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PHY2403H Quantum Field Theory. Lecture 19: Pauli matrices, Weyl spinors, SL(2,c), Weyl action, Weyl equation, Dirac matrix, Dirac action, Dirac Lagrangian. Taught by Prof. Erich Poppitz

DISCLAIMER: Very rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Fermions: $\mathbb{R}^3$ rotations.

Given a real vector $\mathbf{x}$ and the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.1}$$

We may form a Pauli matrix representation of a vector

$$\boldsymbol{\sigma} \cdot \mathbf{x} = \begin{bmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{bmatrix}, \tag{1.2}$$

where $\boldsymbol{\sigma} = \left(\sigma^1, \sigma^2, \sigma^3\right)$. This matrix, like the Pauli matrixes, is a 2×2 Hermitian traceless matrix. We find that the determinant is

$$\begin{aligned} \det(\boldsymbol{\sigma} \cdot \mathbf{x}) &= -x_3^2 - x_1^2 - x_2^2 \\ &= -\mathbf{x}^2. \end{aligned} \tag{1.3}$$

We may form

$$U(\boldsymbol{\sigma} \cdot \mathbf{x})U^\dagger, \tag{1.4}$$

where U is a unitary 2×2 unit determinant matrix, satisfying

$$\begin{aligned} U^\dagger U &= 1 \\ \det U &= 1. \end{aligned} \tag{1.5}$$

Further

$$\det(U\sigma\cdot\mathbf{x})U^\dagger = \det U \det(\sigma\cdot\mathbf{x}) \det U^\dagger \\ = \det(\sigma\cdot\mathbf{x}). \tag{1.6}$$

Moral: $U(\sigma\cdot\mathbf{x})U^\dagger = \sigma\cdot\mathbf{x}'$, where $\mathbf{x}'$ has the same length of $\mathbf{x}$.

We may use this to represent an arbitrary rotation

$$U(\sigma\cdot\mathbf{x})U^\dagger = R^i{}_j x^j \sigma^i \tag{1.7}$$

We say that $U \in SU(2)$ and $R \in SU(3)$, and $SU(2)$ is called the "universal cover of $SO(3)$".

Pauli figured out that, in non-relativistic QM, that this type of transformation also applies to (spin) wave functions (spinors)

$$\Psi(\mathbf{x}) \rightarrow \Psi'(\mathbf{x}') = U\Psi(\mathbf{x}) \tag{1.8}$$

where

$$\mathbf{x} \rightarrow \mathbf{x}' = R\mathbf{x}, \tag{1.9}$$

and $R^{\mathrm{T}}R = 1$. Here Ψ is a two element vector

$$\Psi(\mathbf{x}) = \begin{bmatrix} \Psi_\uparrow(\mathbf{x}) \\ \Psi_\downarrow(\mathbf{x}) \end{bmatrix}, \tag{1.10}$$

so the transformation should be thought of as a matrix operation

$$\begin{bmatrix} \Psi_\uparrow(\mathbf{x}) \\ \Psi_\downarrow(\mathbf{x}) \end{bmatrix} \rightarrow \begin{bmatrix} \Psi'_\uparrow(\mathbf{x}') \\ \Psi'_\downarrow(\mathbf{x}') \end{bmatrix} = U \begin{bmatrix} \Psi_\uparrow(\mathbf{x}) \\ \Psi_\downarrow(\mathbf{x}) \end{bmatrix}. \tag{1.11}$$

Having seen such representations and their $SU(2)$ transformations in NRQM, we want to know what the relativistic generalization is.

1.2 Lorentz group

Let

$$(x^0, \mathbf{x}) = x^0\sigma^0 + \sigma\cdot\mathbf{x} \\ = \begin{bmatrix} x^0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x^0 - x_3 \end{bmatrix}. \tag{1.12}$$

This has determinant

$$\det(x^0, \mathbf{x}) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \\ = x^\mu x_\mu. \tag{1.13}$$

We therefore identify $(x^0, \mathbf{x})$ as a four vector

$$(x^0, \mathbf{x}) = x^\mu \sigma_\mu \tag{1.14}$$

We say that $SL(2, \mathbb{C})$ is a double cover of $SO(1, 3)$.

Note that the matrix U can be built explicitly. For example, it may be built up using Euler angles as sketched in fig. 1.1. or algebraically

$$U = e^{i\psi\sigma_3/2} e^{i\theta\sigma_1/2} e^{i\phi\sigma_3/2}. \tag{1.15}$$

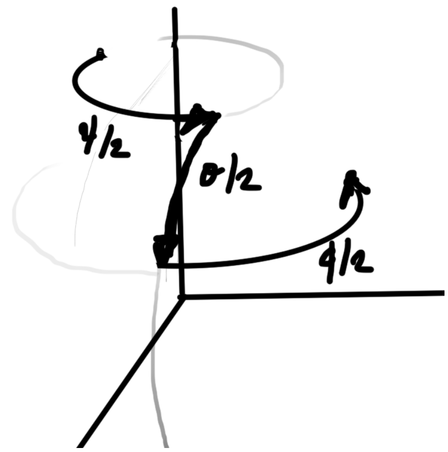

Figure 1.1: Euler angle rotations.

1.3 Weyl spinors.

We will see that there is generalization of Pauli spinors, called Weyl spinors, but we will have to introduce 4 component objects.

We'd like to argue that there is a correspondence (also $2 \rightarrow 1$) between $SL(2, \mathbb{C}) \rightarrow SO(1,3)$. Here:

- S : special
- L : linear
- $2 : 2 \times 2$
- $\mathbb{C}$: complex.

and we say that $M \in SL(2, \mathbb{C})$ if $\det M = 1$, where M is a complex 2×2, but not necessarily unitary. The $SU(2)$ group is a subset of $SL(2, \mathbb{C})$. In this representation $SU(2)$ matrices are $SL(2, \mathbb{C})$ matrices, but not necessarily the opposite.

We introduce a special notation for the identity matrix

$$\sigma^0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.16}$$

and can now form four vectors in a matrix representation

$$\begin{aligned} x \cdot \sigma &\equiv x^\mu \sigma_\mu \\ &\equiv x^0 \sigma^0 + \boldsymbol{\sigma} \cdot \mathbf{x} \\ &= \begin{bmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{bmatrix}. \end{aligned} \tag{1.17}$$

Such 2×2 matrices are Hermitian. Notice that the space of 2×2 Hermitian matrices is 4 dimensional.

We found that

$$\det(x^\mu \sigma_\mu) = (x^0)^2 - \mathbf{x}^2. \tag{1.18}$$

The transformation

$$x^\mu \sigma_\mu \rightarrow M \left(x^\mu \sigma_\mu\right) M^\dagger, \tag{1.19}$$

maps 2×2 Hermitian matrices to 2×2 Hermitian matrices using a unit determinant transformation M. Note that M is not unitary, as it is an arbitrary (Hermitian) matrix. In particular $MM^\dagger \ne 1$! Also note that the determinant of the transformed object is

$$\det\left(M \left(x^\mu \sigma_\mu\right) M^\dagger\right) = 1 \times \det\left(x^\mu \sigma_\mu\right) \times 1, \tag{1.20}$$

since $\det M = 1$, so that we see that the Lorentz invariant length is preserved by such a transformation. This can be expressed as

$$x \cdot \sigma \rightarrow M x \cdot \sigma M^\dagger = x' \cdot \sigma, \tag{1.21}$$

where $(x')^2 = x^2$.

Motivated by this $SL(2,\mathbb{C}) \rightarrow SO(1,3)$ correspondence, postulate that we study two component objects

$$U(x) = \begin{bmatrix} U_1(x) \\ U_2(x) \end{bmatrix}, \tag{1.22}$$

where $x = (x^0, x^1, x^2, x^3)$ is a four-vector, and assume that such objects transform as follows in $SO(1,3)$

$$\begin{aligned} U(x) &\rightarrow U'(x') = M^\dagger U(x) \\ x^\mu &\rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu, \end{aligned} \tag{1.23}$$

where $M^\dagger$ is the one giving rise to Λ. To understand what is meant by "giving rise to", consider

$$\begin{aligned} M x^\mu \sigma_\mu M^\dagger &= x'^\nu \sigma_\nu \\ &= \sigma_\nu \Lambda^\nu{}_\mu x^\mu, \end{aligned} \tag{1.24}$$

and this holds for all x^μ, we must have

$$\boxed{M \sigma_\mu M^\dagger = \sigma_\nu \Lambda^\nu{}_\mu.} \tag{1.25}$$

Theorem 1.1: Transformation of $U^\dagger(x)\sigma_\mu U(x)$

$U^\dagger(x)\sigma_\mu U(x)$ transforms as a four vector.

Proof:

$$
\begin{aligned}
U^\dagger(x)\sigma_\mu U(x) &\rightarrow U'^\dagger(x')\sigma_\mu U'(x') \\
&= (M^\dagger U(x))^\dagger \sigma_\mu M^\dagger U(x) \\
&= U^\dagger(x)\left(M\sigma_\mu M^\dagger\right)U(x) \\
&= U^\dagger(x)\sigma_\nu U(x)\Lambda^\nu{}_\mu
\end{aligned}
\tag{1.26}
$$

so we find that $U^\dagger(x)\sigma_\mu U(x)$ transforms as a four vector as claimed.

Theorem 1.2: Transformation of partials.

The four-gradient coordinates transform as a four vector

$$
(\partial_\mu)' = (\Lambda^{-1})^\sigma{}_\mu \partial_\sigma.
$$

Proof[1]: Inverting the transformation relation

$$
x'^\mu = \Lambda^\mu{}_\nu x^\nu,
\tag{1.27}
$$

gives

$$
x^\sigma = (\Lambda^{-1})^\sigma{}_\mu \Lambda^\mu{}_\nu x^\nu = (\Lambda^{-1})^\sigma{}_\mu x'^\mu,
\tag{1.28}
$$

so

$$
\begin{aligned}
\partial_\mu &\rightarrow (\partial_\mu)' \\
&= \frac{\partial}{\partial x'^\mu} \\
&= \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial}{\partial x^\sigma} \\
&= (\Lambda^{-1})^\sigma{}_\mu \frac{\partial}{\partial x^\sigma} \\
&= (\Lambda^{-1})^\sigma{}_\mu \partial_\sigma.
\end{aligned}
\tag{1.29}
$$

Theorem 1.3: Transformation of $U^\dagger\sigma^\mu\partial_\mu U$

$U^\dagger\sigma^\mu\partial_\mu U$ transforms as a four vector.

[1] In class we proved this by considering the transformation properties of a direction derivative $dx^\mu \cdot \partial_\mu$, but that isn't the method that seems most intuitive to me.

Proof:

$$
\begin{aligned}
U^\dagger(x)\sigma^\mu \frac{\partial}{\partial x^\mu} U(x) \to {U'}^\dagger(x')\sigma^\mu \frac{\partial}{\partial {x'}^\mu} U'(x') \\
&= \Lambda^\mu{}_\nu U^\dagger(x)\sigma^\nu \frac{\partial}{\partial {x'}^\mu} (\Lambda^{-1})^{\mu'}{}_\mu U(x) \\
&= U^\dagger(x)\sigma^\nu \frac{\partial}{\partial {x'}^\mu} \delta^{\mu'}{}_\nu U(x) \\
&= U^\dagger(x)\sigma^\nu \frac{\partial}{\partial x^\nu} U(x)
\end{aligned}
\tag{1.30}
$$

We can now define

Definition 1.1: Weyl action (name?)

We may construct the following Lorentz invariant action

$$
S_{\text{Weyl}} = \int d^4x iU^\dagger(x)\sigma^\mu\partial_\mu U(x),
$$

where $U(x)$ is a Weyl spinor.

The i factor here is so that the action is real. This can be seen by noting that $(i\sigma^\mu)^\dagger = -i\sigma^\mu$ and integrating the Hermitian conjugate by parts

$$
\left(i\sigma^0\right)^\dagger = \left(\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right)^\dagger = -i\sigma^0 \tag{1.31a}
$$

$$
\left(i\sigma^1\right)^\dagger = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}^\dagger = -i\sigma^1 \tag{1.31b}
$$

$$
\left(i\sigma^2\right)^\dagger = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^\dagger = -i\sigma^2 \tag{1.31c}
$$

$$
\left(i\sigma^3\right)^\dagger = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^\dagger = -i\sigma^3 \tag{1.31d}
$$

$$
\begin{aligned}
S^\dagger_{\text{Weyl}} &= \int d^4x\partial_\mu U^\dagger(x)(i\sigma^\mu)^\dagger U(x) \\
&= -\int d^4x\partial_\mu U^\dagger(x) i\sigma^\mu U(x) \\
&= -\int d^4x\partial_\mu \left(U^\dagger(x) i\sigma^\mu U(x)\right) + \int d^4x U^\dagger(x) i\sigma^\mu \partial_\mu U(x) \\
&= \int d^4x U^\dagger(x) i\sigma^\mu\partial_\mu U(x) \\
&= S_{\text{Weyl}},
\end{aligned}
\tag{1.32}
$$

where it was assumed that any boundary terms vanish.

Theorem 1.4: Weyl equation.

Variation of the action definition 1.1 gives rise to the equations of motion

$$\sigma^\mu \frac{\partial}{\partial x^\mu} U = 0$$

which is called the Weyl equation.

Proof:

$$\begin{aligned}
\delta S &= i \int d^4x \left(\delta U^\dagger \sigma^\mu \partial_\mu U + U^\dagger \sigma^\mu \partial_\mu \delta U \right) \\
&= i \int d^4x \left(\delta U^\dagger \sigma^\mu \partial_\mu U + \partial_\mu \left(U^\dagger \sigma^\mu \delta U \right) - (\partial_\mu U^\dagger) \sigma^\mu \delta U \right) \\
&= i \int d^4x \left(\delta U^\dagger \left(\sigma^\mu \partial_\mu U \right) - \left((\partial_\mu U^\dagger) \sigma^\mu \right) \delta U \right) \\
&= \int d^4x \left(\delta U^\dagger \left(i \sigma^\mu \partial_\mu U \right) + \left(\delta U^\dagger \left(i \sigma^\mu \partial_\mu U \right) \right)^\dagger \right).
\end{aligned} \tag{1.33}$$

Requiring this to vanish for all variations $\delta U^\dagger$ proves the result.

Written out explicitly in matrix form, the Weyl equation is

$$\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0, \tag{1.34}$$

or

$$(\partial_0 + \partial_3) U_1 + (\partial_1 - i\partial_2) U_2 = 0 \tag{1.35a}$$

$$(\partial_1 + i\partial_2) U_1 + (\partial_0 - \partial_3) U_2 = 0. \tag{1.35b}$$

Theorem 1.5: Weyl equation relation to the massless KG equation.

The Weyl equation is equivalent to a set of massless KG equations.

$$\partial_\mu \partial^\mu U_k = 0,$$

for $k = 1, 2$.

Proof:
Multiplying eq. (1.35a) by $\partial_1 + i\partial_2$ gives

$$\begin{aligned}
(\partial_1 + i\partial_2)\left((\partial_0 + \partial_3)U_1 + (\partial_1 - i\partial_2)U_2\right) &= (\partial_0 + \partial_3)(\partial_1 + i\partial_2)U_1 + (\partial_1 + i\partial_2)(\partial_1 - i\partial_2)U_2 \\
&= -(\partial_0 + \partial_3)(\partial_0 - \partial_3)U_2 + (\partial_1 + i\partial_2)(\partial_1 - i\partial_2)U_2 \\
&= (-\partial_{00} + \partial_{33} + \partial_{11} + \partial_{22})U_2 \\
&= \left(-\partial_0\partial^0 - \partial_3\partial^3 - \partial_1\partial^1 - \partial_2\partial^2\right)U_2 \\
&= -\partial_\mu\partial^\mu U_2.
\end{aligned} \tag{1.36}$$

Similarly, multiplying eq. (1.35b) by $\partial_1 - i\partial_2$ we find

$$\begin{aligned}
0 &= (\partial_1 - i\partial_2)\left((\partial_1 + i\partial_2)U_1 + (\partial_0 - \partial_3)U_2\right) \\
&= (\partial_{11} + \partial_{22})U_1 + (\partial_0 - \partial_3)\underbrace{(\partial_1 - i\partial_2)U_2}_{=-(\partial_0+\partial_3)U_1} \\
&= (\partial_{11} + \partial_{22} - \partial_{00} + \partial_{33})U_1 \\
&= -\partial_\mu\partial^\mu U_1.
\end{aligned} \tag{1.37}$$

Because S_{Weyl} results in a massless KG equation, this is no good for electrons, and we have to look for a different action.

Claim: $U^{\text{T}}\sigma_2 U$ is the only bilinear Lorentz invariant that we can add to the action.
An action like:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}mU^{\text{T}}\sigma_2 U + \frac{1}{2}m^* U^\dagger\sigma_2(U^\dagger)^{\text{T}}, \tag{1.38}$$

may exist in nature (we don't know), and are called Majorana neutrino masses. The problem with such a Lagrangian density is that it breaks $U(1)$ symmetry. In particular $U \to e^{i\alpha}U$ symmetry of the kinetic term. This means that the particle associated with such a Lagrangian cannot be charged.

Recall that we introduced electromagnetic potentials into NRQM with

$$i\hbar\frac{\partial}{\partial t}\Psi = \frac{1}{2m}(\boldsymbol{\nabla} - e\mathbf{A})^2\,\Psi \tag{1.39}$$

which is a gauge transformation. We'd like to have this capability.

What we can do instead and maintain $U(1)$ symmetries, is to introduce two U's, like

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}mU_1^{\text{T}}\sigma_2 U_2 + \frac{1}{2}m^* U_2^\dagger\sigma_2(U_1^\dagger)^{\text{T}} \tag{1.40}$$

What we are really doing is assembling a four component spinor out of the two U's.

1.4 Lorentz symmetry.

We want to examine the Lorentz invariance of $U^{\mathrm{T}}\sigma_2 U$, but need an intermediate result first.

Lemma 1.1: Transpose of Pauli vector representation

For any $\mathbf{x} \in \mathbb{R}^3$

$$(\boldsymbol{\sigma} \cdot \mathbf{x})^{\mathrm{T}} = -\sigma^2 (\boldsymbol{\sigma} \cdot \mathbf{x}) \sigma^2,$$

or more compactly

$$\boldsymbol{\sigma}^{\mathrm{T}} = -\sigma^2 \boldsymbol{\sigma} \sigma^2.$$

Geometrically, this transposition operation reflects $\mathbf{x}$ about the y-axis.

Proving lemma 1.1 is well suited to software (FIXME: link: diracWeylMatrixRepresentationAndIdentities.nb), but can also be done algebraically with ease. First note that

$$\begin{aligned}\sigma_1^{\mathrm{T}} &= \sigma_1 \\ \sigma_2^{\mathrm{T}} &= -\sigma_2 \\ \sigma_3^{\mathrm{T}} &= \sigma_3\end{aligned} \tag{1.41}$$

which means that

$$\begin{aligned}(\boldsymbol{\sigma} \cdot \mathbf{x})^{\mathrm{T}} &= \sigma^1 x^1 - \sigma^2 x^2 + \sigma^3 x^3 \\ &= \sigma^2 \sigma^2 \left(\sigma^1 x^1 - \sigma^2 x^2 + \sigma^3 x^3 \right) \\ &= \sigma^2 \left(-\sigma^1 x^1 - \sigma^2 x^2 - \sigma^3 x^3 \right) \sigma^2 \\ &= -\sigma^2 (\boldsymbol{\sigma} \cdot \mathbf{x})^{\mathrm{T}} \sigma^2.\end{aligned} \tag{1.42}$$

Now we are ready to proceed.

Theorem 1.6: $U^{\mathrm{T}}\sigma_2 U$ invariance

$U^{\mathrm{T}}\sigma_2 U$ is Lorentz invariant.

Proof:

$$\begin{aligned}U^{\mathrm{T}}\sigma_2 U &\rightarrow U'^{\mathrm{T}} \sigma_2 U' \\ &= U^{\mathrm{T}} M^{\dagger\mathrm{T}} \sigma_2 M^{\dagger} U,\end{aligned} \tag{1.43}$$

where $U' = M^{\dagger} U$ and $U'^{\mathrm{T}} = U^{\mathrm{T}} M^{\dagger T}$.

Note that if we can show that $M^{\dagger\mathrm{T}} \sigma_2 M^{\dagger} = \sigma_2$, then we are done.

It is simple to show that any

$$U = e^{i\sigma\cdot\mathbf{a}}, \tag{1.44}$$

for $\mathbf{a} \in \mathbb{R}^3$, has eigenvalues $\pm i|\mathbf{a}|$. The determinant of such a matrix is

$$\det U = \begin{vmatrix} e^{i|\mathbf{a}|} & 0 \\ 0 & e^{-i|\mathbf{a}|} \end{vmatrix} = 1, \tag{1.45}$$

so we see that such a matrix has the $U^\dagger U = 1$ and $\det U = 1$ properties that we desire for elements of $SU(2)$[2]. We haven't shown that all matrices $U \in SU(2)$ can be written in this form, but let's assume that's the case.

Claim: Generalizing from the exponential form of $SU(2)$ elements seen above, we assume that any $SL(2,\mathbb{C})$ matrix M can be written as

$$M^\dagger = e^{i\sigma\cdot(\mathbf{a}+i\mathbf{b})}, \tag{1.46}$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

The transpose of an exponential of a sigma matrix goes like

$$\begin{aligned}
\left(e^{\sigma\cdot\mathbf{u}}\right)^{\mathrm{T}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left((\sigma\cdot\mathbf{u})^k \right)^{\mathrm{T}} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\sigma_2(\sigma\cdot\mathbf{u})\sigma_2\right)^k \\
&= \sigma_2 \left(\sum_{k=0}^{\infty} \frac{1}{k!} (-\sigma\cdot\mathbf{u})^k \right) \sigma_2 \\
&= \sigma_2 e^{-\sigma\cdot\mathbf{u}} \sigma_2,
\end{aligned} \tag{1.47}$$

so

$$\begin{aligned}
{M^\dagger}^{\mathrm{T}} \sigma_2 M^\dagger &= \left(e^{i\sigma\cdot(\mathbf{a}+i\mathbf{b})}\right)^{\mathrm{T}} \sigma_2 e^{i\sigma\cdot(\mathbf{a}+i\mathbf{b})} \\
&= \left(\sigma_2 e^{-i\sigma\cdot(\mathbf{a}+i\mathbf{b})} \sigma_2\right) \sigma_2 e^{i\sigma\cdot(\mathbf{a}+i\mathbf{b})} \\
&= \sigma_2,
\end{aligned} \tag{1.48}$$

which is the result required to finish the proof of theorem 1.6[3].

1.5 Dirac matrices.

[2]In class the suitability of $e^{i\sigma\cdot\mathbf{a}}$ as an element of $SU(2)$ was demonstrated with an argument that diagonalizable matrices satisfy $\det e^A = e^{\operatorname{tr} A}$

[3]A slightly different derivation was done in class, but this one makes more sense to me.

Definition 1.2: Dirac matrices.

The Dirac matrices γ^μ, $\mu \in \{0,1,2,3\}$ are matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

that is

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},$$

We will use the explicit 4×4 matrix representation

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}.$$

The metric relations can also be written explicitly in the handy form

$$\begin{aligned} \left(\gamma^0\right)^2 &= 1 \\ \left(\gamma^i\right)^2 &= -1. \end{aligned} \tag{1.49}$$

Written out explicitly, these matrices are

$$\begin{aligned} \gamma^0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ \gamma^2 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \qquad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{1.50}$$

We will see (HW4) that Lorentz transformations take the form

$$x' \cdot \gamma = \Lambda_{1/2}^{-1} \left(x \cdot \gamma\right) \Lambda_{1/2}, \tag{1.51}$$

where

$$\Lambda_{1/2} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}, \tag{1.52}$$

where

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]. \tag{1.53}$$

In particular

$$
\begin{aligned}
S^{0k} &= \frac{i}{4}\left[\gamma^0, \gamma^k\right] \\
&= \frac{i}{2}\gamma^0\gamma^k \\
&= \frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \\
&= \frac{i}{2}\begin{bmatrix} -\sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix}
\end{aligned}
\tag{1.54}
$$

will generate boosts, whereas

$$
\begin{aligned}
S^{jk} &= \frac{i}{4}\left[\gamma^j, \gamma^k\right] \\
&= \frac{i}{2}\gamma^j\gamma^k \\
&= \frac{i}{2}\begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \\
&= -\frac{i}{2}\begin{bmatrix} \sigma^k\sigma^j & 0 \\ 0 & \sigma^k\sigma^j \end{bmatrix} \\
&= \frac{1}{2}\epsilon^{jkl}\begin{bmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{bmatrix},
\end{aligned}
\tag{1.55}
$$

are rotations (and in this case, are Hermitian).

The explicit expansion of the half Lorentz transformation operator is

$$
\begin{aligned}
\Lambda_{1/2} &= e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \\
&= e^{-i\omega_{0k}S^{0k}-\frac{i}{2}\omega_{jk}S^{jk}} \\
&= \exp\left(-\frac{1}{2}\begin{bmatrix} \omega_{0k}\sigma^k & 0 \\ 0 & -\omega_{0k}\sigma^k \end{bmatrix} - \frac{i}{4}\begin{bmatrix} \omega_{jk}\epsilon^{jkl}\sigma^l & 0 \\ 0 & \omega_{jk}\epsilon^{jkl}\sigma^l \end{bmatrix}\right) \\
&= \begin{bmatrix} e^{-\left(\frac{1}{2}\omega_{0k}\sigma_0+\frac{i}{4}\omega_{jk}e^{jkl}\sigma^l\right)} & 0 \\ 0 & e^{-\left(-\frac{1}{2}\omega_{0k}\sigma_0+\frac{i}{4}\omega_{jk}e^{jkl}\sigma^l\right)} \end{bmatrix}
\end{aligned}
\tag{1.56}
$$

where the $1/2$ factor of ω_{0i} vanished because we had a sum over $0i$ and $i0$ which have been grouped.

Lemma 1.2: Some Dirac matrix identities.

$$
\begin{aligned}
(\gamma^0)^\dagger &= \gamma^0 \\
(\gamma^k)^\dagger &= -\gamma^k
\end{aligned}
$$

$$\gamma^0 (i\gamma^\mu)^\dagger \gamma^0 = -i\gamma^\mu.$$

The first two are clear from inspection of eq. (1.50). For the last, for $\mu = 0$

$$\begin{aligned}
\gamma^0 (i\gamma^0)^\dagger \gamma^0 &= \gamma^0 (\gamma^0)^\dagger (-i) \gamma^0 \\
&= -i\gamma^0\gamma^0\gamma^0 \\
&= -i\gamma^0,
\end{aligned} \tag{1.57}$$

and for $\mu = k \neq 0$

$$\begin{aligned}
\gamma^0 (i\gamma^k)^\dagger \gamma^0 &= \gamma^0 (-i)(-\gamma^k) \gamma^0 \\
&= +i\gamma^0\gamma^k\gamma^0 \\
&= -i\gamma^0\gamma^0\gamma^k \\
&= -i\gamma^k,
\end{aligned} \tag{1.58}$$

which completes the proof.

1.6 Dirac Lagrangian.

We postulate that there is a four-component object

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \qquad \Psi^\dagger = \left(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^* \right), \tag{1.59}$$

where ψ_μ's are all complex fields, and assume that the fields transform as

$$\Psi(x) \rightarrow \Psi'(x') = \Lambda_{1/2} \Psi(x), \tag{1.60}$$

where our vectors transform in the usual $x \rightarrow x' = \Lambda x$ fashion, where the incremental form of the Lorentz transformation is the usual

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + O(\omega^2). \tag{1.61}$$

Definition 1.3: Overbar operator (name?).

$$\overline{\Psi} = \Psi^\dagger \gamma^0.$$

Definition 1.4: Dirac Lagrangian.

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\Psi(x).$$

Armed with lemma 1.2 we can now show the following.

Theorem 1.7: The Dirac action is a real Lorentz scalar.

The action

$$S = \int d^4x\overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi,$$

is a real scalar and is Lorentz invariant.

Real: To show that the action is real, we compute it's Hermitian conjugate, apply lemma 1.2 and integrate by parts

$$\begin{aligned}
S^\dagger &= \int d^4x\Psi^\dagger\left(-i(\gamma^\mu)^\dagger \overleftarrow{\partial}_\mu - m\right)(\gamma^0)^\dagger\Psi \\
&= \int d^4x\Psi^\dagger\left((i\gamma^\mu)^\dagger \overleftarrow{\partial}_\mu - m\right)\gamma^0\Psi \\
&= \int d^4x\Psi^\dagger\left(-\gamma^0(i\gamma^\mu)\gamma^0 \overleftarrow{\partial}_\mu - m\right)\gamma^0\Psi \\
&= \int d^4x\overline{\Psi}\left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right)\Psi \\
&= -\int d^4x\partial_\mu\left(\overline{\Psi}i\gamma^\mu\Psi\right) + \int d^4x\overline{\Psi}i\gamma^\mu\partial_\mu\Psi - \int d^4x\overline{\Psi}m\Psi \\
&= \int d^4x\overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi \\
&= S,
\end{aligned} \tag{1.62}$$

where ∂_μ without an overarrow means the traditional right acting operator, and assuming that the boundary terms vanish.

To show the Lorentz invariance, we will consider just the transformation of the Dirac Lagrangian density. We need a couple additional pieces of information to do so, the first of which is the transformation property[4]

$$\overline{\Psi} \rightarrow \Psi\Lambda_{1/2}^{-1}, \tag{1.63}$$

and (from HW4)

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\alpha\gamma^\alpha. \tag{1.64}$$

[4] Not proven here, but there's an argument for that in [1] (eq. 3.33).

The Lagrangian transforms as

$$
\begin{aligned}
\overline{\Psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\Psi(x) \to \overline{\Psi'}(x')\gamma^0\left(i\gamma^\mu\frac{\partial}{\partial x'^\mu} - m\right)\Psi'(x') \\
= \overline{\Psi}(x)\Lambda_{1/2}^{-1}\left(i\gamma^\mu{(\Lambda^{-1})^\alpha}_\mu\partial_\alpha - m\right)\Lambda_{1/2}\Psi(x) \\
= \overline{\Psi}(x)\left(i\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}{(\Lambda^{-1})^\alpha}_\mu\partial_\alpha - m\right)\Psi(x) \\
= \overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi
\end{aligned}
\tag{1.65}
$$

We find that $\overline{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi$ is a Lorentz scalar, whereas $\overline{\Psi}\gamma^\mu\Psi$ is a 4 vector.

1.7 Problems:

Exercise 1.1 Show that $\overline{\Psi}\Psi$ is a Lorentz scalar.

Answer for Exercise 1.1

The Lorentz property follows from eq. (1.63)

$$
\begin{aligned}
\overline{\Psi}\Psi &\to \left(\overline{\Psi}\Lambda_{1/2}^{-1}\right)\left(\Lambda_{1/2}\Psi\right) \\
&= \overline{\Psi}\Psi.
\end{aligned}
\tag{1.66}
$$

The scalar nature of this product can be seen easily by expansion.

$$
\begin{aligned}
\overline{\Psi}\Psi &= \Psi^\dagger\gamma^0\Psi \\
&= \begin{bmatrix}\Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^*\end{bmatrix}\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{bmatrix} \\
&= \begin{bmatrix}\Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^*\end{bmatrix}\begin{bmatrix}\Psi_3\\ \Psi_4\\ \Psi_1\\ \Psi_2\end{bmatrix} \\
&= \Psi_1^*\Psi_3 + \Psi_2^*\Psi_4 + \Psi_3^*\Psi_1 + \Psi_4^*\Psi_2 \\
&= 2\,\mathrm{Re}\left(\Psi_1^*\Psi_3 + \Psi_2^*\Psi_4\right).
\end{aligned}
\tag{1.67}
$$

Clearly any individual $\Psi^\dagger\gamma^\mu\Psi$ product will also be a scalar.

Exercise 1.2 Show that $\overline{\Psi}\gamma^\mu\Psi$ transforms as a four vector.

Answer for Exercise 1.2

$$
\begin{aligned}
\overline{\Psi}\gamma^{\mu}\Psi &\to \left(\overline{\Psi}\Lambda_{1/2}^{-1}\right)\gamma^{\mu}\left(\Lambda_{1/2}\Psi\right) \\
&= \overline{\Psi}\left(\Lambda_{1/2}^{-1}\gamma^{\mu}\Lambda_{1/2}\right)\Psi \\
&= \overline{\Psi}\left(\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\right)\Psi \\
&= \Lambda^{\mu}{}_{\nu}\overline{\Psi}\gamma^{\nu}\Psi.
\end{aligned} \tag{1.68}
$$

Bibliography

[1] Michael E Peskin and Daniel V Schroeder. *An introduction to Quantum Field Theory*. Westview, 1995. 4