Graphical representation of the associated Legendre Polynomials for $l = 1$

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Peeter Joot — peeter.joot@gmail.com

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1. First observations.

In Bohm's QT ([1], 14.17), the properties of $l = 1$ associated Legendre polynomials are examined under rotation.

Those eigenfunctions are the normalized versions of following

$$\psi_1 = \sin\theta e^{i\phi} \tag{1}$$
$$\psi_0 = \cos\theta \tag{2}$$
$$\psi_{-1} = \sin\theta e^{-i\phi} \tag{3}$$

The normalization is provided by a surface area inner product

$$(u, v) = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} uv^* \sin\theta d\theta d\phi \tag{4}$$

With the normalization discarded, there is a direct relationship between these normal eigenfunctions with a triple of vectors associated with a point on the unit sphere. Referring to figure (1), observe the three doubled arrow vectors, all associated with a point on the unit sphere $\mathbf{x} = (x, y, z) = (\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta)$.

The normal to the x, y plane from $\mathbf{x}$, designated $\mathbf{n}$ has the vectorial value

$$\mathbf{n} = \cos\theta\mathbf{e}_3 \tag{5}$$

From the origin to the point of of the x, y plane intersection to the normal we have

$$\boldsymbol{\rho} = \sin\theta(\cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2) = \mathbf{e}_1 \sin\theta e^{\mathbf{e}_1\mathbf{e}_2\phi} \tag{6}$$

and finally in the opposite direction also in the plane and mirroring $\boldsymbol{\rho}$ we have the last of this triplet of vectors

$$\boldsymbol{\rho}_- = \sin\theta(\cos\phi\mathbf{e}_1 - \sin\phi\mathbf{e}_2) = \mathbf{e}_1 \sin\theta e^{-\mathbf{e}_1\mathbf{e}_2\phi} \tag{7}$$

So, if we choose to use $i = \mathbf{e}_1\mathbf{e}_2$ (the bivector for the plane normal to the z-axis), then we can in fact vectorize these eigenfunctions. The vectors $\boldsymbol{\rho}$ (i.e. ψ_1), and $\boldsymbol{\rho}_-$ (i.e. ψ_{-1}) are both

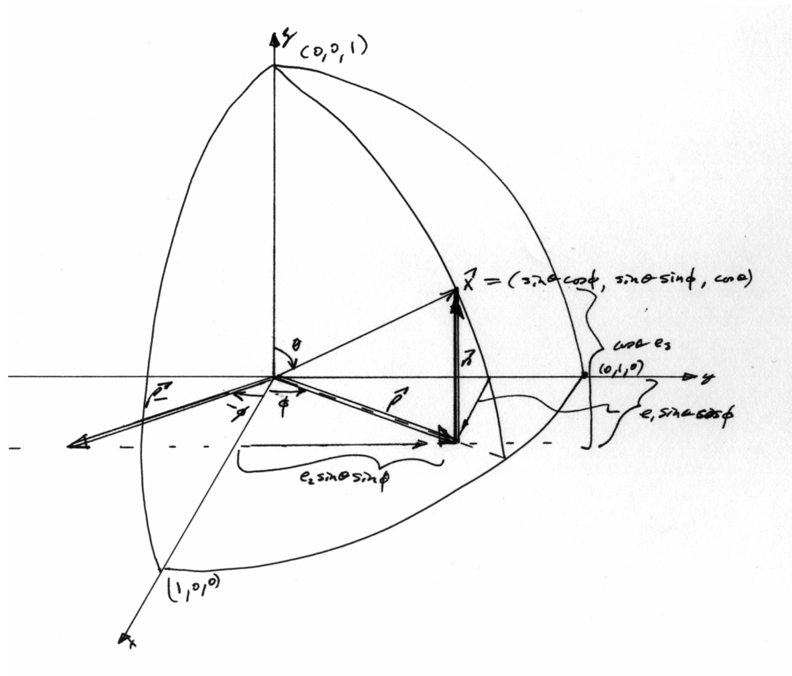

Figure 1: Vectoring the $l = 1$ associated Legendre polynomials.

normal to $\mathbf{n}$ (i.e. ψ_0), but while the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_-$ are both in the plane one is produced with a counterclockwise rotation of $\mathbf{e}_1$ by ϕ in the plane and the other with an opposing rotation.

Summarizing, we can write the unnormalized vectors the relations

$$
\begin{aligned}
\psi_1 &= \mathbf{e}_1\boldsymbol{\rho} &&= \sin\theta e^{\mathbf{e}_1\mathbf{e}_2\phi} \\
\psi_0 &= \mathbf{e}_3\mathbf{n} &&= \cos\theta \\
\psi_{-1} &= \mathbf{e}_1\boldsymbol{\rho}_- &&= \sin\theta e^{-\mathbf{e}_1\mathbf{e}_2\phi}
\end{aligned}
$$

I have no familiarity yet with the $l = 2$ or higher Legendre eigenfunctions. Do they also admit a geometric representation?

2. Expressing Legendre eigenfunctions using rotations.

We can express a point on a sphere with a pair of rotation operators. First rotating $\mathbf{e}_3$ towards $\mathbf{e}_1$ in the z, x plane by θ, then in the x, y plane by ϕ we have the point $\mathbf{x}$ in figure (1)

Writing the result of the first rotation as $\mathbf{e}_3'$ we have

$$\mathbf{e}_3' = \mathbf{e}_3 e^{\mathbf{e}_{31}\theta} = e^{-\mathbf{e}_{31}\theta/2}\mathbf{e}_3 e^{\mathbf{e}_{31}\theta/2} \tag{8}$$

One more rotation takes $\mathbf{e}_3'$ to $\mathbf{x}$. That is

$$\mathbf{x} = e^{-\mathbf{e}_{12}\phi/2}\mathbf{e}_3' e^{\mathbf{e}_{12}\phi/2} \tag{9}$$

All together, writing $R_\theta = e^{\mathbf{e}_{31}\theta/2}$, and $R_\phi = e^{\mathbf{e}_{12}\phi/2}$, we have

$$\mathbf{x} = \tilde{R_\phi}\tilde{R_\theta}\mathbf{e}_3 R_\theta R_\phi \tag{10}$$

It's worth a quick verification that this produces the desired result.

$$\begin{aligned}
\tilde{R_\phi}\tilde{R_\theta}\mathbf{e}_3 R_\theta R_\phi &= \tilde{R_\phi}\mathbf{e}_3 e^{\mathbf{e}_{31}\theta} R_\phi \\
&= e^{-\mathbf{e}_{12}\phi/2}(\mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta) e^{\mathbf{e}_{12}\phi/2} \\
&= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{\mathbf{e}_{12}\phi}
\end{aligned}$$

This is the expected result

$$\mathbf{x} = \mathbf{e}_3 \cos\theta + \sin\theta(\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta) \tag{11}$$

The projections onto the $\mathbf{e}_3$ and the x, y plane are then, respectively,

$$\mathbf{x}_z = \mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{x}) = \mathbf{e}_3 \cos\theta \tag{12}$$

$$\mathbf{x}_{x,y} = \mathbf{e}_3(\mathbf{e}_3 \wedge \mathbf{x}) = \sin\theta(\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta) \tag{13}$$

So if $\mathbf{x}_\pm$ is the point on the unit sphere associated with the rotation angles $\theta, \pm\phi$, then we have for the $l = 1$ associated Legendre polynomials

$$\psi_0 = \mathbf{e}_3 \cdot \mathbf{x} \tag{14}$$

$$\psi_{\pm 1} = \mathbf{e}_1\mathbf{e}_3(\mathbf{e}_3 \wedge \mathbf{x}_\pm) \tag{15}$$

Note that the $\pm$ was omitted from $\mathbf{x}$ for ψ_0 since either produces the same $\mathbf{e}_3$ component. This gives us a nice geometric interpretation of these eigenfunctions. We see that ψ_0 is the biggest when $\mathbf{x}$ is close to straight up, and when this occurs $\psi_{\pm 1}$ are correspondingly reduced, but when $\mathbf{x}$ is close to the x, y plane where $\psi_{\pm 1}$ will be greatest the z-axis component is reduced.

References

[1] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989.