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PHY2403H Quantum Field Theory. Lecture 21, Part I: Dirac equation solutions, orthogonality conditions, direct products. Taught by Prof. Erich Poppitz

DISCLAIMER: Rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Review.

We were studying the Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi} \left(i\gamma^\mu \partial_\mu - m \right) \Psi, \tag{1.1}$$

from which we find

$$\left(i\gamma^\mu \partial_\mu - m \right) \Psi = 0, \tag{1.2}$$

the Dirac equation, and saw that solutions to this equation satisfies the KG equation. We found solution

$$\Psi(x) = u(p) e^{-ip\cdot x}, \tag{1.3}$$

which is automatically a solution to the KG equation. There are actually two linearly independent solutions

$$u^s(p) = \begin{bmatrix} \sqrt{p\cdot\sigma}\zeta^s \\ \sqrt{p\cdot\overline{\sigma}}\zeta^s \end{bmatrix}, \tag{1.4}$$

where $\zeta^1 = (1,0)^\text{T}, \zeta^2 = (0,1)^\text{T}$.

1.2 Normalization.

Theorem 1.1: $u^\dagger u$

$$u^{r\dagger}u^s = 2p_0\delta^{rs}.$$

Proof:

$$\begin{aligned} u^{s\dagger}u^r &= \begin{bmatrix}\zeta^{s\dagger}\sqrt{p\cdot\sigma} & \zeta^{s\dagger}\sqrt{p\cdot\bar{\sigma}}\end{bmatrix}\begin{bmatrix}\sqrt{p\cdot\sigma}\zeta \\ \sqrt{p\cdot\bar{\sigma}}\zeta\end{bmatrix} \\ &= \zeta^{s\dagger}\left(\sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}}\right)\zeta^r \\ &= \zeta^{s\dagger}\left(p\cdot\sigma + p\cdot\bar{\sigma}\right)\zeta^r \\ &= \zeta^{s\dagger}\left(p_0 - \mathbf{p}\cdot\boldsymbol{\sigma} + p_0 + \mathbf{p}\cdot\boldsymbol{\sigma}\right)\zeta^r \\ &= 2p_0\zeta^{s\dagger}\zeta^r. \end{aligned} \tag{1.5}$$

We can easily see that $\zeta^{s\dagger}\zeta^r = \delta^{rs}$ by writing out those products

$$\begin{aligned} \zeta^{1\dagger}\zeta^1 &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} &= 1 \\ \zeta^{1\dagger}\zeta^2 &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} &= 0 \\ \zeta^{2\dagger}\zeta^1 &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} &= 0 \\ \zeta^{2\dagger}\zeta^2 &= \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} &= 1, \end{aligned} \tag{1.6}$$

which completes the proof.

We also want to compute $\bar{u}u$, but need a couple intermediate results.

Lemma 1.1: Products of $p\cdot\sigma, p\cdot\bar{\sigma}$.

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p\cdot\bar{\sigma})(p\cdot\sigma) = m^2.$$

Proof:

$$\begin{aligned} (p\cdot\sigma)(p\cdot\bar{\sigma}) &= \left(p^0 - \mathbf{p}\cdot\boldsymbol{\sigma}\right)\left(p^0 + \mathbf{p}\cdot\boldsymbol{\sigma}\right) \\ &= (p^0)^2 - (\mathbf{p}\cdot\boldsymbol{\sigma})^2 \\ &= (p^0)^2 - \mathbf{p}^2 \\ &= m^2 \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} (p\cdot\bar{\sigma})(p\cdot\sigma) &= \left(p^0 + \mathbf{p}\cdot\boldsymbol{\sigma}\right)\left(p^0 - \mathbf{p}\cdot\boldsymbol{\sigma}\right) \\ &= (p^0)^2 - (\mathbf{p}\cdot\boldsymbol{\sigma})^2 \\ &= (p^0)^2 - \mathbf{p}^2 \\ &= m^2. \end{aligned} \tag{1.8}$$

Theorem 1.2: $\bar{u}u$.

$$\bar{u}^r(\mathbf{p})u^s(\mathbf{p}) = 2m\delta^{rs}.$$

Proof:

$$\begin{aligned}
\bar{u}^r u^s &= u^{r\dagger}\gamma^0 u^s \\
&= \begin{bmatrix} \zeta^{r\dagger}\sqrt{p\cdot\sigma} & \zeta^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{p\cdot\sigma}\zeta^s \\ \sqrt{p\cdot\bar{\sigma}}\zeta^s \end{bmatrix} \\
&= \zeta^{r\dagger}\left(\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} + \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\right)\zeta^s \\
&= 2m\zeta^{r\dagger}\zeta^s \\
&= 2m\delta^{rs},
\end{aligned} \tag{1.9}$$

which completes the proof.

1.3 Other solution.

Now we seek the other plane wave solution

$$\Psi(x) = v(p)e^{ip\cdot x}. \tag{1.10}$$

It can be demonstrated (exercise 1.1) that the solution has the form

$$\boxed{v^s(p) = \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix},} \tag{1.11}$$

where $\eta^1 = (1,0)^{\mathrm{T}}, \eta^2 = (0,1)^{\mathrm{T}}$.

Theorem 1.3: v normalization.

$$\begin{aligned}
\bar{v}^r(p)v^s(p) &= -2m\delta^{rs} \\
v^{r\dagger}(p)v^s(p) &= 2p^0\delta^{rs}.
\end{aligned}$$

Theorem 1.3 is proven in exercise 1.2.

It will also be useful to restate the $2\delta^{rs}p_0$ normalization conditions as

$$\begin{aligned}
u^{r\dagger}(\mathbf{p})u^s(\mathbf{p}) &= 2\omega_{\mathbf{p}}\delta^{sr} \\
v^{r\dagger}(\mathbf{p})v^s(\mathbf{p}) &= 2\omega_{\mathbf{p}}\delta^{sr}.
\end{aligned} \tag{1.12}$$

Various orthogonality conditions exist between the u's and v's

Theorem 1.4: Dirac adjoint orthogonality conditions.

$$\bar{u}^r(p)v^s(p) = 0$$
$$\bar{v}^r(p)u^s(p) = 0.$$

Proof left to exercise 1.3.

Theorem 1.5: Dagger orthogonality conditions.

$$v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0$$
$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = 0.$$

Proof left to exercise 1.4.

Finally, there are a couple tensor products of interest.

Definition 1.1: Tensor product.

Given a pair of vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

the tensor product is the matrix of all elements $x_i y_j$

$$x \otimes y^{\mathrm{T}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \otimes \begin{bmatrix} y_1 \cdots y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ x_3 y_1 & \ddots & & \\ \vdots & & & \\ x_n y_1 & \cdots & & x_n y_n \end{bmatrix}.$$

Theorem 1.6: Direct product relations.

$$\sum_{s=1}^{2} u^s(p) \otimes \bar{u}^s(p) = \gamma \cdot p + m$$
$$\sum_{s=1}^{2} v^s(p) \otimes \bar{v}^s(p) = \gamma \cdot p - m$$

For the v's

$$\begin{aligned}
&\sum_{s=1,2} \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix} \otimes \begin{bmatrix} (\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} & -(\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \sum_{s=1,2} \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ \sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix} \otimes \begin{bmatrix} -(\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} & (\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} \end{bmatrix} \\
&= \sum_{s=1,2} \begin{bmatrix} -\sqrt{p\cdot\sigma}\eta^s \otimes (\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\sigma}\eta^s \otimes (\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\eta^s \otimes (\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} & -\sqrt{p\cdot\bar{\sigma}}\eta^s \otimes (\eta^s)^{\mathrm{T}}\sqrt{p\cdot\sigma} \end{bmatrix},
\end{aligned} \tag{1.13}$$

but

$$\begin{aligned}
\eta^1 \otimes \eta^{1T} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned} \tag{1.14}$$

and

$$\begin{aligned}
\eta^2 \otimes \eta^{2T} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned} \tag{1.15}$$

so $\sum_{s=1,2} \eta^s \otimes \eta^{s\mathrm{T}} = 1$, leaving

$$\begin{aligned}
\sum_{s=1}^{2} v^s(p) \otimes \bar{v}^s(p) &= \begin{bmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} \end{bmatrix} \\
&= \begin{bmatrix} -\sqrt{p\cdot\sigma p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma} p\cdot\bar{\sigma}} & -\sqrt{p\cdot\bar{\sigma} p\cdot\bar{\sigma}} \end{bmatrix} \\
&= \begin{bmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{bmatrix} \\
&= -m\mathbf{1} + p^0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \mathbf{p}\cdot \begin{bmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \\
&= -m + p^\mu \gamma_\mu,
\end{aligned} \tag{1.16}$$

as stated. Proof for the u's is left to exercise 1.5.

1.4 Problems:

Exercise 1.1 Verify $v(p)$ solution.

Show that eq. (1.11) is a solution of the Dirac equation.

Answer for Exercise 1.1

Let $D = \left(i\gamma^\mu \partial_\mu - m\right)$ represent the Dirac operator. Applying to $e^{ip\cdot x}$ we have

$$\begin{aligned}
De^{ip\cdot x} &= \left(i\gamma^\mu \partial_\mu - m\right) e^{ip_\mu x^\mu} \\
&= -\left(\gamma^\mu p_\mu + m\right) e^{ip\cdot x} \\
&= -\left(m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + p_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_k \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix} \right) e^{ip\cdot x} \\
&= -\begin{bmatrix} m & p_0\sigma^0 + p_k\sigma^k \\ p_0\sigma^0 - p_k\sigma^k & m \end{bmatrix} e^{ip\cdot x} \\
&= -\begin{bmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{bmatrix} e^{ip\cdot x}.
\end{aligned} \tag{1.17}$$

We are now set to apply the Dirac operator to eq. (1.11)

$$\begin{aligned}
Dv(p) &= \begin{bmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{bmatrix} \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix}, e^{ip\cdot x} \\
&= -\begin{bmatrix} \left(m\sqrt{p\cdot\sigma} - p\cdot\sigma\sqrt{p\cdot\bar{\sigma}}\right)\eta \\ \left(p\cdot\bar{\sigma}\sqrt{p\cdot\sigma} - m\sqrt{p\cdot\bar{\sigma}}\right)\eta \end{bmatrix} e^{ip\cdot x} \\
&= \begin{bmatrix} \sqrt{p\cdot\sigma}\left(m - \sqrt{p\cdot\sigma p\cdot\bar{\sigma}}\right)\eta \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{p\cdot\bar{\sigma} p\cdot\sigma} - m\right)\eta \end{bmatrix} e^{ip\cdot x} \\
&= \begin{bmatrix} \sqrt{p\cdot\sigma}\left(m - \sqrt{m^2}\right)\eta \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{m^2} - m\right)\eta \end{bmatrix} e^{ip\cdot x} \\
&= 0.
\end{aligned} \tag{1.18}$$

Exercise 1.2 $v(p)$ normalization.

Prove theorem 1.3.

Answer for Exercise 1.2

Expanding the matrices gives

$$
\begin{aligned}
\bar{v}^r v^s &= v^{r\dagger}\gamma^0 v^s \\
&= \begin{bmatrix} \eta^{r\mathrm{T}}\sqrt{p\cdot\sigma} & -\eta^{r\mathrm{T}}\sqrt{p\cdot\bar{\sigma}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix} \\
&= \begin{bmatrix} \eta^{r\mathrm{T}}\sqrt{p\cdot\sigma} & -\eta^{r\mathrm{T}}\sqrt{p\cdot\bar{\sigma}} \end{bmatrix} \begin{bmatrix} -\sqrt{p\cdot\bar{\sigma}}\eta^s \\ \sqrt{p\cdot\sigma}\eta^s \end{bmatrix} \\
&= -\eta^{r\mathrm{T}}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\eta^s - \eta^{r\mathrm{T}}\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\eta^s \\
&= \delta^{rs} 2\sqrt{m^2} \\
&= 2m\delta^{rs},
\end{aligned}
\tag{1.19}
$$

and

$$
\begin{aligned}
v^{r\dagger} v^s &= \begin{bmatrix} \eta^{r\mathrm{T}}\sqrt{p\cdot\sigma} & -\eta^{r\mathrm{T}}\sqrt{p\cdot\bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix} \\
&= \eta^{r\mathrm{T}}(p\cdot\sigma)\eta^s + \eta^{r\mathrm{T}}(p\cdot\bar{\sigma})\eta^s \\
&= \delta^{rs}\left(p_0 - \mathbf{p}\cdot\boldsymbol{\sigma} + p_0 + \mathbf{p}\cdot\boldsymbol{\sigma}\right) \\
&= 2p_0\delta^{rs}.
\end{aligned}
\tag{1.20}
$$

Exercise 1.3 $\bar{u}v, \bar{v}u$ relations.

Prove theorem 1.4.

Answer for Exercise 1.3

We need only expand the matrix products

$$
\begin{aligned}
\bar{u}^r v^s &= \begin{bmatrix} \zeta^{r\mathrm{T}}\sqrt{p\cdot\bar{\sigma}} & \zeta^{r\mathrm{T}}\sqrt{p\cdot\sigma} \end{bmatrix} \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix} \\
&= m\zeta^{r\mathrm{T}}\eta^s - m\zeta^{r\mathrm{T}}\eta^s \\
&= 0,
\end{aligned}
\tag{1.21}
$$

and

$$
\begin{aligned}
\bar{v}^r u^s &= \begin{bmatrix} -\eta^{r\mathrm{T}}\sqrt{p\cdot\bar{\sigma}} & \eta^{r\mathrm{T}}\sqrt{p\cdot\sigma} \end{bmatrix} \begin{bmatrix} \sqrt{p\cdot\sigma}\zeta^s \\ \sqrt{p\cdot\bar{\sigma}}\zeta^s \end{bmatrix} \\
&= -m\eta^{r\mathrm{T}}\zeta^s + m\eta^{r\mathrm{T}}\zeta^s \\
&= 0,
\end{aligned}
\tag{1.22}
$$

Exercise 1.4 Dagger orthonormality conditions.

Prove theorem 1.5.

Answer for Exercise 1.4

$$
\begin{aligned}
u^{r\dagger}(\mathbf{p}) v^s(-\mathbf{p}) &= \begin{bmatrix} \zeta^{r\mathrm{T}} \sqrt{p \cdot \sigma} & \zeta^{r\mathrm{T}} \sqrt{p \cdot \bar{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{q \cdot \sigma} \eta^s \\ -\sqrt{q \cdot \bar{\sigma}} \eta^s \end{bmatrix} \Bigg|_{p=(0,\mathbf{p}),q=(0,-\mathbf{p})} \\
&= \begin{bmatrix} \zeta^{r\mathrm{T}} \sqrt{-\mathbf{p} \cdot \boldsymbol{\sigma}} & \zeta^{r\mathrm{T}} \sqrt{\mathbf{p} \cdot \boldsymbol{\sigma}} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{p} \cdot \boldsymbol{\sigma}} \eta^s \\ -\sqrt{-\mathbf{p} \cdot \boldsymbol{\sigma}} \eta^s \end{bmatrix} \\
&= \zeta^{r\mathrm{T}} \sqrt{-\mathbf{p} \cdot \boldsymbol{\sigma}} \sqrt{\mathbf{p} \cdot \boldsymbol{\sigma}} \eta^s - \zeta^{r\mathrm{T}} \sqrt{\mathbf{p} \cdot \boldsymbol{\sigma}} \sqrt{-\mathbf{p} \cdot \boldsymbol{\sigma}} \eta^s \\
&= 0.
\end{aligned} \tag{1.23}
$$

Exercise 1.5 Direct product relation for the u's.

Prove the u direct product relations of theorem 1.6.

Answer for Exercise 1.5

$$
\begin{aligned}
\sum u^s \otimes \bar{u}^s &= \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^s \\ \sqrt{p \cdot \bar{\sigma}} \zeta^s \end{bmatrix} \otimes \begin{bmatrix} \zeta^{s\mathrm{T}} \sqrt{p \cdot \bar{\sigma}} & \zeta^{s\mathrm{T}} \sqrt{p \cdot \sigma} \end{bmatrix} \\
&= \sum \begin{bmatrix} \sqrt{p \cdot \sigma} \zeta^s \otimes \zeta^{s\mathrm{T}} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \zeta^s \otimes \zeta^{s\mathrm{T}} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \zeta^s \otimes \zeta^{s\mathrm{T}} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \zeta^s \otimes \zeta^{s\mathrm{T}} \sqrt{p \cdot \sigma} \end{bmatrix} \\
&= \begin{bmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{bmatrix} \\
&= m + p \cdot \gamma.
\end{aligned} \tag{1.24}
$$