Peeter Joot
peeterjoot@pm.me

Vector gradients in dyadic notation and geometric algebra.

This is an exploration of the dyadic representation of the gradient acting on a vector in $\mathbb{R}^3$, where we determine a tensor product formulation of a vector differential. Such a tensor product formulation can be split into symmetric and antisymmetric components. The geometric algebra (GA) equivalents of such a split are determined.

1.1 GA gradient of a vector.

In GA we are free to express the product of the gradient and a vector field by adjacency. In coordinates (summation over repeated indexes assumed), such a product has the form

$$\begin{aligned} \boldsymbol{\nabla} \mathbf{v} &= \left(\mathbf{e}_i \partial_i \right) \left(v_j \mathbf{e}_j \right) \\ &= \left(\partial_i v_j \right) \mathbf{e}_i \mathbf{e}_j. \end{aligned} \tag{1.1}$$

In this sum, any terms with $i = j$ are scalars since $\mathbf{e}_i^2 = 1$, and the remaining terms are bivectors. This can be written compactly as

$$\boldsymbol{\nabla} \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla} \wedge \mathbf{v}, \tag{1.2}$$

or for $\mathbb{R}^3$

$$\boldsymbol{\nabla} \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{v} + I \left(\boldsymbol{\nabla} \times \mathbf{v} \right), \tag{1.3}$$

either of which breaks the gradient into into divergence and curl components. In eq. (1.2) this vector gradient is expressed using the bivector valued curl operator ($\boldsymbol{\nabla} \wedge \mathbf{v}$), whereas eq. (1.3) is expressed using the vector valued dual form of the curl ($\boldsymbol{\nabla} \times \mathbf{v}$) from convential vector algebra.

It is worth noting that order matters in the GA coordinate expansion of eq. (1.1). It is not correct to write

$$\boldsymbol{\nabla} \mathbf{v} = \left(\partial_i v_j \right) \mathbf{e}_j \mathbf{e}_i, \tag{1.4}$$

which is only true when the curl, $\boldsymbol{\nabla} \wedge \mathbf{v} = 0$, is zero.

1.2 Dyadic representation.

Given a vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$, the differential of that field can be computed by chain rule

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x_i} dx_i = \left(d\mathbf{x} \cdot \boldsymbol{\nabla}\right) \mathbf{v}, \tag{1.5}$$

where $d\mathbf{x} = \mathbf{e}_i dx_i$. This is a representation invariant form of the differential, where we have a scalar operator $d\mathbf{x} \cdot \boldsymbol{\nabla}$ acting on the vector field $\mathbf{v}$. The matrix representation of this differential can be written as

$$d\mathbf{v} = \left(\left[d\mathbf{x}\right]^\dagger \left[\boldsymbol{\nabla}\right]\right) \left[\mathbf{v}\right], \tag{1.6}$$

where we are using the dagger to designate transposition, and each of the terms on the right are the coordinate matrixes of the vectors with respect to the standard basis

$$\left[d\mathbf{x}\right] = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}, \quad \left[\mathbf{v}\right] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \left[\boldsymbol{\nabla}\right] = \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}. \tag{1.7}$$

In eq. (1.6) the parens are very important, as the expression is meaningless without them. With the parens we have a $(1 \times 3)(3 \times 1)$ matrix (i.e. a scalar) multiplied with a 3×1 matrix. That becomes ill-formed if we drop the parens since we are left with an incompatible product of a $(3 \times 1)(3 \times 1)$ matrix on the right. The dyadic notation, which introducing a tensor product into the mix, is a mechanism to make sense of the possibility of such a product. Can we make sense of an expression like $\boldsymbol{\nabla}\mathbf{v}$ without the geometric product in our toolbox?

Stepping towards that question, let's examine the coordinate expansion of our vector differential eq. (1.5), which is

$$d\mathbf{v} = dx_i \left(\partial_i v_j\right) \mathbf{e}_j. \tag{1.8}$$

If we allow a matrix of vectors, this has a block matrix form

$$d\mathbf{v} = \left[d\mathbf{x}\right]^\dagger \left[\boldsymbol{\nabla} \otimes \mathbf{v}\right] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}. \tag{1.9}$$

Here we introduce the tensor product

$$\boldsymbol{\nabla} \otimes \mathbf{v} = \partial_i v_j \, \mathbf{e}_i \otimes \mathbf{e}_j, \tag{1.10}$$

and designate the matrix of coordinates $\partial_i v_j$, a second order tensor, by $\left[\boldsymbol{\nabla} \otimes \mathbf{v}\right]$.

We have succeeded in factoring out a vector gradient. We can introduce dot product between vectors and a direct product of vectors, by observing that eq. (1.9) has the structure of a quadradic form, and define

$$\mathbf{x} \cdot (\mathbf{a} \otimes \mathbf{b}) \equiv \left[\mathbf{x}\right]^\dagger \left[\mathbf{a} \otimes \mathbf{b}\right] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}, \tag{1.11}$$

so that eq. (1.9) takes the form

$$d\mathbf{v} = d\mathbf{x} \cdot (\boldsymbol{\nabla} \otimes \mathbf{v}) . \tag{1.12}$$

Such a dot product gives operational meaning to the gradient-vector tensor product.

1.3 Symmetrization and antisymmetrization of the vector differential in GA.

Using the dyadic notation, it's possible to split a vector derivative into symmetric and antisymmetric components with respect to the gradient-vector direct product

$$\begin{aligned} d\mathbf{v} &= d\mathbf{x} \cdot \left(\frac{1}{2} \left(\boldsymbol{\nabla} \otimes \mathbf{v} + (\boldsymbol{\nabla} \otimes \mathbf{v})^\dagger \right) + \frac{1}{2} \left(\boldsymbol{\nabla} \otimes \mathbf{v} - (\boldsymbol{\nabla} \otimes \mathbf{v})^\dagger \right) \right) \\ &= d\mathbf{x} \cdot (\mathbf{d} + \boldsymbol{\Omega}) , \end{aligned} \tag{1.13}$$

where $\boldsymbol{\Omega}$ is a traceless antisymmetric tensor.

A question of potential interest is "what GA equvivalent of this expression?". There are two identities that are helpful for extracting this equivalence, the first of which is the k-blade vector product identities. Given a k-blade B_k (i.e.: a product of k orthogonal vectors, or the wedge of k vectors), and a vector $\mathbf{a}$, the dot product of the two is

$$B_k \cdot \mathbf{a} = \frac{1}{2} \left(B_k \mathbf{a} + (-1)^{k+1} \mathbf{a} B_k \right) \tag{1.14}$$

Specifically, given two vectors $\mathbf{a}, \mathbf{b}$, the vector dot product can be written as a symmetric sum

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2} (\mathbf{ab} + \mathbf{ba}) = \mathbf{b} \cdot \mathbf{a}, \tag{1.15}$$

and given a bivector B and a vector $\mathbf{a}$, the bivector-vector dot product can be written as an antisymmetric sum

$$B \cdot \mathbf{a} = \frac{1}{2} (B\mathbf{a} - \mathbf{a}B) = -\mathbf{a} \cdot B. \tag{1.16}$$

We may apply these to expressions where one of the vector terms is the gradient, but must allow for the gradient to act bidirectionally. That is, given multivectors M, N

$$\begin{aligned} M \boldsymbol{\nabla} N &= \partial_i (M \mathbf{e}_i N) \\ &= (\partial_i M) \mathbf{e}_i N + M \mathbf{e}_i (\partial_i N), \end{aligned} \tag{1.17}$$

where parens have been used to indicate the scope of applicibility of the partials. In particular, this means that we may write the divergence as a GA symmetric sum

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{2} (\boldsymbol{\nabla} \mathbf{v} + \mathbf{v} \boldsymbol{\nabla}) , \tag{1.18}$$

which clearly corresponds to the symmetric term $\mathbf{d} = (1/2) \left(\boldsymbol{\nabla} \otimes \mathbf{v} + (\boldsymbol{\nabla} \otimes \mathbf{v})^\dagger \right)$ from eq. (1.13).

Let's assume that we can write our vector differential in terms of a divergence term isomorphic to the symmetric sum in eq. (1.13), and a "something else", $\mathbf{X}$. That is

$$\begin{aligned} d\mathbf{v} &= (d\mathbf{x} \cdot \boldsymbol{\nabla})\,\mathbf{v} \\ &= d\mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{v}) + \mathbf{X}, \end{aligned} \tag{1.19}$$

where

$$\mathbf{X} = (d\mathbf{x} \cdot \boldsymbol{\nabla})\,\mathbf{v} - d\mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{v}), \tag{1.20}$$

is a vector expression to be reduced to something simpler. That reduction is possible using the distribution identity

$$\mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}, \tag{1.21}$$

so we find

$$\mathbf{X} = \boldsymbol{\nabla} \cdot (d\mathbf{x} \wedge \mathbf{v}). \tag{1.22}$$

We find the following GA split of the vector differential into symmetric and antisymmetric terms

$$\boxed{d\mathbf{v} = (d\mathbf{x} \cdot \boldsymbol{\nabla})\mathbf{v} = d\mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{v}) + \boldsymbol{\nabla} \cdot (d\mathbf{x} \wedge \mathbf{v}).} \tag{1.23}$$

Such a split avoids the indeterminant nature of the tensor product, which we only give meaning by introducing the quadratic form based dot product given by eq. (1.11).