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Derivative of a delta function

In the retarded time Green's function for the spacetime gradient of geometric algebra, we end up with terms like

$$\frac{d}{dr}\delta(-r/c + t - t'), \tag{1.1}$$

or

$$\frac{d}{dt}\delta(-r/c + t - t'), \tag{1.2}$$

where t' is the integration variable of the test function that the delta function will be applied to. If these were derivatives with respect to the integration variable, then we could use

$$\int_{-\infty}^{\infty} \left(\frac{d}{dt'}\delta(t') \right) \phi(t') = -\phi'(0), \tag{1.3}$$

which follows by chain rule, and an assumption that $\phi(t')$ is well behaved at the points at infinity. It's not obvious to me that this can be applied to either of eq. (1.1) or eq. (1.2).

Let's go back to square one, and figure out the meaning of these delta functions by their action on a test function. We wish to compute

$$\int_{-\infty}^{\infty} \frac{d}{du}\delta(au + b - t') f(t') dt'. \tag{1.4}$$

Let's start with a change of variables $t'' = au + b - t'$, for which we find

$$\begin{aligned} t' &= au + b - t'' \\ dt'' &= -dt' \\ \frac{d}{du} &= \frac{dt''}{du}\frac{d}{dt''} = a\frac{d}{dt''}. \end{aligned} \tag{1.5}$$

Substitution back into eq. (1.4) gives

$$
\begin{aligned}
a\int_{\infty}^{-\infty}\left(\frac{d}{dt''}\delta(t'')\right)f(au+b-t'')(-dt'') &= a\int_{-\infty}^{\infty}\left(\frac{d}{dt''}\delta(t'')\right)f(au+b-t'')dt'' \\
&= a\delta(t'')f(au+b-t'')\big|_{-\infty}^{\infty} - a\int_{-\infty}^{\infty}\delta(t'')\frac{d}{dt''}f(au+b-t'')dt'' \\
&= -a\frac{d}{dt''}f(au+b-t'')\bigg|_{t''=0} \\
&= a\frac{d}{ds}f(s)\bigg|_{s=au+b}.
\end{aligned}
\tag{1.6}
$$

This shows that the action of the derivative of the delta function (with respect to a non-integration variable parameter u) is

$$
\boxed{\frac{d}{du}\delta(au+b-t') = a\frac{d}{ds}\bigg|_{s=au+b}.}
\tag{1.7}
$$