PEETER JOOT

CONVEX OPTIMIZATION

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Notes and problems from UofT ECE1505H 2017

March 2023 – version V0.1.5

Peeter Joot: *Convex optimization,* Notes and problems from UofT ECE1505H 2017, © March 2023

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export PATH=`pwd`/latex/bin:$PATH

cd ece1505-convex-optimization

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I reserve the right to impose dictatorial control over any editing and content decisions, and may not accept merge requests as-is, or at all. That said, I will probably not refuse reasonable suggestions or merge requests.

Dedicated to:
Aurora and Lance, my awesome kids, and
Sofia, who not only tolerates and encourages my studies, but is also awesome enough to think that math is sexy.

PREFACE

This is a partial set of course notes from the Winter 2017, University of Toronto Convex optimization course (ECE1505H), taught by Prof. Stark Draper, covering nine lectures worth of the material, and the first two problem sets.

THIS DOCUMENT IS REDACTED. THE SOLUTION TO PROBLEM SET 1 IS NOT VISIBLE. PLEASE EMAIL ME FOR THE FULL VERSION IF YOU ARE NOT TAKING ECE1505.

This document contains:

- Lecture notes.
- Personal notes exploring auxiliary details.
- Worked practice problems.
- Problem set 2, without solution.

My thanks go to Professor Draper for teaching the portion of this course that I took.

Peeter Joot peeterjoot@pm.me

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1

INTRODUCTION.

1.1 WHAT'S THIS COURSE ABOUT?

- Science of optimization.
- problem formulation, design, analysis of engineering systems.

1.2 BASIC CONCEPTS

- Basic concepts. convex sets, functions, problems.
- Theory (about 40 % of the material). Specifically Lagrangian duality.
- Algorithms: gradient descent, Newton's, interior point, ...

Homework will involve computational work (solving problems, ...)

1.3 GOALS

- Recognize and formulate engineering problems as convex optimization problems.
- To develop (Matlab) code to solve problems numerically.
- To characterize the solutions via duality theory
- NOT a math course, but lots of proofs.
- NOT a communications course, but lots of ... (?)
- NOT a CS course, but lots of useful algorithms.

Definition 1.1: Mathematical program

$$\min_{\mathbf{x}} F_0(\mathbf{x}) \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ is subject to constraints $F_i : \mathbb{R}^m \to \mathbb{R}^1$

$$F_i(\mathbf{x}) \leq 0, \qquad i = 1, \cdots, m. \tag{1.2}$$

The function $F_0 : \mathbb{R}^m \to \mathbb{R}^1$ is called the "objective function".

Solving a problem produces:

An optimal $\mathbf{x}^*$ is a value $\mathbf{x}$ that gives the smallest value among all the feasible $\mathbf{x}$ for the objective function F_0. Such a function is sketched in fig. 1.1.

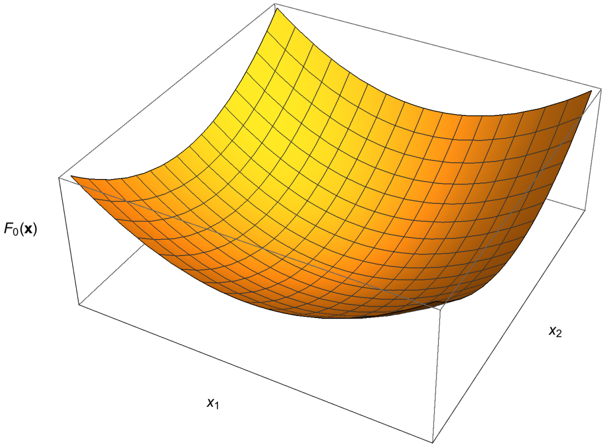


Figure 1.1: Convex objective function.

- A convex objective looks like a bowl, "holds water".
- If connect two feasible points line segment in the ? above bottom of the bowl.

A non-convex function is illustrated in fig. 1.2, which has a number of local minimums.

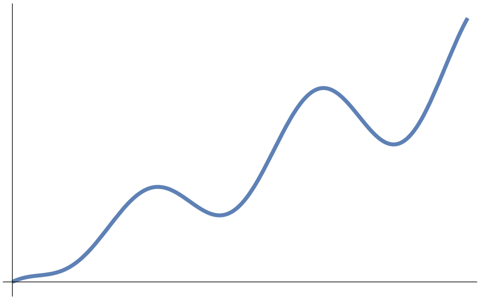


Figure 1.2: Non-convex (wavy) figure with a number of local minimums.

1.4 EXAMPLE: LINE FITTING.

A linear fit of some points distributed around a line $y = ax + b$ is plotted in fig. 1.3. Here a, b are the optimization variables $\mathbf{x} = (a, b)$.

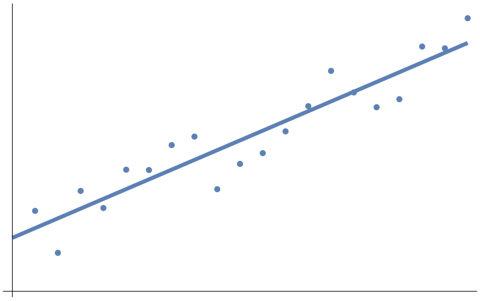


Figure 1.3: Linear fit of points around a line.

How is the solution for such a best fit line obtained?

Approach 1: Calculus minimization of a multivariable error function. Describe an error function, describing how far from the line a given point is.

$$y_i - (ax_i + b), \tag{1.3}$$

Because this can be positive or negative, we can define a squared variant of this, and then sum over all data points.

$$F_0 = \sum_{i=1}^{n} \left(y_i - (ax_i + b)\right)^2 . \tag{1.4}$$

One way to solve (for a, b): Take the derivatives

$$\begin{aligned}\frac{\partial F_0}{\partial a} &= \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0\\ \frac{\partial F_0}{\partial b} &= \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0.\end{aligned} \tag{1.5}$$

This yields

$$\begin{aligned}\sum_{i=1}^{n} y_i &= \left(\sum_{i=1}^{n} x_i\right) a + \left(\sum_{i=1}^{n} 1\right) b\\ \sum_{i=1}^{n} x_i y_i &= \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b.\end{aligned} \tag{1.6}$$

In matrix form, this is

$$\begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix} = \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{1.7}$$

If invertible, have an analytic solution for (a^*, b^*). This is a convex optimization problem because $F(x) = x^2$ is a convex “quadratic program”. In general a quadratic program has the structure

$$F(a, b) = (\cdots)a^2 + (\cdots)ab + (\cdots)b^2. \tag{1.8}$$

Approach 2: Linear algebraic formulation.

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \tag{1.9}$$

or

$$\mathbf{y} = H\mathbf{v} + \mathbf{z}, \tag{1.10}$$

where $\mathbf{z}$ is the error vector. The problem is now reduced to : Fit $\mathbf{y}$ to be as close to $H\mathbf{v} + \mathbf{z}$ as possible, or to minimize the norm of the error vector, or

$$\begin{aligned}\min_{\mathbf{v}} \|\mathbf{y} - H\mathbf{v}\|_2^2 &= \min_{\mathbf{v}} (\mathbf{y} - H\mathbf{v})^{\mathrm{T}} (\mathbf{y} - H\mathbf{v})\\ &= \min_{\mathbf{v}} \left(\mathbf{y}^{\mathrm{T}}\mathbf{y} - \mathbf{y}^{\mathrm{T}} H\mathbf{v} - \mathbf{v}^{\mathrm{T}} H\mathbf{y} + \mathbf{v}^{\mathrm{T}} H^{\mathrm{T}} H\mathbf{v}\right)\\ &= \min_{\mathbf{v}} \left(\mathbf{y}^{\mathrm{T}}\mathbf{y} - 2\mathbf{y}^{\mathrm{T}} H\mathbf{v} + \mathbf{v}^{\mathrm{T}} H^{\mathrm{T}} H\mathbf{v}\right).\end{aligned} \tag{1.11}$$

It is now possible to take the derivative with respect to the **v** vector (i.e. the gradient with respect to the coordinates of the constraint vector)

$$\begin{aligned}\frac{\partial}{\partial \mathbf{v}}\left(\mathbf{y}^{\mathrm{T}}\mathbf{y} - 2\mathbf{y}^{\mathrm{T}}H\mathbf{v} + \mathbf{v}^{\mathrm{T}}H^{\mathrm{T}}H\mathbf{v}\right) &= -2\mathbf{y}^{\mathrm{T}}H + 2\mathbf{v}^{\mathrm{T}}H^{\mathrm{T}}H \\ &= 0,\end{aligned} \tag{1.12}$$

or

$$(H^{\mathrm{T}}H)\mathbf{v} = H^{\mathrm{T}}\mathbf{y}, \tag{1.13}$$

so, assuming that $H^{\mathrm{T}}H$ is invertible, the optimization problem has solution

$$\mathbf{v}^* = (H^{\mathrm{T}}H)^{-1}H^{\mathrm{T}}\mathbf{y}, \tag{1.14}$$

where

$$\begin{aligned}H^{\mathrm{T}}H &= \begin{bmatrix} x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix},\end{aligned} \tag{1.15}$$

as seen in the calculus approach.

1.5 MAXIMUM LIKELYHOOD ESTIMATION (MLE).

It is reasonable to ask why the 2-norm was picked for the objective function?

- One justification is practical: Because we can solve the derivative equation.
- Another justification: In statistics the error vector $\mathbf{z} = \mathbf{y} - H\mathbf{v}$ can be modeled as an IID (Independently and Identically Distributed) Gaussian random variable (i.e. noise). Under this model, the use of the 2-norm can be viewed as a consequence of such an ML estimation problem (see [1] ch. 7).

A Gaussian fig. 1.4 IID model is given by

$$y_i = ax_i + b \tag{1.16a}$$

$$z_i = y_i - ax_i - b \sim N(O, O^2) \tag{1.16b}$$

$$P_Z(z) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2} z^2/\sigma^2\right). \tag{1.16c}$$

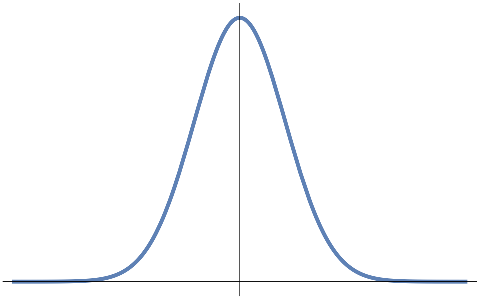


Figure 1.4: Gaussian probability distribution.

MLE: Maximum Likelyhood Estimator Pick (a, b) to maximize the probability of observed data.

$$\begin{aligned}
(a^*, b^*) &= \arg\max P(x, y; a, b) \\
&= \arg\max P_Z(y - (ax + b)) \\
&= \arg\max \prod_{i=1}^{n} \\
&= \arg\max \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}(y_i - ax_i - b)^2/\sigma^2\right).
\end{aligned} \tag{1.17}$$

Taking logs gives

$$\begin{aligned}
(a^*, b^*) &= \arg\max \left(\text{constant} - \frac{1}{2}\sum_i (y_i - ax_i - b)^2/\sigma^2\right) \\
&= \arg\min \frac{1}{2}\sum_i (y_i - ax_i - b)^2/\sigma^2 \\
&= \arg\min \sum_i (y_i - ax_i - b)^2/\sigma^2
\end{aligned} \tag{1.18}$$

Here arg max is not the maximum of the function, but the value of the parameter (the argument) that maximizes the function.

Double sides exponential noise A double sided exponential distribution is plotted in fig. 1.5, and has the mathematical form

$$P_Z(z) = \frac{1}{2c} \exp\left(-\frac{1}{c}|z|\right). \tag{1.19}$$

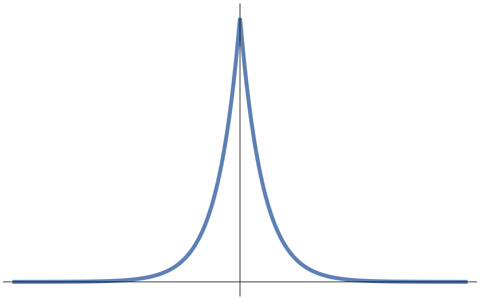


Figure 1.5: Double sided exponential probability distribution.

The optimization problem is

$$\begin{aligned} \max_{a,b} \prod_{i=1}^{n} P_z(z_i) &= \max_{a,b} \prod_{i=1}^{n} \frac{1}{2c} \exp\left(-\frac{1}{c}|z_i|\right) \\ &= \max_{a,b} \prod_{i=1}^{n} \frac{1}{2c} \exp\left(-\frac{1}{c}|y_i - ax_i - b|\right) \\ &= \max_{a,b} \left(\frac{1}{2c}\right)^n \exp\left(-\frac{1}{c}\sum_{i=1}^{n} |y_i - ax_i - b|\right). \end{aligned} \tag{1.20}$$

This is a L1 norm problem

$$\min_{a,b} \sum_{i=1}^{n} |y_i - ax_i - b|. \tag{1.21}$$

i.e.

$$\min_{\mathbf{v}} \|\mathbf{y} - H\mathbf{v}\|_1 . \tag{1.22}$$

This is still convex, but has no analytic solution, and is an example of a linear program.

1.5.1 *Solution of linear program*

Introduce helper variables $t_1, \cdots, t_n$, and minimize $\sum_i t_i$, such that

$$|y_i - ax_i - b| \le t_i. \tag{1.23}$$

This is now an optimization problem for $a, b, t_1, \cdots t_n$. A linear program is defined as

$$\min_{a,b,t_1,\cdots t_n} \sum_i t_i \tag{1.24}$$

such that

$$y_i - ax_i - b \le t_i y_i - ax_i - b \ge -t_i \tag{1.25}$$

Single sided exponential What if your noise doesn't look double sided, with only noise for values $x > 0$. Can define a single sided probability distribution, as that of fig. 1.6.

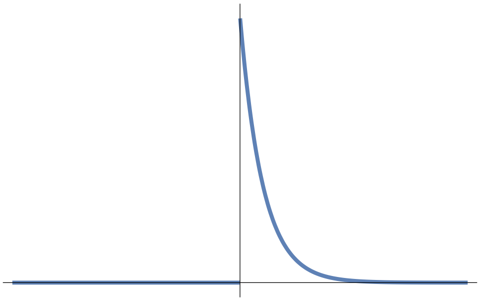


Figure 1.6: Single sided exponential distribution.

$$P_Z(z) = \begin{cases} \frac{1}{c} e^{-z/c} & z \ge 0 \\ 0 & z < 0 \end{cases} \tag{1.26}$$

i.e. all z_i error values are always non-negative.

$$\log P_z(z) = \begin{cases} \text{const} - z/c & z > 0 \\ -\infty & z < 0 \end{cases} \tag{1.27}$$

Problem becomes

$$\min_{a,b} \sum_i \left(y_i - ax_i - b\right), \tag{1.28}$$

such that

$$y_i - ax_i - b \ge t_i \qquad \forall i \tag{1.29}$$

Uniform noise For noise that is uniformly distributed in a range, as that of fig. 1.7, which is constant in the range $[-c, c]$ and zero outside that range.

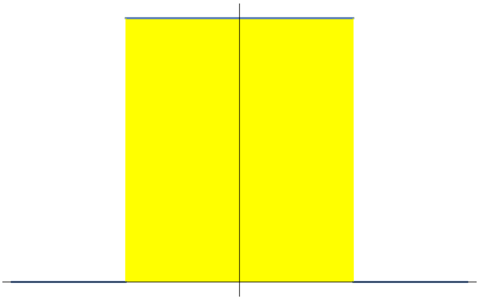


Figure 1.7: Uniform probability distribution.

$$P_Z(z) = \begin{cases} \frac{1}{2c} & |z| \leq c \\ 0 & |z| > c. \end{cases} \tag{1.30}$$

or

$$\log P_Z(z) = \begin{cases} \text{const} & |z| \leq c \\ -\infty & |z| > c. \end{cases} \tag{1.31}$$

MLE solution

$$\max_{a,b} \prod_{i=1}^{n} P(x, y; a, b) = \max_{a,b} \sum_{i=1}^{n} \log P_Z(y_i - ax_i - b). \tag{1.32}$$

Here the argument is constant if $-c \leq y_i - ax_i - b \leq c$, so an ML solution is $\underline{\text{any}}$ (a, b) such that

$$|y_i - ax_i - b| \leq c \qquad \forall i \in 1, \cdots, n. \tag{1.33}$$

This is a linear program known as a "feasibility problem".

$$\min d \tag{1.34}$$

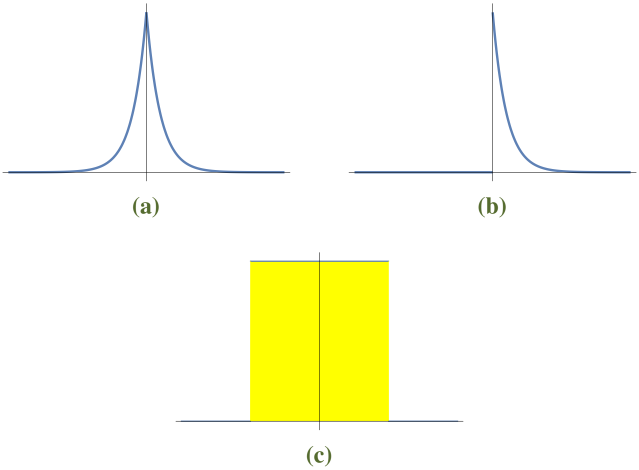
such that

$$\begin{aligned} y_i - ax_i - b &\leq d \\ y_i - ax_i - b &\geq -d \end{aligned} \tag{1.35}$$

If $d^* \leq c$, then the problem is feasible, however, if $d^* > c$ it is infeasible.

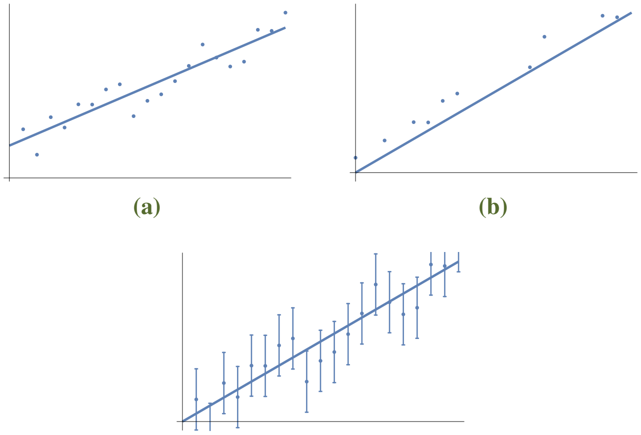
1.5.2 *Method comparison*

The double sided exponential, single sided exponential and uniform probability distributions of fig. 1.8 each respectively represent the point plots of the form fig. 1.9. The double sided exponential samples are distributed on both sides of the line, the single sided strictly above or on the line, and the uniform representing error bars distributed around the line of best fit.



(a) **(b)** **(c)**

Figure 1.8: Distributions



(a)

(b)

(c)

Figure 1.9: Samples

MATHEMATICAL BACKGROUND. 2

Topics

- Calculus: Derivatives and Jacobians, Gradients, Hessians, approximation functions.
- Linear algebra, Matrices, decompositions, ...

2.1 NORMS

Definition 2.1: Vector space

A set of elements (vectors) that is closed under vector addition and scaling.

This generalizes the directed arrow concept of vector space (fig. 2.1) that is familiar from geometry.

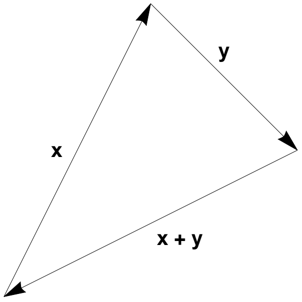


Figure 2.1: Vector addition.

Definition 2.2: Normed vector spaces

A vector space with a notion of length of any single vector, the "norm".

Definition 2.3: Inner product space.

A normed vector space with a notion of a real angle between any pair of vectors.

This course has a focus on optimization in $\mathbb{R}^n$. Complex spaces in the context of this course can be considered with a mapping $\mathbb{C}^n \to \mathbb{R}^{2n}$.

Definition 2.4: Norm.

A norm is a function operating on a vector

$$\mathbf{x} = (x_1, x_2, \cdots, x_n)$$

that provides a mapping

$$\|\cdot\| : \mathbb{R}^n \to \mathbb{R},$$

where

- $\|\mathbf{x}\| \geq 0$
- $\|\mathbf{x}\| = 0 \iff \mathbf{x} = 0$
- $\|t\mathbf{x}\| = |t| \, \|\mathbf{x}\|$
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. This is the triangle inequality.

Example: Euclidean norm

$$\|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{2.1}$$

Example: l_p-norms

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}. \tag{2.2}$$

For $p = 1$, this is

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|, \tag{2.3}$$

For $p = 2$, this is the Euclidean norm eq. (2.1). For $p = \infty$, this is

$$||\mathbf{x}||_\infty = \max_{i=1}^{n} |x_i|. \tag{2.4}$$

Note that it hasn't been proven here that $||\mathbf{x}||_p$ satisfies the triangle inequality. This is only the case for $p \geq 1$, and in general is known as the Minkowski identity [3]. The proof of this also requires Holder's inequality. A very nice treatment of both can be found on Dr. Chris Tisdell's youtube channel ([5], [4]).

Definition 2.5: Unit ball

$$\{\mathbf{x} | ||\mathbf{x}|| \leq 1\}$$

The regions of the unit ball under the $l_1, l_2, and l_\infty$ norms are plotted in fig. 2.2.

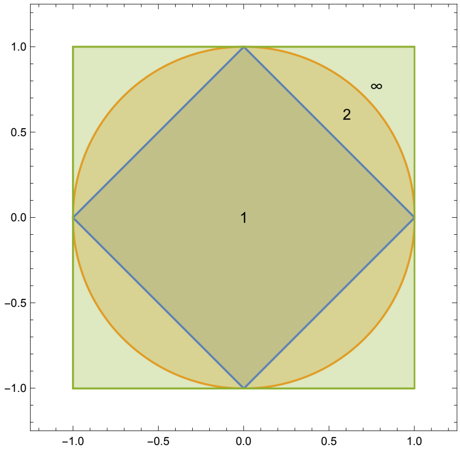


Figure 2.2: Some unit ball regions.

The l_2 norm is not only familiar, but can be "induced" by an inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y} = \sum_{i=1}^{n} x_i y_i, \tag{2.5}$$

which is not true for all norms. The norm induced by this inner product is

$$||\mathbf{x}||_2 = \sqrt{\langle \mathbf{x}, \mathbf{y} \rangle} \tag{2.6}$$

Inner product spaces have a notion of angle (fig. 2.3) given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta, \tag{2.7}$$

and always satisfy the Cauchy-Schwartz inequality

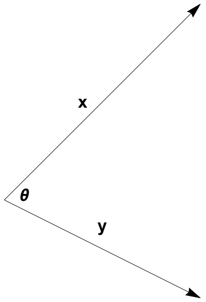


Figure 2.3: Inner product induced angle.

$$\langle \mathbf{x}, \mathbf{y} \rangle \le \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2 \,. \tag{2.8}$$

In an inner product space we say $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors $\mathbf{x} \perp \mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, as sketched in fig. 2.4.

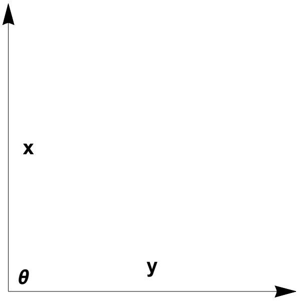


Figure 2.4: Orthogonality.

2.2 DUAL NORM

Definition 2.6: Dual norm

Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$. The "dual" norm $\|\cdot\|_*$ is defined as

$$\|\mathbf{z}\|_* = \sup_{\mathbf{x}} \left\{ \mathbf{z}^\mathrm{T} \mathbf{x} | \|\mathbf{x}\| \le 1 \right\}.$$

where sup is roughly the “least upper bound”.

This is a limit over the unit ball of $\|\cdot\|$.

l_2 dual .

Dual of the l_2 is the l_2 norm.

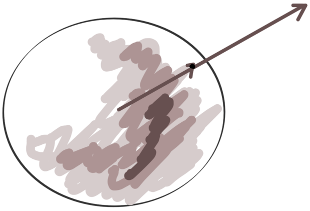


Figure 2.5: l_2 dual norm determination.

Proof:

$$\begin{aligned}
\|\mathbf{z}\|_* &= \sup_{\mathbf{x}} \left\{ \mathbf{z}^{\mathrm{T}}\mathbf{x} \middle| \|\mathbf{x}\|_2 \leq 1 \right\} \\
&= \sup_{\mathbf{x}} \{ \|\mathbf{z}\|_2 \|\mathbf{x}\|_2 \cos\theta | \|\mathbf{x}\|_2 \leq 1 \} \\
&\leq \sup_{\mathbf{x}} \{ \|\mathbf{z}\|_2 \|\mathbf{x}\|_2 | \|\mathbf{x}\|_2 \leq 1 \} \\
&\leq \cancel{\|\mathbf{z}\|_2} \left\| \frac{\mathbf{z}}{\cancel{\|\mathbf{z}\|_2}} \right\|_2 \\
&= \|\mathbf{z}\|_2 .
\end{aligned} \tag{2.9}$$

l_1 dual . For l_1, the dual is the l_∞ norm. Proof:

$$\|\mathbf{z}\|_* = \sup_{\mathbf{x}} \left\{ \mathbf{z}^{\mathrm{T}}\mathbf{x} \middle| \|\mathbf{x}\|_1 \leq 1 \right\}, \tag{2.10}$$

but

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}}\mathbf{x} = \sum_{i=1}^{n} z_i x_i &\leq \left| \sum_{i=1}^{n} z_i x_i \right| \\
&\leq \sum_{i=1}^{n} |z_i x_i|,
\end{aligned} \tag{2.11}$$

so

$$\begin{aligned}
\|\mathbf{z}\|_* &= \sum_{i=1}^{n} |z_i||x_i| \\
&\leq \left(\max_{j=1}^{n} \left|z_j\right|\right) \sum_{i=1}^{n} |x_i| \\
&\leq \left(\max_{j=1}^{n} \left|z_j\right|\right) \\
&= \|\mathbf{z}\|_\infty \,.
\end{aligned} \tag{2.12}$$

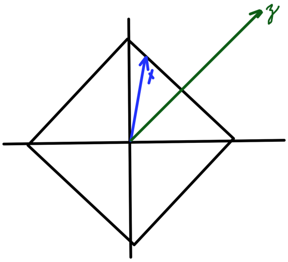


Figure 2.6: l_1 dual norm determination.

l_∞ dual .

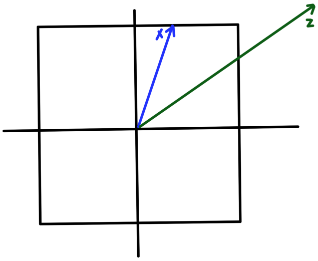


Figure 2.7: l_∞ dual norm determination.

$$\|\mathbf{z}\|_* = \sup_{\mathbf{x}} \left\{ \mathbf{z}^\mathrm{T} \mathbf{x} | \|\mathbf{x}\|_\infty \leq 1 \right\}. \tag{2.13}$$

Here

$$\begin{aligned}
\mathbf{z}^{\mathrm{T}}\mathbf{x} &= \sum_{i=1}^{n} z_i x_i \\
&\leq \sum_{i=1}^{n} |z_i||x_i| \\
&\leq \left(\max_j \left|x_j\right|\right) \sum_{i=1}^{n} |z_i| \\
&= \|\mathbf{x}\|_\infty \sum_{i=1}^{n} |z_i|.
\end{aligned} \tag{2.14}$$

So

$$\|\mathbf{z}\|_* \leq \sum_{i=1}^{n} |z_i| = \|\mathbf{z}\|_1 \,. \tag{2.15}$$

Statement from the lecture: I'm not sure where this fits:

$$x_i^* = \begin{cases} +1 & z_i \geq 0 \\ -1 & z_i \leq 0 \end{cases} \tag{2.16}$$

2.3 MULTIVARIABLE TAYLOR APPROXIMATION

The Taylor series expansion for a scalar function $g : \mathbb{R} \to \mathbb{R}$ about the origin is just

$$g(t) = g(0) + t g'(0) + \frac{t^2}{2} g''(0) + \cdots \tag{2.17}$$

In particular

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(0) + \cdots \tag{2.18}$$

Now consider $g(t) = f(\mathbf{x} + \mathbf{a}t)$, where $f : \mathbb{R}^n \to \mathbb{R}$, $g(0) = f(\mathbf{x})$, and $g(1) = f(\mathbf{x} + \mathbf{a})$. This trick, from [2] allows for a direct expansion of the multivariable Taylor series of a scalar function

$$f(\mathbf{x} + \mathbf{a}) = f(\mathbf{x}) + \left.\frac{d f(\mathbf{x} + \mathbf{a}t)}{dt}\right|_{t=0} + \frac{1}{2} \left.\frac{d^2 f(\mathbf{x} + \mathbf{a}t)}{dt^2}\right|_{t=0} + \cdots \tag{2.19}$$

The first order term is

$$\begin{aligned}\left.\frac{df(\mathbf{x}+\mathbf{a}t)}{dt}\right|_{t=0} &= \sum_{i=1}^{n} \frac{d(x_i+a_it)}{dt} \left.\frac{\partial f(\mathbf{x}+\mathbf{a}t)}{\partial(x_i+a_it)}\right|_{t=0} \\ &= \sum_{i=1}^{n} a_i \frac{\partial f(\mathbf{x})}{\partial x_i} \\ &= \mathbf{a}\cdot\boldsymbol{\nabla} f.\end{aligned} \tag{2.20}$$

Similarly, for the second order term

$$\begin{aligned}\left.\frac{d^2 f(\mathbf{x}+\mathbf{a}t)}{dt^2}\right|_{t=0} &= \left.\left(\frac{d}{dt}\left(\sum_{i=1}^{n} a_i \frac{\partial f(\mathbf{x}+\mathbf{a}t)}{\partial(x_i+a_it)}\right)\right)\right|_{t=0} \\ &= \left.\left(\sum_{j=1}^{n} \frac{d(x_j+a_jt)}{dt} \sum_{i=1}^{n} a_i \frac{\partial^2 f(\mathbf{x}+\mathbf{a}t)}{\partial(x_j+a_jt)\partial(x_i+a_it)}\right)\right|_{t=0} \\ &= \sum_{i,j=1}^{n} a_ia_j \frac{\partial^2 f}{\partial x_i \partial x_j} \\ &= (\mathbf{a}\cdot\boldsymbol{\nabla})^2 f.\end{aligned} \tag{2.21}$$

The complete Taylor expansion of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ is therefore

$$f(\mathbf{x}+\mathbf{a}) = f(\mathbf{x}) + \mathbf{a}\cdot\boldsymbol{\nabla} f + \frac{1}{2}\left(\mathbf{a}\cdot\boldsymbol{\nabla}\right)^2 f + \cdots, \tag{2.22}$$

so the Taylor expansion has an exponential structure

$$f(\mathbf{x}+\mathbf{a}) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\mathbf{a}\cdot\boldsymbol{\nabla}\right)^k f = e^{\mathbf{a}\cdot\boldsymbol{\nabla}} f. \tag{2.23}$$

Should an approximation of a vector valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ be desired it is only required to form a matrix of the components

$$\mathbf{f}(\mathbf{x}+\mathbf{a}) = \mathbf{f}(\mathbf{x}) + [\mathbf{a}\cdot\boldsymbol{\nabla} f_i]_i + \frac{1}{2}[\left(\mathbf{a}\cdot\boldsymbol{\nabla}\right)^2 f_i]_i + \cdots, \tag{2.24}$$

where $[.]_i$ denotes a column vector over the rows $i \in [1, m]$, and f_i are the coordinates of $\mathbf{f}$.

2.4 THE JACOBIAN MATRIX

In [1] the Jacobian $D\mathbf{f}$ of a function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined in terms of the limit of the l_2 norm ratio

$$\frac{\|\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{x}) - (D\mathbf{f})(\mathbf{z} - \mathbf{x})\|_2}{\|\mathbf{z} - \mathbf{x}\|_2}, \tag{2.25}$$

with the statement that the function $\mathbf{f}$ has a derivative if this limit exists. Here the Jacobian $D\mathbf{f} \in \mathbb{R}^{m \times n}$ must be matrix valued.

Let $\mathbf{z} = \mathbf{x} + \mathbf{a}$, so the first order expansion of eq. (2.24) is

$$\mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{x}) + \left[\left(\mathbf{z} - \mathbf{x}\right) \cdot \boldsymbol{\nabla} f_i \right]_i. \tag{2.26}$$

With the (unproven) assumption that this Taylor expansion satisfies the norm limit criteria of eq. (2.25), it is possible to extract the structure of the Jacobian by comparison

$$\begin{aligned}
(D\mathbf{f})(\mathbf{z} - \mathbf{x}) &= \left[\left(\mathbf{z} - \mathbf{x}\right) \cdot \boldsymbol{\nabla} f_i \right]_i \\
&= \left[\textstyle\sum_{j=1}^n (z_j - x_j) \frac{\partial f_i}{\partial x_j} \right]_i \\
&= \left[\frac{\partial f_i}{\partial x_j} \right]_{ij} (\mathbf{z} - \mathbf{x}),
\end{aligned} \tag{2.27}$$

so

$$\boxed{(D\mathbf{f})_{ij} = \frac{\partial f_i}{\partial x_j}} \tag{2.28}$$

Written out explicitly as a matrix the Jacobian is

$$D\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\nabla} f_1)^{\mathrm{T}} \\ (\boldsymbol{\nabla} f_2)^{\mathrm{T}} \\ \vdots \\ (\boldsymbol{\nabla} f_m)^{\mathrm{T}} \end{bmatrix}. \tag{2.29}$$

In particular, when the function is scalar valued

$$D f = (\boldsymbol{\nabla} f)^{\mathrm{T}}. \tag{2.30}$$

With this notation, the first Taylor expansion, in terms of the Jacobian matrix is

$$\boxed{\mathbf{f}(\mathbf{z}) \approx \mathbf{f}(\mathbf{x}) + (D\mathbf{f}) \left(\mathbf{z} - \mathbf{x}\right).} \tag{2.31}$$

Gradient The gradient provides a linear approximation of a function about a point $\mathbf{x}_0 \in \mathbb{R}^n$.

$$\begin{aligned} F(\mathbf{x}) &\approx F(\mathbf{x}_0) + \boldsymbol{\nabla} F(\mathbf{x}_0)^{\mathrm{T}} \left(\mathbf{x} - \mathbf{x}_0\right) \\ &= F(\mathbf{x}_0) + \langle \boldsymbol{\nabla} F(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle, \end{aligned} \tag{2.32}$$

or

$$F(\mathbf{x} + \Delta\mathbf{x}) = F(\mathbf{x}) + \langle \boldsymbol{\nabla} F(\mathbf{x}), \Delta\mathbf{x} \rangle. \tag{2.33}$$

This can be thought of as the definition of the gradient in an inner product space. It will be possible to find the structure of the gradient by considering a perturbation of a function about a point.

When g is a scalar function, the chain rule can be expressed in terms of the gradient

$$\boldsymbol{\nabla}(g(F(\mathbf{x}))) = \left(DF\right)^{\mathrm{T}}\Big|_{\mathbf{x}} \boldsymbol{\nabla} g|_{F(\mathbf{x})}. \tag{2.34}$$

Example 1:

$$\begin{aligned} F &: \mathbb{R}^n \to \mathbb{R} \\ g &: \mathbb{R} \to \mathbb{R}, \end{aligned} \tag{2.35}$$

and let

$$h(\mathbf{x}) = g(F(\mathbf{x})), \tag{2.36}$$

for $\mathbf{x} \in \mathbb{R}^n$, then

$$\boldsymbol{\nabla} h(\mathbf{x}) = g'(F(\mathbf{x}))\boldsymbol{\nabla} F(\mathbf{x}). \tag{2.37}$$

Example 2.1: Quadratic form

$$F(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} P \mathbf{x} = \sum_{i,j=1}^{n} x_i x_j P_{ij}, \tag{2.38}$$

We want to show that

$$\boldsymbol{\nabla} F(\mathbf{x}) = \left(P + P^{\mathrm{T}}\right)\mathbf{x}. \tag{2.39}$$

Consider the k-th derivative

$$\begin{aligned}\frac{\partial}{\partial x_k}F(\mathbf{x}) &= \frac{\partial}{\partial x_k}\left(P_{kk}x_k^2 + \sum_{i\neq k} x_i x_k \left(P_{ik} + P_{ki}\right)\right)\\ &= 2P_{kk}x_k + 2\sum_{i\neq k} x_i \frac{\left(P_{ik} + P_{ki}\right)}{2}\\ &= \sum_{i}^{n} x_i \frac{\left(P_{ik} + P_{ki}\right)}{2}\\ &= \sum_{i}^{n} \left(P_{ik} + P_{ki}\right) x_i,\end{aligned} \tag{2.40}$$

which proves eq. (2.39).

Symmetric matrices Let S^n be the set of symmetric matrices

$$S^n = \left\{P \in \mathbb{R}^{n\times n} | P = P^{\mathrm{T}}\right\}, \tag{2.41}$$

then

$$\boldsymbol{\nabla}(\mathbf{x}^{\mathrm{T}}P\mathbf{x}) = 2P\mathbf{x}. \tag{2.42}$$

2.5 CHAIN RULE

The gradients or Jacobians for compositions of functions can also be calculated

Theorem 2.1: Chain rule

Given functions

$$\begin{aligned}&F : \mathbb{R}^n \rightarrow \mathbb{R}^m\\ &g : \mathbb{R}^m \rightarrow \mathbb{R}^p,\end{aligned} \tag{2.43}$$

$$D(g(F(\mathbf{x}))) = Dg|_{F(\mathbf{x})}\, DF|_{\mathbf{x}}. \tag{2.44}$$

Scalar valued composition To illustrate this, first consider a scalar valued composition

$$\begin{aligned} F &: \mathbb{R}^n \rightarrow \mathbb{R}^n \\ g &: \mathbb{R}^n \rightarrow \mathbb{R}, \end{aligned} \tag{2.45}$$

and let

$$\begin{aligned} h(\mathbf{x}) &= g(F(\mathbf{x})) \\ &= g\left(\begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{bmatrix}\right) \end{aligned} \tag{2.46}$$

for $\mathbf{x} \in \mathbb{R}^n$, then

$$\frac{\partial h(\mathbf{x})}{\partial x_k} = \frac{\partial g}{\partial F_1}\frac{\partial F_1}{\partial x_k} + \frac{\partial g}{\partial F_2}\frac{\partial F_2}{\partial x_k} + \cdots \tag{2.47}$$

With

$$DF(\mathbf{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & & \vdots & \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \tag{2.48}$$

the gradient $\boldsymbol{\nabla} g = (Dg)^{\mathrm{T}}$ is

$$\boldsymbol{\nabla} h(\mathbf{x}) = (DF)^{\mathrm{T}}\Big|_{\mathbf{x}} \boldsymbol{\nabla} g(F(\mathbf{x})), \tag{2.49}$$

or

$$D(g(F(\mathbf{x}))) = Dg|_{F(\mathbf{x})} DF|_{\mathbf{x}}. \tag{2.50}$$

Affine functions An important example are affine functions of $\mathbf{x}$

$$\begin{aligned} F &: \mathbb{R}^n \rightarrow \mathbb{R}^n \\ g &: \mathbb{R}^n \rightarrow \mathbb{R}, \end{aligned} \tag{2.51}$$

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}, \tag{2.52}$$

where A is an $n \times n$ matrix and $\mathbf{b}$ is an $n \times 1$ column vector.

Given a function

$$h(\mathbf{x}) = g(F(\mathbf{x})) = g(A\mathbf{x} + \mathbf{b}). \tag{2.53}$$

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{b} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \sum_{i=1}^{n} a_{2i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{ni}x_i\vdots \end{bmatrix} + \mathbf{b}, \tag{2.54}$$

so

$$DF(\mathbf{x}) = A, \tag{2.55}$$

and

$$\boldsymbol{\nabla}(g(F(\mathbf{x}))) = (A\mathbf{x})^{\mathrm{T}}\, \boldsymbol{\nabla} g|_{F(\mathbf{x})}. \tag{2.56}$$

General case The proof of the general case can be essentially be performed by example, provided that example is sufficiently non-trivial, such as a non-square case such as $n = 4, m = 3, p = 2$

$$F(\mathbf{x}) = \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ F_3(\mathbf{x}) \end{bmatrix}, \tag{2.57}$$

and

$$g(\mathbf{y}) = \begin{bmatrix} g_1(\mathbf{y}) \\ g_2(\mathbf{y}) \end{bmatrix}. \tag{2.58}$$

For such a function

$$\frac{\partial g(F(\mathbf{x}))}{\partial x_1} = \begin{bmatrix} \partial g_1(F(\mathbf{x}))/\partial x_1 \\ \partial g_2(F(\mathbf{x}))/\partial x_1 \end{bmatrix}, \tag{2.59}$$

so

$$\begin{aligned} Dg(F(\mathbf{x})) &= \begin{bmatrix} \partial g_1(F(\mathbf{x}))/\partial x_1 & \partial g_1(F(\mathbf{x}))/\partial x_2 & \cdots & \partial g_1(F(\mathbf{x}))/\partial x_4 \\ \partial g_2(F(\mathbf{x}))/\partial x_1 & \partial g_2(F(\mathbf{x}))/\partial x_2 & \cdots & \partial g_2(F(\mathbf{x}))/\partial x_4 \end{bmatrix} \\ &= \begin{bmatrix} D(g_1(F(\mathbf{x}))) \\ D(g_2(F(\mathbf{x}))) \end{bmatrix}. \end{aligned}$$

(2.60)

This reduces the problem to the composition of a scalar and vector function, such as

$$\begin{aligned} D(g_1(F(\mathbf{x}))) &= \sum_{i=1}^{3}\sum_{j=1}^{4} \left.\frac{\partial g_1}{\partial y_i}\right|_{y_i=F_i(\mathbf{x})} \frac{\partial F_i}{\partial x_j} \\ &= \left.\left(\begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \end{bmatrix}\right)\right|_{\mathbf{y}=F(\mathbf{x})} \begin{bmatrix} \frac{\partial F_i}{\partial x_j} \end{bmatrix}_{ij} \\ &= Dg_1|_{F(\mathbf{x})} DF(\mathbf{x}). \end{aligned} \tag{2.61}$$

The total Jacobian is

$$Dg(F(\mathbf{x})) = \begin{bmatrix} Dg_1|_{F(\mathbf{x})} DF(\mathbf{x}) \\ Dg_2|_{F(\mathbf{x})} DF(\mathbf{x}) \end{bmatrix}, \tag{2.62}$$

which can be factored as

$$D(g(F(\mathbf{x}))) = Dg|_{F(\mathbf{x})} DF(\mathbf{x}). \tag{2.63}$$

.

2.6 THE HESSIAN MATRIX

For scalar valued functions, the text expresses the second order expansion of a function in terms of the Jacobian and Hessian matrices

$$f(\mathbf{z}) \approx f(\mathbf{x}) + (Df)\left(\mathbf{z}-\mathbf{x}\right) + \frac{1}{2}\left(\mathbf{z}-\mathbf{x}\right)^{\mathrm{T}}\left(\boldsymbol{\nabla}^2 f\right)\left(\mathbf{z}-\mathbf{x}\right). \tag{2.64}$$

Because $\boldsymbol{\nabla}^2$ is the usual notation for a Laplacian operator, this $\boldsymbol{\nabla}^2 f \in \mathbb{R}^{n\times n}$ notation for the Hessian matrix is not ideal in my opinion. Ignoring that notational objection for this class, the structure of the Hessian matrix can be extracted by comparison with the coordinate expansion

$$\mathbf{a}^{\mathrm{T}}(\boldsymbol{\nabla}^2 f)\mathbf{a} = \sum_{r,s=1}^{n} a_r a_s \frac{\partial^2 f}{\partial x_r \partial x_s} \tag{2.65}$$

so

$$\boxed{(\boldsymbol{\nabla}^2 f)_{ij} = \frac{\partial^2 f_i}{\partial x_i \partial x_j}.} \tag{2.66}$$

In explicit matrix form the Hessian is

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}. \tag{2.67}$$

Is there a similar nice matrix structure for the Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}^m$?

Example 2.2: Second order scalar function

Given

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{T}} P \mathbf{x} + \mathbf{q}^{\mathrm{T}}\mathbf{x} + \mathbf{c}, \tag{2.68}$$

where P is a symmetric matrix $P = P^{\mathrm{T}}$, then

$$\begin{aligned} \nabla F &= \frac{1}{2}\left(P + P^{\mathrm{T}}\right)\mathbf{x} + \mathbf{q} \\ &= P\mathbf{x} + \mathbf{q}, \end{aligned} \tag{2.69}$$

and

$$\nabla^2 F = P. \tag{2.70}$$

2.7 PROBLEMS.

Exercise 2.1 Taylor series expansion

Consider the function

$$f(\mathbf{x}) = -\sum_{l=1}^{m} \log(b_l - \mathbf{a}_l^{\mathrm{T}}\mathbf{x}), \tag{2.71}$$

where $\mathbf{x} \in \mathbb{R}^n$, $b_l \in \mathbb{R}$ and $\mathbf{a}_l \in \mathbb{R}^n$. Compute $\nabla f(\mathbf{x})$ and $\nabla^2 f(\mathbf{x})$. Write down the first three terms of the Taylor series expansion of $f(\mathbf{x})$ around some $\mathbf{x}_0$.

Answer for Exercise 2.1

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Exercise 2.2 Inversion formula for "small" matrices

Prove the relation

$$(I + A)^{-1} = I - A, \tag{2.72}$$

for A "small". We used this in class to derive the second order expansion of

$$\log \det(I + A). \tag{2.73}$$

Prove this result in two ways:

a. First, prove this for the special case of $A \in S^n_{++}$ where the eigenvalues are small. This is what we needed in class. Use a decomposition of A and Taylor approximation of the eigenvalues.
b. Next prove the general relation: If $A \in \mathbb{R}^{n \times n}$ and $\|A\|_p < 1$ then $I - A$ is non-singular, and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \tag{2.74}$$

where

$$\left\|(I - A)^{-1}\right\|_p \le \frac{1}{1 - \|A\|_p}. \tag{2.75}$$

The p-th matrix norm $\|A\|_p$ is defined in terms of the vector p-norm as

$$\|A\|_p = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \tag{2.76}$$

which, using the scaling property of a norm, can be seen to be equivalent to

$$\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p\,. \tag{2.77}$$

In our derivation in class we used only the zeroth and first-order terms of the expansion.
Some hints that outline one approach to the above result:

(i) One approach to proving the first statement (about non-singularity) is by contradiction: note that if $I - A$ is singular then there exists a vector $\mathbf{v}$ such that $(I - A)\mathbf{v} = 0$ and work from there.

(ii) Next, consider the telescoping sum

$$\sum_{k=0}^{N} A^k (I - A) = I - A^{N+1}, \tag{2.78}$$

and show

$$\lim_{k\to\infty} A^k = 0. \tag{2.79}$$

(iii) To show that $\lim_{k\to\infty} A^k = 0$, it is helpful first to prove that

$$\left\|A^{k+1}\right\|_p \le \|A\|_p \, \|A^k\|_p. \tag{2.80}$$

(iv) Finally, combine your above results and the properties of a norm to show the desired result.

Answer for Exercise 2.2

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3

MATRIX INNER PRODUCT, SVD, AND SET TYPES.

3.1 MATRIX INNER PRODUCT

Given real matrices $X, Y \in \mathbb{R}^{m \times n}$, one possible matrix inner product definition is

$$
\begin{aligned}
\langle X, Y \rangle &= \mathrm{Tr}(X^{\mathrm{T}} Y) \\
&= \mathrm{Tr}\left(\sum_{k=1}^{m} X_{ki} Y_{kj} \right) \\
&= \sum_{k=1}^{m} \sum_{j=1}^{n} X_{kj} Y_{kj} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij}.
\end{aligned}
\tag{3.1}
$$

This inner product induces a norm on the (matrix) vector space, called the Frobenius norm

$$
\begin{aligned}
\|X\|_F &= \mathrm{Tr}(X^{\mathrm{T}} X) \\
&= \sqrt{\langle X, X \rangle} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}^2.
\end{aligned}
\tag{3.2}
$$

3.2 RANGE, NULLSPACE.

Definition 3.1: Range.

Given $A \in \mathbb{R}^{m \times n}$, the range of A is the set:

$$\mathcal{R}(A) = \{ A\mathbf{x} | \mathbf{x} \in \mathbb{R}^n \} .$$

Definition 3.2: Nullspace.

Given $A \in \mathbb{R}^{m \times n}$, the nullspace of A is the set:

$$\mathcal{N}(A) = \{\mathbf{x} | A\mathbf{x} = 0\}.$$

3.3 SVD.

To understand operation of $A \in \mathbb{R}^{m \times n}$, a representation of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, decompose A using the singular value decomposition (SVD).

Definition 3.3: SVD.

Given $A \in \mathbb{R}^{m \times n}$, an operator on $\mathbf{x} \in \mathbb{R}^n$, a decomposition of the following form is always possible

$$\begin{aligned} A &= U \Sigma V^{\mathrm{T}} \\ U &\in \mathbb{R}^{m \times r} \\ V &\in \mathbb{R}^{n \times r}, \end{aligned}$$

where r is the rank of A, and both U and V are orthogonal

$$\begin{aligned} U^{\mathrm{T}} U &= I \in \mathbb{R}^{r \times r} \\ V^{\mathrm{T}} V &= I \in \mathbb{R}^{r \times r}. \end{aligned}$$

Here $\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, is a diagonal matrix of "singular" values, where

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r.$$

For simplicity consider square case $m = n$

$$A\mathbf{x} = \left(U \Sigma V^{\mathrm{T}} \right) \mathbf{x}. \tag{3.3}$$

The first product $V^{\mathrm{T}}\mathbf{x}$ is a rotation, which can be checked by looking at the length

$$\begin{aligned} \left\| V^{\mathrm{T}}\mathbf{x} \right\|_2 &= \sqrt{\mathbf{x}^{\mathrm{T}} V V^{\mathrm{T}} \mathbf{x}} \\ &= \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\ &= \|\mathbf{x}\|_2 , \end{aligned} \tag{3.4}$$

which shows that the length of the vector is unchanged after application of the linear transformation represented by V^{T} so that operation must be a rotation.

Similarly the operation of U on $\Sigma V^{\mathrm{T}}\mathbf{x}$ also must be a rotation. The operation $\Sigma = [\sigma_i]_i$ applies a scaling operation to each component of the vector $V^{\mathrm{T}}\mathbf{x}$.

All linear (square) transformations can therefore be thought of as a rotate-scale-rotate operation. Often the A of interest will be symmetric $A = A^{\mathrm{T}}$.

3.4 SET OF SYMMETRIC MATRICES

Let S^n be the set of real, symmetric $n \times n$ matrices.

Theorem 3.1: Spectral theorem.

When $A \in S^n$ then it is possible to factor A as

$$A = Q \Lambda Q^{\mathrm{T}},$$

where Q is an orthogonal matrix, and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$. Here $\lambda_i \in \mathbb{R}\ \forall i$ are the (real) eigenvalues of A.

A real symmetric matrix $A \in S^n$ is "positive semi-definite" if

$$\mathbf{v}^{\mathrm{T}} A \mathbf{v} \geq 0 \qquad \forall \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq 0,$$

and is "positive definite" if

$$\mathbf{v}^{\mathrm{T}} A \mathbf{v} > 0 \qquad \forall \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq 0.$$

The set of such matrices is denoted S^n_+, and S^n_{++} respectively.

Consider $A \in S^n_+$ (or S^n_{++})

$$A = Q\Lambda Q^{\mathrm{T}}, \tag{3.5}$$

possible since the matrix is symmetric. For such a matrix

$$\begin{aligned} \mathbf{v}^{\mathrm{T}} A \mathbf{v} &= \mathbf{v}^{\mathrm{T}} Q \Lambda A^{\mathrm{T}} \mathbf{v} \\ &= \mathbf{w}^{\mathrm{T}} \Lambda \mathbf{w}, \end{aligned} \tag{3.6}$$

where $\mathbf{w} = A^{\mathrm{T}}\mathbf{v}$. Such a product is

$$\mathbf{v}^{\mathrm{T}} A \mathbf{v} = \sum_{i=1}^{n} \lambda_i w_i^2. \tag{3.7}$$

So, if $\lambda_i \geq 0$ ($\lambda_i > 0$) then $\sum_{i=1}^n \lambda_i w_i^2$ is non-negative (positive) $\forall \mathbf{w} \in \mathbb{R}^n, \mathbf{w} \neq 0$. Since $\mathbf{w}$ is just a rotated version of $\mathbf{v}$ this also holds for all $\mathbf{v}$. A necessary and sufficient condition for $A \in S^n_+$ (S^n_{++}) is $\lambda_i \geq 0$ ($\lambda_i > 0$).

3.5 SQUARE ROOT OF POSITIVE SEMI-DEFINITE MATRIX

Real symmetric matrix power relationships such as

$$\begin{aligned} A^2 &= Q\Lambda Q^{\mathrm{T}} Q \Lambda Q^{\mathrm{T}} \\ &= Q\Lambda^2 Q^{\mathrm{T}}, \end{aligned} \tag{3.8}$$

or more generally $A^k = Q\Lambda^k Q^{\mathrm{T}}$, $k \in \mathbb{Z}$, can be further generalized to non-integral powers. In particular, the square root (non-unique) of a square matrix can be written

$$A^{1/2} = Q \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{bmatrix} Q^{\mathrm{T}}, \tag{3.9}$$

since $A^{1/2}A^{1/2} = A$, regardless of the sign picked for the square roots in question.

3.6 FUNCTIONS OF MATRICES

Consider $F : S^n \to \mathbb{R}$, and define

$$F(X) = \log \det X, \tag{3.10}$$

Here $\operatorname{dom} F = S^n_{++}$. The task is to find ∇F, which can be done by looking at the perturbation $\log\det(X + \Delta X)$

$$\begin{aligned}\log\det(X + \Delta X) &= \log\det(X^{1/2}(I + X^{-1/2}\Delta X X^{-1/2})X^{1/2})\\ &= \log\det(X(I + X^{-1/2}\Delta X X^{-1/2})) \qquad (3.11)\\ &= \log\det X + \log\det(I + X^{-1/2}\Delta X X^{-1/2}).\end{aligned}$$

Let $X^{-1/2}\Delta X X^{-1/2} = M$ where λ_i are the eigenvalues of $M : M\mathbf{v} = \lambda_i\mathbf{v}$ when $\mathbf{v}$ is an eigenvector of M. In particular

$$(I + M)\mathbf{v} = (1 + \lambda_i)\mathbf{v}, \qquad (3.12)$$

where $1 + \lambda_i$ are the eigenvalues of the $I + M$ matrix. Since the determinant is the product of the eigenvalues, this gives

$$\begin{aligned}\log\det(X + \Delta X) &= \log\det X + \log\prod_{i=1}^{n}(1 + \lambda_i)\\ &= \log\det X + \sum_{i=1}^{n}\log(1 + \lambda_i).\end{aligned} \qquad (3.13)$$

If λ_i are sufficiently “small”, then $\log(1 + \lambda_i) \approx \lambda_i$, giving

$$\begin{aligned}\log\det(X + \Delta X) &= \log\det X + \sum_{i=1}^{n}\lambda_i\\ &\approx \log\det X + \operatorname{Tr}(X^{-1/2}\Delta X X^{-1/2}).\end{aligned} \qquad (3.14)$$

Since

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA), \qquad (3.15)$$

this trace operation can be written as

$$\begin{aligned}\log\det(X + \Delta X) &\approx \log\det X + \operatorname{Tr}(X^{-1}\Delta X)\\ &= \log\det X + \langle X^{-1}, \Delta X\rangle,\end{aligned} \qquad (3.16)$$

so

$$\nabla F(X) = X^{-1}. \qquad (3.17)$$

To check this, consider the simplest example with $X \in \mathbb{R}^{1\times 1}$, where we have

$$\frac{d}{dX}\left(\log\det X\right) = \frac{d}{dX}\left(\log X\right) = \frac{1}{X} = X^{-1}. \qquad (3.18)$$

This is a nice example demonstrating how the gradient can be obtained by performing a first order perturbation of the function. The gradient can then be read off from the result.

3.7 SECOND ORDER PERTURBATIONS

- To get first order approximation found the part that varied linearly in ΔX.
- To get the second order part, perturb X^{-1} by ΔX and see how that perturbation varies in ΔX.

For $G(X) = X^{-1}$, this is

$$\begin{aligned} (X+\Delta X)^{-1} &= \left(X^{1/2}(I + X^{-1/2}\Delta X X^{-1/2})X^{1/2}\right)^{-1} \\ &= X^{-1/2}(I + X^{-1/2}\Delta X X^{-1/2})^{-1}X^{-1/2} \end{aligned} \tag{3.19}$$

To be proven in the homework (for “small” A)

$$(I + A)^{-1} \approx I - A. \tag{3.20}$$

This gives

$$\begin{aligned} (X+\Delta X)^{-1} &= X^{-1/2}(I - X^{-1/2}\Delta X X^{-1/2})X^{-1/2} \\ &= X^{-1} - X^{-1}\Delta X X^{-1}, \end{aligned} \tag{3.21}$$

or

$$\begin{aligned} G(X+\Delta X) &= G(X) + (DG)\Delta X \\ &= G(X) + (\boldsymbol{\nabla} G)^{\mathrm{T}}\Delta X, \end{aligned} \tag{3.22}$$

so

$$(\boldsymbol{\nabla} G)^{\mathrm{T}}\Delta X = -X^{-1}\Delta X X^{-1}. \tag{3.23}$$

The Taylor expansion of F to second order is

$$F(X+\Delta X) = F(X) + \mathrm{Tr}\left((\boldsymbol{\nabla} F)^{\mathrm{T}}\Delta X\right) + \frac{1}{2}\left((\Delta X)^{\mathrm{T}}(\boldsymbol{\nabla}^2 F)\Delta X\right). \tag{3.24}$$

The first trace can be expressed as an inner product

$$\begin{aligned} \mathrm{Tr}\left((\boldsymbol{\nabla} F)^{\mathrm{T}}\Delta X\right) &= \langle \boldsymbol{\nabla} F, \Delta X\rangle \\ &= \langle X^{-1}, \Delta X\rangle. \end{aligned} \tag{3.25}$$

The second trace also has the structure of an inner product

$$\begin{aligned} (\Delta X)^{\mathrm{T}}(\boldsymbol{\nabla}^2 F)\Delta X &= \mathrm{Tr}\left((\Delta X)^{\mathrm{T}}(\boldsymbol{\nabla}^2 F)\Delta X\right) \\ &= \langle (\boldsymbol{\nabla}^2 F)^{\mathrm{T}}\Delta X, \Delta X\rangle, \end{aligned} \tag{3.26}$$

where a no-op trace could be inserted in the second order term since that quadratic form is already a scalar. This $(\boldsymbol{\nabla}^2 F)^{\mathrm{T}} \Delta X$ term has essentially been found implicitly by performing the linear variation of $\boldsymbol{\nabla} F$ in ΔX, showing that we must have

$$\operatorname{Tr}\left((\Delta X)^{\mathrm{T}}(\boldsymbol{\nabla}^2 F)\Delta X\right) = \langle -X^{-1}\Delta X X^{-1}, \Delta X\rangle, \tag{3.27}$$

so

$$F(X + \Delta X) = F(X) + \langle X^{-1}, \Delta X\rangle + \frac{1}{2}\langle -X^{-1}\Delta X X^{-1}, \Delta X\rangle, \tag{3.28}$$

or

$$\log\det(X + \Delta X) = \log\det X + \operatorname{Tr}(X^{-1}\Delta X) - \frac{1}{2}\operatorname{Tr}(X^{-1}\Delta X X^{-1}\Delta X). \tag{3.29}$$

3.8 CONVEX SETS

- Types of sets: Affine, convex, cones
- Examples: Hyperplanes, polyhedra, balls, ellipses, norm balls, cone of PSD matrices.

Definition 3.4: Affine set

A set $C \subseteq \mathbb{R}^n$ is affine if $\forall \mathbf{x}_1, \mathbf{x}_2 \in C$ then

$$\theta \mathbf{x}_1 + (1 - \theta)\mathbf{x}_2 \in C, \qquad \forall \theta \in \mathbb{R}.$$

The affine sum above can be rewritten as

$$\mathbf{x}_2 + \theta(\mathbf{x}_1 - \mathbf{x}_2). \tag{3.30}$$

Since θ is a scaling, this is the line containing $\mathbf{x}_2$ in the direction between $\mathbf{x}_1$ and $\mathbf{x}_2$.

Observe that the solution to a set of linear equations

$$C = \{\mathbf{x} | A\mathbf{x} = \mathbf{b}\}, \tag{3.31}$$

is an affine set. To check, note that

$$\begin{aligned} A(\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2) &= \theta A\mathbf{x}_1 + (1 - \theta)A\mathbf{x}_2 \\ &= \theta\mathbf{b} + (1 - \theta)\mathbf{b} \\ &= \mathbf{b}. \end{aligned} \tag{3.32}$$

Definition 3.5: Affine combination.

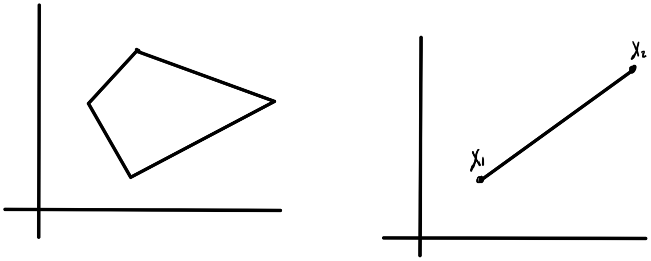
An affine combination of points $\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n$ is

$$\sum_{i=1}^{n} \theta_i \mathbf{x}_i,$$

such that for $\theta_i \in \mathbb{R}$

$$\sum_{i=1}^{n} \theta_i = 1.$$

An affine set contains all affine combinations of points in the set. Examples of a couple affine sets are sketched in fig. 3.1.



(a) (b)

Figure 3.1: Affine.

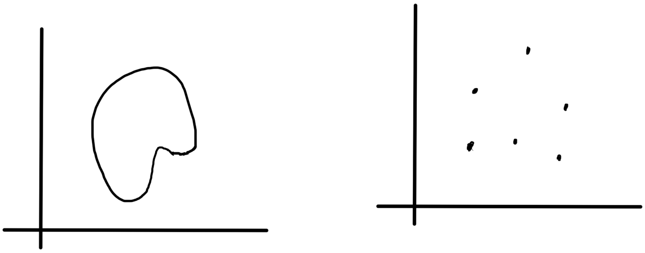
For comparison, a couple of non-affine sets are sketched in fig. 3.2.

Definition 3.6: Convex set

A set $C \subseteq \mathbb{R}^n$ is convex if $\forall \mathbf{x}_1, \mathbf{x}_2 \in C$ and $\forall \theta \in \mathbb{R}, \theta \in [0, 1]$, the combination

$$\theta \mathbf{x}_1 + (1 - \theta)\mathbf{x}_2 \in C. \tag{3.33}$$

Definition 3.7: Convex combination



(a) **(b)**

Figure 3.2: Not affine.

A convex combination of $\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_n$ is

$$\sum_{i=1}^{n} \theta_i \mathbf{x}_i,$$

such that $\forall \theta_i \geq 0$

$$\sum_{i=1}^{n} \theta_i = 1$$

Definition 3.8: Convex hull.

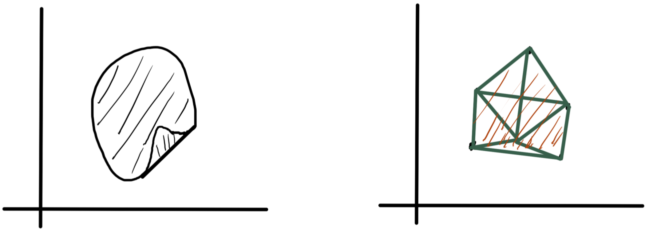
Convex hull of a set C is a set of all convex combinations of points in C, denoted

$$\operatorname{conv}(C) = \left\{ \sum_{i=1}^{n} \theta_i \mathbf{x}_i | \mathbf{x}_i \in C, \theta_i \geq 0, \sum_{i=1}^{n} \theta_i = 1 \right\}.$$

A non-convex set can be converted into a convex hull by filling in all the combinations of points connecting points in the set, as sketched in fig. 3.3.

Definition 3.9: Cones.

A set C is a cone if $\forall \mathbf{x} \in C$ and $\forall \theta \geq 0$ we have $\theta \mathbf{x} \in C$.



(a) **(b)**

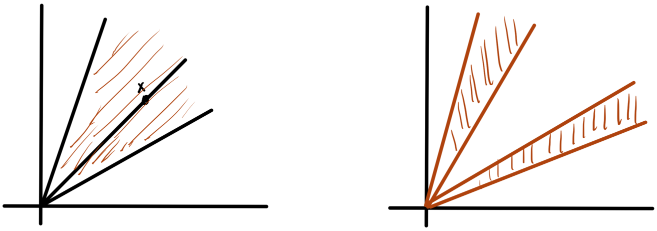
Figure 3.3: Convex hulls.

This scales out if $\theta > 1$ and scales in if $\theta < 1$.

A convex cone is a cone that is also a convex set. A conic combination is

$$\sum_{i=1}^{n} \theta_i \mathbf{x}_i, \theta_i \geq 0.$$

A convex and non-convex 2D cone is sketched in fig. 3.4



(a) **(b)**

Figure 3.4: Convex and non-convex cone.

Like the convex null, it is possible to define affine and conic hulls. These are

Definition 3.10: Affine hull.

Affine hull of a set C is a set of all affine combinations of points in C, denoted

$$\operatorname{affine}(C) = \left\{ \sum_{i=1}^{n} \theta_i \mathbf{x}_i | \mathbf{x}_i \in C, \theta_i \in \mathbb{R}, \sum_{i=1}^{n} \theta_i = 1 \right\}.$$

Definition 3.11: Conic hull.

Conic hull of a set C is a set of all conic combinations of points in C, denoted

$$\operatorname{conic}(C) = \left\{ \sum_{i=1}^{n} \theta_i \mathbf{x}_i | \mathbf{x}_i \in C, \theta_i \geq 0 \right\}.$$

Table 3.1: Affine, Convex, and Conic properties.

	$\theta_i \ge 0$	$\sum \theta_i = 1$
Affine	No	Yes
Convex	Yes	Yes
Conic	Yes	No

A comparison of these three types of hulls are tabulated in table 3.1.

3.9 HYPERPLANES AND HALF SPACES

Definition 3.12: Hyperplane.

A hyperplane is defined by

$$\left\{ \mathbf{x} | \mathbf{a}^\text{T} \mathbf{x} = \mathbf{b}, \mathbf{a} \ne 0 \right\}.$$

A line and plane are examples of this general construct as sketched in fig. 3.5.

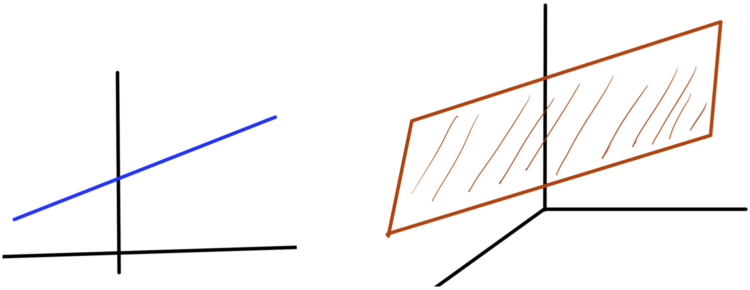

(a) (b)

Figure 3.5: Hyperplanes.

An alternate view is possible should one find any specific $\mathbf{x}_0$ such that $\mathbf{a}^\text{T} \mathbf{x}_0 = \mathbf{b}$

$$\left\{ \mathbf{x} | \mathbf{a}^\text{T} \mathbf{x} = b \right\} = \left\{ \mathbf{x} | \mathbf{a}^\text{T} (\mathbf{x} - \mathbf{x}_0) = 0 \right\} \tag{3.34}$$

This shows that $\mathbf{x} - \mathbf{x}_0 = \mathbf{a}^\perp$ is perpendicular to $\mathbf{a}$, or

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{a}^\perp. \tag{3.35}$$

This is the subspace perpendicular to $\mathbf{a}$ shifted by $\mathbf{x}_0$, subject to $\mathbf{a}^\mathrm{T}\mathbf{x}_0 = \mathbf{b}$. As a set

$$\mathbf{a}^\perp = \left\{ \mathbf{v} | \mathbf{a}^\mathrm{T}\mathbf{v} = 0 \right\}. \tag{3.36}$$

3.10 HALF SPACE

Definition 3.13: Half space.

The half space is defined as

$$\left\{ \mathbf{x} | \mathbf{a}^\mathrm{T}\mathbf{x} = \mathbf{b} \right\} = \left\{ \mathbf{x} | \mathbf{a}^\mathrm{T}(\mathbf{x} - \mathbf{x}_0) \le 0 \right\}.$$

This can also be expressed as $\{\mathbf{x} | \langle \mathbf{a}, \mathbf{x} - \mathbf{x}_0 \rangle \le 0\}$.

3.11 PROBLEMS.

Exercise 3.1 Matrix inner product

a. Verify that $\mathcal{S}^n \subseteq \mathbb{R}^{n \times n}$ is a vector space under the regular matrix addition and scaling (multiplication by scalars in $\mathbb{R}$) operations. Accomplish this by verifying that all properties of a vector space are satisfied.

b. Verify that $\langle A, B \rangle = \mathrm{Tr}(A^\mathrm{T} B)$ where $A, B \in \mathcal{S}^n$ satisfies all the properties of an inner product.

Answer for Exercise 3.1

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT.PLEASE FEEL FREE TO EMAIL ME FOR THE FULL VERSION IF YOU AREN'T TAKING ECE1505.[redacted]

[redacted]. END-REDACTION

4

SETS AND CONVEXITY.

4.1 HYPERPLANES

Find some $\mathbf{x}_0 \in \mathbb{R}^n$ such that $\mathbf{a}^{\mathrm{T}}\mathbf{x}_0 = \mathbf{b}$, so

$$\begin{aligned} \left\{\mathbf{x} | \mathbf{a}^{\mathrm{T}}\mathbf{x} = \mathbf{b}\right\} &= \left\{\mathbf{x} | \mathbf{a}^{\mathrm{T}}\mathbf{x} = \mathbf{a}^{\mathrm{T}}\mathbf{x}_0\right\} \\ &= \left\{\mathbf{x} | \mathbf{a}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0)\right\} \\ &= \mathbf{x}_0 + \mathbf{a}^{\perp}, \end{aligned} \tag{4.1}$$

where

$$\mathbf{a}^{\perp} = \left\{\mathbf{v} | \mathbf{a}^{\mathrm{T}}\mathbf{v} = 0\right\}. \tag{4.2}$$

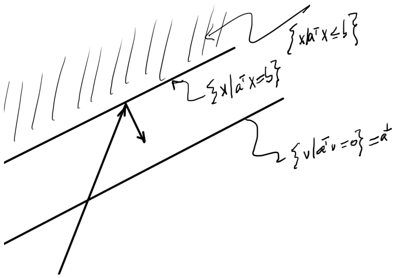


Figure 4.1: Parallel hyperplanes.

Recall

$$\|\mathbf{z}\|_{*} = \sup_{\mathbf{x}} \left\{\mathbf{z}^{\mathrm{T}}\mathbf{x} | \|\mathbf{x}\| \leq 1\right\} \tag{4.3}$$

Denote the optimizer of above as $\mathbf{x}^*$. By definition

$$\mathbf{z}^{\mathrm{T}}\mathbf{x}^* \geq \mathbf{z}^{\mathrm{T}}\mathbf{x} \quad \forall \mathbf{x}, \|\mathbf{x}\| \leq 1 \tag{4.4}$$

This defines a half space in which the unit ball

$$\left\{\mathbf{x} | \mathbf{z}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}^* \leq 0\right\} \tag{4.5}$$

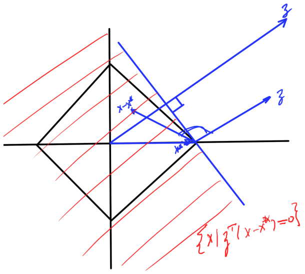


Figure 4.2: Half space containing unit ball.

Start with the l_1 norm, duals of l_1 is l_∞

Similar pic for l_∞, for which the dual is the l_1 norm, as sketched in fig. 4.3. Here the optimizer point is at $(1, 1)$

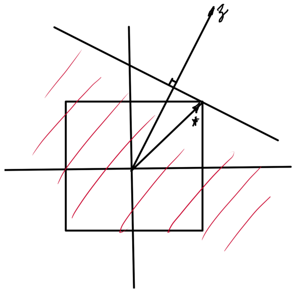


Figure 4.3: Half space containing the unit ball for l_∞.

and a similar pic for l_2, which is sketched in fig. 4.4.

Q: What was this optimizer point?

4.2 POLYHEDRA

$$\mathcal{P} = \left\{ \mathbf{x} | \mathbf{a}_j^\mathrm{T} \mathbf{x} \le \mathbf{b}_j, j \in [1, m], \mathbf{c}_i^\mathrm{T} \mathbf{x} = \mathbf{d}_i, i \in [1, p] \right\} = \{ \mathbf{x} | A\mathbf{x} \le \mathbf{b}, C\mathbf{x} = d \}, \tag{4.6}$$

where the final inequality and equality are component wise.

Proving $\mathcal{P}$ is convex:

- Pick $\mathbf{x}_1 \in \mathcal{P}$, $\mathbf{x}_2 \in \mathcal{P}$
- Pick any $\theta \in [0, 1]$

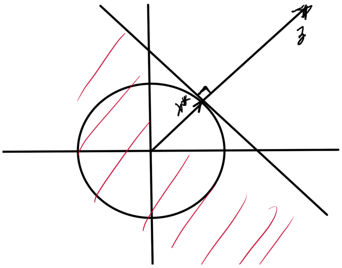


Figure 4.4: Half space containing for l_2 unit ball.

- Test $\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2$. Is it in $\mathscr{P}$?

$$\begin{aligned} A\left(\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2\right) &= \theta A\mathbf{x}_1 + (1-\theta)A\mathbf{x}_2 \\ &\leq \theta \mathbf{b} + (1-\theta)\mathbf{b} \\ &= \mathbf{b}. \end{aligned} \tag{4.7}$$

4.3 BALLS

Euclidean ball for $\mathbf{x}_c \in \mathbb{R}^n, r \in \mathbb{R}$

$$\mathscr{B}(\mathbf{x}_c, r) = \{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_c\|_2 \leq r\}, \tag{4.8}$$

or

$$\mathscr{B}(\mathbf{x}_c, r) = \left\{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_c) \leq r^2\right\}. \tag{4.9}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \theta \in [0, 1]$

$$\begin{aligned} \|\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2 - \mathbf{x}_c\|_2 &= \|\theta(\mathbf{x}_1 - \mathbf{x}_c) + (1-\theta)(\mathbf{x}_2 - \mathbf{x}_c)\|_2 \\ &\leq \|\theta(\mathbf{x}_1 - \mathbf{x}_c)\|_2 + \|(1-\theta)(\mathbf{x}_2 - \mathbf{x}_c)\|_2 \\ &= |\theta| \, \|\mathbf{x}_1 - \mathbf{x}_c\|_2 + |1-\theta| \, \|\mathbf{x}_2 - \mathbf{x}_c\|_2 \\ &= \theta \, \|\mathbf{x}_1 - \mathbf{x}_c\|_2 + \left(1-\theta\right) \|\mathbf{x}_2 - \mathbf{x}_c\|_2 \\ &\leq \theta r + (1-\theta) r \\ &= r. \end{aligned} \tag{4.10}$$

4.4 ELLIPSE

$$\mathscr{E}(\mathbf{x}_c, P) = \left\{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^{\mathrm{T}} P^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\right\}, \tag{4.11}$$

where $P \in S^n_{++}$.

- Euclidean ball is an ellipse with $P = Ir^2$
- Ellipse is image of Euclidean ball $\mathcal{B}(0, 1)$ under affine mapping.

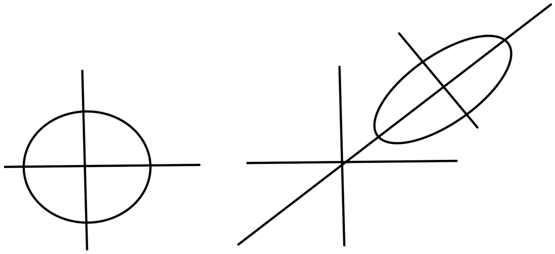


Figure 4.5: Circle and ellipse.

Given

$$F(\mathbf{u}) = P^{1/2}\mathbf{u} + \mathbf{x}_c \tag{4.12}$$

$$\begin{aligned} \{F(\mathbf{u}) | \|\mathbf{u}\|_2 \leq r\} &= \left\{P^{1/2}\mathbf{u} + \mathbf{x}_c | \mathbf{u}^T\mathbf{u} \leq r^2\right\} \\ &= \left\{\mathbf{x} | \mathbf{x} = P^{1/2}\mathbf{u} + \mathbf{x}_c, \mathbf{u}^T\mathbf{u} \leq r^2\right\} \\ &= \left\{\mathbf{x} | \mathbf{u} = P^{-1/2}(\mathbf{x} - \mathbf{x}_c), \mathbf{u}^T\mathbf{u} \leq r^2\right\} \\ &= \left\{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^T P^{-1}(\mathbf{x} - \mathbf{x}_c) \leq r^2\right\} \end{aligned} \tag{4.13}$$

4.5 GEOMETRY OF AN ELLIPSE

Decomposition of positive definite matrix $P \in S^n_{++} \subset S^n$ is:

$$\begin{aligned} P &= Q\,\mathrm{diag}(\lambda_i) Q^T \\ Q^T Q &= 1 \end{aligned}, \tag{4.14}$$

where $\lambda_i \in \mathbb{R}$, and $\lambda_i > 0$.

The ellipse is defined by

$$(\mathbf{x} - \mathbf{x}_c)^T Q\,\mathrm{diag}(1/\lambda_i)(\mathbf{x} - \mathbf{x}_c) Q \leq r^2 \tag{4.15}$$

The term $(\mathbf{x} - \mathbf{x}_c)^T Q$ projects $\mathbf{x} - \mathbf{x}_c$ onto the columns of Q. Those columns are perpendicular since Q is an orthogonal matrix.

Let

$$\tilde{\mathbf{x}} = Q^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_c), \tag{4.16}$$

this shifts the origin around $\mathbf{x}_c$ and Q rotates into a new coordinate system.

The ellipse is therefore

$$\tilde{\mathbf{x}}^{\mathrm{T}} \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n} \end{bmatrix} \tilde{\mathbf{x}} = \sum_{i=1}^{n} \frac{\tilde{x}_i^2}{\lambda_i} \le 1. \tag{4.17}$$

An example is sketched for $\lambda_1 > \lambda_2$ in fig. 4.6.

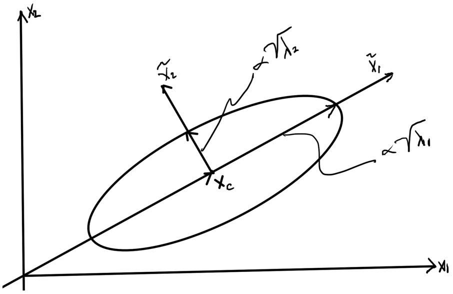


Figure 4.6: Ellipse with $\lambda_1 > \lambda_2$.

- λ_i tells us length of the semi-major axis.
- Larger λ_i means $\tilde{x}_i^2$ can be bigger and still satisfy constraint ≤ 1.
- Volume of ellipse if proportional to $\sqrt{\det P} = \sqrt{\prod_{i=1}^{n} \lambda_i}$.
- When any $\lambda_i \rightarrow 0$ a dimension is lost and the volume goes to zero. That removes the invertibility required.

Ellipses will be seen a lot in this course, since we are interested in “bowl” like geometries (and the ellipse is the image of a Euclidean ball).

4.6 NORM BALL.

The norm ball

$$\mathcal{B} = \{\mathbf{x} \| \|\mathbf{x}\| \le 1\}, \tag{4.18}$$

is a convex set for all norms. Proof:

Take any $\mathbf{x}, \mathbf{y} \in \mathcal{B}$

$$\begin{aligned}\|\theta\mathbf{x} + (1-\theta)\mathbf{y}\| &\le |\theta|\,\|\mathbf{x}\| + |1-\theta|\,\|\mathbf{y}\| \\ &= \theta\,\|\mathbf{x}\| + \left(1-\theta\right)\|\mathbf{y}\|\left(\theta\right) + \left(1-\theta\right) \\ &= 1.\end{aligned} \tag{4.19}$$

This is true for any p-norm $1 \le p$, $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$.

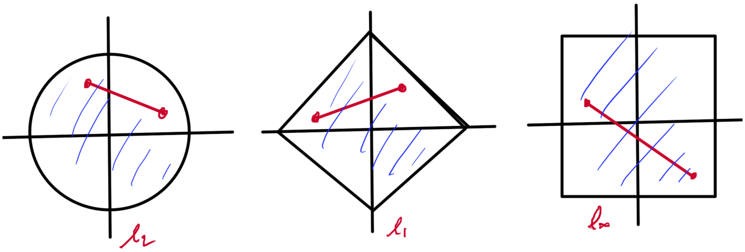


Figure 4.7: Norm ball.

The shape of a $p < 1$ norm unit ball is sketched in fig. 4.8 (lines connecting points in such a region can exit the region).

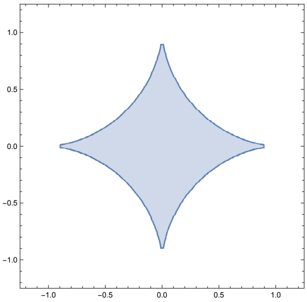


Figure 4.8: Unit ball for $l_{0.6}$ “p-norm”.

4.7 CONES

Recall that C is a cone if $\forall \mathbf{x} \in C, \theta \ge 0, \theta\mathbf{x} \in C$.

Impt cone of PSD matrices

$$\begin{aligned} S^n &= \left\{X \in \mathbb{R}^{n\times n} | X = X^{\mathrm{T}}\right\} \\ S^n_+ &= \left\{X \in S^n | \mathbf{v}^{\mathrm{T}} X \mathbf{v} \geq 0, \quad \forall v \in \mathbb{R}^n\right\} \\ S^n_{++} &= \left\{X \in S^n_+ | \mathbf{v}^{\mathrm{T}} X \mathbf{v} > 0, \quad \forall v \in \mathbb{R}^n\right\} \end{aligned} \tag{4.20}$$

These have respectively

- $\lambda_i \in \mathbb{R}$
- $\lambda_i \in \mathbb{R}_+$
- $\lambda_i \in \mathbb{R}_{++}$

S^n_+ is a cone if:
$X \in S^n_+$, then $\theta X \in S^n_+, \quad \forall \theta \geq 0$

$$\begin{aligned} \mathbf{v}^{\mathrm{T}}(\theta X)\mathbf{v} &= \theta \mathbf{v}^{\mathrm{T}} \mathbf{v} \\ &\geq 0, \end{aligned} \tag{4.21}$$

since $\theta \geq 0$ and because $X \in S^n_+$.
Shorthand:

$$X \in S^n_+ \implies X \geq 0X \quad \in S^n_{++} \implies X > 0. \tag{4.22}$$

Further S^n_+ is a convex cone.
Let $A \in S^n_+, B \in S^n_+, \theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 = 1$, or $\theta_2 = 1 - \theta_1$.
Show that $\theta_1 A + \theta_2 B \in S^n_+$:

$$\mathbf{v}^{\mathrm{T}} \left(\theta_1 A + \theta_2 B\right) \mathbf{v} = \theta_1 \mathbf{v}^{\mathrm{T}} A \mathbf{v} + \theta_2 \mathbf{v}^{\mathrm{T}} B \mathbf{v} \geq 0, \tag{4.23}$$

since $\theta_1 \geq 0, \theta_2 \geq 0, \mathbf{v}^{\mathrm{T}} A \mathbf{v} \geq 0, \mathbf{v}^{\mathrm{T}} B \mathbf{v} \geq 0$.
Inequalities:
Start with a proper cone $K \subseteq \mathbb{R}^n$

- closed, convex
- non-empty interior ("solid")
- "pointed" (contains no lines)

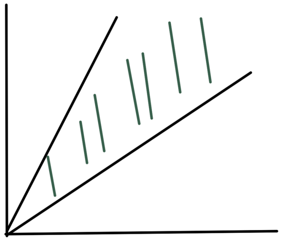


Figure 4.9: Cone.

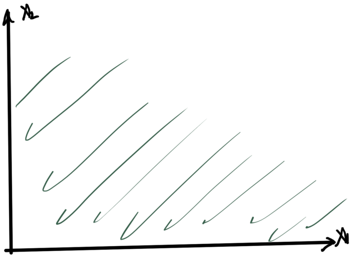


Figure 4.10: K is non-negative "orthant"

The K defines a generalized inequality in $\mathbb{R}^n$ defined as "$\leq_K$" Interpreting

$$\mathbf{x} \leq_K \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in K \mathbf{x} <_K \mathbf{y} \quad \leftrightarrow \mathbf{y} - \mathbf{x} \in \operatorname{int} K \tag{4.24}$$

Why pointed? Want if $\mathbf{x} \leq_K \mathbf{y}$ and $\mathbf{y} \leq_K \mathbf{x}$ with this K is a half space. Example:1: $K = \mathbb{R}^n_+, \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^n$

$$\mathbf{x} \leq_K \mathbf{y} \implies \mathbf{y} - \mathbf{x} \in K \tag{4.25}$$

say:

$$\left[y_1 - x_1 y_2 - x_2 \right] \in R^2_+ \tag{4.26}$$

Also:

$$K = R^1_+ \tag{4.27}$$

(pointed, since it contains no rays)

$$\mathbf{x} \leq_K \mathbf{y}, \tag{4.28}$$

with respect to $K = \mathbb{R}^n_+$ means that $x_i \le y_i$ for all $i \in [1, n]$.

Example:2: For $K = PSD \subseteq S^n$,

$$\mathbf{x} \le_K \mathbf{y}, \tag{4.29}$$

means that

$$\mathbf{y} - \mathbf{x} \in K = S^n_+. \tag{4.30}$$

- Difference $\mathbf{y} - \mathbf{x}$ is always in S
- check if in K by checking if all eigenvalues ≥ 0.
- S^n_{++} is the interior of S^n_+.

Interpretation:

$$\begin{aligned} &\mathbf{x} \le_K \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in K \\ &\mathbf{x} <_K \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in \operatorname{int} K. \end{aligned} \tag{4.31}$$

We'll use these with vectors and matrices so often the K subscript will often be dropped, writing instead (for vectors)

$$\begin{aligned} &\mathbf{x} \le \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in \mathbb{R}^n_+ \\ &\mathbf{x} < \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in \operatorname{int} \mathbb{R}^n_{++} \end{aligned} \tag{4.32}$$

and for matrices

$$\begin{aligned} &\mathbf{x} \le \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in S^n_+ \\ &\mathbf{x} < \mathbf{y} \leftrightarrow \mathbf{y} - \mathbf{x} \in \operatorname{int} S^n_{++}. \end{aligned} \tag{4.33}$$

4.8 INTERSECTION

Take the intersection of (perhaps infinitely many) sets S_α:

If S_α is (affine,convex, conic) for all $\alpha \in A$ then

$$\cap_\alpha S_\alpha \tag{4.34}$$

is

(affine,convex, conic).

To prove in homework:

$$\mathcal{P} = \left\{ \mathbf{x} | \mathbf{a}_i^\mathrm{T} \mathbf{x} \le \mathbf{b}_i, \mathbf{c}_j^\mathrm{T} \mathbf{x} = \mathbf{d}_j, \quad \forall i \cdots j \right\} \tag{4.35}$$

This is convex since the intersection of a bunch of hyperplane and half space constraints.

1. If $S \subseteq \mathbb{R}^n$ is convex then

$$F(S) = \{F(\mathbf{x})|\mathbf{x} \in S\} \tag{4.36}$$

is convex.

2. If $S \subseteq \mathbb{R}^m$ then

$$F^{-1}(S) = \{\mathbf{x}|F(\mathbf{x}) \in S\} \tag{4.37}$$

is convex. Such a mapping is sketched in fig. 4.11.

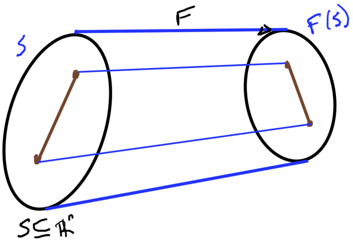


Figure 4.11: Mapping functions of sets.

4.9 PROBLEMS.

Exercise 4.1 Convex, affine, and conic hulls

a. Consider the set

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \subseteq \mathbb{R}^2. \tag{4.38}$$

Sketch conv($\mathcal{S}$), affine($\mathcal{S}$) and conic($\mathcal{S}$), respectively the convex, affine, and conic hulls of the set $\mathcal{S}$. Each is the union of all combinations of the respective type (convex, affine or conic).

b. Repeat part a for the set

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \right\}. \tag{4.39}$$

c. Consider a set $\mathcal{S}$. What are the respective inclusion relations between the convex hull, the affine hull, and the conic hull of $\mathcal{S}$. I.e., which of these three sets are always subsets of the other, regardless of the original $\mathcal{S}$?

Answer for Exercise 4.1

PROBLEM SET RELATED MATERIAL REDACTED IN THIS DOCUMENT.PLEASE FEEL FREE TO EMAIL ME FOR THE FULL VERSION IF YOU AREN'T TAKING ECE1505.[redacted]. END-REDACTION

Exercise 4.2 Distance between two parallel hyperplanes (*[1] pr. 2.5*)

What is the distance between two parallel hyperplanes $\left\{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}^\mathrm{T}\mathbf{x} = b_1\right\}$ and $\left\{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}^\mathrm{T}\mathbf{x} = b_2\right\}$.

Answer for Exercise 4.2

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5

SETS, EPIGRAPHS, QUASI-CONVEXITY, AND SUBLEVEL SETS.

5.1 OPERATIONS THAT PRESERVE CONVEXITY

If S_α is convex $\forall \alpha \in A$, then

$$\cup_{\alpha \in A} S_\alpha, \tag{5.1}$$

is convex.

Example:

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \tag{5.2}$$

$$\begin{aligned} \mathbf{x} &\in \mathbb{R}^n \\ A &\in \mathbb{R}^{m \times n} \\ F &: \mathbb{R}^n \to \mathbb{R}^m \\ \mathbf{b} &\in \mathbb{R}^m \end{aligned} \tag{5.3}$$

(i) If $S \in \mathbb{R}^n$ is convex, then

$$F(S) = \{F(\mathbf{x}) | \mathbf{x} \in S\} \tag{5.4}$$

is convex if F is affine.

(ii) If $S \in \mathbb{R}^m$ is convex, then

$$F^{-1}(S) = \{\mathbf{x} | F(\mathbf{x}) \in S\} \tag{5.5}$$

is convex.

Example:

$$\{\mathbf{y} | \mathbf{y} = A\mathbf{x} + \mathbf{b}, ||\mathbf{x}|| \leq 1\} \tag{5.6}$$

is convex. Here $A\mathbf{x} + \mathbf{b}$ is an affine function ($F(\mathbf{x})$. This is the image of a (convex) unit ball, through an affine map.

Earlier saw when defining ellipses

$$\mathbf{y} = P^{1/2}\mathbf{x} + \mathbf{x}_c \tag{5.7}$$

Example :

$$\{\mathbf{x} | \|A\mathbf{x} + \mathbf{b}\| \leq 1\}, \tag{5.8}$$

is convex. This can be seen by writing

$$\{\mathbf{x} | \|A\mathbf{x} + \mathbf{b}\| \leq 1\} = \{\mathbf{x} | \|F(\mathbf{x})\| \leq 1\} = \{\mathbf{x} | F(\mathbf{x}) \in \mathcal{B}\}, \tag{5.9}$$

where $\mathcal{B} = \{\mathbf{y} | \|\mathbf{y}\| \leq 1\}$. This is the pre-image (under $F()$) of a unit norm ball.

Example:

$$\{\mathbf{x} \in \mathbb{R}^n | x_1 A_1 + x_2 A_2 + \cdots x_n A_n \leq \mathcal{B}\} \tag{5.10}$$

where $A_i \in S^m$ and $\mathcal{B} \in S^m$, and the inequality is a matrix inequality. This is a convex set. The constraint is a “linear matrix inequality” (LMI).

This has to do with an affine map:

$$F(\mathbf{x}) = B - 1x_1 A_1 - x_2 A_2 - \cdots x_n A_n \geq 0 \tag{5.11}$$

(positive semi-definite inequality). This is a mapping

$$F : \mathbb{R}^n \to S^m, \tag{5.12}$$

since all A_i and B are in S^m.

This $F(\mathbf{x}) = B - A(\mathbf{x})$ is a constant and a factor linear in x, so is affine. Can be written

$$\{\mathbf{x} | B - A(\mathbf{x}) \geq 0\} = \{\mathbf{x} | B - A(\mathbf{x}) \in S^m_+\} \tag{5.13}$$

This is a pre-image of a cone of PSD matrices, which is convex. Therefore, this is a convex set.

5.2 SEPARATING HYPERPLANES

Theorem 5.1: Separating hyperplanes

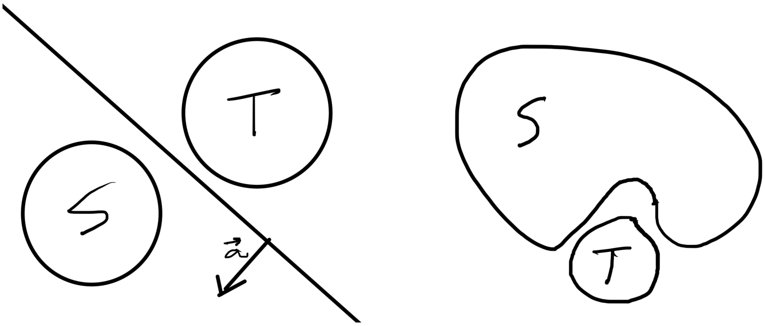
If $S, T \subseteq \mathbb{R}^n$ are convex and disjoint i.e. $S \cup T = 0$, then there exists on $\mathbf{a} \in \mathbb{R}^n$ $\mathbf{a} \neq 0$ and a $\mathbf{b} \in \mathbb{R}^n$ such that

$$\mathbf{a}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}\, \forall \mathbf{x} \in S$$

and

$$\mathbf{a}^{\mathrm{T}}\mathbf{x} < \mathbf{b}\ \forall \mathbf{x} \in T.$$

An example of a hyperplanes that separates two sets and two sets that are not separable is sketched in fig. 5.1.



(a) **(b)**

Figure 5.1: separable and non-separable sets

Proof in the book.

Theorem 5.2: Supporting hyperplane

If S is convex then $\forall x_0 \in \partial S = \mathrm{cl}(S)\ \mathrm{int}(S)$, where ∂S is the boundary of S, then $\exists$ an $\mathbf{a} \neq 0 \in \mathbb{R}^n$ such that $\mathbf{a}^{\mathrm{T}}\mathbf{x} \leq \mathbf{a}^{\mathrm{T}}x_0\ \forall \mathbf{x} \in S$.

Here   denotes “without”.

An example is sketched in fig. 5.2, for which

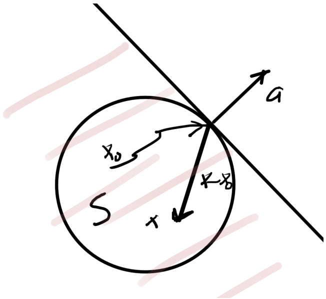


Figure 5.2: Supporting hyperplane.

- The vector **a** perpendicular to tangent plane.
- inner product $\mathbf{a}^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_0) \leq 0$.

A set with a supporting hyperplane is sketched in fig. 5.3, whereas fig. 5.4 shows that there is not necessarily a unique supporting hyperplane at any given point, even if S is convex.

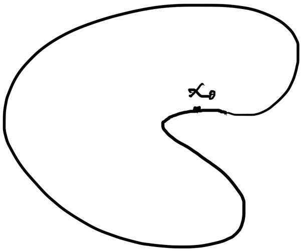


Figure 5.3: Set with supporting hyperplane.

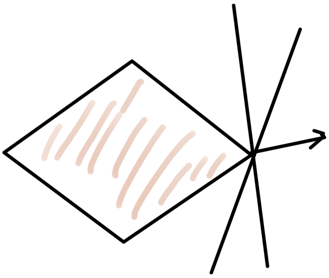


Figure 5.4: No unique supporting hyperplane possible.

5.3 BASIC DEFINITIONS OF CONVEX FUNCTIONS

Theorem 5.3: Convex functions

If $F : \mathbb{R}^n \to \mathbb{R}$ is defined on a convex domain (i.e. $\operatorname{dom} F \subseteq \mathbb{R}^n$ is a convex set), then F is convex if $\forall \mathbf{x}, \mathbf{y} \in \operatorname{dom} F$, $\forall \theta \in [0, 1] \in \mathbb{R}$

$$F(\theta \mathbf{x} + (1 - \theta)\mathbf{y} \leq \theta F(\mathbf{x}) + (1 - \theta)F(\mathbf{y}) \tag{5.14}$$

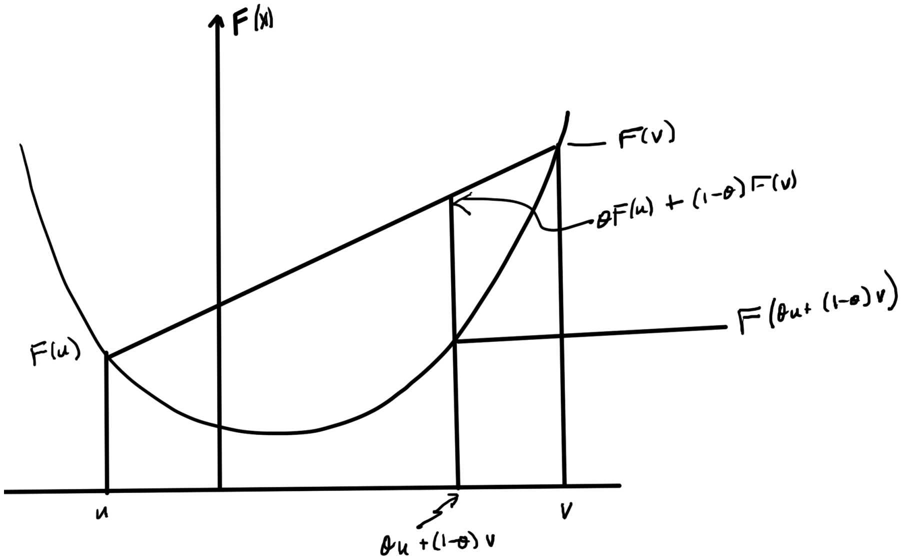


Figure 5.5: Example of convex function.

An example is sketched in fig. 5.5.
Remarks

- Require dom F to be a convex set. This is required so that the function at the point $\theta u + (1-\theta)v$ can be evaluated. i.e. so that $F(\theta u + (1-\theta)v)$ is well defined. Example: dom $F = (-\infty, 0] \cup [1, \infty)$ is not okay, because a linear combination in $(0, 1)$ would be undesirable.

- Parameter θ is "how much up" the line segment connecting $(u, F(u))$ and $(v, F(v))$. This line segment never below the bottom of the bowl. The function is concave, if $-F$ is convex. i.e. If the convex function is flipped upside down. That is

$$F(\theta \mathbf{x} + (1-\theta)\mathbf{y}) \geq \theta F(\mathbf{x}) + (1-\theta)F(\mathbf{y})\ \forall \mathbf{x}, \mathbf{y} \in \operatorname{dom} F, \theta \in [0, 1]. \tag{5.15}$$

- a "strictly" convex function means $\forall \theta \in [0, 1]$

$$F(\theta \mathbf{x} + (1-\theta)\mathbf{y}) < \theta F(\mathbf{x}) + (1 - \mathit{theta})F(\mathbf{y}). \tag{5.16}$$

- Strictly concave function F means $-F$ is strictly convex.

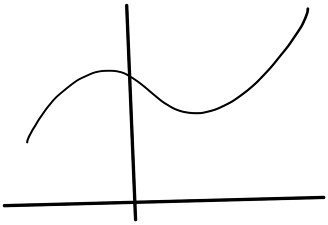


Figure 5.6: Not convex or concave.

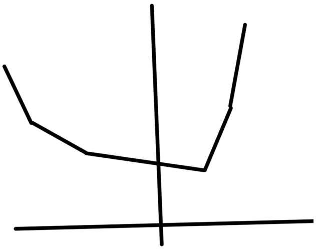


Figure 5.7: Not strictly convex

- Examples:

Definition 5.1: Epigraph of a function

The epigraph epi F of a function $F : \mathbb{R}^n \to \mathbb{R}$ is

$$\text{epi } F = \left\{ (\mathbf{x}, t) \in \mathbb{R}^{n+1} | \mathbf{x} \in \text{dom } F, t \geq F(\mathbf{x}) \right\},$$

where $\mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}$.

Theorem 5.4: Convexity and epigraph.

If F is convex implies epi F is a convex set.

Proof:

For convex function, a line segment connecting any 2 points on function is above the function. i.e. it is epi F.

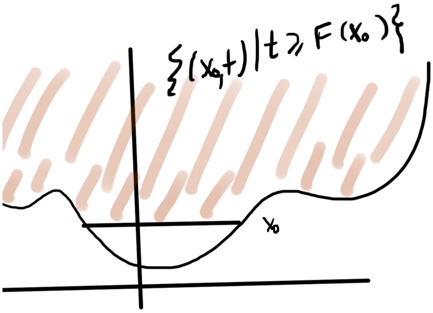


Figure 5.8: Epigraph.

Many authors will go the other way around, showing definition 5.1 from theorem 5.4. That is:

Pick any 2 points in epi F, $(\mathbf{x}, \mu) \in \operatorname{epi} F$ and $(\mathbf{y}, \nu) \in \operatorname{epi} F$. Consider convex combination

$$\theta(\mathbf{x}, \mu) + (1-\theta)(\mathbf{y}, \nu) = (\theta\mathbf{x}(1-\theta)\mathbf{y}, \theta\mu(1-\theta)\nu) \in \operatorname{epi} F, \tag{5.17}$$

since epi F is a convex set.

By definition of epi F

$$F(\theta\mathbf{x}(1-\theta)\mathbf{y}) \leq \theta\mu(1-\theta)\nu. \tag{5.18}$$

Picking $\mu = F(\mathbf{x}), \nu = F(\mathbf{y})$ gives

$$F(\theta\mathbf{x}(1-\theta)\mathbf{y}) \leq \theta F(\mathbf{x})(1-\theta)F(\mathbf{y}). \tag{5.19}$$

5.4 EXTENDED VALUE FUNCTION

Sometimes convenient to work with "extended value function"

$$\tilde{F}(\mathbf{x}) = \begin{cases} F(\mathbf{x}) & \text{If } \mathbf{x} \in \operatorname{dom} F \\ \infty & \text{otherwise.} \end{cases} \tag{5.20}$$

Examples:

- Linear (affine) functions (fig. 5.9) are both convex and concave.

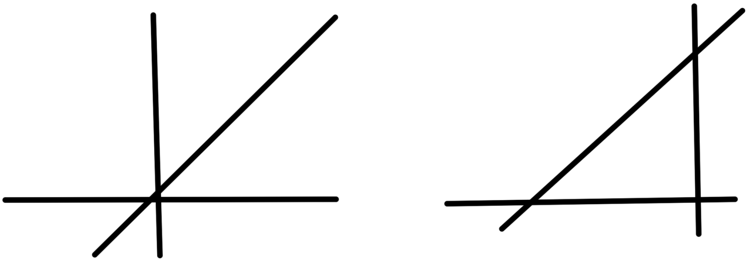


Figure 5.9: Linear functions.

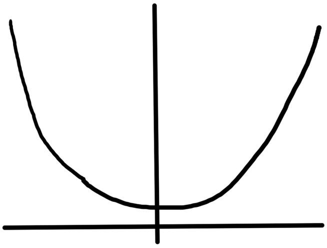


Figure 5.10: Convex (quadratic.)

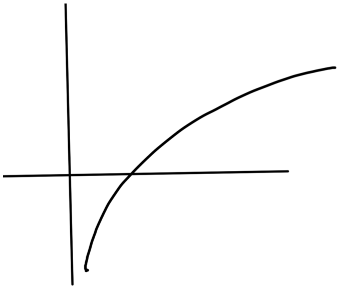


Figure 5.11: Concave (logarithm.)

- x^2 is convex, sketched in fig. 5.10.
- $\log x, \operatorname{dom} F = \mathbb{R}_+$ concave, sketched in fig. 5.11.
- $\|\mathbf{x}\|$ is convex. $\|\theta\mathbf{x} + (1-\theta)\mathbf{y}\| \leq \theta\|\mathbf{x}\| + (1-\theta)\|\mathbf{y}\|$.
- $1/x$ is convex on $\{x|x > 0\} = \operatorname{dom} F$, and concave on $\{x|x < 0\} = \operatorname{dom} F$.

$$\tilde{F}(x) = \begin{cases} \frac{1}{x} & \text{If } x > 0 \\ \infty & \text{else.} \end{cases} \tag{5.21}$$

Definition 5.2: Sublevel

The sublevel set of a function $F : \mathbb{R}^n \to \mathbb{R}$ is

$$C(\alpha) = \{\mathbf{x} \in \operatorname{dom} F | F(\mathbf{x}) \leq \alpha\}$$

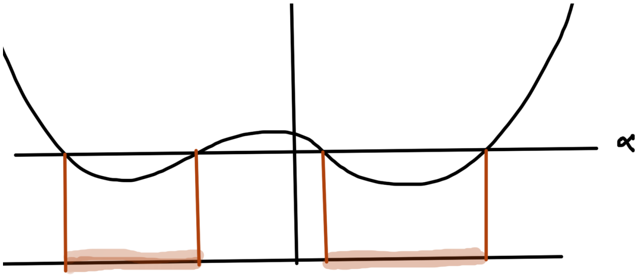


Figure 5.12: Convex sublevel.

Theorem 5.5

If F is convex then $C(\alpha)$ is a convex set $\forall\alpha$.

This is not an if and only if condition, as illustrated in fig. 5.14.
There $C(\alpha)$ is convex, but the function itself is not.
Proof:
Since F is convex, then epi F is a convex set.

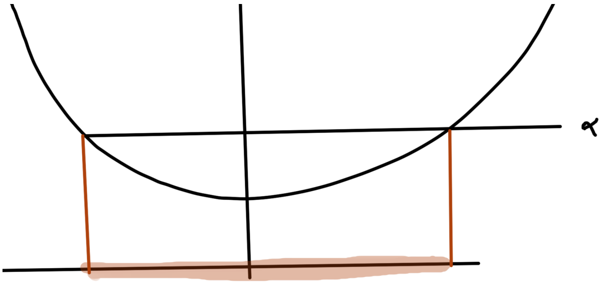


Figure 5.13: Non-convex sublevel.

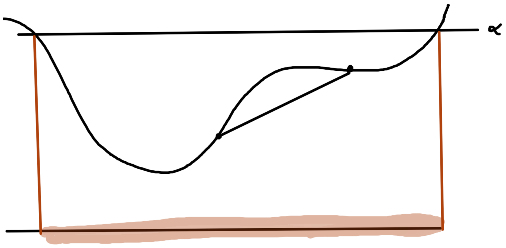


Figure 5.14: Convex sublevel does not imply convexity.

- Let

$$\mathcal{A} = \{(\mathbf{x}, t) | t = \alpha\} \tag{5.22}$$

  is a convex set.

- $\mathcal{A} \cap \operatorname{epi} F$

  is a convex set since it is the intersection of convex sets.

- Project $\mathcal{A} \cap \operatorname{epi} F$ onto $\mathbb{R}^n$ (i.e. domain of F). The projection is an affine mapping. Image of a convex set through affine mapping is a convex set.

Definition 5.3: Quasi-convex.

A function is quasi-convex if all of its sublevel sets are convex.

5.5 COMPOSING CONVEX FUNCTIONS

Properties of convex functions:

- If F is convex, then αF is convex $\forall \alpha > 0$.
- If F_1, F_2 are convex, then the sum $F_1 + F_2$ is convex.
- If F is convex, then $g(\mathbf{x}) = F(A\mathbf{x} + \mathbf{b})$ is convex $\forall \mathbf{x} \in \{\mathbf{x} | A\mathbf{x} + \mathbf{b} \in \operatorname{dom} F\}$.

Note: for the last

$$\begin{aligned} &g : \mathbb{R}^m \to \mathbb{R} \\ &F : \mathbb{R}^n \to \mathbb{R} \\ &\mathbf{x} \in \mathbb{R}^m \\ &A \in \mathbb{R}^{n \times m} \\ &\mathbf{b} \in \mathbb{R}^n \end{aligned} \tag{5.23}$$

Proof (of last):

$$\begin{aligned} g(\theta\mathbf{x} + (1-\theta)\mathbf{y}) &= F(\theta(A\mathbf{x} + \mathbf{b}) + (1-\theta)(A\mathbf{y} + \mathbf{b})) \\ &\leq \theta F(A\mathbf{x} + \mathbf{b}) + (1-\theta)F(A\mathbf{y} + \mathbf{b}) \\ &= \theta g(\mathbf{x}) + (1-\theta) g(\mathbf{y}). \end{aligned} \tag{5.24}$$

5.6 PROBLEMS.

Exercise 5.1 Ellipses, eigenvalues, eigenvectors, and volume

Make neat and clearly-labeled sketches of the ellipsoid $\mathcal{E} = \left\{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^{\mathrm{T}} P^{-1} (\mathbf{x} - \mathbf{x}_c) = 1\right.$ for the following sets of parameters:

a. Center $\mathbf{x}_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$.

b. Center $\mathbf{x}_c = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

c. Center $\mathbf{x}_c = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $P = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$.

For each part (a)-(c) also compute each pair of eigenvalues and corresponding eigenvectors.

d. Recall that the most geometrically meaningful property of the determinant of a square real matrix A is that its magnitude $|\det A|$ is equal to the volume of the parallelepiped $\mathcal{P}$ formed by applying A to the unit cube $\mathcal{C} = \{x | 0 \leq x \leq 1\}$. (Recall that since $\mathbf{x} \in \mathbb{R}^n$ we interpret the inequalities coordinate-wise, i.e., $0 \leq x_i \leq 1$ for all $i = 1, \cdots, n$.) In other words, if $\mathcal{P} = \{A\mathbf{x} | \mathbf{x} \in \mathcal{C}\}$ then $|\det(A)|$ is equal to the volume of $\mathcal{P}$. Furthermore, recall that the determinant of a matrix is zero if any of its eigenvalues are zero. Explain how to interpret this latter fact in terms of the interpretation of $|\det(A)|$ as the volume of $\mathcal{P}$.

Answer for Exercise 5.1

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Exercise 5.2 Proving convexity-preserving operations

a. Prove that the set $\mathcal{S}$ resulting from taking the intersection of a set of convex sets $\mathcal{S}_\alpha$ is itself a convex set. I.e., $\mathcal{S} = \cap_\alpha \mathcal{S}_\alpha$ is a convex set when all the $\mathcal{S}_\alpha$ are convex sets.

b. Consider any affine function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and convex set $\mathcal{S} \subseteq \mathbb{R}^n$. Prove that the image of $\mathcal{S}$ under f, i.e., $f(\mathcal{S}) = \{f(\mathbf{x}) | \mathbf{x} \in \mathcal{S}\}$, is a convex set.

c. Consider any affine function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and convex set $\mathcal{S} \subseteq \mathbb{R}^m$. Prove that the inverse (or pre-) image of $\mathcal{S}$ under f, i.e., $f^{-1}(\mathcal{S}) = \{\mathbf{x} | f(\mathbf{x}) \in \mathcal{S}\}$, is a convex set.

Answer for Exercise 5.2

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Exercise 5.3 Expanded and restricted sets (*[1] pr. 2.14(a)*)

Let $S \subseteq \mathbb{R}^n$, and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

For $a \ge 0$ we define S_a as $\{\mathbf{x} | \operatorname{dist}(\mathbf{x}, S) \le a\}$, where $\operatorname{dist}(\mathbf{x}, S) = \inf_{\mathbf{y} \in S} \|\mathbf{x} - \mathbf{y}\|$. We refer to S_a as S expanded or extended by a. Show that if S is convex, then S_a is convex.

Answer for Exercise 5.3

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FIRST AND SECOND ORDER CONDITIONS. 6

Quasi-convex F_1 and F_2 convex implies $\max(F_1, F_2)$ convex.

Note that $\min(F_1, F_2)$ is NOT convex.

If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, then $F(\mathbf{x}_0 + t\mathbf{v})$ is convex in $t \, \forall t \in \mathbb{R}, \mathbf{x}_0 \in \mathbb{R}^n, \mathbf{v} \in \mathbb{R}^n$, provided $\mathbf{x}_0 + t\mathbf{v} \in \operatorname{dom} F$.

Idea: Restrict to a line (line segment) in dom F. Take a cross section or slice through F alone the line. If the result is a 1D convex function for all slices, then F is convex.

This is nice since it allows for checking for convexity, and is also nice numerically. Attempting to test a given data set for non-convexity with some random lines can help disprove convexity. However, to show that F is convex it is required to test all possible slices (which isn't possible numerically, but is in some circumstances possible analytically).

Differentiable (convex) functions

Definition 6.1: First order condition

If

$$F : \mathbb{R}^n \rightarrow \mathbb{R}$$

is differentiable, then F is convex iff dom F is a convex set and $\forall \mathbf{x}, \mathbf{x}_0 \in \operatorname{dom} F$

$$F(\mathbf{x}) \geq F(\mathbf{x}_0) + \left(\boldsymbol{\nabla} F(\mathbf{x}_0)\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0).$$

This is the first order Taylor expansion. If $n = 1$, this is $F(x) \geq F(x_0) + F'(x_0)(x - x_0)$.

The first order condition says a convex function $\underline{\text{always}}$ lies above its first order approximation, as sketched in fig. 6.1.

When differentiable, the supporting plane is the tangent plane.

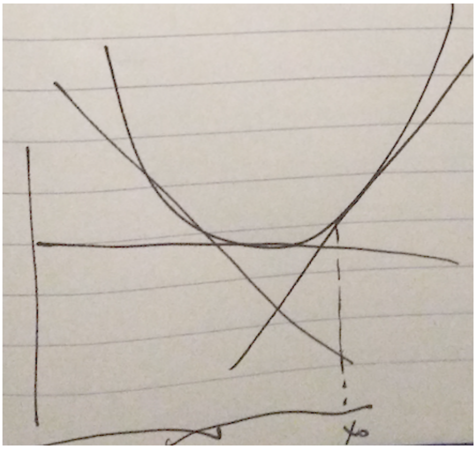


Figure 6.1: First order approximation lies below convex function

Definition 6.2: Second order condition

> If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice differentiable, then F is convex iff $\operatorname{dom} F$ is a convex set and $\boldsymbol{\nabla}^2 F(\mathbf{x}) \ge 0 \,\forall \mathbf{x} \in \operatorname{dom} F$.

The Hessian is always symmetric, but is not necessarily positive. Recall that the Hessian is the matrix of the second order partials $(\boldsymbol{\nabla} F)_{ij} = \partial^2 F/(\partial x_i \partial x_j)$.

The scalar case is $F''(x) \ge 0 \,\forall x \in \operatorname{dom} F$.

An implication is that if F is convex, then $F(x) \ge F(x_0) + F'(x_0)(x - x_0) \,\forall x, x_0 \in \operatorname{dom} F$

Since F is convex, $\operatorname{dom} F$ is convex.

Consider any 2 points $x, y \in \operatorname{dom} F$, and $\theta \in [0, 1]$. Define

$$z = (1 - \theta)x + \theta y \in \operatorname{dom} F, \tag{6.1}$$

then since $\operatorname{dom} F$ is convex

$$\begin{aligned} F(z) &= F((1 - \theta)x + \theta y) \\ &\le (1 - \theta)F(x) + \theta F(y) \end{aligned} \tag{6.2}$$

Reordering

$$\theta F(x) \ge \theta F(x) + F(z) - F(x), \tag{6.3}$$

or

$$F(y) \ge F(x) + \frac{F(x + \theta(y - x)) - F(x)}{\theta}, \tag{6.4}$$

which is, in the limit,

$$F(y) \ge F(x) + F'(x)(y - x) \qquad \square \tag{6.5}$$

To prove the other direction, showing that

$$F(x) \ge F(x_0) + F'(x_0)(x - x_0), \tag{6.6}$$

implies that F is convex. Take any $x, y \in \operatorname{dom} F$ and any $\theta \in [0, 1]$. Define

$$z = \theta x + (1 - \theta)y, \tag{6.7}$$

which is in $\operatorname{dom} F$ by assumption. We want to show that

$$F(z) \le \theta F(x) + (1 - \theta)F(y). \tag{6.8}$$

By assumption

(i) $F(x) \ge F(z) + F'(z)(x - z)$

(ii) $F(y) \ge F(z) + F'(z)(y - z)$

Compute

$$\begin{aligned}
\theta F(x) + (1-\theta)F(y) &\ge \theta \left(F(z) + F'(z)(x-z)\right) + (1-\theta)\left(F(z) + F'(z)(y-z)\right) \\
&= F(z) + F'(z)\left(\theta(x-z) + (1-\theta)(y-z)\right) \\
&= F(z) + F'(z)\left(\theta x + (1-\theta)y - \theta z - (1-\theta)z\right) \\
&= F(z) + F'(z)\left(\theta x + (1-\theta)y - z\right) \\
&= F(z) + F'(z)\left(z - z\right) \\
&= F(z).
\end{aligned} \tag{6.9}$$

Proof of the 2nd order case for $n = 1$ Want to prove that if

$$F : \mathbb{R} \to \mathbb{R} \tag{6.10}$$

is a convex function, then $F''(x) \ge 0 \, \forall x \in \operatorname{dom} F$.

By the first order conditions $\forall x \ne y \in \operatorname{dom} F$

$$F(y) \ge F(x) + F'(x)(y-x)F(x) \quad \ge F(y) + F'(y)(x-y) \tag{6.11}$$

Can combine and get

$$F'(x)(y-x) \le F(y) - F(x) \le F'(y)(y-x) \tag{6.12}$$

Subtract the two derivative terms for

$$\frac{(F'(y) - F'(x))(y-x)}{(y-x)^2} \ge 0, \tag{6.13}$$

or

$$\frac{F'(y) - F'(x)}{y - x} \ge 0. \tag{6.14}$$

In the limit as $y \to x$, this is

$$\boxed{F''(x) \ge 0 \, \forall x \in \operatorname{dom} F.} \tag{6.15}$$

Now prove the reverse condition:
If $F''(x) \ge 0 \, \forall x \in \operatorname{dom} F \subseteq \mathbb{R}$, implies that $F : \mathbb{R} \to \mathbb{R}$ is convex.

Note that if $F''(x) \geq 0$, then $F'(x)$ is non-decreasing in x.
i.e. If $x < y$, where $x, y \in \operatorname{dom} F$, then

$$F'(x) \leq F'(y). \tag{6.16}$$

Consider any $x, y \in \operatorname{dom} F$ such that $x < y$, where

$$F(y) - F(x) = \int_x^y F'(t)dt \geq F'(x) \int_x^y 1dt = F'(x)(y - x). \tag{6.17}$$

This tells us that

$$F(y) \geq F(x) + F'(x)(y - x), \tag{6.18}$$

which is the first order condition. Similarly consider any $x, y \in \operatorname{dom} F$ such that $x < y$, where

$$F(y) - F(x) = \int_x^y F'(t)dt \leq F'(y) \int_x^y 1dt = F'(y)(y - x). \tag{6.19}$$

This tells us that

$$F(x) \geq F(y) + F'(y)(x - y). \tag{6.20}$$

Vector proof: F is convex iff $F(\mathbf{x} + t\mathbf{v})$ is convex $\forall \mathbf{x}, \mathbf{v} \in \mathbb{R}^n, t \in \mathbb{R}$, keeping $\mathbf{x} + t\mathbf{v} \in \operatorname{dom} F$.

Let

$$h(t; \mathbf{x}, \mathbf{v}) = F(\mathbf{x} + t\mathbf{v}) \tag{6.21}$$

then $h(t)$ satisfies scalar first and second order conditions for all $\mathbf{x}, \mathbf{v}$.

$$h(t) = F(\mathbf{x} + t\mathbf{v}) = F(g(t)), \tag{6.22}$$

where $g(t) = \mathbf{x} + t\mathbf{v}$, where

$$\begin{aligned} F &: \mathbb{R}^n \rightarrow \mathbb{R} \\ g &: \mathbb{R} \rightarrow \mathbb{R}^n. \end{aligned} \tag{6.23}$$

This is expressing $h(t)$ as a composition of two functions. By the first order condition for scalar functions we know that

$$h(t) \geq h(0) + h'(0)t. \tag{6.24}$$

Note that

$$h(0) = \left. F(\mathbf{x} + t\mathbf{v}) \right|_{t=0} = F(\mathbf{x}). \tag{6.25}$$

Let’s figure out what $h'(0)$ is. Recall hat for any $\tilde{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$D\tilde{F} \in \mathbb{R}^{m \times n}, \tag{6.26}$$

and

$$D\tilde{F}(\mathbf{x})_{ij} = \frac{\partial \tilde{F}_i(\mathbf{x})}{\partial x_j} \tag{6.27}$$

This is one function per row, for $i \in [1, m]$, $j \in [1, n]$. This gives

$$\begin{aligned} \frac{d}{dt} F(\mathbf{x} + \mathbf{v}t) &= \frac{d}{dt} F(g(t)) \\ &= \frac{d}{dt} h(t) \\ &= Dh(t) \\ &= DF(g(t)) \cdot Dg(t). \end{aligned} \tag{6.28}$$

The first matrix is in $\mathbb{R}^{1 \times n}$ whereas the second is in $\mathbb{R}^{n \times 1}$, since $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}^n$. This gives

$$\frac{d}{dt} F(\mathbf{x} + \mathbf{v}t) = \left. DF(\tilde{\mathbf{x}}) \right|_{\tilde{\mathbf{x}} = g(t)} \cdot Dg(t). \tag{6.29}$$

That first matrix is

$$\begin{aligned} \left. DF(\tilde{\mathbf{x}}) \right|_{\tilde{\mathbf{x}} = g(t)} &= \left. \left(\begin{bmatrix} \frac{\partial F(\tilde{\mathbf{x}})}{\partial \tilde{x}_1} & \frac{\partial F(\tilde{\mathbf{x}})}{\partial \tilde{x}_2} & \cdots & \frac{\partial F(\tilde{\mathbf{x}})}{\partial \tilde{x}_n} \end{bmatrix} \right) \right|_{\tilde{\mathbf{x}} = g(t) = \mathbf{x} + t\mathbf{v}} \\ < F16 > &= \left. \left(\boldsymbol{\nabla} F(\tilde{\mathbf{x}}) \right)^{\mathrm{T}} \right|_{\tilde{\mathbf{x}} = g(t)} \\ &= \left(\boldsymbol{\nabla} F(g(t)) \right)^{\mathrm{T}}. \end{aligned} \tag{6.30}$$

The second Jacobian is

$$Dg(t) = D \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix} = D \begin{bmatrix} x_1 + tv_1 \\ x_2 + tv_2 \\ \vdots \\ x_n + tv_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{v}. \tag{6.31}$$

so

$$h'(t) = Dh(t) = \left(\boldsymbol{\nabla} F(g(t)) \right)^{\mathrm{T}} \mathbf{v}, \tag{6.32}$$

and

$$\begin{aligned} h'(0) &= \left(\boldsymbol{\nabla} F(g(0))\right)^{\mathrm{T}} \mathbf{v} \\ &= \left(\boldsymbol{\nabla} F(\mathbf{x})\right)^{\mathrm{T}} \mathbf{v}. \end{aligned} \tag{6.33}$$

Finally

$$\begin{aligned} F(\mathbf{x}+t\mathbf{v}) \ge h(0) + h'(0)t& \\ &= F(\mathbf{x}) + \left(\boldsymbol{\nabla} F(\mathbf{x})\right)^{\mathrm{T}} (t\mathbf{v}) \\ &= F(\mathbf{x}) + \langle \boldsymbol{\nabla} F(\mathbf{x}), t\mathbf{v}\rangle. \end{aligned} \tag{6.34}$$

Which is true for all $\mathbf{x}, \mathbf{x}+t\mathbf{v} \in \operatorname{dom} F$. Note that the quantity $t\mathbf{v}$ is a shift.

Epigraph Recall that if $(\mathbf{x}, t) \in \operatorname{epi} F$ then $t \ge F(\mathbf{x})$.

$$t \ge F(\mathbf{x}) \ge F(\mathbf{x}_0) + \left(\boldsymbol{\nabla} F(\mathbf{x}_0)\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0), \tag{6.35}$$

or

$$0 \ge -(t - F(\mathbf{x}_0)) + \left(\boldsymbol{\nabla} F(\mathbf{x}_0)\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0), \tag{6.36}$$

In block matrix form

$$0 \ge \begin{bmatrix} \left(\boldsymbol{\nabla} F(\mathbf{x}_0)\right)^{\mathrm{T}} & -1 \end{bmatrix} \begin{bmatrix} \mathbf{x} - \mathbf{x}_0 \\ t - F(\mathbf{x}_0) \end{bmatrix} \tag{6.37}$$

With $\mathbf{w} = \begin{bmatrix} \left(\boldsymbol{\nabla} F(\mathbf{x}_0)\right)^{\mathrm{T}} & -1 \end{bmatrix}$, the geometry of the epigraph relation to the half plane is sketched in fig. 6.2.

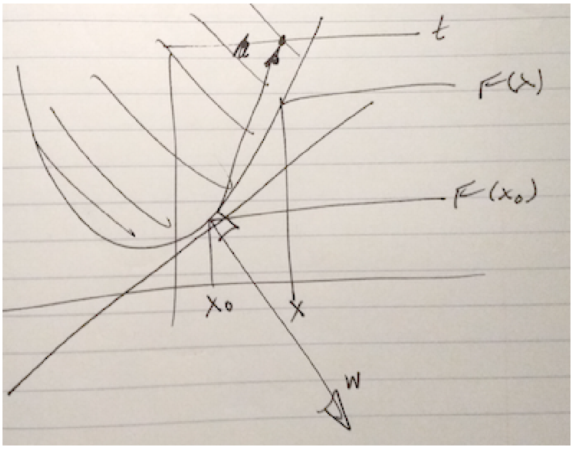

t
F(x)
F(x₀)
x₀
x
w

Figure 6.2: Half planes and epigraph.

7

EXAMPLES OF CONVEX AND CONCAVE FUNCTIONS, LOCAL AND GLOBAL MINIMUMS.

Example:

$$\begin{aligned} F(x) &= x^2 \\ F''(x) &= 2 > 0 \end{aligned} \tag{7.1}$$

strictly convex.

Example:

$$\begin{aligned} F(x) &= x^3 \\ F''(x) &= 6x. \end{aligned} \tag{7.2}$$

Not always non-negative, so not convex. However x^3 is convex on dom $F = \mathbb{R}_+$.

Example:

$$\begin{aligned} F(x) &= x^\alpha \\ F'(x) &= \alpha x^{\alpha-1} \\ F''(x) &= \alpha(\alpha-1)x^{\alpha-2}. \end{aligned} \tag{7.3}$$

This is convex on $\mathbb{R}_+$, if $\alpha \geq 1$, or $\alpha \leq 0$.

Example:

$$\begin{aligned} F(x) &= \log x \\ F'(x) &= \frac{1}{x} \\ F''(x) &= -\frac{1}{x^2} \leq 0 \end{aligned} \tag{7.4}$$

This is concave.

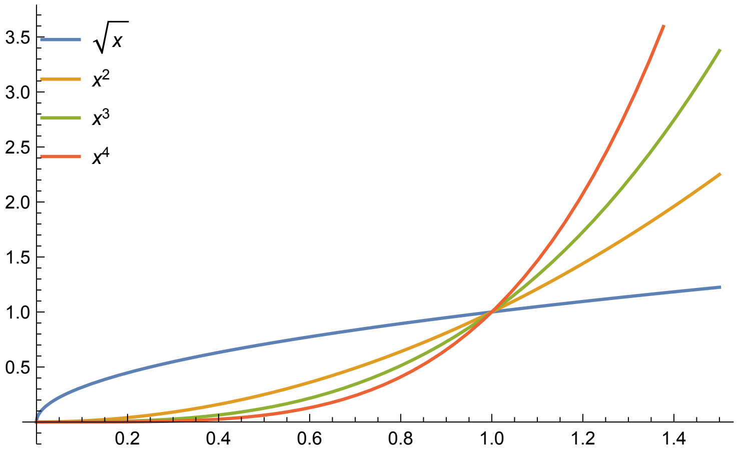


Figure 7.1: Powers of x.

Example:

$$\begin{aligned} F(x) &= x \log x \\ F'(x) &= \log x + x\frac{1}{x} = 1 + \log x \\ F''(x) &= \frac{1}{x} \end{aligned} \tag{7.5}$$

This is strictly convex on $\mathbb{R}_{++}$, where $F''(x) \geq 0$.

Example:

$$\begin{aligned} F(x) &= e^{\alpha x} \\ F'(x) &= \alpha e^{\alpha x} \\ F''(x) &= \alpha^2 e^{\alpha x} \geq 0 \end{aligned} \tag{7.6}$$

Such functions are plotted in fig. 7.2, and are convex function for all α.

Example: For symmetric $P \in S^n$

$$\begin{aligned} F(\mathbf{x}) &= \mathbf{x}^{\mathrm{T}} P \mathbf{x} + 2\mathbf{q}^{\mathrm{T}}\mathbf{x} + r \\ \boldsymbol{\nabla} F &= (P + P^{\mathrm{T}})\mathbf{x} + 2\mathbf{q} = 2P\mathbf{x} + 2\mathbf{q} \\ \boldsymbol{\nabla}^2 F &= 2P. \end{aligned} \tag{7.7}$$

This is convex(concave) if $P \geq 0$ ($P \leq 0$).

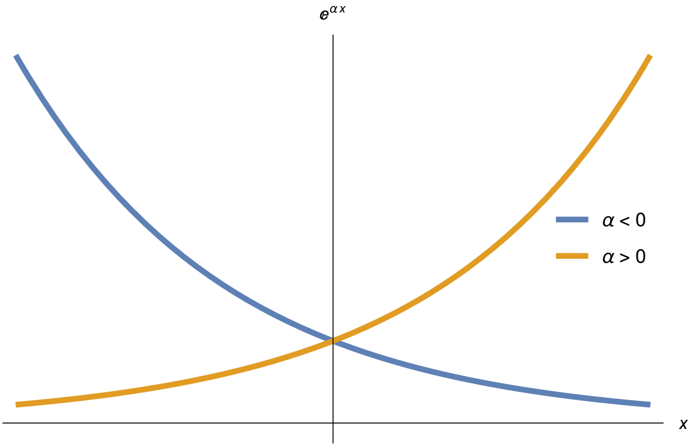
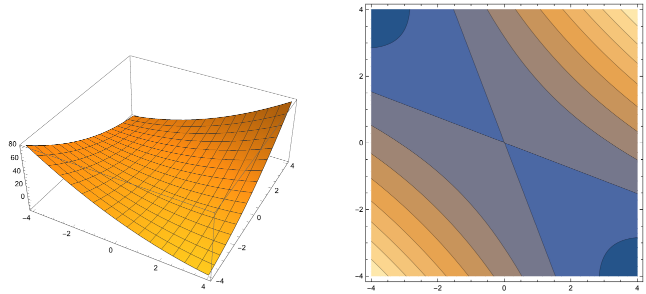


Figure 7.2: Exponential.

Example: A quadratic function

$$F(x,y) = x^2 + y^2 + 3xy, \tag{7.8}$$

that is neither convex nor concave is plotted in fig. 7.3



(a) **(b)**

Figure 7.3: Function with saddle point (3d and contours).

This function can be put in matrix form

$$F(x,y) = x^2 + y^2 + 3xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1.5 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{7.9}$$

and has the Hessian

$$\begin{aligned}\nabla^2 F &= \begin{bmatrix} \partial_{xx}F & \partial_{xy}F \\ \partial_{yx}F & \partial_{yy}F \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \\ &= 2P. \end{aligned} \tag{7.10}$$

From the plot we know that this is not PSD, but this can be confirmed by checking the eigenvalues

$$\begin{aligned} 0 &= \det(P - \lambda I) \\ &= (1-\lambda)^2 - 1.5^2, \end{aligned} \tag{7.11}$$

which has solutions

$$\lambda = 1 \pm \frac{3}{2} = \frac{3}{2}, -\frac{1}{2}. \tag{7.12}$$

This is not PSD nor negative semi-definite, because it has one positive and one negative eigenvalues. This is neither convex nor concave.

Along $y = -x$,

$$\begin{aligned} F(x,y) &= F(x,-x) \\ &= 2x^2 - 3x^2 \\ &= -x^2, \end{aligned} \tag{7.13}$$

so it is concave along this line. Along $y = x$

$$\begin{aligned} F(x,y) &= F(x,x) \\ &= 2x^2 + 3x^2 \\ &= 5x^2, \end{aligned} \tag{7.14}$$

so it is convex along this line.

Example:

$$F(\mathbf{x}) = \sqrt{x_1 x_2}, \tag{7.15}$$

on $\operatorname{dom} F = \{x_1 \ge 0, x_2 \ge 0\}$

For the Hessian

$$\begin{aligned}\frac{\partial F}{\partial x_1} &= \frac{1}{2} x_1^{-1/2} x_2^{1/2} \\ \frac{\partial F}{\partial x_2} &= \frac{1}{2} x_2^{-1/2} x_1^{1/2}\end{aligned} \tag{7.16}$$

The Hessian components are

$$\begin{aligned}\frac{\partial}{\partial x_1}\frac{\partial F}{\partial x_1} &= -\frac{1}{4} x_1^{-3/2} x_2^{1/2} \\ \frac{\partial}{\partial x_1}\frac{\partial F}{\partial x_2} &= \frac{1}{4} x_2^{-1/2} x_1^{-1/2} \\ \frac{\partial}{\partial x_2}\frac{\partial F}{\partial x_1} &= \frac{1}{4} x_1^{-1/2} x_2^{-1/2} \\ \frac{\partial}{\partial x_2}\frac{\partial F}{\partial x_2} &= -\frac{1}{4} x_2^{-3/2} x_1^{1/2}\end{aligned} \tag{7.17}$$

or

$$\boldsymbol{\nabla}^2 F = -\frac{\sqrt{x_1 x_2}}{4} \begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix}. \tag{7.18}$$

Checking this for PSD against $\mathbf{v} = (v_1, v_2)$, we have

$$\begin{aligned}\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \frac{1}{x_1^2} & -\frac{1}{x_1 x_2} \\ -\frac{1}{x_1 x_2} & \frac{1}{x_2^2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \frac{1}{x_1^2} v_1 - \frac{1}{x_1 x_2} v_2 \\ -\frac{1}{x_1 x_2} v_1 + \frac{1}{x_2^2} v_2 \end{bmatrix} \\ &= \left(\frac{1}{x_1^2} v_1 - \frac{1}{x_1 x_2} v_2 \right) v_1 + \left(-\frac{1}{x_1 x_2} v_1 + \frac{1}{x_2^2} v_2 \right) v_2 \\ &= \frac{1}{x_1^2} v_1^2 + \frac{1}{x_2^2} v_2^2 - 2 \frac{1}{x_1 x_2} v_1 v_2 \\ &= \left(\frac{v_1}{x_1} - \frac{v_2}{x_2} \right)^2 \\ &\ge 0,\end{aligned} \tag{7.19}$$

so $\boldsymbol{\nabla}^2 F \le 0$. This is a negative semi-definite function (concave). Observe that this check required checking PSD for all values of $\mathbf{x}$.

This is an example of a more general result

$$F(x) = \left(\prod_{i=1}^{n} x_i \right)^{1/n}, \tag{7.20}$$

which is concave (prove on homework).

Summary. If F is differentiable in $\mathbb{R}^n$, then check the curvature of the function along all lines. i.e. At all locations and in all directions.

If the Hessian is PSD at all $\mathbf{x} \in \operatorname{dom} F$, that is

$$\boldsymbol{\nabla}^2 F \geq 0 \, \forall \mathbf{x} \in \operatorname{dom} F, \tag{7.21}$$

then the function is convex.

more examples of convex, but not necessarily differentiable functions

Example: Over $\operatorname{dom} F = \mathbb{R}^n$

$$F(\mathbf{x}) = \max_{i=1}^{n} x_i \tag{7.22}$$

i.e.

$$\begin{aligned} F((1,2) &= 2 \\ F((3,-1) &= 3 \end{aligned} \tag{7.23}$$

Example:

$$F(\mathbf{x}) = \max_{i=1}^{n} F_i(\mathbf{x}), \tag{7.24}$$

where

$$F_i(\mathbf{x}) = ...? \tag{7.25}$$

max of a set of convex functions is a convex function.

Example:

$$F(x) = x_{[1]} + x_{[2]} + x_{[3]} \tag{7.26}$$

where

$x_{[k]}$ is the k-th largest number in the list

Write

$$F(x) = \max x_i + x_j + x_k \tag{7.27}$$

$$(i, j, k) \in \binom{n}{3} \tag{7.28}$$

Example: For $\mathbf{a} \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$

$$\begin{aligned} F(\mathbf{x}) &= \sum_{i=1}^{n} \log(b_i - \mathbf{a}^{\mathrm{T}}\mathbf{x})^{-1} \\ &= -\sum_{i=1}^{n} \log(b_i - \mathbf{a}^{\mathrm{T}}\mathbf{x}). \end{aligned} \tag{7.29}$$

This $b_i - \mathbf{a}^{\mathrm{T}}\mathbf{x}$ is an affine function of $\mathbf{x}$ so it doesn't affect convexity.

Since log is concave, $-\log$ is convex. Convex functions of affine function of $\mathbf{x}$ is convex function of $\mathbf{x}$.

Example:

$$F(\mathbf{x}) = \sup_{\mathbf{y} \in C} ||\mathbf{x} - \mathbf{y}|| \tag{7.30}$$

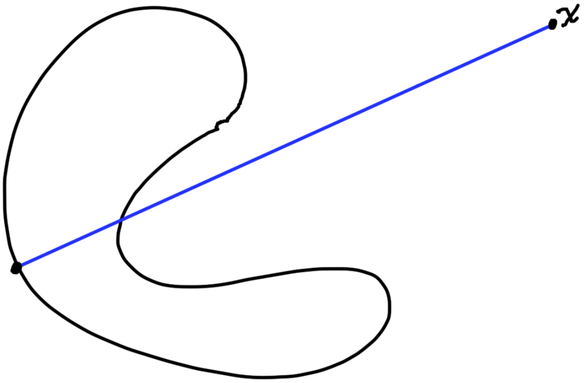


Figure 7.4: Max length function

Here $C \subseteq \mathbb{R}^n$ is not necessarily convex. We are using sup here because the set C may be open. This function is the length of the line from $\mathbf{x}$ to the point in C that is furthest from $\mathbf{x}$.

- $\mathbf{x} - \mathbf{y}$ is linear in $\mathbf{x}$
- $g_{\mathbf{y}}(\mathbf{x}) = ||\mathbf{x} - \mathbf{y}||$ is convex in $\mathbf{x}$ since norms are convex functions.
- $F(\mathbf{x}) = \sup_{\mathbf{y} \in C} ||\mathbf{x} - \mathbf{y}||$. Each $\mathbf{y}$ index is a convex function. Taking max of those.

Example:

$$F(\mathbf{x}) = \inf_{\mathbf{y}\in C} \|\mathbf{x}-\mathbf{y}\|\,. \tag{7.31}$$

Min and max of two convex functions are plotted in fig. 7.5.

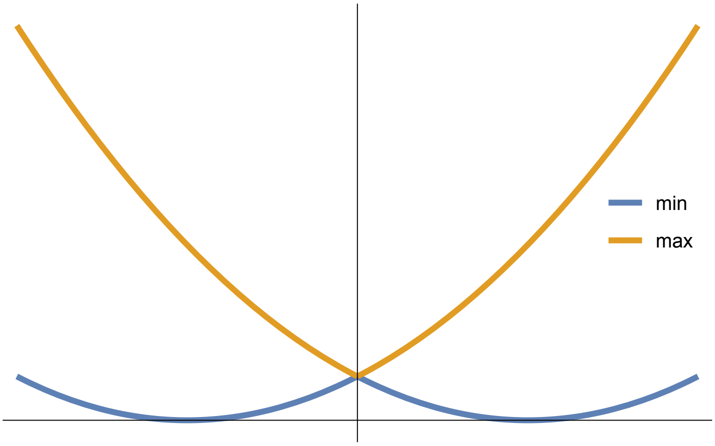


Figure 7.5: Min and max

The max is observed to be convex, whereas the min is not necessarily so.

$$F(\mathbf{z}) = F(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \geq \theta F(\mathbf{x}) + (1-\theta)F(\mathbf{y}). \tag{7.32}$$

This is not necessarily convex for all sets $C \subseteq \mathbb{R}^n$, because the inf of a bunch of convex function is not necessarily convex. However, if C is convex, then $F(\mathbf{x})$ is convex.

Consequences of convexity for differentiable functions

- Think about unconstrained functions $\operatorname{dom} F = \mathbb{R}^n$.
- By first order condition F is convex iff the domain is convex and

$$F(\mathbf{x}) \geq \left(\boldsymbol{\nabla} F(\mathbf{x})\right)^{\mathrm{T}} (\mathbf{y}-\mathbf{x})\, \forall \mathbf{x}, \mathbf{y} \in \operatorname{dom} F. \tag{7.33}$$

If F is convex and one can find an $\mathbf{x}^* \in \operatorname{dom} F$ such that

$$\boldsymbol{\nabla} F(\mathbf{x}^*) = 0, \tag{7.34}$$

then

$$F(\mathbf{y}) \geq F(\mathbf{x}^*)\ \forall \mathbf{y} \in \operatorname{dom} F. \tag{7.35}$$

If you can find the point where the gradient is zero (which can't always be found), then $\mathbf{x}^*$ is a global minimum of F.

Conversely, if $\mathbf{x}^*$ is a global minimizer of F, then $\nabla F(\mathbf{x}^*) = 0$ must hold. If that were not the case, then you would be able to find a direction to move downhill, contracting the optimality of $\mathbf{x}^*$.

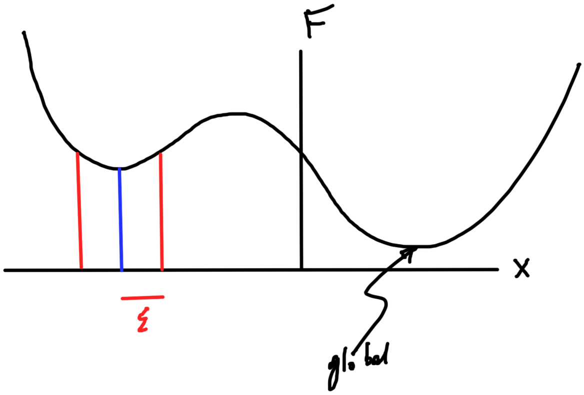


Figure 7.6: Global and local minimums

Local vs Global optimum

Definition 7.1: Local optimum.

$\mathbf{x}^*$ is a local optimum of F if $\exists \epsilon > 0$ such that $\forall \mathbf{x}$, $\|\mathbf{x} - \mathbf{x}^*\| < \epsilon$, we have

$$F(\mathbf{x}^*) \leq F(\mathbf{x})$$

Theorem 7.1

Suppose F is twice continuously differentiable (not necessarily convex)

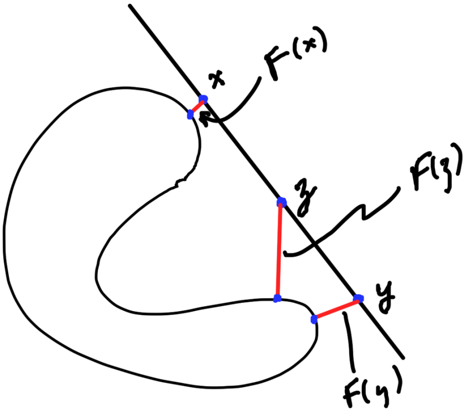


Figure 7.7: min length function.

- If $\mathbf{x}^*$ is a local optimum then

$$\begin{aligned} \boldsymbol{\nabla} F(\mathbf{x}^*) &= 0 \\ \boldsymbol{\nabla}^2 F(\mathbf{x}^*) &\geq 0 \end{aligned}$$

- If

$$\begin{aligned} \boldsymbol{\nabla} F(\mathbf{x}^*) &= 0 \\ \boldsymbol{\nabla}^2 F(\mathbf{x}^*) &\geq 0 \end{aligned},$$

then $\mathbf{x}^*$ is a local optimum.

Proof:

- Let $\mathbf{x}^*$ be a local optimum. Pick any $\mathbf{v} \in \mathbb{R}^n$.

$$\lim_{t \to 0} \frac{F(\mathbf{x}^* + t\mathbf{v}) - F(\mathbf{x}^*)}{t} = \left(\boldsymbol{\nabla} F(\mathbf{x}^*)\right)^{\mathrm{T}} \mathbf{v} \geq 0. \tag{7.36}$$

Here the fraction is ≥ 0 since $\mathbf{x}^*$ is a local optimum.

Since the choice of $\mathbf{v}$ is arbitrary, the only case that you can ensure that $\geq 0, \forall \mathbf{v}$ is

$$\boldsymbol{\nabla} F = 0, \tag{7.37}$$

(or else could pick $\mathbf{v} = -\boldsymbol{\nabla} F(\mathbf{x}^*)$.

This means that $\boldsymbol{\nabla} F(\mathbf{x}^*) = 0$ if $\mathbf{x}^*$ is a local optimum.

Consider the 2nd order derivative

$$\begin{aligned}
&\lim_{t\to 0}\frac{F(\mathbf{x}^* + t\mathbf{v}) - F(\mathbf{x}^*)}{t^2} \\
&\quad = \lim_{t\to 0}\frac{1}{t^2}\left(F(\mathbf{x}^*) + t\left(\boldsymbol{\nabla} F(\mathbf{x}^*)\right)^{\mathrm{T}}\mathbf{v} + \frac{1}{2}t^2\mathbf{v}^{\mathrm{T}}\boldsymbol{\nabla}^2 F(\mathbf{x}^*)\mathbf{v} + O(t^3) - F(\mathbf{x}^*)\right) \\
&\quad = \frac{1}{2}\mathbf{v}^{\mathrm{T}}\boldsymbol{\nabla}^2 F(\mathbf{x}^*)\mathbf{v} \\
&\quad \geq 0.
\end{aligned} \tag{7.38}$$

Here the $\geq$ condition also comes from the fraction, based on the optimiality of $\mathbf{x}^*$. This is true for all choice of $\mathbf{v}$, thus $\boldsymbol{\nabla}^2 F(\mathbf{x}^*)$.

Now we want to prove that if

$$\begin{aligned}
\boldsymbol{\nabla} F(\mathbf{x}^*) &= 0 \\
\boldsymbol{\nabla}^2 F(\mathbf{x}^*) &\geq 0
\end{aligned} ,$$

then $\mathbf{x}^*$ is a local optimum.

Proof:

Again, using Taylor approximation

$$F(\mathbf{x}^* + \mathbf{v}) = F(\mathbf{x}^*) + \left(\boldsymbol{\nabla} F(\mathbf{x}^*)\right)^{\mathrm{T}}\mathbf{v} + \frac{1}{2}\mathbf{v}^{\mathrm{T}}\boldsymbol{\nabla}^2 F(\mathbf{x}^*)\mathbf{v} + o(\|\mathbf{v}\|^2) \tag{7.39}$$

The linear term is zero by assumption, whereas the Hessian term is given as > 0. Any direction that you move in, if your move is small enough, this is going uphill at a local optimum.

7.1 SUMMARIZE:

For twice continuously differentiable functions, at a local optimum $\mathbf{x}^*$, then

$$\begin{aligned}
\boldsymbol{\nabla} F(\mathbf{x}^*) &= 0 \\
\boldsymbol{\nabla}^2 F(\mathbf{x}^*) &\geq 0
\end{aligned} \tag{7.40}$$

If, in addition, F is convex, then $\boldsymbol{\nabla} F(\mathbf{x}^*) = 0$ implies that $\mathbf{x}^*$ is a global optimum. i.e. for (unconstrained) convex functions, local and global optimums are equivalent.

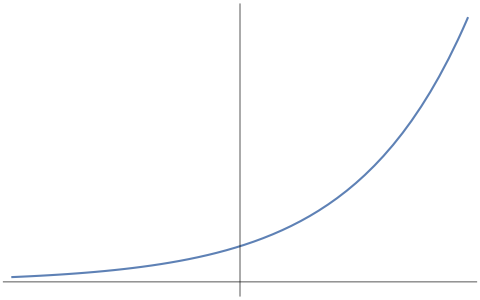


Figure 7.8: Exponential has no global optimum.

- It is possible that a convex function does not have a global optimum. Examples are $F(x) = e^x$ (fig. 7.8) , which has an inf, but no lowest point.
- Our discussion has been for unconstrained functions. For constrained problems (next topic) is not necessarily true that $\nabla F(\mathbf{x}) = 0$ implies that $\mathbf{x}$ is a global optimum, even for F convex.

  As an example of a constrained problem consider

$$\begin{aligned} &\min 2x^2 + y^2 \\ &\quad x \geq 3 \\ &\quad y \geq 5. \end{aligned} \tag{7.41}$$

  The level sets of this objective function are plotted in fig. 7.9. The optimal point is at $\mathbf{x}^* = (3, 5)$, where $\nabla F \neq 0$.

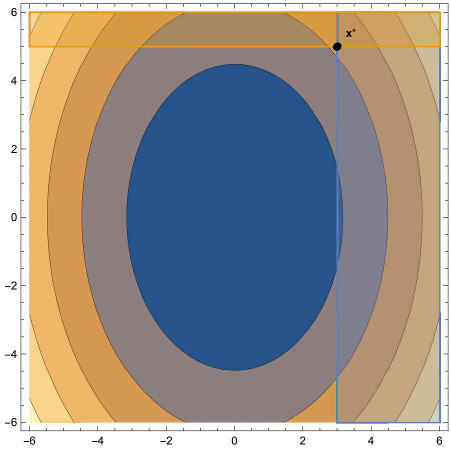


Figure 7.9: Constrained problem with optimum not at the zero gradient point.

8

COMPOSITIONS OF FUNCTIONS.

8.1 PROJECTION

Given $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^p$, if $h(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}, \mathbf{y}$, then

$$F(\mathbf{x}_0) = \inf_{\mathbf{y}} h(\mathbf{x}_0, \mathbf{y}) \tag{8.1}$$

is convex in $\mathbf{x}$, as sketched in fig. 8.1.

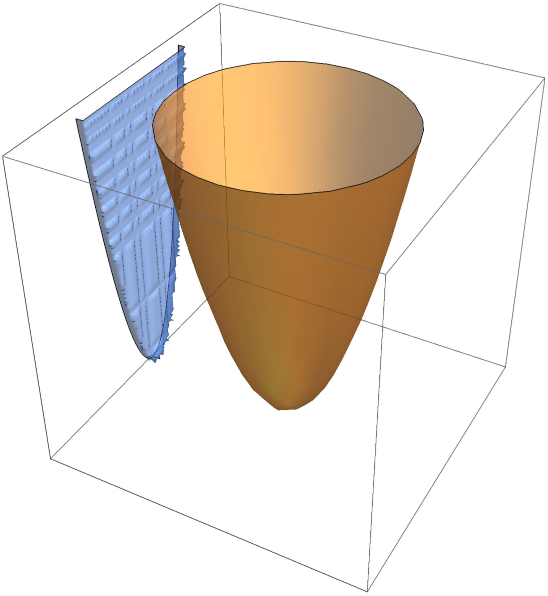


Figure 8.1: Epigraph of h is a filled bowl.

The intuition here is that shining light on the (filled) "bowl". That is, the image of $\operatorname{epi} h$ on the $\mathbf{y} = 0$ screen which we will show is a convex set.

Proof:

Since h is convex in $\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \operatorname{dom} h$, then

$$\operatorname{epi} h = \left\{(\mathbf{x}, \mathbf{y}, t) | t \geq h(\mathbf{x}, \mathbf{y}), \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \operatorname{dom} h\right\}, \tag{8.2}$$

is a convex set.

We also have to show that the domain of F is a convex set. To show this note that

$$\begin{aligned}\operatorname{dom} F &= \left\{\mathbf{x} | \exists \mathbf{y} s.t. \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \operatorname{dom} h\right\}\\ &= \left\{\begin{bmatrix}I_{n\times n} & 0_{n\times p}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} | \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \operatorname{dom} h\right\}.\end{aligned} \tag{8.3}$$

This is an affine map of a convex set. Therefore $\operatorname{dom} F$ is a convex set.

$$\begin{aligned}\operatorname{epi} F &= \left\{\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} | t \geq \inf h(\mathbf{x}, \mathbf{y}), \mathbf{x} \in \operatorname{dom} F, \mathbf{y} : \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \operatorname{dom} h\right\}\\ &= \left\{\begin{bmatrix}I & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ t\end{bmatrix} | t \geq h(\mathbf{x}, \mathbf{y}), \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \operatorname{dom} h\right\}.\end{aligned} \tag{8.4}$$

Example: The function

$$F(\mathbf{x}) = \inf_{\mathbf{y}\in C} \|\mathbf{x} - \mathbf{y}\|, \tag{8.5}$$

over $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in C$, ,is convex if C is a convex set. Reason:

- $\mathbf{x} - \mathbf{y}$ is linear in $(\mathbf{x}, \mathbf{y})$.
- $\|\mathbf{x} - \mathbf{y}\|$ is a convex function if the domain is a convex set
- The domain is $\mathbb{R}^n \times C$. This will be a convex set if C is.
- $h(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is a convex function if $\operatorname{dom} h$ is a convex set. By setting $\operatorname{dom} h = \mathbb{R}^n \times C$, if C is convex, $\operatorname{dom} h$ is a convex set.
- $F()$

8.2 COMPOSITION OF FUNCTIONS

Consider

$$\begin{aligned} F(\mathbf{x}) &= h(g(\mathbf{x})) \\ \operatorname{dom} F &= \{\mathbf{x} \in \operatorname{dom} g | g(\mathbf{x}) \in \operatorname{dom} h\} \\ F &: \mathbb{R}^n \to \mathbb{R} \\ g &: \mathbb{R}^n \to \mathbb{R} \\ h &: \mathbb{R} \to \mathbb{R}. \end{aligned} \tag{8.6}$$

Cases:

(a) g is convex, h is convex and non-decreasing.

(b) g is convex, h is convex and non-increasing.

Show for 1D case ($n = 1$). Get to $n > 1$ by applying to all lines.

(a)

$$\begin{aligned} F'(x) &= h'(g(x))g'(x) \\ F''(x) &= h''(g(x))g'(x)g'(x) + h'(g(x))g''(x) \\ &= h''(g(x))(g'(x))^2 + h'(g(x))g''(x) \\ &= (\geq 0) \cdot (\geq 0)^2 + (\geq 0) \cdot (\geq 0)\,, \end{aligned} \tag{8.7}$$

since h is respectively convex, and non-decreasing.

(b)

$$F'(x) = (\geq 0) \cdot (\geq 0)^2 + (\leq 0) \cdot (\leq 0)\,, \tag{8.8}$$

since h is respectively convex, and non-increasing, and g is concave.

8.3 EXTENDING TO MULTIPLE DIMENSIONS

$$\begin{aligned} F(\mathbf{x}) &= h(g(\mathbf{x})) = h(g_1(\mathbf{x}), g_2(\mathbf{x}), \cdots g_k(\mathbf{x})) \\ g &: \mathbb{R}^n \to \mathbb{R} \\ h &: \mathbb{R}^k \to \mathbb{R}. \end{aligned} \tag{8.9}$$

is convex if g_i is convex for each $i \in [1, k]$ and h is convex and non-decreasing in each argument.

Proof:
again assume $n = 1$, without loss of generality,

$$\begin{aligned} &g : \mathbb{R} \rightarrow \mathbb{R}^k \\ &h : \mathbb{R}^k \rightarrow \mathbb{R} \end{aligned} \tag{8.10}$$

$$F''(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) & g_2(\mathbf{x}) & \cdots & g_k(\mathbf{x}) \end{bmatrix} \boldsymbol{\nabla}^2 h(g(\mathbf{x})) \begin{bmatrix} g_1'(\mathbf{x}) \\ g_2'(\mathbf{x}) \\ \vdots \\ g_k'(\mathbf{x}) \end{bmatrix} + \left(\boldsymbol{\nabla} h(g(x))\right)^{\mathrm{T}} \begin{bmatrix} g_1''(\mathbf{x}) \\ g_2''(\mathbf{x}) \\ \vdots \\ g_k''(\mathbf{x}) \end{bmatrix} \tag{8.11}$$

The Hessian is PSD.

Example:

$$F(x) = \exp(g(x)) = h(g(x)), \tag{8.12}$$

where g is convex is convex, and $h(y) = e^y$. This implies that F is a convex function.

Example:

$$F(x) = \frac{1}{g(x)}, \tag{8.13}$$

is convex if $g(x)$ is concave and positive. The most simple such example of such a function is $h(x) = 1/x$, $\operatorname{dom} h = \mathbb{R}_{++}$, which is plotted in fig. 8.2.

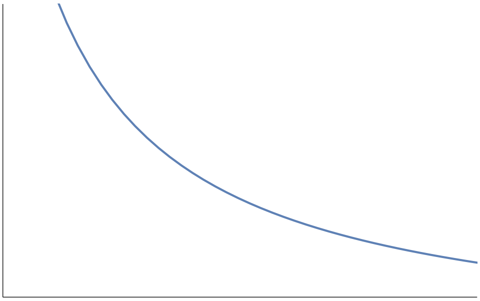


Figure 8.2: Inverse function is convex over positive domain.

Example:

$$F(x) = -\sum_{i=1}^{n} \log(-F_i(x)) \tag{8.14}$$

is convex on $\{x | F_i(x) < 0 \forall i\}$ if all F_i are convex.

- Due to $\operatorname{dom} F$, $-F_i(x) > 0 \; \forall x \in \operatorname{dom} F$
- $\log(x)$ concave on $\mathbb{R}_{++}$ so $-\log$ convex also non-increasing (fig. 8.3).

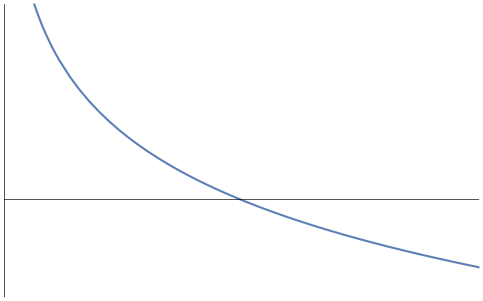


Figure 8.3: Negative logarithm convex over positive domain.

$$F(x) = \sum h_i(x) \tag{8.15}$$

but

$$h_i(x) = -\log(-F_i(x)), \tag{8.16}$$

which is a convex and non-increasing function ($-\log$), of a convex function $-F_i(x)$. Each h_i is convex, so this is a sum of convex functions, and is therefore convex.

Example: Over $\operatorname{dom} F = S^n_{++}$

$$F(X) = \log\det X^{-1} \tag{8.17}$$

To show that this is convex, check all lines in domain. A line in S^n_{++} is a 1D family of matrices

$$\tilde{F}(t) = \log\det((X_0 + tH)^{-1}), \tag{8.18}$$

where $X_0 \in S^n_{++}, t \in \mathbb{R}, H \in S^n$.

F9

For t small enough,

$$X_0 + tH \in S^n_{++} \tag{8.19}$$

$$\begin{aligned}
\tilde{F}(t) &= \log\det((X_0 + tH)^{-1}) \\
&= \log\det\left(X_0^{-1/2}\left(I + tX_0^{-1/2}HX_0^{-1/2}\right)^{-1} X_0^{-1/2}\right) \\
&= \log\det\left(X_0^{-1}\left(I + tX_0^{-1/2}HX_0^{-1/2}\right)^{-1}\right) \\
&= \log\det X_0^{-1} + \log\det\left(I + tX_0^{-1/2}HX_0^{-1/2}\right)^{-1} \\
&= \log\det X_0^{-1} - \log\det\left(I + tX_0^{-1/2}HX_0^{-1/2}\right) \\
&= \log\det X_0^{-1} - \log\det\left(I + tM\right).
\end{aligned} \tag{8.20}$$

If λ_i are eigenvalues of M, then $1 + t\lambda_i$ are eigenvalues of $I + tM$. i.e.:

$$\begin{aligned}
(I + tM)\mathbf{v} &= I\mathbf{v} + t\lambda_i\mathbf{v} \\
&= (1 + t\lambda_i)\mathbf{v}.
\end{aligned} \tag{8.21}$$

This gives

$$\tilde{F}(t) = \log\det X_0^{-1} - \log\prod_{i=1}^{n}(1 + t\lambda_i) = \log\det X_0^{-1} - \sum_{i=1}^{n}\log(1 + t\lambda_i) \tag{8.22}$$

- $1 + t\lambda_i$ is linear in t.
- $-\log$ is convex in its argument.
- sum of convex function is convex.

Example:

$$F(X) = \lambda_{\max}(X), \tag{8.23}$$

is convex on $\operatorname{dom} F \in S^n$

(a)

$$\lambda_{\max}(X) = \sup_{\|\mathbf{v}\|_2 \le 1} \mathbf{v}^{\mathrm{T}} X \mathbf{v}, \tag{8.24}$$

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \tag{8.25}$$

Recall that a decomposition

$$\begin{aligned} X &= Q\Lambda Q^{\mathrm{T}} \\ Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} &= I \end{aligned} \tag{8.26}$$

can be used for any $X \in S^n$.

(b)

Note that $\mathbf{v}^{\mathrm{T}}X\mathbf{v}$ is linear in X. This is a max of a number of linear (and convex) functions, so it is convex.

Last example:

(non-symmetric matrices)

$$F(X) = \sigma_{\max}(X), \tag{8.27}$$

is convex on dom $F = \mathbb{R}^{m\times n}$. Here

$$\sigma_{\max}(X) = \sup_{\|\mathbf{v}\|_2=1} \|X\mathbf{v}\|_2 \tag{8.28}$$

This is called an operator norm of X. Using the SVD

$$\begin{aligned} X &= U\Sigma V^{\mathrm{T}} \\ U &= \mathbb{R}^{m\times r} \\ \Sigma &\in \text{diag} \in \mathbb{R}r \times r \\ V^T &\in \mathbb{R}^{r\times n}. \end{aligned} \tag{8.29}$$

Have

$$\begin{aligned} \|X\mathbf{v}\|_2^2 &= \left\|U\Sigma V^{\mathrm{T}}\mathbf{v}\right\|_2^2 \\ &= \mathbf{v}^{\mathrm{T}}V\Sigma U^{\mathrm{T}}U\Sigma V^{\mathrm{T}}\mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}}V\Sigma\Sigma V^{\mathrm{T}}\mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}}V\Sigma^2 V^{\mathrm{T}}\mathbf{v} \\ &= \tilde{\mathbf{v}}^{\mathrm{T}}\Sigma^2\tilde{\mathbf{v}}, \end{aligned} \tag{8.30}$$

where $\tilde{\mathbf{v}} = \mathbf{v}^{\mathrm{T}} V$, so

$$||X\mathbf{v}||_2^2 = \sum_{i=1}^{r} \sigma_i^2 \, ||\tilde{\mathbf{v}}|| \leq \sigma_{\max}^2 \, ||\tilde{\mathbf{v}}||^2 \, , \tag{8.31}$$

or

$$\begin{aligned} ||X\mathbf{v}||_2 &\leq \sqrt{\sigma_{\max}^2} \, ||\tilde{\mathbf{v}}|| \\ &\leq \sigma_{\max}. \end{aligned} \tag{8.32}$$

Set **v** to the right singular value of X to get equality.

PROBLEM SET II (NOT ATTEMPTED). 9

Exercise 9.1 Identifying convexity (*[1] pr. 3.16 (a)-(c)*)

For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave

a. $f(x) = e^x - 1$ on $\mathbb{R}$.
b. $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
c. $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

Exercise 9.2 Products and ratios of convex functions (*[1] pr. 3.32 (a)*)

In general the product or ration of two convex functions is not convex. However, there are some results that apply to functions on $\mathbb{R}$. Prove the following

Exercise 9.3 Convex-concave functions and saddle-points (*[1] pr. 3.14*)

We way the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex-concave if $f(\mathbf{x}, \mathbf{z})$ is a concave function of $\mathbf{z}$, for each fixed $\mathbf{x}$, and a convex function of $\mathbf{x}$, for each fixed $\mathbf{z}$. We also require its domain to have the product form $\operatorname{dom} f = A \times B$, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are convex.

a. Give a second-order condition for a twice differentable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ to be convex-concave, in terms of its Hessian $\boldsymbol{\nabla}^2 f(\mathbf{x}, \mathbf{z})$.
b. Suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a convex-concave and differentiable, with $\boldsymbol{\nabla} f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$. Show that the saddle-point property holds: for all $\mathbf{x}, \mathbf{z}$, we have

$$f(\tilde{\mathbf{x}}, \mathbf{z}) \leq f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \leq f(\mathbf{x}, \tilde{\mathbf{z}}). \tag{9.1}$$

Show that this implies that f satisfies the strong max-in property:

$$\sup_{\mathbf{z}} \inf_{\mathbf{x}} f(\mathbf{x}, \mathbf{z}) = \inf_{\mathbf{x}} \sup_{\mathbf{z}} f(\mathbf{x}, \mathbf{z}) \tag{9.2}$$

(and their common value is $f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$).

c. Now suppose that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is differentiable, but not necessarily convex-concave, and the saddle-point property holds at $\tilde{\mathbf{x}}, \tilde{\mathbf{z}}$:

$$f(\tilde{\mathbf{x}}, \mathbf{z}) \le f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \le f(\mathbf{x}, \tilde{\mathbf{z}}), \tag{9.3}$$

for all $\mathbf{x}, \mathbf{z}$. Show that $\nabla f(\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) = 0$.

Exercise 9.4 Parameterized convexity

Consider the function

$$f(x, y) = x^2 + y^2 + \beta xy + x + 2y. \tag{9.4}$$

Find (x^*, y^*) for which $\nabla f = 0$. Express your answer as a function of β. For which values of β is the (x^*, y^*) a global minimum of $f(x, y)$?

Exercise 9.5 Maximum likelyhood estimation.

In this problem, we are given a set of data points (x_i, y_i), $i = 1 \cdots 100$. We wish to fit a quadratic model,

$$y_i = ax_i^2 + bx_i + c + n_i, \tag{9.5}$$

to the data. Here, (a, b, c) are the parameters to be determined and n_i is the unknown observation noise. The (x_i, y_i) points are contained in a file `dataForMLest.mat` available on the course webpage. You may load the data to `MATLAB` using the command `load ps01data` and view them using `scatter(x,y,'+')`. Please use the same data set and find the maximum likelihood estimate of (a, b, c) assuming n_i's are i.i.d. when

a. $n_i \sim N(0, 1)$;

b. n_i is always positive and $p_{n_i}(z) = e^{-z}u(z)$ where $u(\cdot)$ is the unit step function.

Please plot the data and the models on the same `MATLAB` figure and submit the figgure as a part of your solution. (`MATLAB` has built-in functions to solve many optimization problems. For example, `linprog` solves a linear programming problem, quadprog solves a quadratic programming problem. You may use `help linprog` to get more details.

Hint: part a has an analytic solution.)

Exercise 9.6 First and second order conditions for convexity

In class we proved the first and second-order conditions for convexity of differentiable scalar functions. In particular, the first-order condition we showed is that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if $\operatorname{dom} f$ is a convex set and

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0), \tag{9.6}$$

for all $x, x_0 \in \operatorname{dom} f$. The second-order condition we showed is that a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if $\operatorname{dom} f$ is a convex set and $f''(x) \geq 0$ for all $x \in \operatorname{dom} f$.

In this problem you are asked to prove the two corresponding vector results:

a. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable then f is a convex function if and only if $\operatorname{dom} f$ is a convex set and

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) + \left(\boldsymbol{\nabla} f(\mathbf{x}_0)\right)^{\mathrm{T}} (\mathbf{x} - \mathbf{x}_0), \tag{9.7}$$

for all $\mathbf{x}, \mathbf{x}_0 \in \operatorname{dom} f$.

b. If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable then f is a convex function if and only if $\operatorname{dom} f$ is a convex set and

$$\boldsymbol{\nabla}^2 f(\mathbf{x}) \geq 0, \tag{9.8}$$

for all $\mathbf{x} \in \operatorname{dom} f$.

To prove the above two results, follow the approach followed in class. Namely, show that the differentiable (twice differentiable) vector function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the first order (second-order) scalar condition holds along all lines in the domain. In other words, show that f is convex if and only if $\operatorname{dom} f$ is a convex set and the first-order (second-order) scalar condition holds for $f(\mathbf{x}_0 + t\mathbf{v})$ for all $\mathbf{x}_0, \mathbf{v} \in \mathbb{R}^n, t \in \mathbb{R}$ and $\mathbf{x}_0 + t\mathbf{v} \in \operatorname{dom} f$. To be clear, we already proved part a in class. For part a you are simply asked to reproduce that proof to ensure you fully understand the proof method. Clearly explain the overall logic and the logic of each step. Then, in part b you are asked to take the same (lines-based) approach to show the second-order condition.

Exercise 9.7 Kullback-Leibler divergence and the information inequality (*[1] pr.*

Let D_{kl} be the Kullback-Liebler divergence, as defined in (3.17). Prove the information inequality:

$$D_{kl}(\mathbf{u}, \mathbf{v}) \ge 0, \tag{9.9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{++}$. Also show that $D_{kl}(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

Hint: The Kullback-Liebler divergence can be expressed as

$$D_{kl}(\mathbf{u}, \mathbf{v}) = f(\mathbf{u}) - f(\mathbf{v}) - \left(\boldsymbol{\nabla} f(\mathbf{v})\right)^{\mathrm{T}} (\mathbf{u} - \mathbf{v}). \tag{9.10}$$

Exercise 9.8 Examples of proving convexity

a. Show that the following function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex where

$$f(\mathbf{x}) = \begin{cases} -(x_1 x_2 \cdots x_n)^{1/n} & \text{if } x_1 > 0, \cdots, x_n > 0 \\ \infty & \text{otherwise} \end{cases} \tag{9.11}$$

Prove the above result by computing the Hessian of the function. (Note, this is a special case of [1] pr. 3.18(b) in which you are asked to show that $(\det X)^{1/n}$ is concave on $\operatorname{dom} f = S^n_{++}$. While for this problem I ask you to compute the Hessian, in that problem you may take any approach you wish.)

Hint: it may prove useful to use the relation that for any real numbers $\alpha_1, \cdots, \alpha_n$, $(\sum_{i=1}^n \alpha_i)^2 \le n(\sum_{i=1}^n \alpha_i^2)$.

b. Show that the following function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex where

$$f(\mathbf{x}) = \exp\left(\beta \mathbf{x}^{\mathrm{T}} A \mathbf{x}\right), \tag{9.12}$$

where $\mathbf{x} \in \mathbb{R}^n$, A is a positive semidefinite symmetric $n \times n$ matrix, and β is a positive scalar.

Exercise 9.9 $(\det X)^{1/n}$ is concave on $\operatorname{dom} f = S^n_{++}$ (*[1] pr. 3.18 (b)*)

Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

$$f(X) = \operatorname{tr}(X^{-1}), \tag{9.13}$$

is convex on $\operatorname{dom} f = S^n_{++}$.

Exercise 9.10 Some functions on the probability simplex. (*[1] pr. 3.24 (a)-(e)*)

Let x be a real-valued random variable which takes values $\{a_1, \cdots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\text{prob}(x = a_i) = p_i \; i \in [1, n]$. For each of the following functions of p (on the probability simplex $\left\{ p \in \mathbb{R}^n_+ | \mathbf{1}^{\mathrm{T}} p = 1 \right\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

a. $\mathbf{E}x$.

b. $\text{prob}(x \geq \alpha)$.

c. $\text{prob}(\alpha \leq x \leq \beta)$.

d. $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

e. $\operatorname{var} x = \mathbf{E}(x - \mathbf{E}x)^2$.

A

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