Peeter Joot
peeterjoot@protonmail.com

Multivector plane wave representation

The geometric algebra form of Maxwell's equations in free space (or source free isotopic media with group velocity c) is the multivector equation

$$\left(\boldsymbol{\nabla} + \frac{1}{c} \frac{\partial}{\partial t} \right) F(\mathbf{x}, t) = 0. \tag{1.1}$$

Here $F = \mathbf{E} + Ic\mathbf{B}$ is a multivector with grades 1 and 2 (vector and bivector components). The velcoty c is called the group velocity since F, or its components $\mathbf{E}, \mathbf{H}$ satisfy the wave equation, which can be seen by pre-multiplying with $\boldsymbol{\nabla} - (1/c)\partial/\partial t$ to find

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) F(\mathbf{x}, t) = 0. \tag{1.2}$$

Let's look at the frequency domain solution of this equation with a presumed phasor representation

$$F(\mathbf{x}, t) = \mathrm{Re} \left(F(\mathbf{k}) e^{-j\mathbf{k} \cdot \mathbf{x} + j\omega t} \right), \tag{1.3}$$

where j is a scalar imaginary, not necessarily with any geometric interpretation.

Maxwell's equation reduces to just

$$0 = -j \left(\mathbf{k} - \frac{\omega}{c} \right) F(\mathbf{k}). \tag{1.4}$$

If $F(\mathbf{k})$ has a left multivector factor

$$F(\mathbf{k}) = \left(\mathbf{k} + \frac{\omega}{c} \right) \tilde{F}, \tag{1.5}$$

where $\tilde{F}$ is a multivector to be determined, then

$$\begin{aligned} \left(\mathbf{k} - \frac{\omega}{c} \right) F(\mathbf{k}) &= \left(\mathbf{k} - \frac{\omega}{c} \right) \left(\mathbf{k} + \frac{\omega}{c} \right) \tilde{F} \\ &= \left(\mathbf{k}^2 - \left(\frac{\omega}{c} \right)^2 \right) \tilde{F}, \end{aligned} \tag{1.6}$$

which is zero if if $\left\lVert \mathbf{k} \right\rVert = \omega/c$.

Let $\hat{\mathbf{k}} = \mathbf{k}/\|\mathbf{k}\|$, and $\|\mathbf{k}\| \tilde{F} = F_0 + F_1 + F_2 + F_3$, where F_0, F_1, F_2, and F_3 are respectively have grades 0,1,2,3. Then

$$\begin{aligned}
F(\mathbf{k}) &= \left(1 + \hat{\mathbf{k}}\right) (F_0 + F_1 + F_2 + F_3) \\
&= F_0 + F_1 + F_2 + F_3 + \hat{\mathbf{k}} F_0 + \hat{\mathbf{k}} F_1 + \hat{\mathbf{k}} F_2 + \hat{\mathbf{k}} F_3 \\
&= F_0 + F_1 + F_2 + F_3 + \hat{\mathbf{k}} F_0 + \hat{\mathbf{k}} \cdot F_1 + \hat{\mathbf{k}} \cdot F_2 + \hat{\mathbf{k}} \cdot F_3 + \hat{\mathbf{k}} \wedge F_1 + \hat{\mathbf{k}} \wedge F_2 \\
&= \left(F_0 + \hat{\mathbf{k}} \cdot F_1\right) + \left(F_1 + \hat{\mathbf{k}} F_0 + \hat{\mathbf{k}} \cdot F_2\right) + \left(F_2 + \hat{\mathbf{k}} \cdot F_3 + \hat{\mathbf{k}} \wedge F_1\right) + \left(F_3 + \hat{\mathbf{k}} \wedge F_2\right).
\end{aligned} \tag{1.7}$$

Since the field F has only vector and bivector grades, the grades zero and three components of the expansion above must be zero, or

$$\begin{aligned}
F_0 &= -\hat{\mathbf{k}} \cdot F_1 \\
F_3 &= -\hat{\mathbf{k}} \wedge F_2,
\end{aligned} \tag{1.8}$$

so

$$\begin{aligned}
F(\mathbf{k}) &= \left(1 + \hat{\mathbf{k}}\right) \left(F_1 - \hat{\mathbf{k}} \cdot F_1 + F_2 - \hat{\mathbf{k}} \wedge F_2\right) \\
&= \left(1 + \hat{\mathbf{k}}\right) \left(F_1 - \hat{\mathbf{k}} F_1 + \hat{\mathbf{k}} \wedge F_1 + F_2 - \hat{\mathbf{k}} F_2 + \hat{\mathbf{k}} \cdot F_2\right).
\end{aligned} \tag{1.9}$$

The multivector $1 + \hat{\mathbf{k}}$ has the projective property of gobbling any leading factors of $\hat{\mathbf{k}}$

$$\begin{aligned}
(1 + \hat{\mathbf{k}})\hat{\mathbf{k}} &= \hat{\mathbf{k}} + 1 \\
&= 1 + \hat{\mathbf{k}},
\end{aligned} \tag{1.10}$$

so for $F_i \in F_1, F_2$

$$(1 + \hat{\mathbf{k}})(F_i - \hat{\mathbf{k}} F_i) = (1 + \hat{\mathbf{k}})(F_i - F_i) = 0, \tag{1.11}$$

leaving

$$F(\mathbf{k}) = \left(1 + \hat{\mathbf{k}}\right) \left(\hat{\mathbf{k}} \cdot F_2 + \hat{\mathbf{k}} \wedge F_1\right). \tag{1.12}$$

For $\hat{\mathbf{k}} \cdot F_2$ to be non-zero F_2 must be a bivector that lies in a plane containing $\hat{\mathbf{k}}$, and $\hat{\mathbf{k}} \cdot F_2$ is a vector in that plane that is perpendicular to $\hat{\mathbf{k}}$. On the other hand $\hat{\mathbf{k}} \wedge F_1$ is non-zero only if F_1 has a non-zero component that does not lie in along the $\hat{\mathbf{k}}$ direction, but $\hat{\mathbf{k}} \wedge F_1$, like F_2 describes a plane that containing $\hat{\mathbf{k}}$. This means that having both bivector and vector free variables F_2 and F_1 provide more degrees of freedom than required. For example, if $\mathbf{E}$ is any vector, and $F_2 = \hat{\mathbf{k}} \wedge \mathbf{E}$, then

$$\begin{aligned}
\left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{k}} \cdot F_2 &= \left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{k}} \cdot \left(\hat{\mathbf{k}} \wedge \mathbf{E}\right) \\
&= \left(1 + \hat{\mathbf{k}}\right) \left(\mathbf{E} - \hat{\mathbf{k}} \left(\hat{\mathbf{k}} \cdot \mathbf{E}\right)\right) \\
&= \left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{k}} \left(\hat{\mathbf{k}} \wedge \mathbf{E}\right) \\
&= \left(1 + \hat{\mathbf{k}}\right) \hat{\mathbf{k}} \wedge \mathbf{E},
\end{aligned} \tag{1.13}$$

which has the form $\left(1+\hat{\mathbf{k}}\right)\left(\hat{\mathbf{k}}\wedge F_1\right)$, so the solution of the free space Maxwell's equation can be written

$$\boxed{F(\mathbf{x},t) = \mathrm{Re}\left(\left(1+\hat{\mathbf{k}}\right)\mathbf{E}\,e^{-j\mathbf{k}\cdot\mathbf{x}+j\omega t}\right),} \tag{1.14}$$

where $\mathbf{E}$ is any vector for which $\mathbf{E}\cdot\mathbf{k} = 0$.