Peeter Joot
peeterjoot@pm.me

PHY2403H Quantum Field Theory. Lecture 23: QED and QCD interaction Lagrangian, Feynman propagator and rules for Fermions, hadron pair production, scattering cross section, quark pair production. Taught by Prof. Erich Poppitz

DISCLAIMER: Notes from class, with auxillary details. These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

These notes cover the final lecture of the course, which followed ch. 1 [1] §5.1 fairly closely (filling in some details, leaving out some others.)

1.1 Review.

Our Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi} \left(i\gamma^\mu \partial_\mu - m \right) \Psi, \tag{1.1}$$

which can be consider solved by fields $\Psi(x), \overline{\Psi}(x) = \Psi^\dagger(x)\gamma^0$

$$\Psi(x) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{-ip\cdot x} u^s(p) a^s_{\mathbf{p}} + e^{ip\cdot x} v^s(p) a^{s\dagger}_{\mathbf{p}} \right) \tag{1.2a}$$

$$\overline{\Psi}(x) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left(e^{ip\cdot x} \overline{u}^s(p) a^{s\dagger}_{\mathbf{p}} + e^{-ip\cdot x} \overline{v}^s(p) a^s_{\mathbf{p}} \right) \tag{1.2b}$$

where the creation and annihilation operators satisfy

$$\left\{ a^s_{\mathbf{p}}, a^{r\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{1.3a}$$

$$\left\{ b^s_{\mathbf{p}}, b^{r\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{1.3b}$$

(plus various relations for the u, v's.)

1.2 Photon.

Recall that we identified a number of symmetries

- $SO(1,3)$
- P, C, T : DIY
- $U(1)_V : \Psi \to e^{i\alpha}\Psi$
- $U(1)_A$: If $m = 0$, then $U(1)_A : \Psi \to e^{i\alpha\gamma_5}\Psi$. If $m \neq 0$ only for $\alpha = \pi : \Psi \to -\Psi$.

Photon interaction can be introduced by utilizing a $U(1)$ gauge field, demanding invariance under $U(1)_V$ with $\alpha = \alpha(x)$. That is

$$\Psi(x) \to e^{i\alpha(x)}\Psi(x), \tag{1.4}$$

which has derivatives

$$\partial_\mu \Psi(x) \to e^{i\alpha(x)}\left(\partial_\mu \Psi(x) + i\partial_\mu \alpha(x)\Psi(x)\right). \tag{1.5}$$

Solution. Introduce $A_\mu(x)$, such that under $U(1)_V$ we have

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \tag{1.6}$$

where "e" is a dimensionless coupling constant

$$\begin{aligned} \partial_\mu \Psi(x) &\to \left(\partial_\mu + ieA_\mu\right)\Psi \\ &\to e^{i\alpha(x)}\left(\partial_\mu \Psi + \cancel{i\partial_\mu \alpha}\Psi - \cancel{i\partial_\mu \alpha}\Psi\right) \end{aligned} \tag{1.7}$$

We've now constructed the QED Lagrangian density

$$\mathcal{L}_{\text{QED}} = \overline{\Psi}\left(i\gamma^\mu\left(\partial_\mu + ieA_\mu\right) - m\right)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.8}$$

We may write this as

$$\mathcal{L}_{\text{QED}} = \overbrace{\boxed{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi}}^{\text{Free Lagrangian}} - \underbrace{\boxed{e\overline{\Psi}\gamma_\mu\Psi A^\mu}}_{\text{interaction Lagrangian}} \tag{1.9}$$

We introduce spinor fields Ψ_e and muon fields Ψ_μ, so that the total Lagrangian is now

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}_e\left(i\gamma^\mu\partial_\mu - m\right)\Psi_e - e\overline{\Psi}_e\gamma_\mu\Psi_e A^\mu + \overline{\Psi}_\mu\left(i\gamma^\mu\partial_\mu - m\right)\Psi_\mu - e\overline{\Psi}_\mu\gamma_\mu\Psi_\mu A^\mu \tag{1.10}$$

- $m_e \sim 0.5\,\text{MeV}$
- $m_\mu \sim 105\,\text{MeV}$

There are also quark fields that we can add into the mix

$$\mathcal{L}_{\text{quarks}} = \sum_q \overline{\Psi}_q \left(i\gamma^\mu - m_q \right) \Psi_q + eQ_q \overline{\Psi}_q \gamma^\nu \Psi_q A_\nu \tag{1.11}$$

Quark charges are $Q_q = (2/3, -1/3)$. It turns out that the only way to produce quarks is through (electron?) interaction?

Can also introduce a Fermi interaction

$$\mathcal{L}_{4-Fermi} = \frac{c}{v^2} \overline{\Psi}_\mu \gamma^\nu (1 - \gamma_5) \Psi_{\nu,\mu} - \overline{\Psi}_e (1 - \gamma_5) \ldots. \tag{1.12}$$

We now want to do some calculations with the photon interactions from eq. (1.10). In particular, we will study the effects of the $-e\overline{\Psi}_e \gamma_\mu \Psi_e A^\mu$ interaction Lagrangian.

1.3 Propagator.

Before we can study the interaction, we need to determine the structure of the propagator. For Grassman (anti-commuting) operators

$$T(O_f(x)O'_f(x)) = \Theta(x_0 - x'_0)O_f(x)O_f(x') + \Theta(x'_0 - x_0)O_f(x')O_f(x) \tag{1.13}$$

The propagator can be determined from

$$\left\langle T(\Psi_\alpha(x)\Psi_\beta(x) \right\rangle_0 = D_{F_{\alpha\beta}}(x - y), \tag{1.14}$$

where $\alpha, \beta = 1, 2, 3, 4$.

Referring back to eq. (1.2a), eq. (1.2b), that propagator is

$$\begin{aligned}\left\langle T(\Psi_\alpha(x)\Psi_\beta(x) \right\rangle_0 &= \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3 \sqrt{2\omega_{\mathbf{q}}}} \left(e^{-ip\cdot x} e^{+iq\cdot y} \Theta(x_0 - y_0) u^s_\alpha(p) \bar{u}^r_\beta(q) \left\langle a^s_{\mathbf{p}} a^{r\dagger}_{\mathbf{q}} \right\rangle \right. \\ &\left. + e^{ip\cdot x} e^{-iq\cdot y} \Theta(y_0 - x_0) \bar{v}^s_\beta(p) v^r_\beta(q) \left\langle b^s_{\mathbf{q}} a^{r\dagger}_{\mathbf{p}} \right\rangle \right) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} \Theta(x_0 - y_0) u^s_\alpha(p) \bar{u}^r_\beta(p) \right. \\ &\left. + e^{ip\cdot(x-y)} \Theta(y_0 - x_0) \bar{v}^s_\beta(p) v^r_\beta(p) \right) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} \left(e^{-ip\cdot x} \Theta(x_0 - y_0) \left(\gamma^\mu_{\alpha\beta} p_\mu + m \right) \right. \\ &\left. + e^{ip\cdot x} \Theta(y_0 - x_0) \left(\gamma^\mu_{\alpha\beta} p_\mu - m \right) \right) =\end{aligned} \tag{1.15}$$

where $\gamma^\mu_{\alpha\beta}$ are the α, β components of the gamma matrices. Now we can replace the p_μ's with derivatives acting on the exponentials

$$
\begin{aligned}
&\left\langle T(\Psi_\alpha(x)\Psi_\beta(x)\right\rangle_0 = \Theta(x_0 - y_0)\left(i\gamma^\mu_{\alpha\beta}\partial_\mu + m\right)\int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip\cdot(x-y)} \\
&- \Theta(y_0 - x_0)\left(-i\gamma^\mu_{\alpha\beta}\partial_\mu - m\right)\int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{-ip\cdot(x-y)} = \Theta(x_0 - y_0)\left(i\gamma^\mu_{\alpha\beta}\partial_\mu + m\right)D(x-y) \\
&- \Theta(y_0 - x_0)\left(-i\gamma^\mu_{\alpha\beta}\partial_\mu - m\right)D(y-x) = \left(\gamma^\mu_{\alpha\beta}\partial^{(x)}_\mu + m\right)\left(\Theta(x_0 - y_0)D(x-y)\right. \\
&\left. + \Theta(y_0 - x_0)D(y-x)\right) - i\gamma^0\delta(x^0 - y^0)\cancel{(D(x-y) - D(y-x))},
\end{aligned}
\tag{1.16}
$$

where we've killed off a factor that is zero (off the light cone?)

We are left with just an action on the Feynman propagator

$$
\left\langle T(\Psi_\alpha(x)\Psi_\beta(x)\right\rangle_0 = \left(\gamma^\mu_{\alpha\beta}\partial^{(x)}_\mu + m\right) D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\gamma^\mu_{\alpha\beta}p_\mu + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}
\tag{1.17}
$$

Now that we have a propagator, let's try

$$
\mathcal{L}_{\text{int}} = \int dt d^3x \left(e\overline{\Psi}\gamma_\mu\Psi A^\mu\right).
\tag{1.18}
$$

1.4 Feynman rules.

We can consider various scattering processes, such as $e^+e^- \to \mu^+\mu^-$ as sketched in fig. 1.1, or $e^+e^- \to e^+e^-$ as sketched in fig. 1.2, or Compton scattering $e^-\gamma \to e^-\gamma$ as sketched in fig. 1.3.

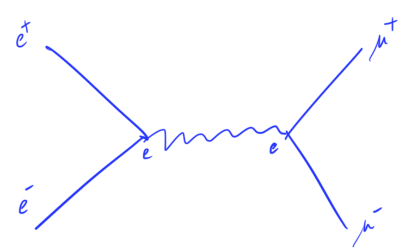

Figure 1.1: Electron, positron decay to muon pairs.

To do so we need to determine the Feynman rules for Fermions. For Fermions Ψ and anti-Fermions $\overline{\Psi}$

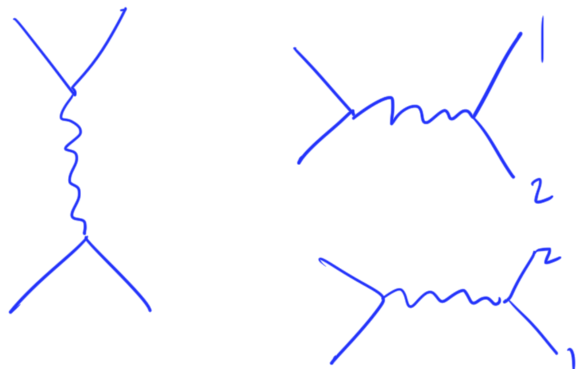

Figure 1.2: Electron, positron collision.

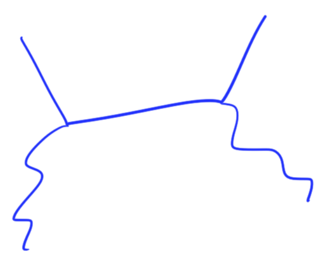

Figure 1.3: Compton scattering.

we have

$$
\begin{aligned}
\overbracket{\Psi\,|\mathbf{p}}, s\rangle &= u^s(p) \\
\overbracket{\bar{\Psi}\,|\mathbf{p}}, s\rangle &= \bar{v}^s(p) \\
\langle \overbracket{\mathbf{p}, s|\Psi} &= \bar{u}^s(p) \\
\langle \overbracket{\mathbf{p}, s|\bar{\Psi}} &= v^s(p),
\end{aligned}
\tag{1.19}
$$

where we mean

$$
|\mathbf{p}, s\rangle = a_{\mathbf{p}}^{s\dagger} |0\rangle \sqrt{2\omega_{\mathbf{p}}},
\tag{1.20}
$$

for Fermions, and

$$
|\mathbf{p}, s\rangle = b_{\mathbf{p}}^{s\dagger} |0\rangle \sqrt{2\omega_{\mathbf{p}}},
\tag{1.21}
$$

for anti-Fermions.

The flow of Fermion and anti-Fermion number charge is designated by arrow direction in the diagram, as in the respective diagrams of fig. 1.4.

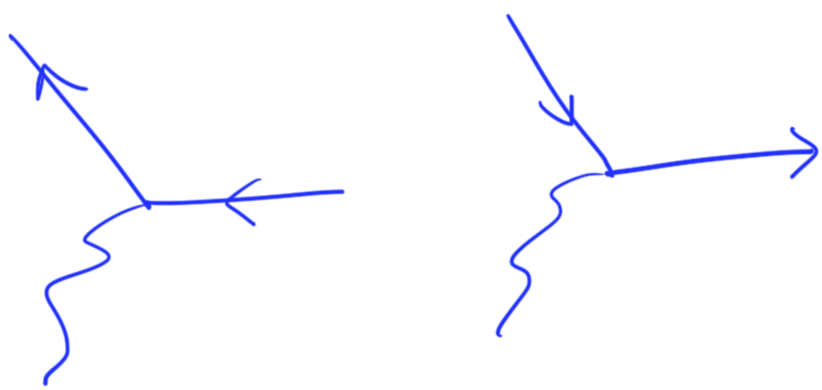

Figure 1.4: Flow of # charge.

The Feynman propagator for Fermions is

$$
\frac{i\left(\not{p} + m\right)}{p^2 - m^2 + i\epsilon},
\tag{1.22}
$$

whereas the photon propagator is

$$
\left\langle A_\mu A_\nu \right\rangle = -i \frac{g_{\mu\nu}}{q^2 + i\epsilon}.
\tag{1.23}
$$

1.5 Example: $e^- e^+ \rightarrow \mu^- \mu^+$.

As an example, consider the process sketched in fig. 1.5. Such a process is "ultra-relativistic", in that the

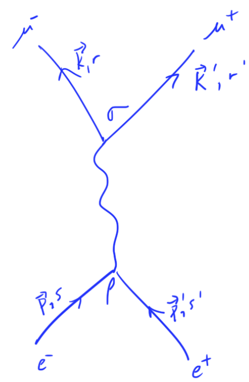

Figure 1.5: $e^- e^+ \rightarrow \mu^- \mu^+$ process.

electron and positron pair must be moving very fast to create muons.

The matrix element is

$$\langle \mu^+ \mu^- | \bar{\Psi} \gamma^\sigma \Psi A^\sigma A^\rho \bar{\Psi} \gamma^\rho \Psi | e^+ e^- \rangle = \overbrace{\boxed{\bar{v}^{s'}(p')}}^{\text{incoming anti-electron}} (-ie\gamma^\rho) \underbrace{\boxed{u^s(p)}}_{\text{incoming electron}} \overbrace{\boxed{\left(\frac{-i g_{\rho\sigma}}{q^2} \right)}}^{\text{ignoring } i\epsilon.} \bar{u}^r(k)(-ie\gamma^\sigma) v^{r'}(k') \tag{1.24}$$

Question: Why are we writing the factors of the matrix element from left to right, corresponding to the right to left reading of the matrix element?

Equation (1.24) reduces to

$$iM = i\frac{e^2}{q^2} \bar{v}^{s'}(p') \gamma^\rho u^s(p) \bar{u}^r(k) \gamma_\rho v^{r'}(k'), \tag{1.25}$$

where the $(2\pi)^4 \delta^{(4)}(\ldots)$ term hasn't been made explicit.

We'd like to compute the absolute square of eq. (1.25), and use the following lemma to do so.

Lemma 1.1: Some conjugates.

$$(\bar{v}\gamma^\mu u)^\dagger = \bar{u}\gamma^\mu v$$
$$(\bar{u}\gamma^\mu v)^\dagger = \bar{v}\gamma^\mu u.$$

The proof is left to exercise 1.1. Employing this, we have

$$|M|^2 = \frac{e^4}{q^4}\left(\bar{v}^{r'}(k')\gamma_\rho u^r(k)\bar{u}^s(p)\gamma^\rho v^{s'}(p')\right) \times \left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right). \tag{1.26}$$

The problem can be simplified by computing the cross section that sums over all spins, assuming that the states are not polarized (i.e. average over all the up, down states).

Digression

Such an average is related to the density matrix

$$\rho_{\text{in}} = \sum_{ss'} |ss'\rangle \frac{1}{4} \langle ss'| . \tag{1.27}$$

$$\text{tr}\left(e^{iHt}\rho_{\text{in}}e^{iHt}\rho_{\text{f}}\,|rr'\rangle\langle rr'|\right) \tag{1.28}$$

That is, We want to sum over all the initial and final state polarizations $\frac{1}{4}\sum_{ss'}\sum_{rr'}|M|^2$

$$\begin{aligned}\frac{1}{4}\sum_{ss',rr'}|M|^2 &= \sum_{ss'rr'}\frac{e^4}{4q^4}\bar{v}^{r'}(k')\gamma_\rho u^r(k)\bar{u}^r(k)\gamma_\mu v^{r'}(k')\bar{u}^s(p)\gamma^\rho v^{s'}(p')\bar{v}^{s'}(p')\gamma^\mu u^s(p)\\ &= \frac{e^4}{4q^4}\sum_{r'}\bar{v}^{r'}(k')\gamma_\rho\left(\not{k}+m_\mu\right)\gamma_\mu v^{r'}(k') \times \sum_s \bar{u}^s(p)\gamma^\rho\left(\not{p}'-m_e\right)\gamma^\mu u^s(p),\end{aligned} \tag{1.29}$$

where we first used the freedom to move the $\bar{u}\gamma v, \bar{v}\gamma u$ terms, which are scalars, and then used **??** to eliminate the sum over s', r indexes.

Temporarily expressing the remaining factors in coordinates exposes a trace structure. For example

$$\begin{aligned}\sum_{r'}\bar{v}^{r'}(k')\gamma_\rho\left(\not{k}+m_\mu\right)\gamma_\mu v^{r'}(k') &= \sum_{r'}(\bar{v}^{r'}(k'))_a(\gamma_\rho)_{ab}\left(\not{k}+m_\mu\right)_{bc}(\gamma_\mu)_{cd}(v^{r'}(k'))_d\\ &= \sum_{r'}(v^{r'}(k'))_d(\bar{v}^{r'}(k'))_a(\gamma_\rho)_{ab}\left(\not{k}+m_\mu\right)_{bc}(\gamma_\mu)_{cd}\\ &= \left(\not{k}'-m_\mu\right)_{da}(\gamma_\rho)_{ab}\left(\not{k}+m_\mu\right)_{bc}(\gamma_\mu)_{cd}\\ &= \text{tr}\left(\left(\not{k}'-m_\mu\right)\gamma_\rho\left(\not{k}+m_\mu\right)\gamma_\mu\right),\end{aligned} \tag{1.30}$$

since the cyclic sum of matrix coordinates can be expressed as a trace, namely $\operatorname{tr} ABC = A_{ab}B_{bc}C_{ca}$. We are left with

$$\frac{1}{4}\sum_{ss',rr'} |M|^2 = \frac{e^4}{4q^4} \operatorname{tr}\left(\left(\not{k}' - m_\mu\right)\gamma_\nu\left(\not{k} + m_\mu\right)\gamma_\mu\right) \times \operatorname{tr}\left(\left(\not{p} + m_e\right)\gamma^\nu\left(\not{p}' - m_e\right)\gamma^\mu\right). \tag{1.31}$$

Each trace is now a product of two, three, or four gamma matrices, which can be reduced using the identities:

Lemma 1.2: Dirac matrix product traces.

$$\begin{aligned}
\operatorname{tr}\left(\gamma_\mu\gamma_\nu\right) &= 4g_{\mu\nu} \\
\operatorname{tr}\left(\gamma_\mu\gamma_\nu\gamma_\alpha\right) &= 0 \\
\operatorname{tr}\left(\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta\right) &= 4\left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\alpha\nu}\right)
\end{aligned}$$

The proof is left to exercise 1.2.

Utilizing the above, and setting $m_e = 0$ (compared to m_μ) the p, p' dependent trace reduces to

$$\begin{aligned}
\operatorname{tr}\left((\not{p} + m_e)\gamma^\nu(\not{p}' - m_e)\gamma^\mu\right) &= \operatorname{tr}\left(\not{p}\gamma^\nu\not{p}'\gamma^\mu\right) \\
&= p_\alpha {p'}_\beta \operatorname{tr}\left(\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right) \\
&= 4p_\alpha {p'}_\beta\left(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\nu\beta}\right) \\
&= 4\left(-p\cdot p' g^{\nu\mu} + p^\nu p'^\mu + p^\mu p'^\nu\right),
\end{aligned} \tag{1.32}$$

and the k, k' dependent trace reduces to

$$\begin{aligned}
\operatorname{tr}\left(\left(\not{k}' - m_\mu\right)\gamma_\nu\left(\not{k} + m_\mu\right)\gamma_\mu\right) &= \operatorname{tr}\left(\not{k}'\gamma_\nu\not{k}\gamma_\mu\right) - m_\mu^2 \operatorname{tr}\left(\gamma_\nu\gamma_\mu\right) + m_\mu \cancel{\operatorname{tr}\left(\not{k}'\gamma_\nu\gamma_\mu\right)} - m_\mu \cancel{\operatorname{tr}\left(\gamma_\nu\not{k}\gamma_\mu\right)} \\
&= 4\left({k'}_\alpha k_\beta\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\nu\beta}\right) - m_\mu^2 g_{\nu\mu}\right) \\
&= 4\left({k'}_\nu k_\mu + {k'}_\mu k_\nu - \left(k\cdot k' + m_\mu^2\right)g_{\nu\mu}\right).
\end{aligned} \tag{1.33}$$

We can now multiply out the traces and simplify (exercise 1.3) to get

$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{8e^4}{q^4}\left(p\cdot k' p'\cdot k + p\cdot k p'\cdot k' + p\cdot p' m_\mu^2\right). \tag{1.34}$$

The next task is to consider these four vector dot products from the center of mass frame for the electrons, as sketched in fig. 1.6. Let q represent the total rest frame four momentum

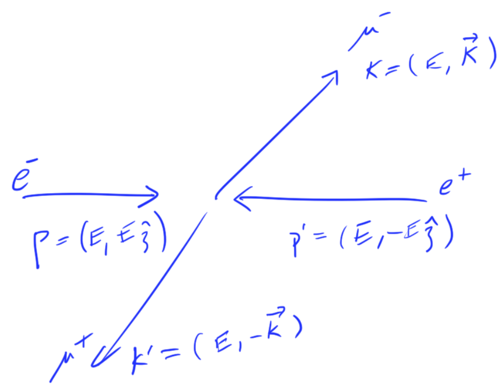

Figure 1.6: Electron center of mass frame.

$$\begin{aligned} q &= p + p' \\ &= (2E, \mathbf{0}), \end{aligned} \tag{1.35}$$

where $q^2 = 4E^2$. We also have

$$\begin{aligned} p \cdot p' &= (E, E\hat{\mathbf{z}}) \cdot (E, -E\hat{\mathbf{z}}) \\ &= E^2 - E^2(\hat{\mathbf{z}} \cdot (-\hat{\mathbf{z}})) \\ &= 2E^2. \end{aligned} \tag{1.36a}$$

$$\begin{aligned} p \cdot k &= (E, E\hat{\mathbf{z}}) \cdot (E, \mathbf{k}) \\ &= E^2 - E \left\lVert \mathbf{k} \right\rVert \cos\theta, \end{aligned} \tag{1.36b}$$

$$\begin{aligned} p \cdot k' &= (E, E\hat{\mathbf{z}}) \cdot (E, -\mathbf{k}) \\ &= E^2 - (E\hat{\mathbf{z}}) \cdot (-\mathbf{k}) \\ &= E^2 + E \left\lVert \mathbf{k} \right\rVert \cos\theta \end{aligned} \tag{1.36c}$$

$$\begin{aligned} p' \cdot k' &= (E, -E\hat{\mathbf{z}}) \cdot (E, -\mathbf{k}) \\ &= E^2 - (-E\hat{\mathbf{z}}) \cdot (-\mathbf{k}) \\ &= E^2 - E \left\lVert \mathbf{k} \right\rVert \cos\theta \end{aligned} \tag{1.36d}$$

$$\begin{aligned} p' \cdot k &= (E, -E\hat{\mathbf{z}}) \cdot (E, \mathbf{k}) \\ &= E^2 - (-E\hat{\mathbf{z}}) \cdot \mathbf{k} \\ &= E^2 + E \left\lVert \mathbf{k} \right\rVert \cos\theta, \end{aligned} \tag{1.36e}$$

but

$$\mathbf{k}^2 = E^2 - m_\mu^2, \tag{1.37}$$

or

$$\|\mathbf{k}\| = E\sqrt{1 - \frac{m_\mu^2}{E^2}}. \tag{1.38}$$

We can now put the pieces back together and almost have the non-polarized cross section

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}} |M|^2 &= \frac{8e^4}{(4E^2)^2}\left(\left(E^2 + E\,\|\mathbf{k}\|\cos\theta\right)^2 + \left(E^2 - E\,\|\mathbf{k}\|\cos\theta\right)^2 + m_\mu^2 2E^2 \right) \\
&= \frac{e^4}{2}\left(\left(1 + \sqrt{1 - \frac{m_\mu^2}{E^2}}\cos\theta\right)^2 + \left(1 - \sqrt{1 - \frac{m_\mu^2}{E^2}}\cos\theta\right)^2 + 2\frac{m_\mu^2}{E^2} \right) \\
&= \frac{e^4}{2}\left(2 + 2\left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta + 2\frac{m_\mu^2}{E^2}\right),
\end{aligned} \tag{1.39}$$

or

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |M|^2 = e^4\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right).} \tag{1.40}$$

The total (average polarization) differential cross section ([1] eq. 4.84), is

$$\frac{d\sigma}{d\Omega}_{\text{CM}} = \frac{1}{2E_A 2E_B |v_A - v_B|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{\text{CM}}}\frac{1}{4}\sum_{\text{spins}} |M|^2. \tag{1.41}$$

Plug in $E_A = E_B = 2E_{\text{CM}}$, $v_A - v_B \sim 2c = 2$, $e^2 = 4\pi\alpha$, and eq. (1.40) for

$$\begin{aligned}
\frac{d\sigma}{d\Omega}_{\text{CM}} &= \frac{1}{E_{\text{CM}}^2(2)}\frac{1}{(4\pi)^2 E_{\text{CM}}}\frac{E_{\text{CM}}}{2}\sqrt{1 - \frac{m_\mu^2}{E^2}}(4\pi\alpha)^2\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right) \\
&= \frac{\alpha^2}{4E_{\text{CM}}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right).
\end{aligned} \tag{1.42}$$

Integrating to find the total cross section we have

$$
\begin{aligned}
\sigma_{\text{total}} &= \int d\Omega \frac{d\sigma}{d\Omega} \\
&= 2\pi \int_{-1}^{1} d\cos\theta \frac{\alpha^2}{4E_{\text{CM}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right) \\
&= \frac{2\pi\alpha^2}{4E_{\text{CM}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(2\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\int_{-1}^{1} u^2 du\right) \\
&= \frac{4\pi\alpha^2}{4E_{\text{CM}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{E^2}+\frac{1}{3}\left(1-\frac{m_\mu^2}{E^2}\right)\right),
\end{aligned}
\tag{1.43}
$$

or

$$
\boxed{\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{\text{CM}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right),}
\tag{1.44}
$$

where $E_{\text{CM}} = 2E$.

At the start of the year dimensional analysis was used to state the total cross section, which was determined to have the form

$$
\sigma_{\text{total}} \sim \frac{\alpha^2}{s},
\tag{1.45}
$$

whereas for $E \gg m_\mu$ we've now found

$$
\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{\text{CM}}^2}.
\tag{1.46}
$$

Three months of work has gained us an additional factor of $4/3$!

1.6 Measurement of intermediate quark scattering processes.

In the diagram that we are working from for the $e^- e^+ \rightarrow \mu^- \mu^+$ process, we can replace the muon half of the interaction (fig. 1.7) with anything else that is charged, as sketched in fig. 1.8. In particular, quark pairs from QCD are possible at high energies ($m_\mu \sim 105\,\text{MeV}$) and such products can be measured indirectly. Quarks were the theorized to be strong force carriers, an intermediate stage similar to the photon propagators of QED, connecting two branches of a diagram, as sketched in fig. 1.9. If one hypothesizes a proportionality relationship between the hadron (i.e. muon) and quark scattering cross sections

$$
\sigma_{\text{total}}(e^- e^+ \rightarrow \text{hadrons}) \propto \sigma_{\text{total}}(e^- e^+ \rightarrow \text{quarks}),
\tag{1.47}
$$

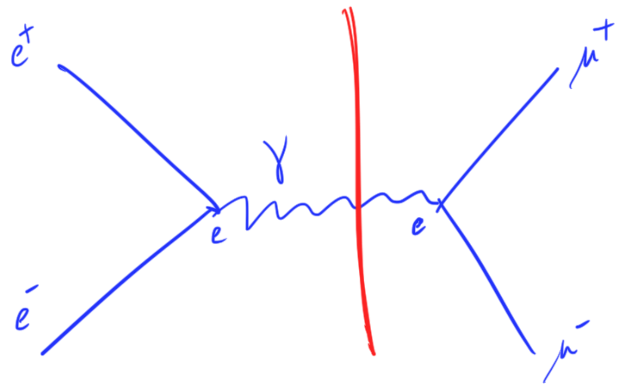

Figure 1.7: Electron and muon halves of the diagram

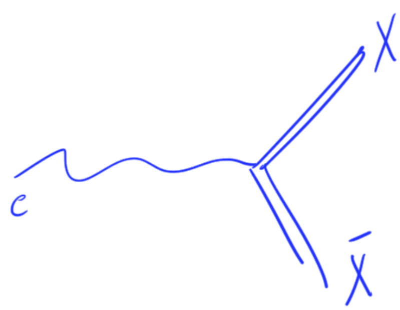

Figure 1.8: Alternate charged pair production.

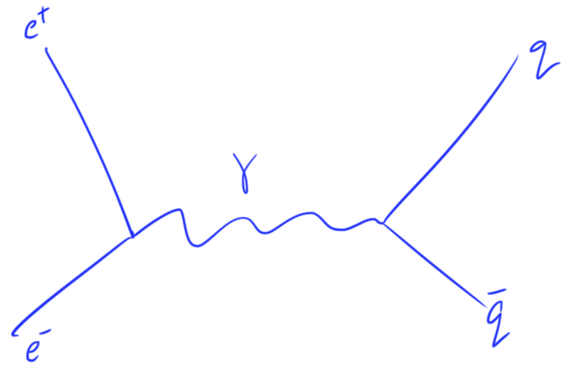

Figure 1.9: Quark pair production.

the ratio between the two

$$\begin{aligned} R &= \frac{\sigma_{\text{total}}(e^- e^+ \rightarrow \text{quarks})}{\sigma_{\text{total}}(e^- e^+ \rightarrow \text{hadrons})} \\ &= 3 \sum_q (Q_q)^4, \end{aligned} \tag{1.48}$$

can be measured, and such measurement was deemed to be one of the validations of the QCD theory. The $3 \sum_q (Q_q)^4$ expression includes a 3 that is related to quark "color", and a sum over only the quark charges q that are light enough to be produced. [1] fig. 5.3 includes an experimental depiction of such a measurement, which has a step function form roughly like fig. 1.10, where the steps occur at the energy levels that are sufficient to produce new quarks.

1.7 Problems.

Exercise 1.1 Prove lemma 1.1

Answer for Exercise 1.1

We will prove only the first, which is representative

$$\begin{aligned} (\bar{v} \gamma^\mu u)^\dagger &= u^\dagger (\gamma^\mu)^\dagger (v^\dagger \gamma^0)^\dagger \\ &= u^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^0 v \\ &= \bar{u} \gamma^\mu v. \end{aligned} \tag{1.49}$$

Exercise 1.2 Prove lemma 1.2

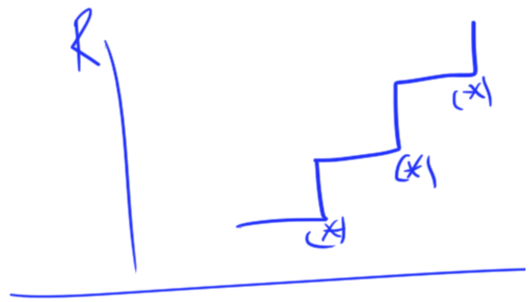

Figure 1.10: R quark step function.

Answer for Exercise 1.2

For the two matrix trace, consider

$$\begin{aligned}\operatorname{tr}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) &= 2g_{\mu\nu}\operatorname{tr}(1) \\ &= 8g_{\mu\nu},\end{aligned} \tag{1.50}$$

but

$$\begin{aligned}\operatorname{tr}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) &= \operatorname{tr}\left(\gamma_\mu\gamma_\nu\right) + \operatorname{tr}\left(\gamma_\nu\gamma_\mu\right) \\ &= 2\operatorname{tr}\left(\gamma_\mu\gamma_\nu\right),\end{aligned} \tag{1.51}$$

so $\operatorname{tr}\left(\gamma_\mu\gamma_\nu\right) = 4g_{\mu\nu}$ as claimed. For the traces of the three matrix products, there are three possible products of interest (for $r \ne s$)

$$\gamma^0\gamma^r\gamma^s = -i\epsilon^{rst}\begin{bmatrix} 0 & \sigma^t \\ \sigma^t & 0 \end{bmatrix}, \tag{1.52}$$

which is traceless. We also have (for distinct r, s, t)

$$\gamma^r\gamma^s\gamma^t = -\begin{bmatrix} 0 & \sigma^r\sigma^s\sigma^t \\ \sigma^r\sigma^s\sigma^t & 0 \end{bmatrix}, \tag{1.53}$$

which is also traceless. All other three matrix products (except permutations of the two above) are proportional to a single γ^μ, which is traceless. A lazier, brute force proof by Mathematica (tracesOfDiracMatrixProducts.nb) is also possible. For the four matrix traces, the trace will be zero unless we have two matching pairs of gamma matrices (since $\gamma^0\gamma^1\gamma^2\gamma^3$ or its permutations is traceless.) Assuming such matched pairs, we can reduce the product like so

- $\mu = \nu \implies \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}$

- $\mu = \alpha, \nu \neq \alpha \implies \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = -4g^{\nu\beta}$
- $\mu = \beta(\mu \neq \nu, \mu \neq \alpha) \implies \operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4g^{\nu\alpha}$

It's clear that we can summarize these possibilities as stated in lemma 1.2.

Exercise 1.3

Show that

$$\left(p^\beta p'^\alpha + p^\alpha p'^\beta - p\cdot p' g^{\alpha\beta}\right) \times \left(k'_\beta k_\alpha + k'_\alpha k_\beta - \left(k\cdot k' + m_\mu^2\right) g_{\alpha\beta}\right) = 2\left(p\cdot k p'\cdot k' + p\cdot k' p'\cdot k + m_\mu^2 p\cdot p'\right)$$

Answer for Exercise 1.3

Proceeding mechanically, but carefully, we have

$$\begin{aligned}
&p^\beta p'^\alpha k'_\beta k_\alpha + p^\beta p'^\alpha k'_\alpha k_\beta - p^\beta p'^\alpha \left(k\cdot k' + m_\mu^2\right) g_{\alpha\beta} \\
&\quad + p^\alpha p'^\beta k'_\beta k_\alpha + p^\alpha p'^\beta k'_\alpha k_\beta - p^\alpha p'^\beta \left(k\cdot k' + m_\mu^2\right) g_{\alpha\beta} \\
&\quad - p\cdot p' g^{\alpha\beta} k'_\beta k_\alpha - p\cdot p' g^{\alpha\beta} k'_\alpha k_\beta + p\cdot p' g^{\alpha\beta}\left(k\cdot k' + m_\mu^2\right) g_{\alpha\beta} \\
&= p\cdot k' p'\cdot k + p\cdot k p'\cdot k' - p\cdot p'\left(k\cdot k' + m_\mu^2\right) \\
&\quad + p\cdot k p'\cdot k' + p\cdot k' p'\cdot k - p\cdot p'\left(k\cdot k' + m_\mu^2\right) \\
&\quad - p\cdot p' k\cdot k' - p\cdot p' k\cdot k' + 4p\cdot p'\left(k\cdot k' + m_\mu^2\right) \\
&= 2p\cdot k' p'\cdot k + 2p\cdot k p'\cdot k' - 2p\cdot p' k\cdot k' + 2p\cdot p'\left(k\cdot k' + m_\mu^2\right) \\
&= 2p\cdot k' p'\cdot k + 2p\cdot k p'\cdot k' + 2p\cdot p' m_\mu^2 \\
&= 2\left(p\cdot k' p'\cdot k + p\cdot k p'\cdot k' + p\cdot p' m_\mu^2\right).
\end{aligned} \tag{1.54}$$

Bibliography

[1] Michael E Peskin and Daniel V Schroeder. *An introduction to Quantum Field Theory*. Westview, 1995. 1, 1.5, 1.6