# Taylor's theorem derviation.

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Feb. 2, 2008. Last Revision: $Date: 2008/10/1401:13:59$

[Hestenes(1999)] presents a very simple derivation of Taylor's Theorem, but I feel a slightly different (dumber, but longer) presentation would be more effective.

In the same fashion, form the integral

$$I = \int_t^{t+s} F'(u)du$$

Now, observe that the this first order derviative can be written in terms of it's second order derivative

$$(uF'(u))' = F'(u) + uF''(u)$$

So we could write

$$\begin{aligned}
I &= \int_t^{t+s} ((uF'(u))' - uF''(u))du \\
&= uF'(u)|_{u=t}^{t+s} - \int_t^{t+s} uF''(u))du \\
&= (t+s)F'(t+s) - tF'(t) - \int_t^{t+s} uF''(u))du
\end{aligned}$$

This is true, but not the Taylor expansion we are used to. Adjusting things slightly leaves a zero term at $u = t + s$, as follows:

$$\big((t+s-u)F'(u)\big)' = -F'(u) + (t+s-u)F''(u)$$

$$
\begin{aligned}
I &= F(t+s) - F(t) \\
&= \int_t^{t+s} (-((t+s-u)F'(u))' + (t+s-u)F''(u))du \\
&= -(t+s-u)F'(u)|_{u=t}^{t+s} + \int_t^{t+s} (t+s-u)F''(u)du \\
&= sF'(t) + \int_t^{t+s} (t+s-u)F''(u)du
\end{aligned}
$$

This results in the first two order terms of the Tailor series, with an explicit remainder term

$$
F(t+s) = F(t) + sF'(t) + \int_t^{t+s} (t+s-u)F''(u)du
$$

This process can be continued for as many terms as desired. Doing the calculation for the second order term yields

$$
\left( \frac{(t+s-u)^2}{2} F''(u) \right)' = -(t+s-u)F''(u) + \frac{(t+s-u)}{2} F'''(u)
$$

And substituting that into the first order expansion above we have

$$
F(t+s) = F(t) + sF'(t) + \frac{s^2}{2} F''(t) + \int_t^{t+s} \frac{1}{2} (t+s-u)^2 F'''(u)du
$$

Induction produces n terms of Taylor's series with an explicit remainder

$$
F(t+s) = \sum_{k=0}^{n} \frac{s^k}{k!} \frac{d^k}{dt^k} F(t) + \int_t^{t+s} \frac{1}{n!} (t+s-u)^n \frac{d^{n+1}}{dt^{n+1}} F(u)du
$$

To truely prove the infinite series result one would have to show that the remainder term tends to zero.

# References


[Hestenes(1999)] D. Hestenes. *New Foundations for Classical Mechanics*. Kluwer Academic Publishers, 1999.