Attempts at solutions for some Goldstein Mechanics problems.

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1 Motivation.

Now have the so often sited [Goldstein(1951)] book to study (an ancient version from the 50's). Here's an attempt at a few of the problems. Some problems were tackled but omitted here since they overlapped with those written up in [Joot(b)] before getting this book.

2 Problem 1.7

Barbell shape, equal masses. center of rod between masses constrained to circular motion.

Assuming motion in a plane, the equation for the center of the rod is:

$$c = ae^{i\theta}$$

and the two mass points positions are:

$$\begin{aligned}
q_1 &= c + (l/2)e^{i\alpha} \\
q_2 &= c - (l/2)e^{i\alpha}
\end{aligned}$$

taking derivatives:

$$\begin{aligned}
\dot{q}_1 &= ai\dot{\theta}e^{i\theta} + (l/2)i\dot{\alpha}e^{i\alpha} \\
\dot{q}_2 &= ai\dot{\theta}e^{i\theta} - (l/2)i\dot{\alpha}e^{i\alpha}
\end{aligned}$$

and squared magnitudes:

$$\begin{aligned}
\dot{q}_\pm &= \left| a\dot{\theta} \pm (l/2)\dot{\alpha}e^{i(\alpha-\theta)} \right|^2 \\
&= \left(a\dot{\theta} \pm \frac{1}{2} l\dot{\alpha}\cos(\alpha-\theta) \right)^2 + \left(\frac{1}{2} l\dot{\alpha}\sin(\alpha-\theta) \right)^2
\end{aligned}$$

Summing the kinetic terms yeilds

$$K = m\left(a\dot{\theta}\right)^2 + m\left(\frac{1}{2}l\dot{\alpha}\right)^2$$

Summing the potential energies, presuming that the motion is verticle, we have:

$$V = mg(l/2)\cos\theta - mg(l/2)\cos\theta$$

So, the Lagrangian is just the Kinetic energy.
Taking derivatives to get the OEMs we have:

$$(ma^2\dot{\theta})' = 0$$
$$\left(\frac{1}{4}ml^2\dot{\alpha}\right)' = 0$$

This is suprising seeming. Is this correct?

3 Problem 1.8

3.1 Problem statement.

Hopefully, not a copyright violation, but here is the problem verbatim:
A system is composed of three particles of equal mass m. Between any two of them there are forces derivable from a potential

$$V = -ge^{-\mu r}$$

where r is the disance between the two particles. In addition, two of the particles each exert a force on the third which can be obtained from a generalized potential of the form

$$U = -f\mathbf{v}\cdot\mathbf{r}$$

v being the relative velocity of the interacting particles and f a constant. Set up the Lagragian for the system, using as coordinates the radius vector **R** of the center of mass and the two vectors

$$\boldsymbol{\rho}_1 = \mathbf{r}_1 - \mathbf{r}_3$$
$$\boldsymbol{\rho}_2 = \mathbf{r}_2 - \mathbf{r}_3$$

Is the total angular momentum of the system conserved?

3.2 Solution attempt.

The center of mass vector is:

$$\mathbf{R} = \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)$$

This can be used to express each of the position vectors in terms of the $\boldsymbol{\rho}_i$ vectors:

$$\begin{aligned}
3m\mathbf{R} &= m(\boldsymbol{\rho}_1 + \mathbf{r}_3) + m(\boldsymbol{\rho}_2 + \mathbf{r}_3) + m\mathbf{r}_3 \\
&= 2m(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2) + 3m\mathbf{r}_3 \\
\mathbf{r}_3 &= \mathbf{R} - \frac{1}{3}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2) \\
\mathbf{r}_2 = \boldsymbol{\rho}_2 + \mathbf{r}_3 = \boldsymbol{\rho}_2 + \mathbf{r}_3 &= \frac{2}{3}\boldsymbol{\rho}_2 - \frac{1}{2}\boldsymbol{\rho}_1 + \mathbf{R} \\
\mathbf{r}_1 = \boldsymbol{\rho}_1 + \mathbf{r}_3 &= \frac{2}{3}\boldsymbol{\rho}_1 - \frac{1}{2}\boldsymbol{\rho}_2 + \mathbf{R}
\end{aligned}$$

Now, that is enough to specify the part of the Lagrangian from the potentials that act between all the particles

$$\mathcal{L}_V = \sum -V_{ij} = g\left(e^{-\mu|\boldsymbol{\rho}_1|} + e^{-\mu|\boldsymbol{\rho}_2|} + e^{-\mu|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|}\right)$$

Now, we need to calculate the two U potential terms. If we consider with positions $\mathbf{r}_1$, and $\mathbf{r}_2$ to be the ones that can exert a force on the third, the velocities of those masses relative to $\mathbf{r}_3$ are:

$$(\mathbf{r}_3 - \mathbf{r}_k)' = \dot{\boldsymbol{\rho}}_k$$

So, the potential parts of the Lagrangian are

$$\mathcal{L}_{U+V} = g\left(e^{-\mu|\boldsymbol{\rho}_1|} + e^{-\mu|\boldsymbol{\rho}_2|} + e^{-\mu|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|}\right) + f\left(\mathbf{R} - \frac{1}{3}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)\right) \cdot (\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2)$$

The Kinetic part (omitting the m/2 factor), in terms of the CM and relative vectors is

$$\begin{aligned}
(\mathbf{v}_1)^2 + (\mathbf{v}_2)^2 + (\mathbf{v}_3)^2 &= \left(\frac{2}{3}\dot{\boldsymbol{\rho}}_1 - \frac{1}{2}\dot{\boldsymbol{\rho}}_2 + \dot{\mathbf{R}}\right)^2 + \left(\frac{2}{3}\dot{\boldsymbol{\rho}}_2 - \frac{1}{2}\dot{\boldsymbol{\rho}}_1 + \dot{\mathbf{R}}\right)^2 + \left(\dot{\mathbf{R}} - \frac{1}{3}(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2)\right)^2 \\
&= 3\dot{\mathbf{R}}^2 + (5/9 + 1/4)((\dot{\boldsymbol{\rho}}_1)^2 + (\dot{\boldsymbol{\rho}}_2)^2) \\
&+ 2(-2/3 + 1/9)\dot{\boldsymbol{\rho}}_1 \cdot \dot{\boldsymbol{\rho}}_1 + 2(1/3 - 1/2)(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2) \cdot \dot{\mathbf{R}}
\end{aligned}$$

So the Kinetic part of the Lagrangian is

$$\mathcal{L}_K = \frac{3m}{2}\dot{\mathbf{R}}^2 + \frac{29m}{72}((\dot{\boldsymbol{\rho}}_1)^2 + (\dot{\boldsymbol{\rho}}_2)^2) - \frac{5m}{9}\dot{\boldsymbol{\rho}}_1 \cdot \dot{\boldsymbol{\rho}}_2 - \frac{m}{6}(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2) \cdot \dot{\mathbf{R}}$$

and finally, the total Lagrangian is

$$\mathcal{L} = \frac{3m}{2}\dot{\mathbf{R}}^2 + \frac{29m}{72}((\dot{\boldsymbol{\rho}}_1)^2 + (\dot{\boldsymbol{\rho}}_2)^2) - \frac{5m}{9}\dot{\boldsymbol{\rho}}_1 \cdot \dot{\boldsymbol{\rho}}_2 - \frac{m}{6}(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2) \cdot \dot{\mathbf{R}}$$
$$+ g\left(e^{-\mu|\boldsymbol{\rho}_1|} + e^{-\mu|\boldsymbol{\rho}_2|} + e^{-\mu|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|}\right) + f\left(\mathbf{R} - \frac{1}{3}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)\right) \cdot (\dot{\boldsymbol{\rho}_1} + \dot{\boldsymbol{\rho}_2})$$

3.3 Angular momentum conservation?

How about the angular momentum conservation question? How to answer that? One way would be to compute the forces from the Lagrangian, and take cross products but is that really the best way? Perhaps the answer is as simple as observing that there are no external torque's on the system, thus $d\mathbf{L}/dt = 0$, or angular momentum for the system is constant (conserved). Is that actually the case with these velocity dependent potentials?

It was suggested to me on PF that I should look at how this Lagrangian transforms under rotation, and use Noether's theorem. The goldstein book doesn't explicitly mention this theorem that I can see, and I don't think it was covered yet if it did.

Suppose we did know about Noether's theorem for this problem (as I know do now that I'm revisiting it), we'd have to see if the Lagrangian is invariant under rotation. Suppose that a rigid rotation is introduced, which we can write in GA formalism using dual sided quaternion products

$$\mathbf{x} \rightarrow \mathbf{x}' = e^{-i\hat{\mathbf{n}}\alpha/2}\mathbf{x}e^{i\hat{\mathbf{n}}\alpha/2}$$

(could probably also use a matrix formulation, but the parameterization is messier).

For all the relative vectors $\boldsymbol{\rho}_k$ we have

$$|\boldsymbol{\rho}_k'| = |\boldsymbol{\rho}_k|$$

So all the V potential interactions are invariant.

Since the rotation quaternion here is a fixed non-time dependent quantity we have

$$\dot{\boldsymbol{\rho}}_k' = e^{-i\hat{\mathbf{n}}\alpha/2}\dot{\boldsymbol{\rho}}_k e^{i\hat{\mathbf{n}}\alpha/2}$$

,so for the dot product in the the remaining potential term we have

$$\begin{aligned}
\left(\mathbf{R}' - \frac{1}{3}\left(\boldsymbol{\rho}_1' + \boldsymbol{\rho}_2'\right)\right) \cdot \left(\dot{\boldsymbol{\rho}}_1' + \dot{\boldsymbol{\rho}}_2'\right) &= \left(e^{-i\hat{\mathbf{n}}\alpha/2}\left(\mathbf{R} - \frac{1}{3}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right)\right) e^{i\hat{\mathbf{n}}\alpha/2}\right) \cdot \left(e^{-i\hat{\mathbf{n}}\alpha/2}\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2 e^{i\hat{\mathbf{n}}\alpha/2}\right) \\
&= \left\langle e^{-i\hat{\mathbf{n}}\alpha/2}\left(\mathbf{R} - \frac{1}{3}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right)\right) e^{i\hat{\mathbf{n}}\alpha/2} e^{-i\hat{\mathbf{n}}\alpha/2}\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2 e^{i\hat{\mathbf{n}}\alpha/2}\right\rangle \\
&= \left\langle e^{-i\hat{\mathbf{n}}\alpha/2}\left(\mathbf{R} - \frac{1}{3}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right)\right)\left(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2\right) e^{i\hat{\mathbf{n}}\alpha/2}\right\rangle \\
&= \left\langle e^{i\hat{\mathbf{n}}\alpha/2} e^{-i\hat{\mathbf{n}}\alpha/2}\left(\mathbf{R} - \frac{1}{3}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right)\right)\left(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2\right)\right\rangle \\
&= \left\langle \left(\mathbf{R} - \frac{1}{3}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right)\right)\left(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2\right)\right\rangle \\
&= \left(\mathbf{R} - \frac{1}{3}\left(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2\right)\right) \cdot \left(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2\right)
\end{aligned}$$

So, presuming I interpretted the $\mathbf{r}$ in $\mathbf{v} \cdot \mathbf{r}$ correctly, all the vector quantites in the Lagrangian are rotation invariant, and by Noether's we should have system angular momentum conservation.

3.4 Application of Noether's

Invoking Noether's here seems like cheating, at least without computing the conserved current, so let's do this.

To make this easier, suppose we generalize the Lagrangian slightly to get rid of all the peculiar and specific numerical constants. Let $\rho_3 = \mathbf{R}$, then our Lagrangian has the functional form

$$\mathcal{L} = \alpha^{ij}\dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_j + g^i e^{-\mu|\boldsymbol{\rho}_i|} + g^{ij} e^{-\mu\left|\boldsymbol{\rho}_i - \boldsymbol{\rho}_j\right|} + f^i \boldsymbol{\rho}_i \cdot \left(\dot{\boldsymbol{\rho}}_1 + \dot{\boldsymbol{\rho}}_2\right)$$

We can then pick specific α^{ij}, f^i, and g^{ij} (not all non-zero), to match the Lagrangian of this problem. This could be expanded in terms of coordinates, producing nine generalized coordinates and nine corresponding velocity terms, but since our Lagrangian transformation is so naturally expressed in vector form this doens't seem like a reasonable thing to do.

Let's step up the abstraction one more level instead and treat the Noether symmetry in the more general case, supposing that we have a Lagrangian that is invariant under the same rotational transformation applied above, but has the following general form with explicit vector parameterization, where as above, all our vectors come in functions of the dot products (either explicit or implied by absolute values) of our vectors or their time derivatives

$$\mathcal{L} = f(\mathbf{x}_k \cdot \mathbf{x}_j, \mathbf{x}_k \cdot \dot{\mathbf{x}}_j, \dot{\mathbf{x}}_k \cdot \dot{\mathbf{x}}_j)$$

Having all the parameterization being functions of dot products gives the desired rotational symmetry for the Lagrangian. This must be however, not a dot product with an arbitrary vector, but one of the generalized vector parameters of the Lagrangian. Something like the $\mathbf{A} \cdot \mathbf{v}$ term in the Lorentz force Lagrangian doesn't have this invariance since $\mathbf{A}$ doens't transform along with $\mathbf{v}$. Also Note that the absolute values of the $\boldsymbol{\rho}_k$ vectors are functions of dot products.

Now we are in shape to compute the conserved "current" for a rotational symmetry. Our vectors and their derivatives are explicitly rotated

$$
\begin{aligned}
\mathbf{x}'_k &= e^{-i\hat{\mathbf{n}}\alpha/2}\mathbf{x}_k e^{i\hat{\mathbf{n}}\alpha/2} \\
\dot{\mathbf{x}}'_k &= e^{-i\hat{\mathbf{n}}\alpha/2}\dot{\mathbf{x}}_k e^{i\hat{\mathbf{n}}\alpha/2}
\end{aligned}
$$

and our Lagrangian is assumed, as above with all vectors coming in dot product pairs, to have rotational invariance when all the vectors in the system are rotated

$$
\mathcal{L} \rightarrow \mathcal{L}'(\mathbf{x}'_k, \dot{\mathbf{x}}'_j) = \mathcal{L}(\mathbf{x}_k, \dot{\mathbf{x}}_j)
$$

The essence of Noether's theorem was applied chain rule, looking at how the transformed Lagrangian changes with respect to the transformation. In this case we want to calculate

$$
\left.\frac{d\mathcal{L}'}{d\alpha}\right|_{\alpha=0}
$$

First seeing the Noether's derivation, I didn't understand why the evaluation at $\alpha = 0$ was required, even after doing this derivation for myself in [Joot(a)] (after an initial botched attempt), but the reason for it actually became clear with this application, as writing it up will show.

Anyways, back to the derivative. One way to evaluate this would be in terms of coordinates, writing $\mathbf{x}'_k = \mathbf{e}^m x'_{km}$,

$$
\frac{d\mathcal{L}'}{d\alpha}(\mathbf{x}'_k, \dot{\mathbf{x}}'_j) = \sum_{k,m} \frac{\partial \mathcal{L}'}{\partial x'_{km}} \frac{\partial x'_{km}}{\partial \alpha} + \frac{\partial \mathcal{L}'}{\partial \dot{x}'_{km}} \frac{\partial \dot{x}'_{km}}{\partial \alpha}
$$

This is a bit of a mess however, and begs for some shorthand. Let's write

$$\nabla_{\mathbf{x}'_k} \mathcal{L}' = e^m \frac{\partial \mathcal{L}'}{\partial x'_{km}}$$
$$\nabla_{\dot{\mathbf{x}}'_k} \mathcal{L}' = e^m \frac{\partial \mathcal{L}'}{\partial \dot{x}'_{km}}$$

Then the chain rule application above becomes

$$\frac{d\mathcal{L}'}{d\alpha}(\mathbf{x}'_k, \dot{\mathbf{x}}'_j) = \sum_k \left(\nabla_{\mathbf{x}'_k} \mathcal{L}' \right) \cdot \frac{\partial \mathbf{x}'_k}{\partial \alpha} + \left(\nabla_{\dot{\mathbf{x}}'_k} \mathcal{L}' \right) \cdot \frac{\partial \dot{\mathbf{x}}'_k}{\partial \alpha}$$

Now, while this notational sugar unfortunately has an obscuring effect, it is also practical since we can now work with the transformed position and velocity vectors directly

$$\begin{aligned}
\frac{\partial \mathbf{x}'_k}{\partial \alpha} &= (-i\hat{\mathbf{n}}/2) e^{-i\hat{\mathbf{n}}\alpha/2} \mathbf{x}_k e^{i\hat{\mathbf{n}}\alpha/2} + e^{-i\hat{\mathbf{n}}\alpha/2} \mathbf{x}_k e^{i\hat{\mathbf{n}}\alpha/2} (i\hat{\mathbf{n}}/2) \\
&= (-i\hat{\mathbf{n}}/2)\mathbf{x}'_k + \mathbf{x}'_k (i\hat{\mathbf{n}}/2) \\
&= i(\hat{\mathbf{n}} \wedge \mathbf{x}'_k)
\end{aligned}$$

So we have

$$\frac{d\mathcal{L}'}{d\alpha}(\mathbf{x}'_k, \dot{\mathbf{x}}'_j) = \sum_k \left(\nabla_{\mathbf{x}'_k} \mathcal{L}' \right) \cdot \left(i(\hat{\mathbf{n}} \wedge \mathbf{x}'_k) \right) + \sum_k \left(\nabla_{\dot{\mathbf{x}}'_k} \mathcal{L}' \right) \cdot \left(i(\hat{\mathbf{n}} \wedge \dot{\mathbf{x}}'_k) \right)$$

Next step is to reintroduce the notational sugar noting that we can vectorize the Euler-Lagrange equations by writing

$$\nabla_{\mathbf{x}_k} \mathcal{L} = \frac{d}{dt} \nabla_{\dot{\mathbf{x}}_k} \mathcal{L}$$

We have now a three fold reduction in the number of Euler-Lagrange equations. For each of the generalized vector parameters, we have the Lagrangian gradient with respect to that vector parameter (a generalized force) equals the time rate of change of the velocity gradient.

Inserting this we have

$$\frac{d\mathcal{L}'}{d\alpha}(\mathbf{x}'_k, \dot{\mathbf{x}}'_j) = \sum_k \left(\frac{d}{dt} \nabla_{\dot{\mathbf{x}}'_k} \mathcal{L}' \right) \cdot \left(i(\hat{\mathbf{n}} \wedge \mathbf{x}'_k) \right) + \sum_k \left(\nabla_{\dot{\mathbf{x}}'_k} \mathcal{L}' \right) \cdot \left(i(\hat{\mathbf{n}} \wedge \dot{\mathbf{x}}'_k) \right)$$

Now we can drop the primes in gradient terms because of the Lagrangian invariance for this symmetry, and are left almost with a perfect differential

$$\frac{d\mathcal{L}'}{d\alpha}(\mathbf{x}'_k, \dot{\mathbf{x}}'_j) = \sum_k \left(\frac{d}{dt}\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) \cdot \left(i(\hat{\mathbf{n}} \wedge \mathbf{x}'_k)\right) + \sum_k \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) \cdot \left(i(\hat{\mathbf{n}} \wedge \dot{\mathbf{x}}'_k)\right)$$

Here's where the evaluation at $\alpha = 0$ comes in, since $\mathbf{x}'_k(\alpha = 0) = \mathbf{x}_k$, and we can now antidifferentiate

$$\begin{aligned}
\left.\frac{d\mathcal{L}'}{d\alpha}(\mathbf{x}'_k, \dot{\mathbf{x}}'_j)\right|_{\alpha=0} &= \sum_k \left(\frac{d}{dt}\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) \cdot \left(i(\hat{\mathbf{n}} \wedge \mathbf{x}_k)\right) + \sum_k \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) \cdot \left(i(\hat{\mathbf{n}} \wedge \dot{\mathbf{x}}_k)\right) \\
&= \sum_k \frac{d}{dt}\left(\left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) \cdot \left(i(\hat{\mathbf{n}} \wedge \mathbf{x}_k)\right)\right) \\
&= \sum_k \frac{d}{dt}\left\langle\left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) i(\hat{\mathbf{n}} \wedge \mathbf{x}_k)\right\rangle \\
&= \sum_k \frac{d}{dt}\frac{1}{2}\left\langle\left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) i(\hat{\mathbf{n}}\mathbf{x}_k - \mathbf{x}_k\hat{\mathbf{n}})\right\rangle \\
&= \sum_k \frac{d}{dt}\frac{1}{2}\left\langle\hat{\mathbf{n}} i\left(\mathbf{x}_k\left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) - \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right)\mathbf{x}_k\right)\right\rangle \\
&= \sum_k \frac{d}{dt}\frac{1}{2}\left\langle\hat{\mathbf{n}} i\left(\mathbf{x}_k\left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right) - \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right)\mathbf{x}_k\right)\right\rangle \\
&= \sum_k \frac{d}{dt}\left\langle\hat{\mathbf{n}} i\left(\mathbf{x}_k \wedge \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right)\right)\right\rangle \\
&= \sum_k \frac{d}{dt}\left\langle\hat{\mathbf{n}} i^2\left(\mathbf{x}_k \times \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right)\right)\right\rangle \\
&= \sum_k \frac{d}{dt} - \hat{\mathbf{n}} \cdot \left(\mathbf{x}_k \times \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right)\right)
\end{aligned}$$

Because of the symmetry this entire derivative is zero, so we have

$$\hat{\mathbf{n}} \cdot \sum_k \left(\mathbf{x}_k \times \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}\right)\right) = \text{constant}$$

The Lagrangian velocity gradient can be identified as the momentum (ie: the canonical momentum conjugate to $\mathbf{x}_k$)

$$\mathbf{p}_k \equiv \boldsymbol{\nabla}_{\dot{\mathbf{x}}_k}\mathcal{L}$$

Also noting that this is constant for any $\hat{\mathbf{n}}$, we finally have the conserved "current" for a rotational symmetry of a system of particles

$$\sum_k \mathbf{x}_k \times \mathbf{p}_k = \text{constant}$$

This digression to Noether's appears to be well worth it for answering the angular momentum question of the problem. Glibly saying "yes angular momentum is conserved", just because the Lagrangian has a rotational symmetry is not enough. We've seen in this particular problem that the Lagrangian, having only dot products has the rotational symmetry, but because of the velocity dependent potential terms $f^i \dot{\boldsymbol{\rho}}_k \cdot \dot{\boldsymbol{\rho}}_j$, the normal Kinetic energy momentum vectors are not equal to the canonican congugate momentum vectors. Only when the angular momentum is generalized, and written in terms of the canonical conguate momentum is the total system angular momentum conserved. Namely, the generalized angular momentum for this problem is conserved

$$\sum_k \mathbf{x}_k \times \left(\boldsymbol{\nabla}_{\dot{\mathbf{x}}_k} \mathcal{L}\right) = \text{constant}$$

but the "traditional" angular momentum $\sum_k \mathbf{x}_k \times m\dot{\mathbf{x}}_k$, is not.

4 Problem 2.1

Prove that the shortest length curve between two points in space is a straight line.

A first attempt of this I used:

$$ds = \sqrt{1 + (dy/dx)^2 + (dz/dx)^2} dx$$

Application of the Euler-Lagrange equations does show that one ends up with a linear relation between the y and z coordinates, but no mention of x. Rather than write that up, consider instead a parameterization of the coordinates:

$$\begin{aligned}
x &= x_1(\lambda) \\
y &= x_2(\lambda) \\
z &= x_3(\lambda)
\end{aligned}$$

in terms of this arbitrary parameterization we have a segment length of:

$$ds = \sqrt{\sum \left(\frac{dx_i}{d\lambda}\right)^2} d\lambda = f\left(x_i\right) d\lambda$$

Application of the Euler-Lagrange equation to f we have:

$$
\begin{aligned}
\frac{\partial f}{\partial x_i} &= 0 \\
&= \frac{d}{d\lambda}\frac{\partial}{\partial \dot{x}_i}\sqrt{\sum \dot{x}_j^2} \\
&= \frac{d}{d\lambda}\frac{\dot{x}_i}{\sqrt{\sum \dot{x}_j^2}}
\end{aligned}
$$

Therefore each of these quotients can be equated to a constant:

$$
\begin{aligned}
\frac{\dot{x}_i}{\sqrt{\sum \dot{x}_j^2}} &= {c_i}^{-2} \\
{c_i}^2 \dot{x}_i^2 &= \sum \dot{x}_j^2 \\
({c_i}^2 - 1)\dot{x}_i^2 &= \sum_{j \neq i} \dot{x}_j^2 \\
(1 - {c_i}^2)\dot{x}_i^2 + \sum_{j \neq i} \dot{x}_j^2 &= 0
\end{aligned}
$$

This last form shows explicitly that not all of these squared derivative terms can be linearly independent. In particular, we have a zero determinant:

$$
0 = \begin{vmatrix}
1 - c_1^2 & 1 & 1 & 1 & \dots \\
1 & 1 - c_2^2 & 1 & 1 & \vdots \\
1 & 1 & 1 - c_3^2 & 1 & \\
 & & & \ddots & \\
 & & & & 1 - {c_n}^2
\end{vmatrix}
$$

Now, expanding this for a couple specific cases isn't too hard. For $n = 2$ we have:

$$
\begin{aligned}
0 &= (1 - c_1^2)(1 - c_2^2) - 1 \\
c_1^2 + c_2^2 &= c_1^2 c_2^2 \\
c_1^2 &= \frac{c_2^2}{c_2^2 - 1} \\
c_2^2 - 1 &= \frac{c_2^2}{c_1^2}
\end{aligned}
$$

This can be substuited back into one our c_2^2 equation:

$$
\begin{aligned}
({c_2}^2 - 1)\dot{x}_2^2 &= \dot{x}_1^2 \\
\frac{c_2^2}{c_1^2}\dot{x}_2^2 &= \dot{x}_1^2 \\
\pm\frac{c_2}{c_1}\dot{x}_2 &= \dot{x}_1 \\
\pm\frac{c_2}{c_1}x_2 &= x_1 + \kappa
\end{aligned}
$$

This is precisely the straight line that was desired, but we have setup for proving that consideration of all path variations from two points in $\mathbb{R}^N$ space has the shortest distance when that path is a straight line.

Despite the general setup, I'm going to chicken out and show this only for the $\mathbb{R}^3$ case. In that case our determinant expands to:

$$
c_1^2 + c_2^2 + c_3^2 = c_1^2 c_2^2 c_3^2
$$

Since not all of the $\dot{x}_i^2$ can be linearly independent, one can be eliminated:

$$
\begin{aligned}
(1 - c_1^2)\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 &= 0 \\
(1 - c_2^2)\dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_1^2 &= 0 \\
(1 - c_3^2)\dot{x}_3^2 + \dot{x}_1^2 + \dot{x}_2^2 &= 0
\end{aligned}
$$

Let's pick $\dot{x}_1^2$ to eliminate, and subst 2 into 3:

$$
\begin{aligned}
(1 - c_3^2)\dot{x}_3^2 + (-(1 - c_2^2)\dot{x}_2^2 - \dot{x}_3^2) + \dot{x}_2^2 &= 0 \implies \\
-c_3^2\dot{x}_3^2 + c_2^2\dot{x}_2 &= 0 \\
\pm c_3\dot{x}_3 &= c_2\dot{x}_2
\end{aligned}
$$

Since these equations are symmetric, we can do this for all, with the result:

$$
\begin{aligned}
\pm c_3\dot{x}_3 &= c_2\dot{x}_2 \\
\pm c_3\dot{x}_3 &= c_1\dot{x}_1 \\
\pm c_2\dot{x}_2 &= c_1\dot{x}_1
\end{aligned}
$$

Since the c_i constants are arbitrary, then we can for example pick the negative sign for $\pm c_2$, and the positive for the rest, then add all of these, and scale by two:

$$c_3\dot{x}_3 - c_2\dot{x}_2 = c_1\dot{x}_1$$

and integrating:

$$c_3 x_3 - c_2 x_2 = c_1 x_1 + \kappa$$

Again, we have the general equation of a line, subject to the desired constraints on the end points. In the end we didn't need to evaluate the determinant after all, as done in the $\mathbb{R}^2$ case.

5 Problem 2.2

Prove that the geodesics (shortest length paths) on a spherical surface are great circles.

As a variational problem, the first step is to formulate an element of length on the surface. If we write our vector in spherical coordinates (ϕ on the equator, and θ measuring from the north pole) we have:

FIXME: Scan picture.

$$\mathbf{r} = (x, y, z) = R(\sin\theta cos\phi, \sin\theta\sin\phi, \cos\theta)$$

A differential vector element on the surface is (set $R = 1$ without loss of generality) :

$$\begin{aligned}
d\mathbf{r} &= \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{d\lambda}d\lambda + \frac{d\mathbf{r}}{d\phi}\frac{d\phi}{d\lambda}d\lambda \\
&= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\dot{\theta}d\lambda + (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)\dot{\phi}d\lambda \\
&= (\cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi}, \cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi}, -\sin\theta\dot{\theta})d\lambda
\end{aligned}$$

Calculation of the length ds of this vector yields:

$$ds = |d\mathbf{r}| = \sqrt{\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2}d\lambda$$

This completes the setup for the minimization problem, and we want to minimize:

$$s = \int \sqrt{\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2}d\lambda$$

and can therefore apply the Euler-Lagrange equations to the function

$$f(\theta, \phi, \dot{\theta}, \dot{\phi}, \lambda) = \sqrt{\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2}$$

The ϕ is cyclic, and we have:

$$\frac{\partial f}{\partial\phi} = 0 = \frac{d}{d\lambda}\frac{\dot{\phi}\sin^2\theta}{f}$$

Thus we have:

$$\begin{aligned}
\dot{\phi}^2 \sin^4 \theta &= K^2 \left(\dot{\theta}^2 + (\dot{\phi} \sin \theta)^2 \right) \\
\dot{\phi}^2 \sin^2 \theta (\sin^2 \theta - K^2) &= K^2 \dot{\theta}^2 \\
\dot{\phi}^2 &= \frac{K^2 \dot{\theta}^2}{\sin^2 \theta (\sin^2 \theta - K^2)} \\
\dot{\phi} &= \frac{K \dot{\theta}}{\sin \theta \sqrt{\sin^2 \theta - K^2}}
\end{aligned}$$

This is in a nicely separated form, but it is not obvious that this describes paths that are great circles.

Let's have a look at the second equation.

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{d}{d\lambda} \frac{\partial f}{\partial \dot{\theta}} \\
\frac{\sin \theta \cos \theta \dot{\phi}^2}{f} &= \frac{d}{d\lambda} \frac{\dot{\theta}}{f} \\
&= \frac{\ddot{\theta}}{f} - \frac{1}{2} \frac{(\dot{\theta}^2 + (\dot{\phi} \sin \theta)^2)'}{f^3} \\
&= \frac{\ddot{\theta}}{f} - \frac{\dot{\theta}\ddot{\theta} + \dot{\phi} \sin \theta (\ddot{\phi} \sin \theta + \dot{\phi} \cos \theta \dot{\theta})}{f^3} \\
\implies - \sin \theta \cos \theta \dot{\phi}^2 (\dot{\theta}^2 + (\dot{\phi} \sin \theta)^2) &= -\ddot{\theta} (\dot{\theta}^2 + (\dot{\phi} \sin \theta)^2) + \dot{\theta}\ddot{\theta} + \dot{\phi} \sin \theta (\ddot{\phi} \sin \theta + \dot{\phi} \cos \theta \dot{\theta})
\end{aligned}$$

Or,

$$-\ddot{\theta}\dot{\theta}^2 - \ddot{\theta}\dot{\phi}^2 \sin^2 \theta + \dot{\theta}\ddot{\theta} + \dot{\phi}\ddot{\phi} \sin^2 \theta + \dot{\phi}^2 \dot{\theta} \sin \theta \cos \theta + \dot{\phi}^2 \dot{\theta}^2 \sin \theta \cos \theta + \dot{\phi}^4 \sin^3 \theta \cos \theta = 0$$

What a mess! I don't feel inclined to try to reduce this at the moment. I'll come back to this problem later. Perhaps there's a better parameterization?

Did come back to this later, in [Joot(c)], but still didn't get the problem fully solved. Maybe the third time, some time later, will be the charm.

6 Problem 2.3

For $f = f(y, \dot{y}, \ddot{y}, x)$, find the equations for extreme values of

$$I = \int_a^b f dx$$

Here we want y and $\dot{y}$ fixed at the end points. Following the first derivative derivation write the functions in terms of the desired extremum functions plus a set of arbitrary functions:

$$y(x,\alpha) = y(x,0) + \alpha n(x)$$
$$\dot{y}(x,\alpha) = \dot{y}(x,0) + \alpha m(x)$$

Here we specify that these arbitrary variational functions vanish at the endpoints:

$$n(a) = n(b) = m(a) = m(b) = 0$$

The functions $y(x,0)$, and $\dot{y}(x,0)$ are the functions we are looking for as solutions to the min/max problem.

Calculating derivatives we have:

$$\frac{dI}{d\alpha} = \int \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha} + \frac{\partial f}{\partial \ddot{y}}\frac{\partial \ddot{y}}{\partial \alpha} \right) dx$$

Assuming sufficient continuity look at the second term where we have:

$$\begin{aligned}
\frac{\partial \dot{y}}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{\partial y}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial y}{\partial \alpha} \\
&= \frac{\partial}{\partial x} n(x) \\
&= \frac{d}{dx} n(x) \\
&= \frac{d}{dx}\frac{\partial y}{\partial \alpha}
\end{aligned}$$

Similarily for the third term we have:

$$\frac{\partial \ddot{y}}{\partial \alpha} = \frac{d}{dx}\frac{\partial \dot{y}}{\partial \alpha}$$

$$\frac{dI}{d\alpha} = \int \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \underbrace{\frac{\partial f}{\partial \dot{y}}\frac{d}{dx}\frac{\partial y}{\partial \alpha}}_{uv' = (uv)' - u'v} dx + \frac{\partial f}{\partial \ddot{y}}\frac{d}{dx}\frac{\partial \dot{y}}{\partial \alpha} dx$$

Now integrating by parts:

$$\frac{dI}{d\alpha} = \int \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \int \frac{\partial f}{\partial \dot{y}}\frac{d}{dx}\frac{\partial y}{\partial \alpha} dx + \int \frac{\partial f}{\partial \ddot{y}}\frac{d}{dx}\frac{\partial \dot{y}}{\partial \alpha} dx$$

$$\frac{dI}{d\alpha} = \int \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} dx + \left(\frac{\partial f}{\partial \dot{y}} \underbrace{\frac{\partial y}{\partial \alpha}}_{n(x)} \right)_a^b - \int \frac{\partial y}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} dx + \left(\frac{\partial f}{\partial \ddot{y}} \underbrace{\frac{\partial \dot{y}}{\partial \alpha}}_{m(x)} \right)_a^b - \int \frac{\partial \dot{y}}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \ddot{y}} dx$$

Because $m(a) = m(b) = n(a) = n(b)$ the non-integral terms are all zero, leaving:

$$\frac{dI}{d\alpha} = \int \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}dx - \int \frac{\partial y}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}dx - \int \frac{\partial \dot{y}}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \ddot{y}}dx$$

Now consider just this last integral, which we can again integrate by parts:

$$\begin{aligned}
\int \frac{\partial \dot{y}}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \ddot{y}}dx &= \int \underbrace{\frac{d}{dx}\frac{\partial y}{\partial \alpha}}_{u'}\underbrace{\frac{d}{dx}\frac{\partial f}{\partial \ddot{y}}}_{v}dx \\
&= \left(\underbrace{\frac{\partial y}{\partial \alpha}}_{n(x)}\frac{d}{dx}\frac{\partial f}{\partial \ddot{y}}\right)\Bigg|_a^b - \int \frac{\partial y}{\partial \alpha}\frac{d}{dx}\frac{d}{dx}\frac{\partial f}{\partial \ddot{y}}dx \\
&= -\int \frac{\partial y}{\partial \alpha}\frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{y}}dx
\end{aligned}$$

This gives:

$$\begin{aligned}
\frac{dI}{d\alpha} &= \int \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}dx - \int \frac{\partial y}{\partial \alpha}\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}dx + \int \frac{\partial y}{\partial \alpha}\frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{y}}dx \\
\frac{dI}{d\alpha} &= \int dx\frac{\partial y}{\partial \alpha}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{y}}\right) \\
&= \int dx n(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{y}}\right)
\end{aligned}$$

So, if we want this derivative to equal zero for any $n(x)$ we require the inner quantity to by zero:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{y}} = 0 \tag{1}$$

Question. Goldstein writes this in total differential form instead of a derivative:

$$\begin{aligned}
dI &= \frac{dI}{d\alpha}d\alpha \\
&= \int dx\left(\frac{\partial y}{\partial \alpha}d\alpha\right)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} + \frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{y}}\right)
\end{aligned}$$

and then calls this quantity $\frac{\partial y}{\partial \alpha} d\alpha = \delta y$, the variation of y. There must be a mathematical subtlety which motivates this but it isn't clear to me what that is. Since the variational calculus texts go a different route, with norms on functional spaces and so forth, perhaps understanding that motivation isn't worthwhile. In the end, the conclusion is the same, namely that the inner expression must equal zero for the extremum condition.

References

[Goldstein(1951)] H. Goldstein. Classical mechanics. 1951.

[Joot(a)] Peeter Joot. Euler lagrange equations. "http://sites.google.com/site/peeterjoot/geometric-algebra/euler_lagrange.pdf", a.

[Joot(b)] Peeter Joot. Solutions to lagrangian problem set for david tongs mechanics. "http://sites.google.com/site/peeterjoot/geometric-algebra/tong_mf1.pdf", b.

[Joot(c)] Peeter Joot. Worked calculus of variations problems from byron and fuller. "http://sites.google.com/site/peeterjoot/math2009/byron_fuller_calc_var.pdf", c.