Peeter Joot
peeterjoot@protonmail.com

Lorentz boosts in GA paravector notation.

1.1 Motivation.

The notation I prefer for relativistic geometric algebra uses Hestenes' space time algebra (STA) [3], where the basis is a four dimensional space $\{\gamma_\mu\}$, subject to Dirac matrix like relations $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$.
In this formalism a four vector is just the sum of the products of coordinates and basis vectors, for example, using summation convention

$$x = x^\mu \gamma_\mu. \tag{1.1}$$

The invariant for a four-vector in STA is just the square of that vector

$$\begin{aligned} x^2 &= (x^\mu \gamma_\mu) \cdot (x^\nu \gamma_\nu) \\ &= \sum_\mu (x^\mu)^2 (\gamma_\mu)^2 \\ &= (x^0)^2 - \sum_{k=1}^3 (x^k)^2 \\ &= (ct)^2 - \mathbf{x}^2. \end{aligned} \tag{1.2}$$

Recall that a four-vector is time-like if this squared-length is positive, spacelike if negative, and light-like when zero.
Time-like projections are possible by dotting with the "lab-frame" time like basis vector γ_0

$$ct = x \cdot \gamma_0 = x^0, \tag{1.3}$$

and space-like projections are wedges with the same

$$\mathbf{x} = x \cdot \gamma_0 = x^k \sigma_k, \tag{1.4}$$

where sums over Latin indexes $k \in \{1, 2, 3\}$ are implied, and where the elements σ_k

$$\sigma_k = \gamma_k \gamma_0. \tag{1.5}$$

which are bivectors in STA, can be viewed as an Euclidean vector basis $\{\sigma_k\}$.
Rotations in STA involve exponentials of space like bivectors $\theta = a_{ij} \gamma_i \wedge \gamma_j$

$$x' = e^{\theta/2} x e^{-\theta/2}. \tag{1.6}$$

Boosts, on the other hand, have exactly the same form, but the exponentials are with respect to space-time bivectors arguments, such as $\theta = a \wedge \gamma_0$, where a is any four-vector.

Observe that both boosts and rotations necessarily conserve the space-time length of a four vector (or any multivector with a scalar square).

$$\begin{aligned} \left(x'\right)^2 &= \left(e^{\theta/2} x e^{-\theta/2}\right) \left(e^{\theta/2} x e^{-\theta/2}\right) \\ &= e^{\theta/2} x \left(e^{-\theta/2} e^{\theta/2}\right) x e^{-\theta/2} \\ &= e^{\theta/2} x^2 e^{-\theta/2} \\ &= x^2 e^{\theta/2} e^{-\theta/2} \\ &= x^2. \end{aligned} \tag{1.7}$$

1.2 Paravectors.

Paravectors, as used by Baylis [1], represent four-vectors using a Euclidean multivector basis $\{\mathbf{e}_\mu\}$, where $\mathbf{e}_0 = 1$. The conversion between STA and paravector notation requires only multiplication with the timelike basis vector for the lab frame γ_0

$$\begin{aligned} X &= x\gamma_0 \\ &= \left(x^0\gamma_0 + x^k\gamma_k\right)\gamma_0 \\ &= x^0 + x^k\gamma_k\gamma_0 \\ &= x^0 + \mathbf{x} \\ &= ct + \mathbf{x}, \end{aligned} \tag{1.8}$$

We need a different structure for the invariant length in paravector form. That invariant length is

$$\begin{aligned} x^2 &= \left((ct + \mathbf{x})\gamma_0\right)\left((ct + \mathbf{x})\gamma_0\right) \\ &= \left((ct + \mathbf{x})\gamma_0\right)\left(\gamma_0 (ct - \mathbf{x})\right) \\ &= (ct + \mathbf{x})(ct - \mathbf{x}). \end{aligned} \tag{1.9}$$

Baylis introduces an involution operator $\overline{M}$ which toggles the sign of any vector or bivector grades of a multivector. For example, if $M = a + \mathbf{a} + I\mathbf{b} + Ic$, where $a, c \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is a multivector with all grades $0, 1, 2, 3$, then the involution of M is

$$\overline{M} = a - \mathbf{a} - I\mathbf{b} + Ic. \tag{1.10}$$

Utilizing this operator, the invariant length for a paravector X is $X\overline{X}$.

Let's consider how boosts and rotations can be expressed in the paravector form. The half angle operator for a boost along the spacelike $\mathbf{v} = v\hat{\mathbf{v}}$ direction has the form

$$L = e^{-\hat{\mathbf{v}}\phi/2}, \tag{1.11}$$

$$
\begin{aligned}
X' &= ct' + \mathbf{x}' \\
&= x'\gamma_0 \\
&= L x L^\dagger \\
&= e^{-\hat{\mathbf{v}}\phi/2} x^\mu \gamma_\mu e^{\hat{\mathbf{v}}\phi/2} \gamma_0 \\
&= e^{-\hat{\mathbf{v}}\phi/2} x^\mu \gamma_\mu \gamma_0 e^{-\hat{\mathbf{v}}\phi/2} \\
&= e^{-\hat{\mathbf{v}}\phi/2} \left(x^0 + \mathbf{x} \right) e^{-\hat{\mathbf{v}}\phi/2} \\
&= L X L.
\end{aligned}
\tag{1.12}
$$

Because the involution operator toggles the sign of vector grades, it is easy to see that the required invariance is maintained

$$
\begin{aligned}
X' \overline{X'} &= L X L \overline{L X L} \\
&= L X L \overline{L} \overline{X} \overline{L} \\
&= L X \overline{X} \overline{L} \\
&= X \overline{X} L \overline{L} \\
&= X \overline{X}.
\end{aligned}
\tag{1.13}
$$

Let's explicitly expand the transformation of eq. (1.12), so we can relate the rapidity angle ϕ to the magnitude of the velocity. This is most easily done by splitting the spacelike component $\mathbf{x}$ of the four vector into its projective and rejective components

$$
\begin{aligned}
\mathbf{x} &= \hat{\mathbf{v}} \hat{\mathbf{v}} \mathbf{x} \\
&= \hat{\mathbf{v}} \left(\hat{\mathbf{v}} \cdot \mathbf{x} + \hat{\mathbf{v}} \wedge \mathbf{x} \right) \\
&= \hat{\mathbf{v}} \left(\hat{\mathbf{v}} \cdot \mathbf{x} \right) + \hat{\mathbf{v}} \left(\hat{\mathbf{v}} \wedge \mathbf{x} \right) \\
&= \mathbf{x}_\parallel + \mathbf{x}_\perp.
\end{aligned}
\tag{1.14}
$$

The exponential

$$
e^{-\hat{\mathbf{v}}\phi/2} = \cosh\left(\phi/2\right) - \hat{\mathbf{v}} \sinh\left(\phi/2\right),
\tag{1.15}
$$

commutes with any scalar grades and with $\mathbf{x}_\parallel$, but anticommutes with $\mathbf{x}_\perp$, so

$$
\begin{aligned}
X' &= \left(ct + \mathbf{x}_\parallel \right) e^{-\hat{\mathbf{v}}\phi/2} e^{-\hat{\mathbf{v}}\phi/2} + \mathbf{x}_\perp e^{\hat{\mathbf{v}}\phi/2} e^{-\hat{\mathbf{v}}\phi/2} \\
&= \left(ct + \mathbf{x}_\parallel \right) e^{-\hat{\mathbf{v}}\phi} + \mathbf{x}_\perp \\
&= \left(ct + \hat{\mathbf{v}} \left(\hat{\mathbf{v}} \cdot \mathbf{x} \right) \right) \left(\cosh\phi - \hat{\mathbf{v}} \sinh\phi \right) + \mathbf{x}_\perp \\
&= \mathbf{x}_\perp + \left(ct \cosh\phi - \left(\hat{\mathbf{v}} \cdot \mathbf{x} \right) \sinh\phi \right) + \hat{\mathbf{v}} \left(\left(\hat{\mathbf{v}} \cdot \mathbf{x} \right) \cosh\phi - ct \sinh\phi \right) \\
&= \mathbf{x}_\perp + \cosh\phi \left(ct - \left(\hat{\mathbf{v}} \cdot \mathbf{x} \right) \tanh\phi \right) + \hat{\mathbf{v}} \cosh\phi \left(\hat{\mathbf{v}} \cdot \mathbf{x} - ct \tanh\phi \right).
\end{aligned}
\tag{1.16}
$$

Employing the argument from [4], we want ϕ defined so that this has structure of a Galilean transformation in the limit where $\phi \rightarrow 0$. This means we equate

$$
\tanh\phi = \frac{v}{c},
\tag{1.17}
$$

so that for small ϕ

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t. \tag{1.18}$$

We can solving for $\sinh^2\phi$ and $\cosh^2\phi$ in terms of v/c using

$$\tanh^2\phi = \frac{v^2}{c^2} = \frac{\sinh^2\phi}{1 + \sinh^2\phi} = \frac{\cosh^2\phi - 1}{\cosh^2\phi}. \tag{1.19}$$

which after picking the positive root required for Galilean equivalence gives

$$\begin{aligned}\cosh\phi &= \frac{1}{\sqrt{1 - (\mathbf{v}/c)^2}} \equiv \gamma \\ \sinh\phi &= \frac{v/c}{\sqrt{1 - (\mathbf{v}/c)^2}} = \gamma v/c.\end{aligned} \tag{1.20}$$

The Lorentz boost, written out in full is

$$ct' + \mathbf{x}' = \mathbf{x}_\perp + \gamma\left(ct - \frac{\mathbf{v}}{c} \cdot \mathbf{x}\right) + \gamma\left(\hat{\mathbf{v}}\left(\hat{\mathbf{v}} \cdot \mathbf{x}\right) - \mathbf{v}t\right). \tag{1.21}$$

Authors like Chappelle, et al., that also use paravectors [2], specify the form of the Lorentz transformation for the electromagnetic field, but for that transformation reversion is used instead of involution. I plan to explore that in a later post, starting from the STA formalism that I already understand, and see if I can make sense of the underlying rationale.

Bibliography

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