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Curvilinear coordinates and gradient in spacetime, and reciprocal frames.

1.1 Motivation.

I started pondering some aspects of spacetime integration theory, and found that there were some aspects of the concepts of reciprocal frames that were not clear to me. In the process of sorting those ideas out for myself, I wrote up the following notes.

In the notes below, I will introduce the many of the prerequisite ideas that are needed to express and apply the fundamental theorem of geometric calculus in a 4D relativistic context. The focus will be the Dirac's algebra of special relativity, known as STA (Space Time Algebra) in geometric algebra parlance. If desired, it should be clear how to apply these ideas to lower or higher dimensional spaces, and to plain old Euclidean metrics.

On notation.   In Euclidean space we use bold face reciprocal frame vectors $\mathbf{x}^i \cdot \mathbf{x}_j = \delta^i{}_j$, which nicely distinguishes them from the generalized coordinates x_i, x^j associated with the basis or the reciprocal frame, that is

$$\mathbf{x} = x^i \mathbf{x}_i = x_j \mathbf{x}^j. \tag{1.1}$$

On the other hand, it is conventional to use non-bold face for both the four-vectors and their coordinates in STA, such as the following standard basis decomposition

$$x = x^\mu \gamma_\mu = x_\mu \gamma^\mu. \tag{1.2}$$

If we use non-bold face x^μ, x_ν for the coordinates with respect to a specified frame, then we cannot also use non-bold face for the curvilinear basis vectors. To resolve this notational ambiguity, I've chosen to use bold face $\mathbf{x}^\mu, \mathbf{x}_\nu$ symbols as the curvilinear basis elements in this relativistic context, as we do for Euclidean spaces.

1.2 Basis and coordinates.

Definition 1.1: Standard Dirac basis.

The Dirac basis elements are $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, satisfying

$$\gamma_0^2 = 1 = -\gamma_k^2, \quad \forall k = 1, 2, 3,$$

and

$$\gamma_\mu \cdot \gamma_\nu = 0, \quad \forall \mu \neq \nu. \tag{1.3}$$

A conventional way of summarizing these orthogonality relationships is $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ are the elements of the metric $G = \text{diag}(+, -, -, -)$.

Definition 1.2: Reciprocal basis for the standard Dirac basis.

We define a reciprocal basis $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$ satisfying $\gamma^\mu \cdot \gamma_\nu = \delta^\mu{}_\nu, \forall \mu, \nu \in 0, 1, 2, 3$.

Theorem 1.1: Reciprocal basis uniqueness.

This reciprocal basis is unique, and for our choice of metric has the values

$$\gamma^0 = \gamma_0, \quad \gamma^k = -\gamma_k, \quad \forall k = 1, 2, 3.$$

Proof is left to the reader.

Definition 1.3: Coordinates.

We define the coordinates of a vector with respect to the standard basis as x^μ satisfying

$$x = x^\mu \gamma_\mu,$$

and define the coordinates of a vector with respect to the reciprocal basis as x_μ satisfying

$$x = x_\mu \gamma^\mu,$$

Theorem 1.2: Coordinates.

Given the definitions above, we may compute the coordinates of a vector, simply by dotting with the basis elements

$$x^\mu = x \cdot \gamma^\mu,$$

and

$$x_\mu = x \cdot \gamma_\mu,$$

Proof. This follows by straightforward computation

$$\begin{aligned}
x \cdot \gamma^\mu &= (x^\nu \gamma_\nu) \cdot \gamma^\mu \\
&= x^\nu (\gamma_\nu \cdot \gamma^\mu) \\
&= x^\nu \delta_\nu{}^\mu \\
&= x^\mu,
\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}
x \cdot \gamma_\mu &= (x_\nu \gamma^\nu) \cdot \gamma_\mu \\
&= x_\nu (\gamma^\nu \cdot \gamma_\mu) \\
&= x_\nu \delta^\nu{}_\mu \\
&= x_\mu.
\end{aligned} \tag{1.5}$$

$\square$

1.3   Derivative operators.

We'd like to determine the form of the (spacetime) gradient operator. The gradient can be defined in terms of coordinates directly, but we choose an implicit definition, in terms of the directional derivative.

Definition 1.4: Directional derivative and gradient.

Let $F = F(x)$ be a four-vector parameterized multivector. The directional derivative of F with respect to the (four-vector) direction a is denoted

$$(a \cdot \nabla) F = \lim_{\epsilon \to 0} \frac{F(x + \epsilon a) - F(x)}{\epsilon},$$

where ∇ is called the space time gradient.

Theorem 1.3: Gradient.

The standard basis representation of the gradient is

$$\nabla = \gamma^\mu \partial_\mu,$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu}.$$

Proof. The Dirac gradient pops naturally out of the coordinate representation of the directional derivative, as we can see by expanding $F(x + \epsilon a)$ in Taylor series

$$
\begin{aligned}
F(x + \epsilon a) &= F(x) + \epsilon \frac{dF(x + \epsilon a)}{d\epsilon} + O(\epsilon^2) \\
&= F(x) + \epsilon \frac{\partial F}{\partial (x^\mu + \epsilon a^\mu)} \frac{\partial (x^\mu + \epsilon a^\mu)}{\partial \epsilon} \\
&= F(x) + \epsilon \frac{\partial F}{\partial (x^\mu + \epsilon a^\mu)} a^\mu.
\end{aligned}
\tag{1.6}
$$

The directional derivative is

$$
\begin{aligned}
\lim_{\epsilon \to 0} \frac{F(x + \epsilon a) - F(x)}{\epsilon} &= \lim_{\epsilon \to 0} a^\mu \frac{\partial F}{\partial (x^\mu + \epsilon a^\mu)} \\
&= a^\mu \frac{\partial F}{\partial x^\mu} \\
&= (a^\nu \gamma_\nu) \cdot \gamma^\mu \frac{\partial F}{\partial x^\mu} \\
&= a \cdot \left(\gamma^\mu \partial_\mu \right) F.
\end{aligned}
\tag{1.7}
$$

$\square$

1.4 Curvilinear bases.

Curvilinear bases are the foundation of the fundamental theorem of multivector calculus. This form of integral calculus is defined over parameterized surfaces (called manifolds) that satisfy some specific non-degeneracy and continuity requirements.

A parameterized vector $x(u, v, \cdots w)$ can be thought of as tracing out a hypersurface (curve, surface, volume, ...), where the dimension of the hypersurface depends on the number of parameters. At each point, a bases can be constructed from the differentials of the parameterized vector. Such a basis is called the tangent space to the surface at the point in question. Our curvilinear bases will be related to these differentials. We will also be interested in a dual basis that is restricted to the span of the tangent space. This dual basis will be called the reciprocal frame, and line the basis of the tangent space itself, also varies from point to point on the surface.

One and two parameter spaces are illustrated in fig. 1.1.

(a)   (b)
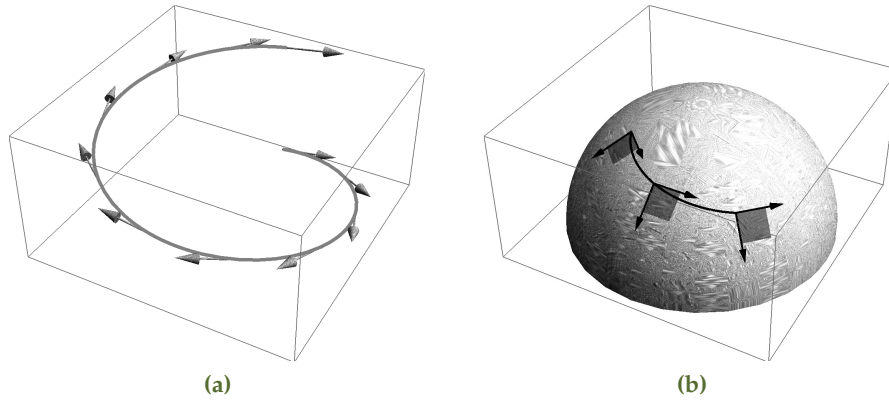
Figure 1.1: One and two parameter curves, with illustration of tangent spaces.

The tangent space basis at a specific point of a two parameter surface, $x(u^0, u^1)$, is illustrated in fig. 1.2. The differential directions that span the tangent space are

$$dx_0 = \frac{\partial x}{\partial u^0} du^0$$

$$dx_1 = \frac{\partial x}{\partial u^1} du^1, \tag{1.8}$$

and the tangent space itself is span $\{dx_0, dx_1\}$. We may form an oriented surface area element $dx_0 \wedge dx_1$ over this surface. Tangent spaces associated with 3 or more parameters cannot be easily visualized in
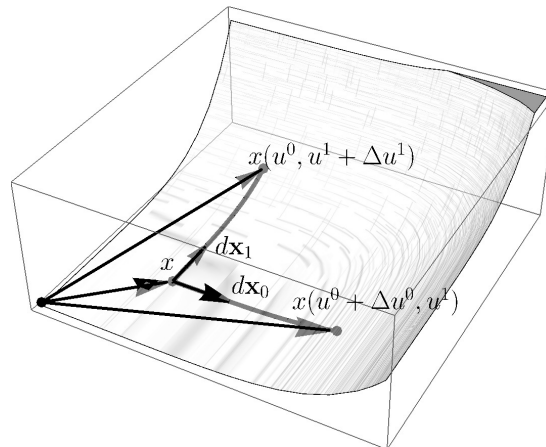
Figure 1.2: Two parameter surface.

three dimensions, but the idea generalizes algebraically without trouble.

> **Definition 1.5: Tangent basis and space.**
>
> Given a parameterization $x = x(u^0, \cdots, u^N)$, where $N < 4$, the span of the vectors
>
> $$\mathbf{x}_\mu = \frac{\partial x}{\partial u^\mu},$$
>
> is called the tangent space for the hypersurface associated with the parameterization, and it's basis is $\{\mathbf{x}_\mu\}$.

Later we will see that parameterization constraints must be imposed, as not all surfaces generated by a set of parameterizations are useful for integration theory. In particular, degenerate parameterizations for which the wedge products of the tangent space basis vectors are zero, or those wedge products cannot be inverted, are not physically meaningful. We require the functional form of parameterizations to be differentiable, to provide a one to one invertible mapping from the parameter space to the vector space. Properly behaved surfaces of this sort are called manifolds.

Having introduced curvilinear coordinates associated with a parameterization, we can now determine the form of the gradient with respect to a parameterization of spacetime.

> **Theorem 1.4: Gradient, curvilinear representation.**
>
> Given a spacetime parameterization $x = x(u^0, u^1, u^2, u^3)$, the gradient with respect to the parameters u^μ is
>
> $$\nabla = \sum_\mu \mathbf{x}^\mu \frac{\partial}{\partial u^\mu},$$
>
> where
>
> $$\mathbf{x}^\mu = \nabla u^\mu.$$
>
> The vectors $\mathbf{x}^\mu$ are called the reciprocal frame vectors, and the ordered set $\{\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ is called the reciprocal basis.
>
> It is convenient to define $\partial_\mu \equiv \partial/\partial u^\mu$, so that the gradient can be expressed in mixed index representation
>
> $$\nabla = \mathbf{x}^\mu \partial_\mu.$$
>
> This introduces some notational ambiguity, since we used $\partial_\mu = \partial/\partial x^\mu$ for the standard basis derivative operators too, but we will be careful to be explicit when there is any doubt about what is intended.

Proof. The proof follows by application of the chain rule.

$$
\begin{aligned}
\nabla F &= \gamma^\alpha \frac{\partial F}{\partial x^\alpha} \\
&= \gamma^\alpha \frac{\partial u^\mu}{\partial x^\alpha} \frac{\partial F}{\partial u^\mu} \\
&= (\nabla u^\mu) \frac{\partial F}{\partial u^\mu} \\
&= \mathbf{x}^\mu \frac{\partial F}{\partial u^\mu}.
\end{aligned}
\tag{1.9}
$$

$\square$

Theorem 1.5: Reciprocal relationship.

The vectors $\mathbf{x}^\mu = \nabla u^\mu$, and $\mathbf{x}_\mu = \partial x / \partial u^\mu$ satisfy the reciprocal relationship

$$
\mathbf{x}^\mu \cdot \mathbf{x}_\nu = \delta^\mu{}_\nu.
$$

Proof.

$$
\begin{aligned}
\mathbf{x}^\mu \cdot \mathbf{x}_\nu &= \nabla u^\mu \cdot \frac{\partial x}{\partial u^\nu} \\
&= \left(\gamma^\alpha \frac{\partial u^\mu}{\partial x^\alpha} \right) \cdot \left(\frac{\partial x^\beta}{\partial u^\nu} \gamma_\beta \right) \\
&= \delta^\alpha{}_\beta \frac{\partial u^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial u^\nu} \\
&= \frac{\partial u^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial u^\nu} \\
&= \frac{\partial u^\mu}{\partial u^\nu} \\
&= \delta^\mu{}_\nu.
\end{aligned}
\tag{1.10}
$$

$\square$

The following example parameterization scales the proper time parameter, and uses polar coordinates in the x, y plane.

Exercise 1.1   Compute the curvilinear and reciprocal basis.

Given $x(t, \rho, \theta, z) = ct\gamma_0 + \gamma_1 \rho e^{-i\theta} + z\gamma_3$, where $i = \gamma_1 \gamma_2$, compute the curvilinear frame vectors and their reciprocals.
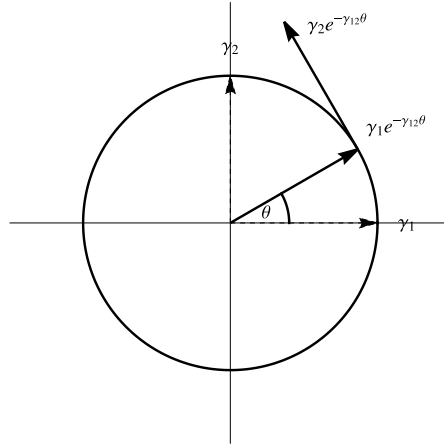
Answer for Exercise 1.1

The frame vectors are all easy to compute

$$
\begin{aligned}
\mathbf{x}_0 &= \frac{\partial x}{\partial t} = c\gamma_0 \\
\mathbf{x}_1 &= \frac{\partial x}{\partial \rho} = \gamma_1 e^{-i\theta} \\
\mathbf{x}_2 &= \frac{\partial x}{\partial \theta} = -\rho\gamma_1\gamma_1\gamma_2 e^{-i\theta} = \rho\gamma_2 e^{-i\theta} \\
\mathbf{x}_3 &= \frac{\partial x}{\partial z} = \gamma_3.
\end{aligned}
\tag{1.11}
$$

The $\mathbf{x}_1$ vector is radial, $\mathbf{x}^2$ is perpendicular to that tangent to the same unit circle, as plotted in fig. 1.3. All

Figure 1.3: Tangent space direction vectors.

of these particular frame vectors happen to be mutually perpendicular, something that will not generally be true for a more arbitrary parameterization.

To compute the reciprocal frame vectors, we must express our parameters in terms of x^μ coordinates, and use implicit integration techniques to deal with the coupling of the rotational terms. First observe that

$$
\begin{aligned}
\gamma_1 e^{-i\theta} &= \gamma_1 \left(\cos\theta - \gamma_1\gamma_2 \sin\theta \right) \\
&= \gamma_1 \cos\theta + \gamma_2 \sin\theta,
\end{aligned}
\tag{1.12}
$$

so $x = x^\mu \gamma_\mu = ct\gamma_0 + \gamma_1\rho e^{-i\theta} + z\gamma_3$ is equivalent to the following set of coordinate equations

$$
\begin{aligned}
x^0 &= ct \\
x^1 &= \rho\cos\theta \\
x^2 &= \rho\sin\theta \\
x^3 &= z.
\end{aligned}
\tag{1.13}
$$

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We can easily evaluate the t, z gradients

$$\nabla t = \frac{\gamma^1}{c}$$

$$\nabla z = \gamma^3,$$

(1.14)

but the ρ, θ gradients are not as easy. First writing

$$\rho^2 = \left(x^1\right)^2 + \left(x^2\right)^2,$$

(1.15)

we find

$$\begin{aligned}
2\rho\nabla\rho &= 2\left(x^1\nabla x^1 + x^2\nabla x^2\right) \\
&= 2\rho\left(\cos\theta\gamma^1 + \sin\theta\gamma^2\right) \\
&= 2\rho\gamma^1\left(\cos\theta + \gamma_1\gamma^2\sin\theta\right) \\
&= 2\rho\gamma^1 e^{-i\theta},
\end{aligned}$$

(1.16)

so

$$\nabla\rho = \gamma^1 e^{-i\theta}.$$

(1.17)

For the θ gradient, we can write

$$\tan\theta = \frac{x^2}{x^1},$$

(1.18)

so

$$\begin{aligned}
\frac{1}{\cos^2\theta}\nabla\theta &= \frac{\gamma^2}{x^1} - x^2\frac{\gamma^1}{\left(x^1\right)^2} \\
&= \frac{1}{\left(x^1\right)^2}\left(\gamma^2 x^1 - \gamma^1 x^2\right) \\
&= \frac{\rho}{\rho^2\cos^2\theta}\left(\gamma^2\cos\theta - \gamma^1\sin\theta\right) \\
&= \frac{1}{\rho\cos^2\theta}\gamma^2\left(\cos\theta - \gamma_2\gamma^1\sin\theta\right) \\
&= \frac{\gamma^2 e^{-i\theta}}{\rho\cos^2\theta},
\end{aligned}$$

(1.19)

or

$$\nabla\theta = \frac{1}{\rho}\gamma^2 e^{-i\theta}.$$

(1.20)

In summary,

$$\begin{aligned}
\mathbf{x}^0 &= \frac{\gamma^0}{c} \\
\mathbf{x}^1 &= \gamma^1 e^{-i\theta} \\
\mathbf{x}^2 &= \frac{1}{\rho}\gamma^2 e^{-i\theta} \\
\mathbf{x}^3 &= \gamma^3.
\end{aligned} \tag{1.21}$$

Despite being a fairly simple parameterization, it was still fairly difficult to solve for the gradients when the parameterization introduced coupling between the coordinates. Because all the tangent space vectors are mutually orthogonal, we didn't need to go through all this trouble, since we could have just computed the inverses of all the tangent space vectors

$$\begin{aligned}
\mathbf{x}^0 &= \frac{1}{\mathbf{x}_0} = \frac{1}{c\gamma_0} = \frac{1}{c}\gamma^0 \\
\mathbf{x}^1 &= \frac{1}{\mathbf{x}_1} = \frac{1}{\gamma_1 e^{-i\alpha}} = \frac{1}{e^{-i\alpha}}\frac{1}{\gamma_1} = e^{i\alpha}\gamma^1 = \gamma^1 e^{-i\alpha} \\
\mathbf{x}^2 &= \frac{1}{\mathbf{x}_2} = \frac{1}{\rho\gamma_2 e^{-i\alpha}} = \frac{1}{\rho}\frac{1}{e^{-i\alpha}}\frac{1}{\gamma_2} = \frac{1}{\rho}e^{i\alpha}\gamma^2 = \frac{1}{\rho}\gamma^2 e^{-i\alpha} \\
\mathbf{x}^3 &= \frac{1}{\mathbf{x}_3} = \frac{1}{\gamma_3} = \gamma^3.
\end{aligned} \tag{1.22}$$

We will not generally be this lucky, and want a less labor intensive strategy to find the reciprocal frame that works for the general case.

There is one additional special cawe. When we have a full parameterization of spacetime, then we can do this with nothing more than a matrix inversion.

Theorem 1.6: Reciprocal frame matrix equations.

Given a spacetime basis $\{\mathbf{x}_0, \cdots \mathbf{x}_3\}$, let $[\mathbf{x}_\mu]$ and $[\mathbf{x}^\nu]$ be column matrices with the coordinates of these vectors and their reciprocals, with respect to the standard basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$. Let

$$A = \begin{bmatrix} [\mathbf{x}_0] & \cdots & [\mathbf{x}_3] \end{bmatrix}, \qquad X = \begin{bmatrix} [\mathbf{x}^0] & \cdots & [\mathbf{x}^3] \end{bmatrix}.$$

The coordinates of the reciprocal frame vectors can be found by solving

$$A^{\mathrm{T}} G X = 1,$$

where $G = \mathrm{diag}(1, -1, -1, -1)$ and the RHS is an 4×4 identity matrix.

Proof. Let $\mathbf{x}_\mu = a_\mu{}^\alpha \gamma_\alpha$, $\mathbf{x}^\nu = b^{\nu\beta}\gamma_\beta$, so that

$$A = \begin{bmatrix} a_\nu{}^\mu \end{bmatrix}, \tag{1.23}$$

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and

$$X = \begin{bmatrix} b^{\nu\mu} \end{bmatrix}, \tag{1.24}$$

where $\mu \in [0, 3]$ are the row indexes and $\nu \in [0, N-1]$ are the column indexes. The reciprocal frame satisfies $\mathbf{x}_\mu \cdot \mathbf{x}^\nu = \delta_\mu{}^\nu$, which has the coordinate representation of

$$\begin{aligned}
\mathbf{x}_\mu \cdot \mathbf{x}^\nu &= \left(a_\mu{}^\alpha \gamma_\alpha \right) \cdot \left(b^{\nu\beta} \gamma_\beta \right) \\
&= a_\mu{}^\alpha \eta_{\alpha\beta} b^{\nu\beta} \\
&= [A^{\mathsf{T}} G B]_\mu{}^\nu,
\end{aligned} \tag{1.25}$$

where μ is the row index and ν is the column index. $\qquad\square$

Exercise 1.2   Matrix inversion reciprocals.

For the parameterization of exercise 1.1, find the reciprocal frame vectors by matrix inversion.

Answer for Exercise 1.2

We expanded $\mathbf{x}_1$ explicitly in eq. (1.12). Doing the same for $\mathbf{x}_2$, we have

$$\begin{aligned}
\mathbf{x}_2 &= \rho \gamma_2 e^{i\theta} \\
&= \rho \gamma_2 \left(\cos\theta - \gamma_1 \gamma_2 \sin\theta \right) \\
&= \rho \left(\gamma_2 \cos\theta - \gamma_1 \sin\theta \right).
\end{aligned} \tag{1.26}$$

Reading off the coordinates of our frame vectors, we have

$$X = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & C & -\rho S & 0 \\ 0 & S & \rho C & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1.27}$$

where $C = \cos\theta$ and $S = \sin\theta$. We want

$$Y = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & -C & -S & 0 \\ 0 & \rho S & -\rho C & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{c} & 0 & 0 & 0 \\ 0 & -C & \frac{S}{\rho} & 0 \\ 0 & -S & -\frac{C}{\rho} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1.28}$$

We can read off the coordinates of the reciprocal frame vectors

$$\begin{aligned}
\mathbf{x}^0 &= \frac{1}{c} \gamma_0 \\
\mathbf{x}^1 &= -\cos\theta\, \gamma_1 - \sin\theta\, \gamma_2 \\
\mathbf{x}^2 &= \frac{1}{\rho} \left(\sin\theta\, \gamma_1 + \cos\theta\, \gamma_2 \right) \\
\mathbf{x}^3 &= -\gamma_3.
\end{aligned} \tag{1.29}$$

Factoring out γ^1 from the $\mathbf{x}^1$ terms, we find

$$\begin{aligned}
\mathbf{x}^1 &= -\cos\theta\gamma_1 - \sin\theta\gamma_2 \\
&= \gamma^1(\cos\theta - \gamma_1\gamma_2\sin\theta) \\
&= \gamma^1 e^{-i\theta}.
\end{aligned} \tag{1.30}$$

Similarly for $\mathbf{x}^2$,

$$\begin{aligned}
\mathbf{x}^2 &= \frac{1}{\rho}(\sin\theta\gamma_1 - \cos\theta\gamma_2) \\
&= \frac{\gamma^2}{\rho}(\sin\theta\gamma_2\gamma_1 + \cos\theta) \\
&= \frac{\gamma^2}{\rho}e^{-i\theta}.
\end{aligned} \tag{1.31}$$

This matches eq. (1.21), as expected, but required only algebraic work to compute.

There will be circumstances where we parameterize only a subset of spacetime, and are interested in calculating quantities associated with such a surface. For example, suppose that

$$x(\rho, \theta) = \gamma_1 \rho e^{-i\theta}, \tag{1.32}$$

where $i = \gamma_1\gamma_2$ as before. We are now parameterizing only the $x - y$ plane. We will still find

$$\begin{aligned}
\mathbf{x}_1 &= \gamma_1 e^{-i\theta} \\
\mathbf{x}_2 &= \gamma_2 \rho e^{-i\theta}.
\end{aligned} \tag{1.33}$$

We can compute the reciprocals of these vectors using the gradient method. It's possible to state matrix equations representing the reciprocal relationship of theorem 1.5, which, in this case, is $X^{\mathsf{T}}GY = 1$, where the RHS is a 2×2 identity matrix, and X, Y are 4×2 matrices of coordinates, with

$$X = \begin{bmatrix} 0 & 0 \\ C & -\rho S \\ S & \rho C \\ 0 & 0 \end{bmatrix}. \tag{1.34}$$

We no longer have a square matrix problem to solve, and our solution set is multivalued. In particular, this matrix equation has solutions

$$\begin{aligned}
\mathbf{x}^1 &= \gamma^1 e^{-i\theta} + \alpha\gamma^0 + \beta\gamma^3 \\
\mathbf{x}^2 &= \frac{\gamma^2}{\rho}e^{-i\theta} + \alpha'\gamma^0 + \beta'\gamma^3.
\end{aligned} \tag{1.35}$$

where $\alpha, \alpha', \beta, \beta'$ are arbitrary constants. In the example we considered, we saw that our ρ, θ parameters were functions of only x^1, x^2, so taking gradients could not introduce any γ^0, γ^3 dependence in $\mathbf{x}^1, \mathbf{x}^2$. It

seems reasonable to assert that we seek an algebraic method of computing a set of vectors that satisfies the reciprocal relationships, where that set of vectors is restricted to the tangent space. We will need to figure out how to prove that this reciprocal construction is identical to the parameter gradients, but let's start with figuring out what such a tangent space restricted solution looks like.

> **Theorem 1.7: Reciprocal frame for two parameter subspace.**
>
> Given two vectors, $\mathbf{x}_1, \mathbf{x}_2$, the vectors $\mathbf{x}^1, \mathbf{x}^2 \in \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ such that $\mathbf{x}^\mu \cdot \mathbf{x}_\nu = \delta^\mu{}_\nu$ are given by
>
> $$\mathbf{x}^1 = \mathbf{x}_2 \cdot \frac{1}{\mathbf{x}_1 \wedge \mathbf{x}_2}$$
> $$\mathbf{x}^2 = -\mathbf{x}_1 \cdot \frac{1}{\mathbf{x}_1 \wedge \mathbf{x}_2},$$
>
> provided $\mathbf{x}_1 \wedge \mathbf{x}_2 \neq 0$ and $(\mathbf{x}_1 \wedge \mathbf{x}_2)^2 \neq 0$.

Proof. The most general set of vectors that satisfy the span constraint are

$$\mathbf{x}^1 = a\mathbf{x}_1 + b\mathbf{x}_2$$
$$\mathbf{x}^2 = c\mathbf{x}_1 + d\mathbf{x}_2. \tag{1.36}$$

We can use wedge products with either $\mathbf{x}_1$ or $\mathbf{x}_2$ to eliminate the other from the RHS

$$\mathbf{x}^1 \wedge \mathbf{x}_2 = a\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)$$
$$\mathbf{x}^1 \wedge \mathbf{x}_1 = -b\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)$$
$$\mathbf{x}^2 \wedge \mathbf{x}_2 = c\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)$$
$$\mathbf{x}^2 \wedge \mathbf{x}_1 = -d\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right), \tag{1.37}$$

and then dot both sides with $\mathbf{x}_1 \wedge \mathbf{x}_2$ to produce four scalar equations

$$
\begin{aligned}
a\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2 &= \left(\mathbf{x}^1 \wedge \mathbf{x}_2\right) \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right) \\
&= (\mathbf{x}_2 \cdot \mathbf{x}_1)\,\cancel{\mathbf{x}^1 \cdot \mathbf{x}_2} - (\mathbf{x}_2 \cdot \mathbf{x}_2)\left(\mathbf{x}^1 \cdot \mathbf{x}_1\right) \\
&= -\mathbf{x}_2 \cdot \mathbf{x}_2
\end{aligned} \tag{1.38a}
$$

$$
\begin{aligned}
-b\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2 &= \left(\mathbf{x}^1 \wedge \mathbf{x}_1\right) \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right) \\
&= \cancel{\mathbf{x}^1 \cdot \mathbf{x}_2}\,(\mathbf{x}_1 \cdot \mathbf{x}_1) - \left(\mathbf{x}^1 \cdot \mathbf{x}_1\right)(\mathbf{x}_1 \cdot \mathbf{x}_2) \\
&= -\mathbf{x}_1 \cdot \mathbf{x}_2
\end{aligned} \tag{1.38b}
$$

$$c \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2 = \left(\mathbf{x}^2 \wedge \mathbf{x}_2\right) \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)$$
$$= \left(\mathbf{x}_2 \cdot \mathbf{x}_1\right) \left(\mathbf{x}^2 \cdot \mathbf{x}_2\right) - \left(\mathbf{x}_2 \cdot \mathbf{x}_2\right) \cancel{\mathbf{x}^2 \cdot \mathbf{x}_1}$$
$$= \mathbf{x}_2 \cdot \mathbf{x}_1 \tag{1.38c}$$

$$-d \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2 = \left(\mathbf{x}^2 \wedge \mathbf{x}_1\right) \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)$$
$$= \left(\mathbf{x}_1 \cdot \mathbf{x}_1\right) \left(\mathbf{x}^2 \cdot \mathbf{x}_2\right) - \left(\mathbf{x}_1 \cdot \mathbf{x}_2\right) \cancel{\mathbf{x}^2 \cdot \mathbf{x}_1}$$
$$= \mathbf{x}_1 \cdot \mathbf{x}_1. \tag{1.38d}$$

Putting the pieces together we have

$$\mathbf{x}^1 = \frac{- \left(\mathbf{x}_2 \cdot \mathbf{x}_2\right) \mathbf{x}_1 + \left(\mathbf{x}_1 \cdot \mathbf{x}_2\right) \mathbf{x}_2}{\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2}$$
$$= \frac{\mathbf{x}_2 \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)}{\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2} \tag{1.39a}$$
$$= \mathbf{x}_2 \cdot \frac{1}{\mathbf{x}_1 \wedge \mathbf{x}_2}$$

$$\mathbf{x}^2 = \frac{\left(\mathbf{x}_1 \cdot \mathbf{x}_2\right) \mathbf{x}_1 - \left(\mathbf{x}_1 \cdot \mathbf{x}_1\right) \mathbf{x}_2}{\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2}$$
$$= \frac{-\mathbf{x}_1 \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)}{\left(\mathbf{x}_1 \wedge \mathbf{x}_2\right)^2} \tag{1.39b}$$
$$= -\mathbf{x}_1 \cdot \frac{1}{\mathbf{x}_1 \wedge \mathbf{x}_2}$$

$\square$

Lemma 1.1: Distribution identity.

Given k-vectors B, C and a vector a, where the grade of C is greater than that of B, then

$$\left(a \wedge B\right) \cdot C = a \cdot \left(B \cdot C\right).$$

See [1] for a proof.

Theorem 1.8: Higher order tangent space reciprocals.

Given an N parameter tangent space with basis $\{\mathbf{x}_0, \mathbf{x}_1, \cdots \mathbf{x}_{N-1}\}$, the reciprocals are given by

$$\mathbf{x}^\mu = (-1)^\mu \left(\mathbf{x}_0 \wedge \cdots \widetilde{\mathbf{x}_\mu} \cdots \wedge \mathbf{x}_{N-1}\right) \cdot I_N^{-1},$$

where the checked term $(\widetilde{\mathbf{x}_\mu})$ indicates that all terms are included in the wedges except the $\mathbf{x}_\mu$ term, and $I_N = \mathbf{x}_0 \wedge \cdots \mathbf{x}_{N-1}$ is the pseudoscalar for the tangent space.

Proof. I'll outline the proof for the three parameter tangent space case, from which the pattern will be clear. The motivation for this proof is a reexamination of the algebraic structure of the two vector solution. Suppose we have a tangent space basis $\{\mathbf{x}_0, \mathbf{x}_1\}$, for which we've shown that

$$\mathbf{x}^0 = \mathbf{x}_1 \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1} = \frac{\mathbf{x}_1 \cdot (\mathbf{x}_0 \wedge \mathbf{x}_1)}{(\mathbf{x}_0 \wedge \mathbf{x}_1)^2}. \tag{1.40}$$

If we dot with $\mathbf{x}_0$ and $\mathbf{x}_1$ respectively, we find

$$\mathbf{x}_0 \cdot \mathbf{x}^0 = \mathbf{x}_0 \cdot \frac{\mathbf{x}_1 \cdot (\mathbf{x}_0 \wedge \mathbf{x}_1)}{(\mathbf{x}_0 \wedge \mathbf{x}_1)^2} = (\mathbf{x}_0 \wedge \mathbf{x}_1) \cdot \frac{\mathbf{x}_0 \wedge \mathbf{x}_1}{(\mathbf{x}_0 \wedge \mathbf{x}_1)^2}. \tag{1.41}$$

We end up with unity as expected. Here the "factored" out vector is reincorporated into the pseudoscalar using the distribution identity lemma 1.1. Similarly, dotting with $\mathbf{x}_1$, we find

$$\mathbf{x}_1 \cdot \mathbf{x}^0 = \mathbf{x}_1 \cdot \frac{\mathbf{x}_1 \cdot (\mathbf{x}_0 \wedge \mathbf{x}_1)}{(\mathbf{x}_0 \wedge \mathbf{x}_1)^2} = (\mathbf{x}_1 \wedge \mathbf{x}_1) \cdot \frac{\mathbf{x}_0 \wedge \mathbf{x}_1}{(\mathbf{x}_0 \wedge \mathbf{x}_1)^2}. \tag{1.42}$$

This is zero, since wedging a vector with itself is zero. We can perform such an operation in reverse, taking the square of the tangent space pseudoscalar, and factoring out one of the basis vectors. After this, division by that squared pseudoscalar will normalize things.

For a three parameter tangent space with basis $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2\}$, we can factor out any of the tangent vectors like so

$$\begin{aligned}
(\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)^2 &= \mathbf{x}_0 \cdot ((\mathbf{x}_1 \wedge \mathbf{x}_2) \cdot (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)) \\
&= (-1)\mathbf{x}_1 \cdot ((\mathbf{x}_0 \wedge \mathbf{x}_2) \cdot (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)) \\
&= (-1)^2 \mathbf{x}_2 \cdot ((\mathbf{x}_0 \wedge \mathbf{x}_1) \cdot (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)).
\end{aligned} \tag{1.43}$$

The toggling of sign reflects the number of permutations required to move the vector of interest to the front of the wedge sequence. Having factored out any one of the vectors, we can rearrange to find that vector that is it's inverse and perpendicular to all the others.

$$\begin{aligned}
\mathbf{x}^0 &= (-1)^0 (\mathbf{x}_1 \wedge \mathbf{x}_2) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2} \\
\mathbf{x}^1 &= (-1)^1 (\mathbf{x}_0 \wedge \mathbf{x}_2) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2} \\
\mathbf{x}^2 &= (-1)^2 (\mathbf{x}_0 \wedge \mathbf{x}_1) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2}.
\end{aligned} \tag{1.44}$$

$\square$

In the fashion above, should we want the reciprocal frame for all of spacetime given dimension 4 tangent space, we can state it trivially

$$\mathbf{x}^0 = (-1)^0 (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3}$$
$$\mathbf{x}^1 = (-1)^1 (\mathbf{x}_0 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3}$$
$$\mathbf{x}^2 = (-1)^2 (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_3) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3}$$
$$\mathbf{x}^3 = (-1)^3 (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2) \cdot \frac{1}{\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3}.$$

(1.45)

This is probably not an efficient way to compute all these reciprocals, since we can utilize a single matrix inversion to solve them in one shot. However, there are theoretical advantages to this construction that will be useful when we get to integration theory.

On degeneracy.   A small mention of degeneracy was mentioned above. Regardless of metric, $\mathbf{x}_0 \wedge \mathbf{x}_1 = 0$ means that this pair of vectors are colinear. A tangent space with such a pseudoscalar is clearly undesirable, and we must construct parameterizations for which the area element is non-zero in all regions of interest.

Things get more interesting in mixed signature spaces where we can have vectors that square to zero (i.e. lightlike). If the tangent space pseudoscalar has a lightlike factor, then that pseudoscalar will not be invertible. Such a degeneracy will will likely lead to many other troubles, and parameterizations of this sort should be avoided.

This following problem illustrates an example of this sort of degenerate parameterization.

Exercise 1.3   Degenerate surface parameterization.

Given a spacetime plane parameterization $x(u, v) = ua + vb$, where

$$a = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3,$$ (1.46)

$$b = \gamma_0 - \gamma_1 + \gamma_2 - \gamma_3,$$ (1.47)

show that this is a degenerate parameterization, and find the bivector that represents the tangent space. Are these vectors lightlike, spacelike, or timelike? Comment on whether this parameterization represents a physically relevant spacetime surface.

Answer for Exercise 1.3

To characterize the vectors, we square them

$$a^2 = b^2 = \gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 - 3 = -2,$$ (1.48)

so a, b are both spacelike vectors. The tangent space is clearly just span $\{a, b\}$ = span $\{e, f\}$ where

$$
\begin{aligned}
e &= \gamma_0 + \gamma_2 \\
f &= \gamma_1 + \gamma_3.
\end{aligned}
\tag{1.49}
$$

Observe that $a = e + f$, $b = e - f$, and e is lightlike ($e^2 = 0$), whereas f is spacelike ($f^2 = -2$), and $e \cdot f = 0$, so $ef = -fe$. The bivector for the tangent plane is

$$
\begin{aligned}
\langle ab \rangle_2 &= \langle (e + f)(e - f) \rangle_2 \\
&= \langle e^2 - f^2 - 2ef \rangle_2 \\
&= -2ef,
\end{aligned}
\tag{1.50}
$$

where
$$
ef = \gamma_{01} + \gamma_{21} + \gamma_{23} + \gamma_{03}.
\tag{1.51}
$$

Because e is lightlike (zero square), and $ef = -fe$, the bivector ef squares to zero

$$
(ef)^2 = -e^2 f^2 = 0,
\tag{1.52}
$$

which shows that the parameterization is degenerate.

This parameterization can also be expressed as

$$
\begin{aligned}
x(u, v) &= u(e + f) + v(e - f) \\
&= (u + v)e + (u - v)f,
\end{aligned}
\tag{1.53}
$$

a linear combination of a lightlike and spacelike vector. Intuitively, we expect that a physically meaningful spacetime surface involves linear combinations spacelike vectors, or combinations of a timelike vector with spacelike vectors. This beastie is something entirely different.

Final notes. There are a few loose ends above. In particular, we haven't conclusively proven that the set of reciprocal vectors $x^\mu = \nabla u^\mu$ are exactly those obtained through algebraic means. For a full parameterization of spacetime, they are necessarily the same, since both are unique. So we know that eq. (1.45) must equal the reciprocals obtained by evaluating the gradient for a full parameterization (and this must also equal the reciprocals that we can obtain through matrix inversion.) We have also not proved explicitly that the three parameter construction of the reciprocals in eq. (1.44) is in the tangent space, but that is a fairly trivial observation, so that can be left as an exercise for the reader dismissal. Some additional thought about this is probably required, but it seems reasonable to put that on the back burner and move on to some applications. XX

1.5 More examples.

new blog post: Here are a few additional examples of reciprocal frame calculations.

Exercise 1.4   Unidirectional arbitrary functional dependence.

Let

$$x = af(u),$$

where a is a constant vector and $f(u)$ is some arbitrary differentiable function with a non-zero derivative in the region of interest.

Answer for Exercise 1.4

Here we have just a single tangent space direction (a line in spacetime) with tangent vector

$$\mathbf{x}_u = a\frac{\partial f}{\partial u} \tag{1.54}$$
$$= af_u,$$

so we see that the tangent space vectors are just rescaled values of the direction vector a. This is a simple enough parameterization that we can compute the reciprocal frame vector explicitly using the gradient. We expect that $\mathbf{x}^u = 1/\mathbf{x}_u$, and find

$$\frac{1}{a} \cdot x = f(u), \tag{1.55}$$

but for constant a, we know that $\nabla a \cdot x = a$, so taking gradients of both sides we find

$$\frac{1}{a} = \nabla f = \frac{\partial f}{\partial u}\nabla u, \tag{1.56}$$

so the reciprocal vector is

$$\mathbf{x}^u = \nabla u = \frac{1}{af_u}, \tag{1.57}$$

as expected.

Exercise 1.5   Linear two variable parameterization.

Let $x = au + bv$, where $x \wedge a \wedge b = 0$ represents spacetime plane (also the tangent space.) Find the curvilinear coordinates and their reciprocals.

Answer for Exercise 1.5

The frame vectors are easy to compute, as they are just

$$\mathbf{x}_u = \frac{\partial x}{\partial u} = a$$
$$\mathbf{x}_v = \frac{\partial x}{\partial v} = b. \tag{1.58}$$

This is an example of a parametric equation that we can easily invert, as we have

$$x \wedge a = -v \, (a \wedge b)$$
$$x \wedge b = u \, (a \wedge b) \,, \tag{1.59}$$

so

$$u = \frac{1}{a \wedge b} \cdot (x \wedge b)$$
$$= \frac{1}{(a \wedge b)^2} \, (a \wedge b) \cdot (x \wedge b) \tag{1.60}$$
$$= \frac{(b \cdot x)(a \cdot b) - (a \cdot x)(b \cdot b)}{(a \wedge b)^2}$$

$$v = -\frac{1}{a \wedge b} \cdot (x \wedge a)$$
$$= -\frac{1}{(a \wedge b)^2} \, (a \wedge b) \cdot (x \wedge a) \tag{1.61}$$
$$= -\frac{(b \cdot x)(a \cdot a) - (a \cdot x)(a \cdot b)}{(a \wedge b)^2}$$

Recall that $\nabla (a \cdot x) = a$, if a is a constant, so our gradients are just

$$\nabla u = \frac{b \, (a \cdot b) - a \, (b \cdot b)}{(a \wedge b)^2}$$
$$= b \cdot \frac{1}{a \wedge b}, \tag{1.62}$$

and

$$\nabla v = -\frac{b \, (a \cdot a) - a \, (a \cdot b)}{(a \wedge b)^2}$$
$$= -a \cdot \frac{1}{a \wedge b}. \tag{1.63}$$

Expressed in terms of the frame vectors, this is just

$$\mathbf{x}^u = \mathbf{x}_v \cdot \frac{1}{\mathbf{x}_u \wedge \mathbf{x}_v}$$
$$\mathbf{x}^v = -\mathbf{x}_u \cdot \frac{1}{\mathbf{x}_u \wedge \mathbf{x}_v}, \tag{1.64}$$

so we were able to show, for this special two parameter linear case, that the explicit evaluation of the gradients has the exact structure that we intuited that the reciprocals must have, provided they are constrained to the spacetime plane $a \wedge b$. It is interesting to observe how this structure falls out of the linear system solution so directly. Also note that these reciprocals are not defined at the origin of the (u, v) parameter space.

Now consider a variation of the previous problem, with $x = au^2 + bv^2$. Find the curvilinear coordinates and their reciprocals.

Answer for Exercise 1.6

$$\mathbf{x}_u = \frac{\partial x}{\partial u} = 2ua$$
$$\mathbf{x}_v = \frac{\partial x}{\partial v} = 2vb. \tag{1.65}$$

Our tangent space is still the $a \wedge b$ plane (as is the surface itself), but the spacing of the cells starts getting wider in proportion to u, v. Utilizing the work from the previous problem, we have

$$2u\nabla u = b \cdot \frac{1}{a \wedge b}$$
$$2v\nabla v = -a \cdot \frac{1}{a \wedge b}. \tag{1.66}$$

A bit of rearrangement, shows that we recover eq. (1.64) easily. This is a second demonstration that the gradient and the algebraic formulations for the reciprocals match, at least for these special cases of linear non-coupled parameterizations.

Exercise 1.7   Reciprocal frame for generalized cylindrical parameterization.

Let the vector parameterization be $x(\rho, \theta) = \rho e^{-i\theta/2} x(\rho_0, \theta_0) e^{i\theta}$, where $i^2 = \pm 1$ is a unit bivector (+1 for a boost, and -1 for a rotation), and where θ, ρ are scalars. Find the tangent space vectors and their reciprocals.

Note that this is cylindrical parameterization for the rotation case, and traces out hyperbolic regions for the boost case. The boost case is illustrated in fig. 1.4 where hyperbolas in the light cone are found for boosts of γ_0 with various values of ρ, and the spacelike hyperbolas are boosts of γ_1, again for various values of ρ.

Answer for Exercise 1.7

The tangent space vectors are

$$\mathbf{x}_\rho = \frac{x}{\rho}, \tag{1.67}$$

and

$$\mathbf{x}_\theta = -\frac{i}{2}x + x\frac{i}{2} \tag{1.68}$$
$$= x \cdot i.$$

Recall that $x \cdot i$ lies perpendicular to x (in the plane i), as illustrated in fig. 1.5. This means that $\mathbf{x}_\rho$ and $\mathbf{x}_\theta$ are orthogonal, so we can find the reciprocal vectors by just inverting them
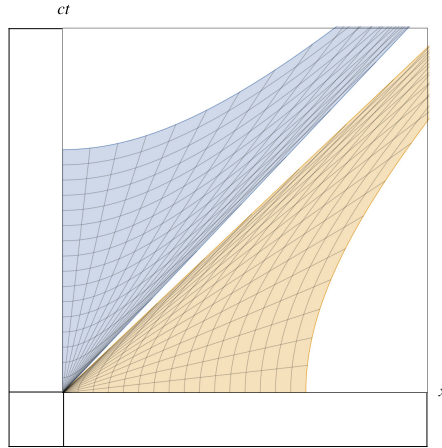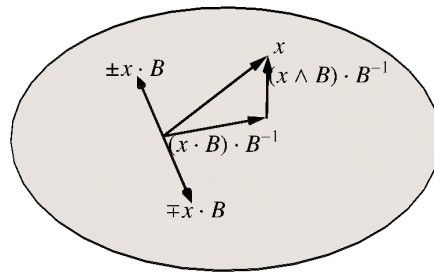
Figure 1.4: "Cylindrical" boost parameterization.

Figure 1.5: Projection and rejection geometry.

$$\mathbf{x}^\rho = \frac{\rho}{x}$$

$$\mathbf{x}^\theta = \frac{1}{x \cdot i}. \tag{1.69}$$

1.6 Parameterization of a general linear transformation.

Given N parameters $u^0, u^1, \cdots u^{N-1}$, a general linear transformation from the parameter space to the vector space has the form

$$x = a^\alpha{}_\beta \gamma_\alpha u^\beta, \tag{1.70}$$

where $\beta \in [0, \cdots, N-1]$ and $\alpha \in [0, 3]$. For such a general transformation, observe that the curvilinear basis vectors are

$$\begin{aligned}
\mathbf{x}_\mu &= \frac{\partial x}{\partial u^\mu} \\
&= \frac{\partial}{\partial u^\mu} a^\alpha{}_\beta \gamma_\alpha u^\beta \\
&= a^\alpha{}_\mu \gamma_\alpha.
\end{aligned} \tag{1.71}$$

We find an interpretation of $a^\alpha{}_\mu$ by dotting $\mathbf{x}_\mu$ with the reciprocal frame vectors of the standard basis

$$\begin{aligned}
\mathbf{x}_\mu \cdot \gamma^\nu &= a^\alpha{}_\mu \left(\gamma_\alpha \cdot \gamma^\nu \right) \\
&= a^\nu{}_\mu,
\end{aligned} \tag{1.72}$$

so

$$x = \mathbf{x}_\mu u^\mu. \tag{1.73}$$

We are able to reinterpret eq. (1.70) as a contraction of the tangent space vectors with the parameters, scaling and summing these direction vectors to characterize all the points in the tangent plane.

Theorem 1.9: Projecting onto the tangent space.

Let T represent the tangent space. The projection of a vector onto the tangent space has the form

$$\mathrm{Proj}_T y = (y \cdot \mathbf{x}^\mu) \, \mathbf{x}_\mu = (y \cdot \mathbf{x}_\mu) \, \mathbf{x}^\mu.$$

Proof. Let's designate a as the portion of the vector y that lies outside of the tangent space

$$y = y^\mu \mathbf{x}_\mu + a. \tag{1.74}$$

If we knew the coordinates y^μ, we would have a recipe for the projection. Algebraically, requiring that a lies outside of the tangent space, is equivalent to stating $a \cdot \mathbf{x}_\mu = a \cdot \mathbf{x}^\mu = 0$. We use that fact, and then take dot products

$$\begin{aligned}
y \cdot \mathbf{x}^\nu &= \left(y^\mu \mathbf{x}_\mu + a \right) \cdot \mathbf{x}^\nu \\
&= y^\nu,
\end{aligned} \tag{1.75}$$

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so

$$y = \left(y \cdot \mathbf{x}^\mu \right) \mathbf{x}_\mu + a. \tag{1.76}$$

Similarly, the tangent space projection can be expressed as a linear combination of reciprocal basis elements

$$y = y_\mu \mathbf{x}^\mu + a. \tag{1.77}$$

Dotting with $\mathbf{x}_\mu$, we have

$$\begin{aligned} y \cdot \mathbf{x}^\mu &= \left(y_\alpha \mathbf{x}^\alpha + a \right) \cdot \mathbf{x}_\mu \\ &= y_\mu, \end{aligned} \tag{1.78}$$

so

$$y = \left(y \cdot \mathbf{x}^\mu \right) \mathbf{x}_\mu + a. \tag{1.79}$$

We find the two stated ways of computing the projection.

Observe that, for the special case that all of $\{\mathbf{x}_\mu\}$ are orthogonal, the equivalence of these two projection methods follows directly, since

$$\begin{aligned} \left(y \cdot \mathbf{x}^\mu \right) \mathbf{x}_\mu &= \left(y \cdot \frac{1}{\mathbf{x}_\mu} \right) \frac{1}{\mathbf{x}^\mu} \\ &= \left(y \cdot \frac{\mathbf{x}_\mu}{\left(\mathbf{x}_\mu \right)^2} \right) \frac{\mathbf{x}^\mu}{\left(\mathbf{x}^\mu \right)^2} \\ &= \left(y \cdot \mathbf{x}_\mu \right) \mathbf{x}^\mu. \end{aligned} \tag{1.80}$$

$\square$

Bibliography

[1] Peeter Joot. *Geometric Algebra for Electrical Engineers*. Kindle Direct Publishing, 2019. 1.4