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# Momentum of scalar field

Way back in lecture 8, it was claimed that

$$P^k = \int d^3 x \hat{\pi} \partial^k \hat{\phi} = \int \frac{d^3 p}{(2\pi)^3} p^k a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{1.1}$$

If I compute this, I get a normal ordered variation of this operator, but also get some time dependent terms. Here's the computation (dropping hats)

$$\begin{aligned} P^k &= \int d^3 x \hat{\pi} \partial^k \phi \\ &= \int d^3 x \partial_0 \phi \partial^k \phi \\ &= \int d^3 x \frac{d^3 p d^3 q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p 2\omega_q}} \partial_0 \left( a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right) \partial^k \left( a_{\mathbf{q}} e^{-iq\cdot x} + a^\dagger_{\mathbf{q}} e^{iq\cdot x} \right). \end{aligned} \tag{1.2}$$

The exponential derivatives are

$$\begin{aligned} \partial_0 e^{\pm ip\cdot x} &= \partial_0 e^{\pm i p_\mu x^\mu} \\ &= \pm i p_0 \partial_0 e^{\pm ip\cdot x}, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \partial^k e^{\pm ip\cdot x} &= \partial^k e^{\pm i p^\mu x_\mu} \\ &= \pm i p^k e^{\pm ip\cdot x}, \end{aligned} \tag{1.4}$$

so

$$\begin{aligned} P^k &= -\int d^3 x \frac{d^3 p d^3 q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p 2\omega_q}} p_0 q^k \left( -a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right) \left( -a_{\mathbf{q}} e^{-iq\cdot x} + a^\dagger_{\mathbf{q}} e^{iq\cdot x} \right) \\ &= -\frac{1}{2} \int d^3 x \frac{d^3 p d^3 q}{(2\pi)^6} \sqrt{\frac{\omega_p}{\omega_q}} q^k \left( a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)\cdot x} + a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(p+q)\cdot x} - a_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(q-p)\cdot x} - a^\dagger_{\mathbf{p}} a_{\mathbf{q}} e^{i(p-q)\cdot x} \right) \\ &= \frac{1}{2} \int \frac{d^3 p d^3 q}{(2\pi)^3} \sqrt{\frac{\omega_p}{\omega_q}} q^k \Big( -a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(\omega_{\mathbf{p}}+\omega_{\mathbf{q}})t} \delta^3(\mathbf{p}+\mathbf{q}) - a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(\omega_{\mathbf{p}}+\omega_{\mathbf{q}})t} \delta^3(-\mathbf{p}-\mathbf{q}) \\ &\qquad + a_{\mathbf{p}} a^\dagger_{\mathbf{q}} e^{i(\omega_{\mathbf{q}}-\omega_{\mathbf{p}})t} \delta^3(\mathbf{p}-\mathbf{q}) + a^\dagger_{\mathbf{p}} a_{\mathbf{q}} e^{i(\omega_{\mathbf{p}}-\omega_{\mathbf{q}})t} \delta^3(\mathbf{q}-\mathbf{p}) \Big) \\ &= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} p^k \left( a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}} - a_{\mathbf{p}} a_{-\mathbf{p}} e^{-2i\omega_{\mathbf{p}} t} - a^\dagger_{\mathbf{p}} a^\dagger_{-\mathbf{p}} e^{2i\omega_{\mathbf{p}} t} \right). \end{aligned} \tag{1.5}$$

What is the rationale for ignoring those time dependent terms? Does normal ordering also implicitly drop any non-paired creation/annihilation operators? If so, why?