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Gauge transformation in the Lorentz force Lagrangian.

Exercise 1.1 Lorentz force gauge transformation.

Show that the gauge transformation $A \rightarrow A + \nabla\psi$ applied to the Lorentz force Lagrangian

$$L = \frac{1}{2}mv^2 + qA \cdot v/c, \tag{1.1}$$

does not change the equations of motion.

Answer for Exercise 1.1

The gauge transformed Lagrangian is

$$L = \frac{1}{2}mv^2 + qA \cdot v/c + \frac{qv}{c} \cdot \nabla\phi. \tag{1.2}$$

We know that the Lorentz force equations are obtained from the first two terms, so need only consider the effects of the new ϕ dependent term on the action. First observe that

$$\begin{aligned} v \cdot \nabla\phi &= \frac{dx^\mu}{d\tau}\frac{\partial\phi}{\partial x^\mu} \\ &= \frac{d\phi}{d\tau}. \end{aligned} \tag{1.3}$$

This means that the action is transformed to

$$\begin{aligned} S &\rightarrow S + \frac{q}{c}\int d\tau \frac{d\phi}{d\tau} \\ &= S + \frac{q}{c}\phi|_{\Delta\tau}. \end{aligned} \tag{1.4}$$

As the action is evaluated over a fixed interval, the gauge transformation only changes the action by a constant, so the equations of motion are unchanged.