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Hamiltonian for the non-homogeneous Klein-Gordon equation

In class we derived the field for the non-homogeneous Klein-Gordon equation

$$\begin{aligned}\phi(x) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip\cdot x} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{ip\cdot x} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right) \Bigg|_{p^0=\omega_{\mathbf{p}}} \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{i\omega_{\mathbf{p}} t - i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right).\end{aligned} \tag{1.1}$$

This means that we have

$$\begin{aligned}\pi = \dot{\phi} &= \int \frac{d^3 p}{(2\pi)^3} \frac{i\omega_{\mathbf{p}}}{\sqrt{2\omega_{\mathbf{p}}}} \left(-e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + e^{i\omega_{\mathbf{p}} t - i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right) \\ (\boldsymbol{\nabla}\phi)_k = &= \int \frac{d^3 p}{(2\pi)^3} \frac{i p_k}{\sqrt{2\omega_{\mathbf{p}}}} \left(e^{-i\omega_{\mathbf{p}} t + i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) - e^{i\omega_{\mathbf{p}} t - i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}}^\dagger - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right),\end{aligned} \tag{1.2}$$

and could plug these into the Hamiltonian

$$H = \int d^3 p \left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{m^2}{2}\phi^2 \right), \tag{1.3}$$

to find H in terms of $\tilde{j}$ and $a_{\mathbf{p}}^\dagger, a_{\mathbf{p}}$. The result was mentioned in class, and it was left as an exercise to verify.

There's an easy way and a dumb way to do this exercise. I did it the dumb way, and then after suffering through two long pages, where the equations were so long that I had to write on the paper sideways, I realized the way I should have done it.

The easy way is to observe that we've already done exactly this for the case $\tilde{j} = 0$, which had the answer

$$H = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger \right). \tag{1.4}$$

To handle this more general case, all we have to do is apply a transformation

$$a_{\mathbf{p}} \rightarrow a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}}, \tag{1.5}$$

to eq. (1.4), which gives

$$\begin{aligned} H &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}} \left(\left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right)^{\dagger} \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right)^{\dagger} \right) \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}} \left(\left(a^{\dagger}_{\mathbf{p}} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) + \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a^{\dagger}_{\mathbf{p}} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \right). \end{aligned} \tag{1.6}$$

Like the $\tilde{j} = 0$ case, we can use normal ordering. This is easily seen by direct expansion:

$$\begin{aligned} \left(a^{\dagger}_{\mathbf{p}} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) &= a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} - \frac{i\tilde{j}^*(p) a_{\mathbf{p}}}{\sqrt{2\omega_{\mathbf{p}}}} + \frac{a^{\dagger}_{\mathbf{p}} i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} + \frac{|j|^2}{2\omega_{\mathbf{p}}} \\ \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a^{\dagger}_{\mathbf{p}} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) &= a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} + \frac{i\tilde{j}^*(p) a^{\dagger}_{\mathbf{p}}}{\sqrt{2\omega_{\mathbf{p}}}} - \frac{a_{\mathbf{p}} i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} + \frac{|j|^2}{2\omega_{\mathbf{p}}}. \end{aligned} \tag{1.7}$$

Because $\tilde{j}$ is just a complex valued function, it commutes with $a_{\mathbf{p}}, a^{\dagger}_{\mathbf{p}}$, and these are equal up to the normal ordering, allowing us to write

$$: H := \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}} \left(a^{\dagger}_{\mathbf{p}} - \frac{i\tilde{j}^*(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right) \left(a_{\mathbf{p}} + \frac{i\tilde{j}(p)}{\sqrt{2\omega_{\mathbf{p}}}} \right), \tag{1.8}$$

which is the result mentioned in class.