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PHY2403H Quantum Field Theory. Lecture 21, Part II: Dirac Hamiltonian, Hamiltonian eigenvalues, general solution, creation and annihilation operators, Dirac Sea, anti-electrons. Taught by Prof. Erich Poppitz

DISCLAIMER: Very rough notes from class, with some additional side notes. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

1.1 Lagrangian.

Theorem 1.1: Dirac Hamiltonian.

The Dirac Hamiltonian is

$$H = \int d^3 x \Psi^\dagger \left(-i\gamma^0 \gamma^j \partial_j \Psi + m\gamma^0 \right) \Psi.$$

To prove theorem 1.1, we start with the spacetime expansion of the Dirac Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{Dirac}} &= \overline{\Psi} i \gamma^0 \partial_0 \Psi + i \overline{\Psi} \gamma^j \partial_j \Psi - m \overline{\Psi} \Psi \\ &= \Psi^\dagger \gamma^0 i \gamma^0 \partial_0 \Psi + i \Psi \gamma^0 \gamma^j \partial_j \Psi - m \Psi^\dagger \gamma^0 \Psi \\ &= \Psi^\dagger i \dot{\Psi} + i \Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi - m \Psi^\dagger \gamma^0 \Psi. \end{aligned} \tag{1.1}$$

We see that the momentum conjugate to Ψ is

$$\begin{aligned} \pi_\psi &= \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} \\ &= i \Psi^\dagger. \end{aligned} \tag{1.2}$$

Computing the Hamiltonian density in the usual way, we have

$$\begin{aligned}\mathcal{H}_{\text{Dirac}} &= \pi_\Psi \dot{\Psi} - \mathcal{L} \\ &= i\Psi^\dagger \dot{\Psi} - \left(\Psi^\dagger i\dot{\Psi} + i\Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi - m\Psi^\dagger \gamma^0 \Psi\right) \\ &= -i\Psi^\dagger \gamma^0 \gamma^j \partial_j \Psi + m\Psi^\dagger \gamma^0 \Psi.\end{aligned} \tag{1.3}$$

Integrating over a 3-volume provides the Dirac Hamiltonian of theorem 1.1.

Now we want to examine the action of $-i\gamma^0\gamma^j\partial_j + m\gamma^0 = \gamma^0 \left(-i\gamma^j\partial_j + m\right)$ on the plane wave solutions we have found.

Theorem 1.2: Hamiltonian action on Dirac plane wave solutions.

For $\Psi_u = u(p)e^{-ip\cdot x}$, and $\Psi_v = v(p)e^{ip\cdot x}$, we have

$$\begin{aligned}-\gamma^0 \left(i\gamma^j\partial_j - m\right)\Psi_u &= p_0\Psi_u \\ -\gamma^0 \left(i\gamma^j\partial_j - m\right)\Psi_v &= -p_0\Psi_v.\end{aligned}$$

Theorem 1.2, which shows that Ψ_u, Ψ_v are eigenvectors of the operator $\gamma^0 \left(-i\gamma^j\partial_j + m\right)$ with eigenvalues $\pm\omega_{\mathbf{p}}$. These eigenvalue equations follow from the Dirac equation for Ψ_u, Ψ_v. These are

$$\begin{aligned}\left(i\gamma^\mu\partial_\mu - m\right)ue^{-ip\cdot x} &= \left(i\gamma^j\partial_j + i\gamma^0\partial_0 - m\right)ue^{-ip\cdot x} \\ &= \left(i\gamma^j\partial_j + i(-i)\gamma^0 p_0 - m\right)ue^{-ip\cdot x}\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}\left(i\gamma^\mu\partial_\mu - m\right)ve^{ip\cdot x} &= \left(i\gamma^j\partial_j + i\gamma^0\partial_0 - m\right)ve^{ip\cdot x} \\ &= \left(i\gamma^j\partial_j + i(i)\gamma^0 p_0 - m\right)ve^{ip\cdot x}.\end{aligned} \tag{1.5}$$

Rearranging gives

$$\begin{aligned}\left(i\gamma^j\partial_j - m\right)ue^{-ip\cdot x} &= -\gamma^0 p_0 ue^{-ip\cdot x} \\ \left(i\gamma^j\partial_j - m\right)ve^{ip\cdot x} &= +\gamma^0 p_0 ue^{-ip\cdot x},\end{aligned} \tag{1.6}$$

and theorem 1.2 follows immediately.

1.2 General solution.

As with the KG equation, let's introduce a generic solution formed from linear combinations of our specific $u^s(p) = u^s_{\mathbf{p}}, v^s(p) = v^s_{\mathbf{p}}$ solutions

$$\Psi(\mathbf{x}, t) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip\cdot x} u^s_{\mathbf{p}} a^s_{\mathbf{p}} + e^{ip\cdot x} v^s_{\mathbf{p}} b^s_{\mathbf{p}} \right). \tag{1.7}$$

Theorem 1.3: Dirac Hamiltonian in terms of creation and annihilation operators.

Substitution of the superposition eq. (1.7) into the Dirac Hamiltonian of theorem 1.1 results in

$$H_{\text{Dirac}} = \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a^{r\dagger}_{\mathbf{p}} a^r_{\mathbf{p}} - b^{r\dagger}_{-\mathbf{p}} b^r_{-\mathbf{p}} \right).$$

Deferring interpretation slightly, we first prove theorem 1.3, making the somewhat lazy guess that all the time dependent terms will be wiped out. This assumption allows us to use the zero time fields of our superposition solution

$$\Psi(\mathbf{x}, 0) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(u^s_{\mathbf{p}} a^s_{\mathbf{p}} + v^s_{-\mathbf{p}} b^s_{-\mathbf{p}} \right) \tag{1.8a}$$

$$\Psi^\dagger(\mathbf{x}, 0) = \sum_{r=1}^{2} \int \frac{d^3 q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} e^{-i\mathbf{q}\cdot\mathbf{x}} \left(u^{r\dagger}_{\mathbf{q}} a^{r\dagger}_{\mathbf{q}} + v^{r\dagger}_{-\mathbf{q}} b^{r\dagger}_{-\mathbf{q}} \right). \tag{1.8b}$$

Making use of the eigenvalue equations theorem 1.2 the Hamiltonian is reduced to

$$\begin{aligned}
H_{\text{Dirac}} &= \sum_{r,s=1}^{2} \int \frac{d^3 x d^3 p d^3 q}{(2\pi)^6 2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \left(u^{r\dagger}_{\mathbf{q}} a^{r\dagger}_{\mathbf{q}} + v^{r\dagger}_{-\mathbf{q}} b^{r\dagger}_{-\mathbf{q}} \right) \omega_{\mathbf{p}} \left(u^s_{\mathbf{p}} a^s_{\mathbf{p}} - v^s_{-\mathbf{p}} b^s_{-\mathbf{p}} \right) \\
&= \sum_{r,s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 2\cancel{\omega_{\mathbf{p}}}} \left(u^{r\dagger}_{\mathbf{p}} a^{r\dagger}_{\mathbf{p}} + v^{r\dagger}_{-\mathbf{p}} b^{r\dagger}_{-\mathbf{p}} \right) \cancel{\omega_{\mathbf{p}}} \left(u^s_{\mathbf{p}} a^s_{\mathbf{p}} - v^s_{-\mathbf{p}} b^s_{-\mathbf{p}} \right) \\
&= \frac{1}{2} \sum_{r,s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \left(u^{r\dagger}_{\mathbf{p}} u^s_{\mathbf{p}} a^{r\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} - u^{r\dagger}_{\mathbf{p}} v^s_{-\mathbf{p}} a^{r\dagger}_{\mathbf{p}} b^s_{-\mathbf{p}} + v^{r\dagger}_{-\mathbf{p}} u^s_{\mathbf{p}} b^{r\dagger}_{-\mathbf{p}} a^s_{\mathbf{p}} - v^{r\dagger}_{-\mathbf{p}} v^s_{-\mathbf{p}} b^{r\dagger}_{-\mathbf{p}} b^s_{-\mathbf{p}} \right),
\end{aligned} \tag{1.9}$$

where care was taken not to commute any a, b's. Recall that

$$u^{r\dagger}_{\mathbf{p}} u^s_{\mathbf{p}} = v^{r\dagger}_{\mathbf{p}} v^s_{\mathbf{p}} = 2\omega_{\mathbf{p}} \delta^{rs} \tag{1.10a}$$

$$u^{r\dagger}_{\mathbf{p}} v^{s}_{-\mathbf{p}} = v^{r\dagger}_{-\mathbf{p}} u^{s}_{\mathbf{p}} = 0. \tag{1.10b}$$

Equation (1.10b) kills off our cross terms, and eq. (1.10a) wipes out one of the summation indexes

$$\begin{aligned} H_{\text{Dirac}} &= \frac{1}{2} \sum_{r,s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \left(u^{r\dagger}_{\mathbf{p}} u^{s}_{\mathbf{p}} a^{r\dagger}_{\mathbf{p}} a^{s}_{\mathbf{p}} - \cancel{u^{r\dagger}_{\mathbf{p}} v^{s}_{-\mathbf{p}}} a^{r\dagger}_{\mathbf{p}} b^{s}_{-\mathbf{p}} + \cancel{v^{r\dagger}_{-\mathbf{p}} u^{s}_{\mathbf{p}}} b^{r\dagger}_{-\mathbf{p}} a^{s}_{\mathbf{p}} - v^{r\dagger}_{-\mathbf{p}} v^{s}_{-\mathbf{p}} b^{r\dagger}_{-\mathbf{p}} b^{s}_{-\mathbf{p}} \right) \\ &= \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a^{r\dagger}_{\mathbf{p}} a^{r}_{\mathbf{p}} - b^{r\dagger}_{-\mathbf{p}} b^{r}_{-\mathbf{p}} \right). \end{aligned} \tag{1.11}$$

We see above how the mixed terms were killed off nicely by eq. (1.10b). That also justifies the use of the zero-time fields in this derivation, which can also be seen explicitly without use of the zero-time fields exercise 1.1.

Interpretation. With a minus sign in the Hamiltonian, there is no bound to the energy from below! This makes it troublesome to interpret the $a_{\mathbf{p}}$'s and $b_{\mathbf{p}}$'s as the familiar raising and lowering operators that we know.

We can save the day, making the "Dirac sea" argument, roughly speaking that we can consider a set of completely full negative energy states, where creation of a particle makes a hole in one of those states[1], as sketched roughly in fig. 1.1. Such an argument does not work for Bosons (photons, ...) since an

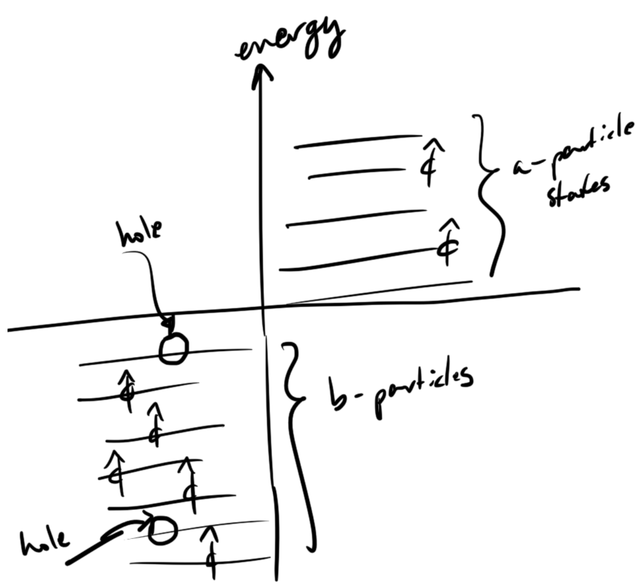

Figure 1.1: Dirac Sea.

[1] There was a long discussion of this topic in class that I was not able to capture in my notes.

arbitrary number of such particles can be stuffed into any given state. It will turn out that our operators are Fermions, which gets us out of this trouble.

We can also get out of this hole algebraically. For $X = a, b$, let

$$\begin{aligned} X^{s\dagger}_{\mathbf{p}} &= \tilde{X}^{s}_{\mathbf{p}} \\ X^{s}_{\mathbf{p}} &= \tilde{X}^{s\dagger}_{\mathbf{p}}. \end{aligned} \tag{1.12}$$

It turns out that some properties of our creation and annihilation operators are

$$\begin{aligned} (a^{s}_{p})^2 &= 0 \\ (a^{s\dagger}_{p})^2 &= 0 \\ (b^{s}_{p})^2 &= 0 \\ (b^{s\dagger}_{p})^2 &= 0, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} \left\{ a^{s}_{\mathbf{p}}, a^{r\dagger}_{\mathbf{q}} \right\} &= \delta^{sr}\delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ \left\{ b^{s}_{\mathbf{p}}, b^{r\dagger}_{\mathbf{q}} \right\} &= \delta^{sr}\delta^{(3)}(\mathbf{p} - \mathbf{q}), \end{aligned} \tag{1.14}$$

where all other anticommutators are zero

$$\begin{aligned} \{a^r, b^s\} &= \left\{ a^r, b^{s\dagger} \right\} \\ &= \left\{ a^{r\dagger}, b^{s} \right\} \\ &= \left\{ a^{r\dagger}, b^{s\dagger} \right\} \\ &= 0. \end{aligned} \tag{1.15}$$

Such a substitution gives

$$\begin{aligned} H_{\text{Dirac}} &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(\tilde{a}^{s}_{\mathbf{p}} a^{s}_{\mathbf{p}} - \tilde{b}^{s}_{\mathbf{p}} b^{s}_{\mathbf{p}} \right) \\ &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(\tilde{a}^{s}_{\mathbf{p}} a^{s}_{\mathbf{p}} + b^{s}_{\mathbf{p}} \tilde{b}^{s}_{\mathbf{p}} + \delta^{ss}\delta^{(3)}(\mathbf{p} - \mathbf{p}) \right) \\ &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(\omega_{\mathbf{p}} \left(\tilde{a}^{s}_{\mathbf{p}} a^{s}_{\mathbf{p}} + b^{s}_{\mathbf{p}} \tilde{b}^{s}_{\mathbf{p}} \right) - 4V_3 \frac{\omega_{\mathbf{p}}}{2} \right). \end{aligned} \tag{1.16}$$

We'll end up dropping the vacuum energy term. We'll end up labelling the a's as the operators associated with electrons, and the b's with anti-electrons.

1.3 Problems.

Exercise 1.1 Derive the Dirac Hamiltonian without using the zero-time field substitution.

Answer for Exercise 1.1

With time left in the mix the fields are

$$\begin{aligned}\Psi(x) &= \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\mathbf{p}}}}\left(e^{-ip\cdot x}u^s_{\mathbf{p}}a^s_{\mathbf{p}} + e^{ip\cdot x}v^s_{\mathbf{p}}b^s_{\mathbf{p}}\right)\\ \Psi^\dagger(x) &= \sum_{r=1}^{2}\int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_{\mathbf{q}}}}\left(e^{iq\cdot x}u^{r\dagger}_{\mathbf{q}}a^{r\dagger}_{\mathbf{q}} + e^{-iq\cdot x}v^{r\dagger}_{\mathbf{q}}b^{r\dagger}_{\mathbf{q}}\right),\end{aligned} \tag{1.17}$$

and the Hamiltonian is

$$\begin{aligned}H_{\text{Dirac}} &= \sum_{r,s=1}^{2}\int \frac{d^3x d^3p d^3q}{(2\pi)^6 2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\left(e^{iq\cdot x}u^{r\dagger}_{\mathbf{q}}a^{r\dagger}_{\mathbf{q}} + e^{-iq\cdot x}v^{r\dagger}_{\mathbf{q}}b^{r\dagger}_{\mathbf{q}}\right)\omega_{\mathbf{p}}\left(e^{-ip\cdot x}u^s_{\mathbf{p}}a^s_{\mathbf{p}} - e^{ip\cdot x}v^s_{\mathbf{p}}b^s_{\mathbf{p}}\right)\\ &= \sum_{r,s=1}^{2}\int \frac{d^3x d^3p d^3q}{(2\pi)^6 2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\left(e^{i\omega_{\mathbf{q}}t - i\mathbf{q}\cdot\mathbf{x}}u^{r\dagger}_{\mathbf{q}}a^{r\dagger}_{\mathbf{q}} + e^{-i\omega_{\mathbf{q}}t + i\mathbf{q}\cdot\mathbf{x}}v^{r\dagger}_{\mathbf{q}}b^{r\dagger}_{\mathbf{q}}\right)\omega_{\mathbf{p}}\left(e^{-i\omega_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}u^s_{\mathbf{p}}a^s_{\mathbf{p}} - e^{i\omega_{\mathbf{p}}t - i\mathbf{p}\cdot\mathbf{x}}v^s_{\mathbf{p}}b^s_{\mathbf{p}}\right)\\ &= \sum_{r,s=1}^{2}\int \frac{d^3x d^3p d^3q}{(2\pi)^6 2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\left(e^{i\omega_{\mathbf{q}}t - i\mathbf{q}\cdot\mathbf{x}}u^{r\dagger}_{\mathbf{q}}a^{r\dagger}_{\mathbf{q}} + e^{-i\omega_{\mathbf{q}}t - i\mathbf{q}\cdot\mathbf{x}}v^{r\dagger}_{-\mathbf{q}}b^{r\dagger}_{-\mathbf{q}}\right)\omega_{\mathbf{p}}\left(e^{-i\omega_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}u^s_{\mathbf{p}}a^s_{\mathbf{p}}\right.\\ &\qquad\left. - e^{i\omega_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}}v^s_{-\mathbf{p}}b^s_{-\mathbf{p}}\right)\\ &= \sum_{r,s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2}\left(e^{i\omega_{\mathbf{q}}t}u^{r\dagger}_{\mathbf{q}}a^{r\dagger}_{\mathbf{q}} + e^{-i\omega_{\mathbf{q}}t}v^{r\dagger}_{-\mathbf{q}}b^{r\dagger}_{-\mathbf{q}}\right)\left(e^{-i\omega_{\mathbf{p}}t}u^s_{\mathbf{p}}a^s_{\mathbf{p}} - e^{i\omega_{\mathbf{p}}t}v^s_{-\mathbf{p}}b^s_{-\mathbf{p}}\right)\\ &= \sum_{r,s=1}^{2}\int \frac{d^3p}{(2\pi)^3 2}\left(u^{r\dagger}_{\mathbf{q}}u^s_{\mathbf{p}}a^{r\dagger}_{\mathbf{q}}a^s_{\mathbf{p}} - v^{r\dagger}_{-\mathbf{q}}v^s_{-\mathbf{p}}b^{r\dagger}_{-\mathbf{q}}b^s_{-\mathbf{p}}\right),\end{aligned} \tag{1.18}$$

where a $\delta^{(3)}(\mathbf{p}-\mathbf{q})$ was factored out and evaluated, and the remaining $v^{r\dagger}_{-\mathbf{p}}u^s, u^{r\dagger}v^s_{-\mathbf{p}}$ terms were killed off. A final use of eq. (1.10a) completes the proof.