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# PHY2403H Quantum Field Theory. Lecture 23: XXX. Taught by Prof. Erich Poppitz

*DISCLAIMER: Very rough notes from class, with some additional side notes.* These are notes for the UofT course PHY2403H, Quantum Field Theory, taught by Prof. Erich Poppitz, fall 2018.

## 1.1 Review

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi} \left( i\gamma^\mu \partial_\mu - m \right) \Psi, \tag{1.1}$$

$$\Psi(x) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left( e^{-ip\cdot x} u^s(p) a^s_{\mathbf{p}} + e^{ip\cdot x} v^s(p) a^{s\dagger}_{\mathbf{p}} \right) \tag{1.2}$$

$$\begin{aligned}\overline{\Psi}(x) &= \Psi^\dagger(x)\gamma^0 \\ &= \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_{\mathbf{p}}}} \left( e^{ip\cdot x} \overline{u}^s(p) a^{s\dagger}_{\mathbf{p}} + e^{-ip\cdot x} \overline{v}^s(p) a^{s}_{\mathbf{p}} \right)\end{aligned} \tag{1.3}$$

$$\left\{ a^s_{\mathbf{p}}, a^{r\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \delta^{sr} \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{1.4}$$

There are some symmetries

- $SO(1,3)$
- $P, C, T$ : DIY
- $U(1)_V : \Psi \rightarrow e^{i\alpha}\Psi$
- $U(1)_A$ : If $m = 0$, then $U(1)_A : \Psi \rightarrow e^{i\alpha\gamma_5}\Psi$. If $m \neq 0$ only for $\alpha = \pi : \Psi \rightarrow -\Psi$.

Now introduce interaction with photon by using a $U(1)$ gauge field, and demand invariance under $U(1)_V$ with $\alpha = \alpha(x)$. Now

$$\Psi(x) \to e^{i\alpha(x)}\Psi(x) \tag{1.5}$$

$$\partial_\mu \Psi(x) \to e^{i\alpha(x)} \left(\partial_\mu \Psi(x) + i\partial_\mu \alpha(x)\right) \Psi \tag{1.6}$$

Solution. Introduce $A_\mu(x)$, such that under $U(1)_V$ we have

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \tag{1.7}$$

where "e" is a dimensionless coupling constant

$$\begin{aligned} \partial_\mu \Psi(x) &\to \left(\partial_\mu + ieA_\mu\right) \Psi \\ &\to e^{i\alpha(x} \left(\partial_\mu \Psi + \cancel{i\partial_\mu \alpha} - \cancel{i\partial_\mu \alpha}...\right) \end{aligned} \tag{1.8}$$

We've now constructed the QED Lagrangian density

$$\mathcal{L}_Q ED = \overline{\Psi} \left(i\gamma^\mu \left(\partial_\mu + ieA_\mu\right) - m\right) \Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.9}$$

We may write this as

$$\mathcal{L}_Q ED = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi} \left(i\gamma^\mu \partial_\mu - m\right) \Psi}_{\text{Free Lagrangian}} - \underbrace{e\overline{\Psi}\gamma_\mu \Psi A^\mu}_{\text{interaction Lagrangian}} \tag{1.10}$$

We introduce spinor fields $\Psi_e$ and muon fields $\Psi_\mu$, so that the total Lagrangian is now

$$\mathcal{L}_Q ED = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}_e \left(i\gamma^\mu \partial_\mu - m\right) \Psi_e - e\overline{\Psi}_e\gamma_\mu \Psi_e A^\mu + \overline{\Psi}_\mu \left(i\gamma^\mu \partial_\mu - m\right) \Psi_\mu - e\overline{\Psi}_\mu \gamma_\mu \Psi_\mu A^\mu \tag{1.11}$$

- $m_e \sim 0.5\,\text{MeV}$
- $m_\mu \sim 105\,\text{MeV}$

There are also quark fields that we can add into the mix

$$\mathcal{L}_q uarks = \sum_q \overline{\Psi}_q \left(i\gamma^\mu - m_q\right) \Psi_q + eQ_q \overline{\Psi}_q \gamma^\nu \Psi_q A_\nu \tag{1.12}$$

Quark charges are $Q_q = (2/3, -1/3)$. It turns out that the only way to produce quarks is through (electron?) interaction?

Can also introduce a Fermi interaction

$$\mathcal{L}_{4-Fermi} = \frac{c}{v^2}\overline{\Psi}_\mu \gamma^\nu (1 - \gamma_5) \Psi_{\nu,\mu} - \overline{\Psi}_e (1 - \gamma_5) .... \tag{1.13}$$

We are now going to do some calculations with eq. (1.11)

In that interaction the $-e\overline{\Psi}_e\gamma_\mu\Psi_e A^\mu$ can create and anhillate (what?)

F1 F2 F3

For Grassman (anti-commuting) operators

$$T(O_f(x)O'_f(x)) = \Theta(x_0 - x'_0)O_f(x)O_f(x') + \Theta(x'_0 - x_0)O_f(x')O_f(x) \tag{1.14}$$

$$\left\langle T(\Psi_\alpha(x)\Psi_\beta(x)\right\rangle_0 = D_{F_{\alpha\beta}}(x-y), \tag{1.15}$$

where $\alpha, \beta = 1, 2, 3, 4$.

Referring back to

$$\begin{aligned}\left\langle T(\Psi_\alpha(x)\Psi_\beta(x)\right\rangle_0 &= \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\mathbf{p}}}}\int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_{\mathbf{q}}}}\left(e^{-ip\cdot x}e^{+iq\cdot y}\Theta(x_0-y_0)u^s_\alpha(p)\bar{u}^r_\beta(q)\left\langle a^s_{\mathbf{p}}a^{r\dagger}_{\mathbf{q}}\right\rangle\right.\\ &\left.+ e^{ip\cdot x}e^{-iq\cdot y}\Theta(y_0-x_0)\bar{v}^s_\beta(p)v^r_\beta(q)\left\langle b^s_{\mathbf{q}}a^{r\dagger}_{\mathbf{p}}\right\rangle\right) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)}\Theta(x_0-y_0)u^s_\alpha(p)\bar{u}^r_\beta(p)\right.\\ &\left.+ e^{ip\cdot(x-y)}\Theta(y_0-x_0)\bar{v}^s_\beta(p)v^r_\beta(p)\right) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}\left(e^{-ip\cdot x}\Theta(x_0-y_0)\left(\gamma^\mu_{\alpha\beta}p_\mu + m\right)\right.\\ &\left.+ e^{ip\cdot x}\Theta(y_0-x_0)\left(\gamma^\mu_{\alpha\beta}p_\mu - m\right)\right) =\end{aligned} \tag{1.16}$$

where $\gamma^\mu_{\alpha\beta}$ are the $\alpha, \beta$ components of the gamma matrices. Now we can replace the $p_\mu$'s with derivatives acting on the exponentials

$$\begin{aligned}\left\langle T(\Psi_\alpha(x)\Psi_\beta(x)\right\rangle_0 &= \Theta(x_0-y_0)\left(i\gamma^\mu_{\alpha\beta}\partial_\mu + m\right)\int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}e^{-ip\cdot(x-y)}\\ &- \Theta(y_0-x_0)\left(-i\gamma^\mu_{\alpha\beta}\partial_\mu - m\right)\int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}}e^{-ip\cdot(x-y)} = \Theta(x_0-y_0)\left(i\gamma^\mu_{\alpha\beta}\partial_\mu + m\right)D(x-y)\\ &- \Theta(y_0-x_0)\left(-i\gamma^\mu_{\alpha\beta}\partial_\mu - m\right)D(y-x) = \left(\gamma^\mu_{\alpha\beta}\partial^{(x)}_\mu + m\right)\left(\Theta(x_0-y_0)D(x-y)\right.\\ &\left.+ \Theta(y_0-x_0)D(y-x)\right) - i\gamma^0\delta(x^0-y^0)\cancel{(D(x-y) - D(y-x))},\end{aligned} \tag{1.17}$$

where we've killed off a factor that is zero (off the light cone?)

We are left with just an action on the Feynman propagator

$$\left\langle T(\Psi_\alpha(x)\Psi_\beta(x)\right\rangle_0 = \left(\gamma^\mu_{\alpha\beta}\partial^{(x)}_\mu + m\right)D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\gamma^\mu_{\alpha\beta}p_\mu + m)}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)} \tag{1.18}$$

Now that we have a propagator, let's try

$$\mathcal{L}_{\text{int}} = \int dt d^3x\left(e\overline{\Psi}\gamma_\mu\Psi A^\mu\right) \tag{1.19}$$