbol{\phi}}\partial_\phi\psi\right) \\
&= -((\partial_\phi\hat{\mathbf{r}})S_\theta + (\partial_\phi\hat{\boldsymbol{\theta}})C_\theta)\psi - (\hat{\mathbf{r}}S_\theta + \hat{\boldsymbol{\theta}}C_\theta)\partial_\phi\psi + (\partial_\phi\hat{\boldsymbol{\phi}}\partial_\phi\psi + \hat{\boldsymbol{\phi}}\partial_{\phi\phi}\psi \\
&= -((S_\theta\hat{\boldsymbol{\phi}})S_\theta + (C_\theta\hat{\boldsymbol{\phi}})C_\theta)\psi - (\hat{\mathbf{r}}S_\theta + \hat{\boldsymbol{\theta}}C_\theta)\partial_\phi\psi + (-\hat{\mathbf{r}}S_\theta - \hat{\boldsymbol{\theta}}C_\theta)\partial_\phi\psi + \hat{\boldsymbol{\phi}}\partial_{\phi\phi}\psi \\
&= -2\hat{\mathbf{r}}S_\theta\partial_\phi\psi - 2\hat{\boldsymbol{\theta}}C_\theta\partial_\phi\psi + \hat{\boldsymbol{\phi}}\left(\partial_{\phi\phi}\psi - \psi\right),
\end{aligned}
\tag{1.9c}
$$

which gives

$$
\begin{aligned}
\boldsymbol{\nabla}^2\left(\hat{\boldsymbol{\phi}}A_\phi\right) &= -2\hat{\mathbf{r}}\frac{1}{r^2 S_\theta}\partial_\phi A_\phi - 2\hat{\boldsymbol{\theta}}\frac{1}{r^2 S_\theta^2}C_\theta\partial_\phi A_\phi \\
&\quad + \hat{\boldsymbol{\phi}}\left(\frac{1}{r^2}\partial_r\left(r^2\partial_r A_\phi\right) + \frac{1}{r^2 S_\theta}\partial_\theta\left(S_\theta\partial_\theta A_\phi\right) + \frac{1}{r^2 S_\theta^2}\left(\partial_{\phi\phi}A_\phi - A_\phi\right)\right) \\
&= -2\hat{\mathbf{r}}\frac{1}{r^2 S_\theta}\partial_\phi A_\phi - 2\hat{\boldsymbol{\theta}}\frac{1}{r^2 S_\theta^2}C_\theta\partial_\phi A_\phi + \hat{\boldsymbol{\phi}}\left(\boldsymbol{\nabla}^2 A_\phi - \frac{1}{r^2}A_\phi\right).
\end{aligned}
\tag{1.10}
$$

The vector Laplacian resolves into three augmented scalar wave equations, all highly coupled

$$
\begin{aligned}
\hat{\mathbf{r}} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{A}\right) &= \boldsymbol{\nabla}^2 A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 S_\theta} \partial_\theta (S_\theta A_\theta) - \frac{2}{r^2 S_\theta} \partial_\phi A_\phi \\
\hat{\boldsymbol{\theta}} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{A}\right) &= \frac{1}{r^2} \frac{C_\theta}{S_\theta} A_r + \frac{2}{r^2} \partial_\theta A_r - \frac{1}{r^2} C_\theta A_r + \boldsymbol{\nabla}^2 A_\theta - \frac{1}{r^2} A_\theta - \frac{1}{r^2 S_\theta^2} C_\theta^2 A_\theta - 2 \frac{1}{r^2 S_\theta^2} C_\theta \partial_\phi A_\phi \\
\hat{\boldsymbol{\phi}} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{A}\right) &= \frac{2}{r^2 S_\theta} \partial_\phi A_r + \frac{2}{r^2 S_\theta^2} C_\theta \partial_\phi A_\theta + \boldsymbol{\nabla}^2 A_\phi - \frac{1}{r^2} A_\phi .
\end{aligned}
\tag{1.11}
$$

I'd guess one way to decouple these equations would be to impose a constraint that allows all the non-wave equation terms in one of the component equations to be killed, and then substitute that constraint into the remaining equations. Let's try one such constraint

$$
A_r = -\frac{1}{S_\theta} \partial_\theta (S_\theta A_\theta) - \frac{1}{S_\theta} \partial_\phi A_\phi . \tag{1.12}
$$

This gives

$$
\begin{aligned}
\hat{\mathbf{r}} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{A}\right) &= \boldsymbol{\nabla}^2 A_r \\
\hat{\boldsymbol{\theta}} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{A}\right) &= \left(\frac{1}{r^2} \frac{C_\theta}{S_\theta} + \frac{2}{r^2} \partial_\theta - \frac{1}{r^2} C_\theta \right) \left(-\frac{1}{S_\theta} \partial_\theta (S_\theta A_\theta) - \frac{1}{S_\theta} \partial_\phi A_\phi \right) \\
&\quad + \boldsymbol{\nabla}^2 A_\theta - \frac{1}{r^2} A_\theta - \frac{1}{r^2 S_\theta^2} C_\theta^2 A_\theta - \frac{2}{r^2 S_\theta^2} C_\theta \partial_\phi A_\phi \\
\hat{\boldsymbol{\phi}} \cdot \left(\boldsymbol{\nabla}^2 \mathbf{A}\right) &= -\frac{2}{r^2 S_\theta} \partial_\phi \left(\frac{1}{S_\theta} \partial_\theta (S_\theta A_\theta) + \frac{1}{S_\theta} \partial_\phi A_\phi \right) + \frac{2}{r^2 S_\theta^2} C_\theta \partial_\phi A_\theta + \boldsymbol{\nabla}^2 A_\phi - \frac{1}{r^2} A_\phi \\
&= -\frac{2}{r^2 S_\theta} \partial_\theta A_\theta - \frac{2}{r^2 S_\theta^2} \partial_{\phi\phi} A_\theta + \boldsymbol{\nabla}^2 A_\phi - \frac{1}{r^2} A_\phi
\end{aligned}
\tag{1.13}
$$

It looks like some additional cancellations may be had in the $\hat{\boldsymbol{\theta}}$ projection of this constrained vector Laplacian. I'm not inclined to try to take this reduction any further without a thorough check of all the algebra (using Mathematica to do so would make sense).

I also guessing that such a solution might be how the TE^r and TM^r modes were defined, but that doesn't appear to be the case according to [1]. There the wave equation is formulated in terms of the vector potentials (picking one to be zero and the other to be radial only). The solution obtained from such a potential wave equation then directly defines the TE^r and TM^r modes. It would be interesting to see how the modes derived in that analysis transform with application of the vector Laplacian derived above.

Bibliography

[1] Constantine A Balanis. *Advanced engineering electromagnetics*. Wiley New York, 1989. 1