QM notes and problems for Bohm, chapter 11.

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May 8, 2009. Last Revision: $Date : 2009/05/1501 : 59 : 51$

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1 Bohm Chapter 11 problems.

Problems and additional details from reading of [Bohm(1989)], chapter 11.

1.1 Problem 1. Probability currents for step potential.

For this problem I include repetition of material covered in the text. Aspects of the treatment were not clear, so attempting this myself should clarify things.

1.1.1 Setup.

This problem and the associated text has a step potential V for $x > 0$. The idea is that we have a left to right stream of particles with an associated wave function, with reflected and transmitted coefficients.

Solutions to the stationary equation are sought in each of the intervals

$$-\frac{\hbar^2}{2m}\psi'' + (V - E)\psi = 0 \tag{1}$$

That is

$$\psi'' = -\frac{2m}{\hbar^2}(E - V)\psi \tag{2}$$

With an assumption of exponential solutions on the left of the barrier (ie: no decay and associated hyperbolic solutions in the $V = 0$ interval), we must have $E > 0$.

So, the solution can be written as the sum

$$\psi_1 = \sum A_\pm \exp\left(\pm i\sqrt{2mE}x/\hbar\right)$$

Similarly for $x > 0$, non-hyperbolic solutions are

$$\psi_t = \sum B_\pm \exp\left(\pm i\sqrt{2m(E - V)}x/\hbar\right)$$

Mathematically, there isn't anything that prevents picking $E - V < 0$ solutions

$$\psi_t = \sum B'_\pm \exp\left(\pm\sqrt{-2m(E - V)}x/\hbar\right)$$

The book considers this case next, and this is also the subject of problem 3.

Back to the $E > V$ case, continuity at $x = 0$ ($\psi_1(0) = \psi_t(0)$) requires

$$A_+ + A_- = B_+ + B_-$$

Whereas derivative continuity, with $p_1 = \sqrt{2mE}$, and $p_2 = \sqrt{2m(E - V)}$ requires

$$\frac{ip_1}{\hbar}(A_+ - A_-) = \frac{ip_2}{\hbar}(B_+ - B_-)$$

This is

$$\begin{bmatrix} 1 & 1 \\ p_1 & -p_1 \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ p_2 & -p_2 \end{bmatrix} \begin{bmatrix} B_+ \\ B_- \end{bmatrix}$$

Matrix inversion gives us the B coefficients in terms of A

$$
\begin{aligned}
\begin{bmatrix} B_+ \\ B_- \end{bmatrix} &= \frac{1}{-2p_2} \begin{bmatrix} -p_2 & -1 \\ -p_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ p_1 & -p_1 \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} \\
&= \frac{1}{2p_2} \begin{bmatrix} p_2 & 1 \\ p_2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ p_1 & -p_1 \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix} \\
&= \frac{1}{2p_2} \begin{bmatrix} (p_1 + p_2) & (p_2 - p_1) \\ (p_2 - p_1) & (p_1 + p_2) \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix}
\end{aligned}
$$

Or equivalently

$$
\begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \frac{1}{2p_1} \begin{bmatrix} (p_1 + p_2) & (p_1 - p_2) \\ (p_1 - p_2) & (p_1 + p_2) \end{bmatrix} \begin{bmatrix} B_+ \\ B_- \end{bmatrix}
$$

Using this last, the assumption of no further barriers in the $x > 0$ interval (ie: no reflection at $x = \infty$), allows the physical situation to dictate $B_- = 0$. Then we have

$$
\begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \frac{B_+}{2p_1} \begin{bmatrix} p_1 + p_2 \\ p_1 - p_2 \end{bmatrix}
$$

The first is

$$
A_+ = \frac{B_+}{2p_1}(p_1 + p_2)
$$

Or

$$
B_+ = \frac{2p_1 A_+}{p_1 + p_2}
$$

and the second is

$$
\begin{aligned}
A_- &= \frac{B_+}{2p_1}(p_1 - p_2) \\
&= \frac{2p_1 A_+}{p_1 + p_2} \frac{1}{2p_1}(p_1 - p_2) \\
&= A_+ \left(\frac{p_1 - p_2}{p_1 + p_2} \right)
\end{aligned}
$$

This reduces the free parameters in the wave functions to the single amplitude

$$\psi_1 = A_+ e^{ip_1 x/\hbar} + A_+ \left(\frac{p_1 - p_2}{p_1 + p_2}\right) e^{-ip_1 x/\hbar} \tag{3}$$

$$\psi_t = A_+ \frac{2p_1}{p_1 + p_2} e^{ip_2 x/\hbar} \tag{4}$$

Inspection provides a check that these do in fact satisfy the desired continuity requirements at $x = 0$ (both the wave function and it's derivative).

Also observe that a sign error typo in the text is implicitly fixed above (sign factor of p_2 in ψ_t). That typo is corrected for right afterward when A, B, C is calculated, since it would otherwise result in a negative sign in the resulting linear equations there too.

1.1.2 Probability currents.

From 3 the probability currents can be calculated. The probability conservation relation follows by taking time derivatives of $\psi^*\psi$, as done in [Joot(a)]. We have, for the x component of the current

$$\frac{\partial \psi\psi^*}{\partial t} + \nabla \cdot \left(\frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)\right) = 0$$

Or in short

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

For ψ_t we have

$$\begin{aligned} J_t &= \frac{\hbar}{2mi}\left(B_+^* e^{-ip_2 x/\hbar} B_+ (ip_2/\hbar) e^{ip_2 x/\hbar} - B_+ e^{-ip_2 x/\hbar} B_+^* (-ip_2/\hbar) e^{-ip_2 x/\hbar}\right) \\ &= \frac{p_2 |B_+|^2}{m} \end{aligned}$$

In terms of A_+, this is

$$J_t = \frac{4p_1^2 p_2 |A_+|^2}{m(p_1 + p_2)^2} \tag{5}$$

For ψ_1 the x component of the current is

$$
\begin{aligned}
J_1 &= \frac{\hbar}{2mi} \left((A_+^* e^{-ip_1x/\hbar} + A_-^* e^{ip_1x/\hbar})(A_+(ip_1/\hbar)e^{ip_1x/\hbar} + A_-(-ip_1/\hbar)e^{-ip_1x/\hbar}) \right) \\
&- \frac{\hbar}{2mi} \left((A_+ e^{ip_1x/\hbar} + A_- e^{-ip_1x/\hbar})(A_+^*(-ip_1/\hbar)e^{-ip_1x/\hbar} + A_-^*(ip_1/\hbar)e^{ip_1x/\hbar}) \right) \\
&= \frac{p_1}{2m} \left(2|A_+|^2 + 2|A_-|^2 + A_-^* A_+ e^{2ip_1x/\hbar} - A_+^* A_- e^{-2ip_1x/\hbar} - A_+ A_-^* e^{2ip_1x/\hbar} + A_- A_+^* e^{-2ip_1x/\hbar} \right) \\
&= \frac{p_1}{m} \left(|A_+|^2 - |A_-|^2 \right)
\end{aligned}
$$

Again in terms of A_+, we have

$$
\begin{aligned}
|A_+|^2 - |A_-|^2 &= |A_+|^2 \left(1 - \left(\frac{p_1 - p_2}{p_1 + p_2} \right)^2 \right) \\
&= |A_+|^2 \frac{4p_1p_2}{(p_1 + p_2)^2} \\
&= 4|A_+|^2 \frac{p_1p_2}{(p_1 + p_2)^2}
\end{aligned}
$$

Which provides the probability current for the sum of the incident and reflected wave functions

$$
J_1 = |A_+|^2 \frac{p_1}{m} \frac{4p_1p_2}{(p_1 + p_2)^2} \tag{6}
$$

So we have $J_1 = J_t$, and this completes the part of the problem that was to show that the currents in the $x < 0$, and $x > 0$ intervals are equal.

The individual incident and reflected currents can also be calculated. These are

$$
\begin{aligned}
J_i &= \frac{p_1}{m} |A_+|^2 \\
J_r &= \frac{p_1}{m} |A_-|^2 \\
&= |A_+|^2 \frac{p_1}{m} \left(\frac{p_1 - p_2}{p_1 + p_2} \right)^2
\end{aligned}
$$

1.1.3 Transmission and reflection coefficients.

As in the text we can calculate the reflection coefficient, the ratio of the magnitudes of the reflected and the incident currents,

$$
\begin{aligned}
R &= \frac{J_r}{J_i} \\
&= \frac{|A_+|^2 \frac{p_1}{m} \left(\frac{p_1-p_2}{p_1+p_2}\right)^2}{\frac{p_1}{m}|A_+|^2} \\
&= \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2
\end{aligned}
$$

and can calculate the transmission coefficient, the ratio of the transmitted current J_t, to the incident current J_i. This is

$$
\begin{aligned}
T &= \frac{J_t}{J_i} \\
&= \frac{\frac{4p_1^2 p_2 |A_+|^2}{m(p_1+p_2)^2}}{\frac{p_1}{m}|A_+|^2} \\
&= \frac{4p_1 p_2}{(p_1 + p_2)^2}
\end{aligned}
$$

Without first calculating the currents explicitly it wasn't clear to me how T was calculated in the text, and doing this first makes that a bit more sensible.

1.1.4 Compare the currents to the probability density.

From 3, the probability density in the two intervals can be calculated. For $x < 0$, we have

$$
\begin{aligned}
\rho &= \psi_1^* \psi_1 \\
&= |A_+|^2 \left(e^{-ip_1x/\hbar} + \left(\frac{p_1 - p_2}{p_1 + p_2}\right) e^{ip_1x/\hbar}\right) \left(e^{ip_1x/\hbar} + \left(\frac{p_1 - p_2}{p_1 + p_2}\right) e^{-ip_1x/\hbar}\right) \\
&= |A_+|^2 \left(1 + \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2 + 2\left(\frac{p_1 - p_2}{p_1 + p_2}\right) \cos\left(2p_1x/\hbar\right)\right)
\end{aligned}
$$

And in the $x > 0$ interval we have

$$
\begin{aligned}
\rho &= \psi_t^* \psi_t \\
&= |A_+|^2 \frac{4p_1^2}{(p_1 + p_2)^2}
\end{aligned}
$$

Comparing to 5, we see that

$$J_t = \frac{p_2}{m}\rho$$

Writing $v_t = p_2/m$ for the velocity associated with the current, we have the desired relation between the current and probability density in the $x > 0$ interval (the transmitted current and probability density).

$$J_t = v_t \rho$$

There doesn't appear to be an such simple relationship between the currents in the $x < 0$ interval where we have probability interference.

1.2 Problem 2. Continuity and probability current conservation.

Show that at $x = 0$ the continuity of these wave functions and their derivatives imply current conservation at $x = 0$. This follows from the fact that $J_1 = J_t$, which is true not just at $x = 0$ since these are both constant.

1.3 Problem 3. Calculate probability current for $E < V$

For the $E < V$ case in the step potential problem above, the solution in the text is found to be for the $x < 0$ interval

$$\begin{aligned}
\psi &= I \cos\left(\sqrt{2mE}x/\hbar + \phi\right) \\
\tan\phi &= \sqrt{(V-E)/E}
\end{aligned}$$

and in the $x > 0$ region is

$$\psi = I \cos(\phi) \exp\left(-\sqrt{2m(V-E)}x/\hbar\right)$$

A quick check of the derivatives of these shows that we have continuity as desired.

That the probability current for this wave function is zero follows from the real nature of this solution, since for any real wave function we have the one dimensional probabilty current as

$$\begin{aligned}
J &= \frac{1}{2mi}\left(\psi^*\psi' - \psi(\psi^*)'\right) \\
&= \frac{1}{2mi}\left(\psi\psi' - \psi\psi'\right) \\
&= 0
\end{aligned}$$

1.4 Problem 4. Rectangular Barrier Tunnelling.

This one is tackled separately in [Joot(b)].

1.5 Problem 5. Reflection coefficient for square well.

End result (straightforward calculation) was

$$\left|\frac{E}{D}\right|^2 = \frac{\frac{1}{4}(p_1/p_2 - p_2/p_1)^2 \sin^2(2p_2a/\hbar)}{\cos^2(2p_2a/\hbar) + \frac{1}{4}(p_1/p_2 + p_2/p_1)\sin^2(2p_2a/\hbar)}$$

1.6 Problem 6.

References

[Bohm(1989)] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989.

[Joot(a)] Peeter Joot. Schrödinger equation probability conservation. "`http://sites.google.com/site/peeterjoot/math2009/sch_current.pdf`", a.

[Joot(b)] Peeter Joot. One dimensional rectangular quantum barrier penetration problem. "`http://sites.google.com/site/peeterjoot/math2009/qm_barrier.pdf`", b.