## Bivector grades of the squared angular momentum operator.

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## 1. Motivation

The aim here is to extract the bivector grades of the squared angular momentum operator

$$
\begin{equation*}
\left\langle(x \wedge \nabla)^{2}\right\rangle_{2} \stackrel{?}{=} \cdots \tag{1}
\end{equation*}
$$

I'd tried this before and believe gotten it wrong. Take it super slow and dumb and careful.

## 2. Non-operator expansion.

Suppose $P$ is a bivector, $P=\left(\gamma^{k} \wedge \gamma^{m}\right) P_{k m}$, the grade two product with a different unit bivector is

$$
\begin{aligned}
\left\langle\left(\gamma_{a} \wedge \gamma_{b}\right)\left(\gamma^{k} \wedge \gamma^{m}\right)\right\rangle_{2} P_{k m} & =\left\langle\left(\gamma_{a} \gamma_{b}-\gamma_{a} \cdot \gamma_{b}\right)\left(\gamma^{k} \wedge \gamma^{m}\right)\right\rangle_{2} P_{k m} \\
& =\left\langle\gamma_{a}\left(\gamma_{b} \cdot\left(\gamma^{k} \wedge \gamma^{m}\right)\right)\right\rangle_{2} P_{k m}+\left\langle\gamma_{a}\left(\gamma_{b} \wedge\left(\gamma^{k} \wedge \gamma^{m}\right)\right)\right\rangle_{2} P_{k m}-\left(\gamma_{a} \cdot \gamma_{b}\right)\left(\gamma^{k} \wedge \gamma^{m}\right) P_{k m} \\
& =\left(\gamma_{a} \wedge \gamma^{m}\right) P_{b m}-\left(\gamma_{a} \wedge \gamma^{k}\right) P_{k b}-\left(\gamma_{a} \cdot \gamma_{b}\right)\left(\gamma^{k} \wedge \gamma^{m}\right) P_{k m} \\
& +\left(\gamma_{a} \cdot \gamma_{b}\right)\left(\gamma^{k} \wedge \gamma^{m}\right) P_{k m}-\left(\gamma_{b} \wedge \gamma^{m}\right) P_{a m}+\left(\gamma_{b} \wedge \gamma^{k}\right) P_{k a} \\
& =\left(\gamma_{a} \wedge \gamma^{c}\right)\left(P_{b c}-P_{c b}\right)+\left(\gamma_{b} \wedge \gamma^{c}\right)\left(P_{c a}-P_{a c}\right)
\end{aligned}
$$

This same procedure will be used for the operator square, but we have the complexity of having the second angular momentum operator change the first bivector result.

## 3. Operator expansion.

In the first few lines of the bivector product expansion above, a blind replacement $\gamma_{a} \rightarrow x$, and $\gamma_{b} \rightarrow \nabla$ gives us

$$
\begin{aligned}
\left\langle(x \wedge \nabla)\left(\gamma^{k} \wedge \gamma^{m}\right)\right\rangle_{2} P_{k m} & =\left\langle(x \nabla-x \cdot \nabla)\left(\gamma^{k} \wedge \gamma^{m}\right)\right\rangle_{2} P_{k m} \\
& =\left\langle x\left(\nabla \cdot\left(\gamma^{k} \wedge \gamma^{m}\right)\right)\right\rangle_{2} P_{k m}+\left\langle x\left(\nabla \wedge\left(\gamma^{k} \wedge \gamma^{m}\right)\right)\right\rangle_{2} P_{k m}-(x \cdot \nabla)\left(\gamma^{k} \wedge \gamma^{m}\right) P_{k m}
\end{aligned}
$$

Using $P_{k m}=x_{k} \partial_{m}$, eliminating the coordinate expansion we have an intermediate result that gets us partway to the desired result

$$
\begin{equation*}
\left\langle(x \wedge \nabla)^{2}\right\rangle_{2}=\langle x(\nabla \cdot(x \wedge \nabla))\rangle_{2}+\langle x(\nabla \wedge(x \wedge \nabla))\rangle_{2}-(x \cdot \nabla)(x \wedge \nabla) \tag{2}
\end{equation*}
$$

An expansion of the first term should be easier than the second. Dropping back to coordinates we have

$$
\begin{aligned}
\langle x(\nabla \cdot(x \wedge \nabla))\rangle_{2} & =\left\langle x\left(\nabla \cdot\left(\gamma^{k} \wedge \gamma^{m}\right)\right)\right\rangle_{2} x_{k} \partial_{m} \\
& =\left\langle x\left(\gamma_{a} \partial^{a} \cdot\left(\gamma^{k} \wedge \gamma^{m}\right)\right)\right\rangle_{2} x_{k} \partial_{m} \\
& =\left\langle x \gamma^{m} \partial^{k}\right\rangle_{2} x_{k} \partial_{m}-\left\langle x \gamma^{k} \partial^{m}\right\rangle_{2} x_{k} \partial_{m} \\
& =x \wedge\left(\partial^{k} x_{k} \gamma^{m} \partial_{m}\right)-x \wedge\left(\partial^{m} \gamma^{k} x_{k} \partial_{m}\right)
\end{aligned}
$$

Okay, a bit closer. Backpedaling with the reinsertion of the complete vector quantities we have

$$
\begin{equation*}
\langle x(\nabla \cdot(x \wedge \nabla))\rangle_{2}=x \wedge\left(\partial^{k} x_{k} \nabla\right)-x \wedge\left(\partial^{m} x \partial_{m}\right) \tag{3}
\end{equation*}
$$

Expanding out these two will be conceptually easier if the functional operation is made explicit. For the first

$$
\begin{aligned}
x \wedge\left(\partial^{k} x_{k} \nabla\right) \phi & =x \wedge x_{k} \partial^{k}(\nabla \phi)+x \wedge\left(\left(\partial^{k} x_{k}\right) \nabla\right) \phi \\
& =x \wedge((x \cdot \nabla)(\nabla \phi))+n(x \wedge \nabla) \phi
\end{aligned}
$$

In operator form this is

$$
\begin{equation*}
x \wedge\left(\partial^{k} x_{k} \nabla\right)=n(x \wedge \nabla)+x \wedge((x \cdot \nabla) \nabla) \tag{4}
\end{equation*}
$$

Now consider the second half of (3). For that we expand

$$
x \wedge\left(\partial^{m} x \partial_{m}\right) \phi=x \wedge\left(x \partial_{m} \partial^{m} \phi\right)+x \wedge\left(\left(\partial^{m} x\right) \partial_{m} \phi\right)
$$

Since $x \wedge x=0$, and $\partial^{m} x=\partial^{m} x_{k} \gamma^{k}=\gamma^{m}$, we have

$$
\begin{aligned}
x \wedge\left(\partial^{m} x \partial_{m}\right) \phi & =x \wedge\left(\gamma^{m} \partial_{m}\right) \phi \\
& =(x \wedge \nabla) \phi
\end{aligned}
$$

Putting things back together we have for (3)

$$
\begin{equation*}
\langle x(\nabla \cdot(x \wedge \nabla))\rangle_{2}=(n-1)(x \wedge \nabla)+x \wedge((x \cdot \nabla) \nabla) \tag{5}
\end{equation*}
$$

This now completes a fair amount of the bivector selection, and a substitution back into (2) yields

$$
\begin{equation*}
\left\langle(x \wedge \nabla)^{2}\right\rangle_{2}=(n-1-x \cdot \nabla)(x \wedge \nabla)+x \wedge((x \cdot \nabla) \nabla)+x \cdot(\nabla \wedge(x \wedge \nabla)) \tag{6}
\end{equation*}
$$

The remaining task is to explicitly expand the last vector-trivector dot product. To do that we use the basic alternation expansion identity

$$
\begin{equation*}
a \cdot(b \wedge c \wedge d)=(a \cdot b)(c \wedge d)-(a \cdot c)(b \wedge d)+(a \cdot d)(b \wedge c) \tag{7}
\end{equation*}
$$

To see how to apply this to the operator case lets write that explicitly but temporarily in coordinates

$$
\begin{aligned}
x \cdot((\nabla \wedge(x \wedge \nabla)) \phi & =\left(x^{\mu} \gamma_{\mu}\right) \cdot\left(\left(\gamma^{v} \partial_{\nu}\right) \wedge\left(x_{\alpha} \gamma^{\alpha} \wedge\left(\gamma^{\beta} \partial_{\beta}\right)\right)\right) \phi \\
& =x \cdot \nabla(x \wedge \nabla) \phi-x \cdot \gamma^{\alpha} \nabla \wedge x_{\alpha} \nabla \phi+x^{\mu} \nabla \wedge x \gamma_{\mu} \cdot \gamma^{\beta} \partial_{\beta} \phi \\
& =x \cdot \nabla(x \wedge \nabla) \phi-x^{\alpha} \nabla \wedge x_{\alpha} \nabla \phi+x^{\mu} \nabla \wedge x \partial_{\mu} \phi
\end{aligned}
$$

Considering this term by term starting with the second one we have

$$
\begin{aligned}
x^{\alpha} \nabla \wedge x_{\alpha} \nabla \phi & =x_{\alpha}\left(\gamma^{\mu} \partial_{\mu}\right) \wedge x^{\alpha} \nabla \phi \\
& =x_{\alpha} \gamma^{\mu} \wedge\left(\partial_{\mu} x^{\alpha}\right) \nabla \phi+x_{\alpha} \gamma^{\mu} \wedge x^{\alpha} \partial_{\mu} \nabla \phi \\
& =x_{\mu} \gamma^{\mu} \wedge \nabla \phi+x_{\alpha} x^{\alpha} \gamma^{\mu} \wedge \partial_{\mu} \nabla \phi \\
& =x \wedge \nabla \phi+x^{2} \nabla \wedge \nabla \phi
\end{aligned}
$$

The curl of a gradient is zero, since summing over an product of antisymmetric and symmetric indexes $\gamma^{\mu} \wedge \gamma^{v} \partial_{\mu v}$ is zero. Only one term remains to evaluate in the vector-trivector dot product now

$$
\begin{equation*}
x \cdot(\nabla \wedge x \wedge \nabla)=(-1+x \cdot \nabla)(x \wedge \nabla)+x^{\mu} \nabla \wedge x \partial_{\mu} \tag{8}
\end{equation*}
$$

Again, a completely dumb and brute force expansion of this is

$$
\begin{aligned}
x^{\mu} \nabla \wedge x \partial_{\mu} \phi & =x^{\mu}\left(\gamma^{v} \partial_{\nu}\right) \wedge\left(x^{\alpha} \gamma_{\alpha}\right) \partial_{\mu} \phi \\
& =x^{\mu} \gamma^{v} \wedge\left(\partial_{v}\left(x^{\alpha} \gamma_{\alpha}\right)\right) \partial_{\mu} \phi+x^{\mu} \gamma^{v} \wedge\left(x^{\alpha} \gamma_{\alpha}\right) \partial_{\nu} \partial_{\mu} \phi \\
& =x^{\mu}\left(\gamma^{\alpha} \wedge \gamma_{\alpha}\right) \partial_{\mu} \phi+x^{\mu} \gamma^{v} \wedge x \partial_{\nu} \partial_{\mu} \phi
\end{aligned}
$$

With $\gamma^{\mu}= \pm \gamma_{\mu}$, the wedge in the first term is zero, leaving

$$
\begin{aligned}
x^{\mu} \nabla \wedge x \partial_{\mu} \phi & =-x^{\mu} x \wedge \gamma^{\nu} \partial_{\nu} \partial_{\mu} \phi \\
& =-x^{\mu} x \wedge \gamma^{v} \partial_{\mu} \partial_{\nu} \phi \\
& =-x \wedge x^{\mu} \partial_{\mu} \gamma^{\nu} \partial_{\nu} \phi
\end{aligned}
$$

In vector form we have finally

$$
\begin{equation*}
x^{\mu} \nabla \wedge x \partial_{\mu} \phi=-x \wedge(x \cdot \nabla) \nabla \phi \tag{9}
\end{equation*}
$$

The final expansion of the vector-trivector dot product is now

$$
\begin{equation*}
x \cdot(\nabla \wedge x \wedge \nabla)=(-1+x \cdot \nabla)(x \wedge \nabla)-x \wedge(x \cdot \nabla) \nabla \phi \tag{10}
\end{equation*}
$$

This was the last piece we needed for the bivector grade selection. Incorporating this into (6), both the $x \cdot \nabla x \wedge \nabla$, and the $x \wedge(x \cdot \nabla) \nabla$ terms cancel leaving the surprising simple result

$$
\begin{equation*}
\left\langle(x \wedge \nabla)^{2}\right\rangle_{2}=(n-2)(x \wedge \nabla) \tag{11}
\end{equation*}
$$

The power of this result is that it allows us to write the scalar angular momentum operator from the Laplacian as

$$
\begin{aligned}
\left\langle(x \wedge \nabla)^{2}\right\rangle & =(x \wedge \nabla)^{2}-\left\langle(x \wedge \nabla)^{2}\right\rangle_{2}-(x \wedge \nabla) \wedge(x \wedge \nabla) \\
& =(x \wedge \nabla)^{2}-(n-2)(x \wedge \nabla)-(x \wedge \nabla) \wedge(x \wedge \nabla) \\
& =(-(n-2)+(x \wedge \nabla)-(x \wedge \nabla) \wedge)(x \wedge \nabla)
\end{aligned}
$$

The complete Laplacian is

$$
\begin{equation*}
\nabla^{2}=\frac{1}{x^{2}}(x \cdot \nabla)^{2}+(n-2) \frac{1}{x} \cdot \nabla-\frac{1}{x^{2}}\left((x \wedge \nabla)^{2}-(n-2)(x \wedge \nabla)-(x \wedge \nabla) \wedge(x \wedge \nabla)\right) \tag{12}
\end{equation*}
$$

In particular in less than four dimensions the quad-vector term is necessarily zero. The 3D Laplacian becomes

$$
\begin{equation*}
\boldsymbol{\nabla}^{2}=\frac{1}{\mathbf{x}^{2}}(1+\mathbf{x} \cdot \boldsymbol{\nabla})(\mathbf{x} \cdot \boldsymbol{\nabla})+\frac{1}{\mathbf{x}^{2}}(1-\mathbf{x} \wedge \boldsymbol{\nabla})(\mathbf{x} \wedge \boldsymbol{\nabla}) \tag{13}
\end{equation*}
$$

So any eigenfunction of the bivector angular momentum operator $\mathbf{x} \wedge \nabla$ is necessarily a simultaneous eigenfunction of the scalar operator.

