Dot product linearity by construction.

Peeter Joot peeter.joot@gmail.com

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1 Motivation.

Reading of [Byron and Fuller(1992)] it is observed that the dot product when defined via geometrical constructs such as

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \theta$$

is linear

$$\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$$

Despite the fact that this is obvious when the dot product is given in algebraic form, this doesn't look obvious geometrically, so my immediate thought was "how would you show this geometrically". Sure enough, in the next paragraph is the statement that the reader will want to show this by construction. Here's such a demonstration and construction.

2 Info from the figure.

2.1 Law of cosines.

From figure 1, with $\mathbf{e}_1 = \hat{\mathbf{x}}$ we have

$$\begin{aligned}
\mathbf{y} &= |\mathbf{y}| \cos \theta \mathbf{e}_1 + |\mathbf{y}| \sin \theta \mathbf{e}_2 \\
\mathbf{z} &= |\mathbf{z}| \cos \alpha \mathbf{e}_1 + |\mathbf{z}| \sin \alpha \mathbf{e}_2
\end{aligned}$$

The vector sum $\mathbf{y} + \mathbf{z}$ is therefore

$$\mathbf{y} + \mathbf{z} = (|\mathbf{y}| \cos \theta + |\mathbf{z}| \cos \alpha)\mathbf{e}_1 + (|\mathbf{y}| \sin \theta + |\mathbf{z}| \sin \alpha)\mathbf{e}_2$$

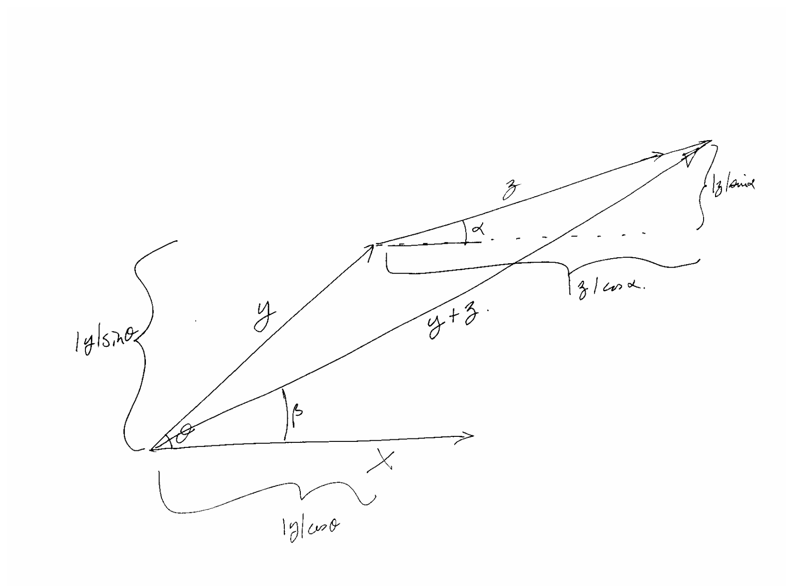

Figure 1: Sum of two vectors and their angles with another.

By using Pythagorus's law, a calculation of the squared length, produces the law of cosines

$$
\begin{aligned}
|\mathbf{y}+\mathbf{z}|^2 &= (|\mathbf{y}|\cos\theta + |\mathbf{z}|\cos\alpha)^2 + (|\mathbf{y}|\sin\theta + |\mathbf{z}|\sin\alpha)^2 \\
&= |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2|\mathbf{y}||\mathbf{z}|(\cos\theta\cos\alpha + \sin\theta\sin\alpha) \\
&= |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2|\mathbf{y}||\mathbf{z}|\cos(\theta-\alpha)
\end{aligned}
$$

2.2 Linearity by construction.

Okay, that is a digression, ... back to the original problem. We get the dot product linearity follows from direct calculation of the cosine of the angle between $\hat{\mathbf{x}}$ and $\mathbf{y}+\mathbf{z}$. Again from the figure we have

$$
\cos\beta = \frac{|\mathbf{y}|\cos\theta + |\mathbf{y}|\cos\alpha}{|\mathbf{y}+\mathbf{z}|}
$$

or

$$
|\mathbf{y}+\mathbf{z}|\cos\beta = |\mathbf{y}|\cos\theta + |\mathbf{y}|\cos\alpha
$$

But $\hat{\mathbf{x}} \cdot \mathbf{y} = |\mathbf{y}| \cos\theta$, and $\hat{\mathbf{x}} \cdot \mathbf{z} = |\mathbf{y}| \cos\alpha$, so we have

$$|\mathbf{y} + \mathbf{z}| \cos\beta = \hat{\mathbf{x}} \cdot \mathbf{y} + \hat{\mathbf{x}} \cdot \mathbf{z}$$

Multiplying by $|\mathbf{x}|$ we have

$$|\mathbf{x}||\mathbf{y} + \mathbf{z}| \cos\beta = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$$

The left hand side is $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z})$, which completes the desired demonstration.

References

[Byron and Fuller(1992)] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992.