1.1 1st Noether theorem.

Recall that, given a transformation
\[ \phi(x) \rightarrow \phi(x) + \delta \phi(x), \]
(1.1)
such that the transformation of the Lagrangian is only changed by a total derivative
\[ \mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu J_\mu^\epsilon, \]
(1.2)
then there is a conserved current
\[ j_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta_\epsilon \phi - J_\mu^\epsilon. \]
(1.3)

Here \( \epsilon \) is an \( x \)-independent quantity (i.e. a global symmetry). This is in contrast to “gauge symmetries”, which can be more accurately be categorized as a redundancy in the description.

As an example, for \( \mathcal{L} = (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)/2 \), let
\[ \phi(x) \rightarrow \phi(x) - a_\nu \partial_\nu \phi \]
(1.4)
\[ \mathcal{L}(\phi, \partial_\mu \phi) \rightarrow \mathcal{L}(\phi, \partial_\mu \phi) - a_\nu \partial_\nu \mathcal{L} \]
(1.5)
\[ = \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu (\delta_\nu^{\mu} a_\nu \mathcal{L}) \]

Here \( j_\mu^\epsilon = J_\mu^\epsilon \big|_{\epsilon=a_\nu} \), and the current is
\[ J_\mu^\nu = (\partial^\nu \phi)(-a_\nu \partial_\nu \phi) + \delta_\nu^{\mu} a_\nu \mathcal{L}. \]
(1.6)

In particular, we have one such current for each \( \nu \), and we write
\[ T_\nu^\mu = -(\partial^\nu \phi)(\partial_\nu \phi) + \delta_\nu^{\mu} \mathcal{L}. \]
(1.7)

By Noether’s theorem, we must have
\[ \partial_\mu T_\nu^\mu = 0, \quad \forall \nu. \]
(1.8)
Check:

\[ \partial_\mu T^\mu_\nu = - (\partial_\mu \partial^\mu \phi)(\partial_\nu \phi) - (\partial^\mu \phi)(\partial_\mu \partial_\nu \phi) + \delta^\mu_\nu \partial_\mu \left( \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{m^2}{2} \phi^2 \right) \]

\[ = - (\partial_\mu \partial^\mu \phi)(\partial_\nu \phi) - (\partial^\mu \phi)(\partial_\mu \partial_\nu \phi) + \frac{1}{2} (\partial_\mu \partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} (\partial_\mu \phi)(\partial_\nu \phi)(\partial_\mu \partial^\mu \phi) - m^2 (\partial_\nu \phi \phi) \]

\[ = - \left( \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 \phi - \frac{\lambda}{4} \phi^4 \right) \]

\[ = 0. \quad (1.9) \]

Example: our potential Lagrangian

\[ L = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 \quad (1.10) \]

Written with upper indexes

\[ T^{\mu\nu} = -(\partial^\mu \phi)(\partial^\nu \phi) + g^{\mu\nu} L \]

\[ = -(\partial^\mu \phi)(\partial^\nu \phi) + g^{\mu\nu} \left( \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 \right) \quad (1.11) \]

There are 4 conserved currents \( J^{\mu\nu} = T^{\mu\nu} \). Observe that this is symmetric (\( T^{\mu\nu} = T^{\nu\mu} \)). We have four associated charges

\[ Q^\nu = \int d^3x T^{0\nu}. \quad (1.12) \]

We call

\[ Q^0 = \int d^3x T^{00}, \quad (1.13) \]

the energy density, and call

\[ P^i = \int d^3x T^{0i}, \quad (1.14) \]

\((i = 1, 2, 3)\) the momentum density.

writing this out explicitly the energy density is

\[ T^{00} = -\phi^2 + \frac{1}{2} \left( \phi^2 - (\nabla \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4 \right) \]

\[ = - \left( \frac{1}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right), \quad (1.15) \]

and

\[ T^{0i} = \partial^0 \phi \partial^i \phi, \quad (1.16) \]

\[ P^i = - \int d^3x \partial^0 \phi \partial^i \phi \quad (1.17) \]

2
Since the energy density is negative definite (due to an arbitrary choice of translation sign), let’s redefine $T^{\mu\nu}$ to have a positive sign

$$T^{00} \equiv \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4,$$

and

$$p^i = \int d^3 x \dot{\phi}^0 \dot{\phi}^i \phi$$

As an operator we have

$$\hat{Q} = \int d^3 x \hat{T}^{00}$$

$$= \int d^3 x \left( \frac{1}{2} \hat{\dot{\phi}}^2 + \frac{1}{2} (\hat{\nabla} \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right).$$

$$\hat{p}^i = \int d^3 x \hat{T}^{0i} \phi$$

We showed that

$$\frac{d\hat{O}}{dt} = i [\hat{H}, \hat{O}]$$

This implied that $\hat{\phi}, \hat{\pi}$ obey the classical EOMs

$$\frac{d\hat{\phi}}{dt} = i [\hat{H}, \hat{\phi}] = \frac{d\hat{\pi}}{dt}$$

$$\frac{d\hat{\pi}}{dt} = i [\hat{H}, \hat{\pi}] = ...$$

In terms of creation and annihilation operators (for the $\lambda = 0$ free field), up to a constant

$$\hat{H} = \int d^3 x \hat{T}^{00}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \omega_p \hat{a}_p^\dagger \hat{a}_p$$

Can show that:

$$\hat{p}^i = \int d^3 x \hat{T}^{0i} \phi$$

$$= ...$$

$$= \int \frac{d^3 p}{(2\pi)^3} p^i \hat{a}_p^\dagger \hat{a}_p$$

Now we see the energy and momentum as conserved quantities associated with spacetime translation.
1.2 Unitary operators

In QM we say that $\hat{p}$ “generates translations”.

With $\hat{p} \equiv -i\hbar \nabla$ that translation is

$$\hat{U} = e^{ia\hat{p}} = e^{a\nabla}$$

(1.27)

In particular

$$\langle x | \hat{U} | \psi \rangle = e^{a\hat{p}} \psi(x) = \psi(x + a).$$

(1.28)

In one dimension

$$\hat{U}\hat{x}\hat{U}^\dagger = e^{a\hat{p}} \psi(x)e^{-a\hat{p}}$$

$$= \hat{x} + a\hat{1}.$$

(1.29)

This uses the Baker-Campbell-Hausdorff formula.

**Theorem 1.1: Baker-Campbell-Hausdorff**

$$e^B Ae^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[B \cdots, [B, A]\right],$$

(1.30)

where the n-th commutator is denoted above

- $n = 1 : [B, A]$
- $n = 2 : [B, [B, A]]$
- $n = 3 : [B, [B, [B, A]]]$

**Proof:**

$$f(t) = e^{tB} Ae^{-tB}$$

$$= f(0) + tf'(0) + \frac{t^2}{2}f''(0) + \cdots + \frac{t^n}{n!} f^{(n)}(0)$$

(1.31)

$$f(0) = A$$

(1.32)

$$f'(t) = e^{tB} B Ae^{-tB} + e^{tB} A(-B) e^{-tB}$$

$$= e^{tB} [B, A] e^{-tB}$$

(1.33)

$$f''(t) = e^{tB} B [B, A] e^{-tB} + e^{tB} [B, A] (-B)e^{-tB}$$

$$= e^{tB} [B, [B, A]] e^{-tB}.$$
From
\[ f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \cdots + \frac{1}{n!}f^{(n)}(0) \]  (1.35)
we have
\[ e^B A e^{-B} = A + [B, A] + \frac{1}{2} [B, [B, A]] + \cdots \]  (1.36)

Example:
\[ e^{a\partial_x} e^{-a\partial_x} = e^{a\partial_x} e^{-a\partial_x} = x + a [\partial_x, x] + \cdots = x + a. \]  (1.37)

Application:
\[ e^{i\text{Hermitian}} = \text{unitary} \]  (1.38)
\[ e^{i\text{Hermitian}} \times e^{-i\text{Hermitian}} = 1 \]  (1.39)
So
\[ \hat{U}(a) = e^{ia\hat{p}} \]  (1.40)
is a unitary operator representing finite translations in a Hilbert space.

\[ \hat{U}(a)\hat{\phi}(x)\hat{U}^\dagger(a) = \hat{\phi}(x) + ia\frac{\hat{p}}{\hbar} \hat{\phi}(x) + \cdots = \hat{\phi}(x + a) \]  (1.41)

1.3 Continuous symmetries

For all infinitesimal transformations, continuous symmetries lead to conserved charges \( Q \). In QFT we map these charges to Hermitian operators \( Q \to \hat{Q} \). We say that these charges are “generators of the corresponding symmetry” through unitary operators
\[ \hat{U} = e^{iparameter\hat{Q}}. \]  (1.44)
These represent the action of the symmetry in the Hilbert space.
Example: spatial translation
\[
\hat{U}(a) = e^{ia \hat{P}}
\] (1.45)

Example: time translation
\[
\hat{U}(t) = e^{iHt}.
\] (1.46)

1.4 Classical scalar theory

For \( d > 2 \) let’s look at
\[
S = \int d^d x \left( \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 - \lambda \phi^{d-2} \right)
\] (1.47)

Take \( m^2, \lambda \to 0 \), the free massless scalar field. We have a shift symmetry in this case since \( \phi(x) \to \phi(x) + \text{constant} \). The current is just
\[
j^\mu = \frac{\partial \phi}{\partial (\partial_\mu \phi)} \delta \phi - \mathcal{J}^\mu
\]
(1.48)
\[
= \text{constant} \times \partial^\mu \phi
\]
where the constant factor has been set to one. This current is clearly conserved since \( \partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = 0 \) (the equation of motion).

These are called “Goldstein Bosons”.

With \( m = \lambda = 0, d = 4 \) we have

NOTE: We did this in class differently with \( d \neq 4, m, \lambda \neq 0 \), and then switched to \( m = \lambda = 0, d = 4 \), which was confusing. I’ve reworked my notes to \( d = 4 \) like the supplemental handout that did the same.

\[
S = \int d^4 x \left( \frac{1}{2} \partial^\mu \phi \partial_\mu \phi \right)
\] (1.49)

Here we have a scale or dilatation invariance
\[
x \to x' = e^\lambda x,
\] (1.50)
\[
\phi(x) \to \phi'(x') = e^{-\lambda} \phi,
\] (1.51)
\[
d^4 x \to d^4 x' = e^{4\lambda} d^4 x,
\] (1.52)

The partials transform as
\[
\partial^\mu \to \frac{\partial}{\partial x'_\mu}
\]
\[
= \frac{\partial x_\mu}{\partial x'_\mu} \frac{\partial}{\partial x_\mu}
\] (1.53)
\[
= e^{-\lambda} \frac{\partial}{\partial x_\mu}
\]
so the partial of the field transforms as

$$
\partial^\mu \phi(x) \rightarrow \frac{\partial \phi'(x')}{\partial x'^\mu} = e^{-2\lambda} \partial^\mu \phi(x),
$$

(1.54)

and finally

$$
(\partial_\mu \phi)^2 \rightarrow e^{-4\lambda} (\partial_\mu \phi(x))^2.
$$

(1.55)

With a $-4\lambda$ power in the transformed quadratic term, and $4\lambda$ in the volume element, we see that the action is invariant. To find Noether current, we need to vary the field and its derivatives

$$
\delta \lambda \phi = \phi'(x) - \phi(x)
$$

$$
\approx \phi'(e^{-\lambda} x') - \phi(x)
$$

$$
\approx \phi'(x' - \lambda x') - \phi(x)
$$

(1.56)

$$
\approx (1 - \lambda)\phi(x) - \lambda x'^\alpha \partial_\alpha \phi'(x') - \phi(x)
$$

$$
= -\lambda (1 + x'^\alpha \partial_\alpha) \phi,
$$

where the last step assumes that $x' \rightarrow x, \phi' \rightarrow \phi$, effectively weeding out any terms that are quadratic or higher in $\lambda$.

Now we need the variation of the derivatives of $\phi$

$$
\delta \partial_\mu \phi(x) = \partial'_\mu \phi'(x) - \partial_\mu \phi(x),
$$

(1.57)

By eq. (1.54)

$$
\partial'_\mu \phi'(x') = e^{-2\lambda} \partial_\mu \phi(x)
$$

$$
= e^{-2\lambda} \partial_\mu \phi(e^{-\lambda} x')
$$

$$
\approx e^{-2\lambda} \partial_\mu \left( \phi(x') - \lambda x'^\alpha \partial_\alpha \phi(x') \right)
$$

$$
\approx (1 - 2\lambda) \partial_\mu \left( \phi(x') - \lambda x'^\alpha \partial_\alpha \phi(x') \right),
$$

(1.58)

so

$$
\delta \partial_\mu \phi = -\lambda x'^\alpha \partial_\alpha \partial_\mu \phi(x) - 2\lambda \partial_\mu \phi(x) + O(\lambda^2)
$$

$$
= -\lambda \left( x'^\alpha \partial_\alpha + 2 \right) \partial_\mu \phi(x),
$$

(1.59)

$$
\delta \mathcal{L} = \left( \partial'^\mu \phi \right) \delta \partial_\mu \phi
$$

$$
= -\lambda \left( 2 \partial_\mu \phi + x'^\alpha \partial_\alpha \partial_\mu \phi \right) \partial'^\mu \phi,
$$

(1.60)

or

$$
\frac{\delta \mathcal{L}}{-\lambda} = 4 \mathcal{L} + x'^\alpha \left( \partial_\alpha \partial_\mu \phi \right) \partial'^\mu \phi
$$

$$
= 4 \mathcal{L} + x'^\alpha \partial_\alpha \left( \mathcal{L} \right)
$$

$$
= 4 \mathcal{L} + \partial_\alpha \left( x'^\alpha \mathcal{L} \right) - \mathcal{L} \partial_\alpha x'^\alpha.
$$

(1.61)
The variation in the Lagrangian density is thus

$$\delta \mathcal{L} = \partial_\mu J^\mu_\lambda = \partial_\mu ( - \lambda x^\mu \mathcal{L}) ,$$

(1.62)

and the current is

$$J^\mu_\lambda = - \lambda x^\mu \mathcal{L} .$$

(1.63)

The Noether current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi - J^\mu$$

$$= - \partial^\mu \phi (1 + x^\nu \partial_\nu) \phi + \frac{1}{2} x^\mu \partial_\nu \phi \delta^\nu \phi ,$$

(1.64)

or after flipping signs

$$j^\mu_{\text{dil}} = \partial^\mu \phi (1 + x^\nu \partial_\nu) \phi - \frac{1}{2} x^\mu \partial_\nu \phi \delta^\nu \phi$$

$$= x_\nu \left( \partial^\mu \phi \delta^\nu \phi - \frac{1}{2} \delta^\nu_\rho \partial_\lambda \phi \delta^\lambda \phi \right) + \frac{1}{2} \partial^\mu (\phi^2) ,$$

(1.65)

$$j^\mu_{\text{dil conformal}} = - x_\nu T^{\nu \mu} + \frac{1}{2} \delta^\mu (\phi^2) ,$$

(1.66)

The current and $T^{\mu \nu}$ can both be redefined $j^\mu_\nu = j^\mu_\nu + \partial_\nu C^{\mu \nu}$ adding an antisymmetric $C^{\mu \nu} = -C^{\nu \mu}$

$$j^\mu_{\text{dil conformal}} = - x_\nu T^{\nu \mu}_{\text{conformal}}$$

(1.67)

$$\partial_\mu j^\mu_{\text{dil conformal}} = - T_{\text{conformal}}^{\nu \mu}$$

(1.68)

consequence: $0 = T^{00} - T^{11} - T^{22} - T^{33} , \text{ which is essentially}$

$$0 = \rho - 3 p = 0 .$$

(1.69)