Lorentz force rotor formulation.

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Contents

1 Motivation.

Both [Baylis et al.(2007)Baylis, Cabrera, and Keselica] and [Doran and Lasenby(2003)] cover rotor formulations of the Lorentz force equation. Work through some of this on my own to better understand it.

2 In terms of GA.

An active Lorentz transformation can be used to translate from the rest frame of a particle with worldline x to an observer frame, as in

$$y = \Lambda x \tilde{\Lambda} \tag{1}$$

Here Lorentz transformation is used in the general sense, and can include both spatial rotation and boost effects, but satisfies $\Lambda\tilde{\Lambda} = 1$. Taking proper time derivatives we have

$$\begin{aligned}
\dot{y} &= \dot{\Lambda} x \tilde{\Lambda} + \Lambda x \dot{\tilde{\Lambda}} \\
&= \Lambda \left(\tilde{\Lambda} \dot{\Lambda} \right) x \tilde{\Lambda} + \Lambda x \left(\dot{\tilde{\Lambda}} \Lambda \right) \tilde{\Lambda}
\end{aligned}$$

Since $\tilde{\Lambda}\Lambda = \Lambda\tilde{\Lambda} = 1$ we also have

$$\begin{aligned}
0 &= \dot{\Lambda}\tilde{\Lambda} + \Lambda\dot{\tilde{\Lambda}} \\
0 &= \tilde{\Lambda}\dot{\Lambda} + \dot{\tilde{\Lambda}}\Lambda
\end{aligned}$$

Here's where a bivector variable

$$\Omega/2 = \tilde{\Lambda}\dot{\Lambda} \tag{2}$$

is introduced, from which we have $\dot{\tilde{\Lambda}}\Lambda = -\Omega/2$, and

$$\dot{y} = \frac{1}{2} \left(\Lambda \Omega x \tilde{\Lambda} - \Lambda x \Omega \tilde{\Lambda} \right)$$

Or

$$\tilde{\Lambda} \dot{y} \Lambda = \frac{1}{2} \left(\Omega x - x \Omega \right)$$

The inclusion of the factor of two in the definition of Ω was cheating, so that we get the bivector vector dot product above. Presuming Ω is really a bivector (return to this in a bit), we then have

$$\tilde{\Lambda} \dot{y} \Lambda = \Omega \cdot x \tag{3}$$

We can express the time evolution of y using this as a stepping stone, since we have

$$\tilde{\Lambda} y \Lambda = x$$

Using this we have

$$
\begin{aligned}
0 &= \left\langle \tilde{\Lambda}\dot{y}\Lambda - \Omega \cdot x \right\rangle_1 \\
&= \left\langle \tilde{\Lambda}\dot{y}\Lambda - \Omega x \right\rangle_1 \\
&= \left\langle \tilde{\Lambda}\dot{y}\Lambda - \Omega\tilde{\Lambda}y\Lambda \right\rangle_1 \\
&= \left\langle \left(\tilde{\Lambda}\dot{y} - \tilde{\Lambda}\Lambda\Omega\tilde{\Lambda}y\right)\Lambda \right\rangle_1 \\
&= \left\langle \tilde{\Lambda}\left(\dot{y} - \Lambda\Omega\tilde{\Lambda}y\right)\Lambda \right\rangle_1
\end{aligned}
$$

So we have the complete time evolution of our observer frame worldline for the particle, as a sort of an eigenvalue equation for the proper time differential operator

$$
\dot{y} = \left(\Lambda\Omega\tilde{\Lambda}\right) \cdot y = \left(2\dot{\Lambda}\tilde{\Lambda}\right) \cdot y
$$

Now, what Baylis did in his lecture, and what Doran/Lasenby did as well in the text (but I didn't understand it then when I read it the first time) was to identify this time evolution in terms of Lorentz transform change with the Lorentz force.

Recall that the Lorentz force equation is

$$
\dot{v} = \frac{e}{mc} F \cdot v \tag{4}
$$

where $F = \mathbf{E} + ic\mathbf{B}$, like $\dot{\Lambda}\tilde{\Lambda}$ is also a bivector. If we write the velocity worldline of the particle in the lab frame in terms of the rest frame particle worldline as

$$
v = \Lambda ct\gamma_0\tilde{\Lambda}
$$

Then for the field F observed in the lab frame we are left with a differential equation $2\dot{\Lambda}\tilde{\Lambda} = eF/mc$ for the Lorentz transformation that produces the observed motion of the particle given the field that acts on it

$$
\dot{\Lambda} = \frac{e}{2mc} F\Lambda \tag{5}
$$

Okay, good. I understand now well enough what they've done to reproduce the end result (with the exception of my result including a factor of c since they've worked with $c = 1$).

2.1 Omega bivector.

It's been assumed above that $\Omega = 2\tilde{\Lambda}\dot{\Lambda}$ is a bivector. One way to confirm this is by examining the grades of this product. Two bivectors, not neccessarily related can only have grades 0, 2, and 4. Because $\Omega = -\tilde{\Omega}$, as seen above, it can have no grade 0 or grade 4 parts.

While this is a powerful way to verify the bivector nature of this object it is fairly abstract. To get a better feel for this, let's consider this object in detail for a purely spatial rotation, such as

$$\begin{aligned}
R_\theta(x) &= \Lambda x \tilde{\Lambda} \\
\Lambda &= \exp(-in\theta/2) = \cos(\theta/2) - in\sin(\theta/2)
\end{aligned}$$

where n is a spatial unit bivector, $n^2 = 1$, in the span of $\{\sigma_k = \gamma_k\gamma_0\}$.

2.1.1 Verify rotation form.

To verify that this has the appropriate action, by linearily two two cases must be considered. First is the action on n or the components of any vector in this direction.

$$\begin{aligned}
R_\theta(n) &= \Lambda n \tilde{\Lambda} \\
&= (\cos(\theta/2) - in\sin(\theta/2))\, n\tilde{\Lambda} \\
&= n\,(\cos(\theta/2) - in\sin(\theta/2))\,\tilde{\Lambda} \\
&= n\Lambda\tilde{\Lambda} \\
&= n
\end{aligned}$$

The rotation operator does not change any vector colinear with the axis of rotation (the normal). For a vector m that is perpendicular to axis of rotation n (ie: $2(m \cdot n) = mn + nm = 0$), we have

$$\begin{aligned}
R_\theta(m) &= \Lambda m \tilde{\Lambda} \\
&= (\cos(\theta/2) - in\sin(\theta/2))\, m\tilde{\Lambda} \\
&= (m\cos(\theta/2) - i(nm)\sin(\theta/2))\,\tilde{\Lambda} \\
&= (m\cos(\theta/2) + i(mn)\sin(\theta/2))\,\tilde{\Lambda} \\
&= m(\tilde{\Lambda})^2 \\
&= m\exp(in\theta)
\end{aligned}$$

This is a rotation of the vector m that lies in the in plane by θ as desired.

2.1.2 The rotation bivector.

We want derivatives of the Λ object.

$$\begin{aligned}
\dot{\Lambda} &= \frac{\dot{\theta}}{2}\left(-\sin(\theta/2) - in\cos(\theta/2)\right) - i\dot{n}\cos(\theta/2) \\
&= \frac{in\dot{\theta}}{2}\left(in\sin(\theta/2) - \cos(\theta/2)\right) - i\dot{n}\cos(\theta/2) \\
&= -\frac{1}{2}\exp(-in\theta/2)in\dot{\theta} - i\dot{n}\cos(\theta/2)
\end{aligned}$$

So we have

$$\begin{aligned}
\Omega &= 2\tilde{\Lambda}\dot{\Lambda} \\
&= -in\dot{\theta} - 2\exp(in\theta/2)i\dot{n}\cos(\theta/2) \\
&= -in\dot{\theta} - 2\cos(\theta/2)\left(\cos(\theta/2) - in\sin(\theta/2)\right)i\dot{n} \\
&= -in\dot{\theta} - 2\cos(\theta/2)\left(\cos(\theta/2)i\dot{n} + n\dot{n}\sin(\theta/2)\right)
\end{aligned}$$

Since $n \cdot \dot{n} = 0$, we have $n\dot{n} = n \wedge \dot{n}$, and sure enough all the terms are bivectors. Specifically we have

$$\Omega = -\dot{\theta}(in) - (1 + \cos\theta)(i\dot{n}) - \sin\theta(n \wedge \dot{n})$$

2.2 Omega bivector for boost.

TODO.

3 Tensor variation of the Rotor Lorentz force result.

There isn't anything in the initial Lorentz force rotor result that intrinsically requires geometric algebra. At least until one actually wants to express the Lorentz transformation consisely in terms of half angle or boost rapidity exponentials.

In fact the logic above is not much different than the approach used in [Tong()] for rigid body motion. Let's try this in matrix or tensor form and see how it looks.

3.1 Tensor setup.

Before anything else some notation for the tensor work must be established. Similar to 1 write a Lorentz transformed vector as a linear transformation. Since we want only the matrix of this linear transformation with respect to a specific observer frame, the details of the transformation can be omitted for now. Write

$$y = \mathcal{L}(x) \tag{6}$$

and introduce an orthonormal frame $\{\gamma_\mu\}$, and the corresponding reciprocal frame $\{\gamma^\mu\}$, where $\gamma_\mu \cdot \gamma^\nu = \delta_\mu{}^\nu$. In this basis, the relationship between the vectors becomes

$$\begin{aligned} y^\mu \gamma_\mu &= \mathcal{L}(x^\nu \gamma_\nu) \\ &= x^\nu \mathcal{L}(\gamma_\nu) \end{aligned}$$

Or

$$y^\mu = x^\nu \mathcal{L}(\gamma_\nu) \cdot \gamma^\mu$$

The matrix of the linear transformation can now be written as

$$\Lambda_\nu{}^\mu = \mathcal{L}(\gamma_\nu) \cdot \gamma^\mu \tag{7}$$

and this can now be used to express the coordinate transformation in abstract index notation

$$y^\mu = x^\nu \Lambda_\nu{}^\mu \tag{8}$$

Similarily, for the inverse transformation, we can write

$$x = \mathcal{L}^{-1}(y) \tag{9}$$

$$\Pi_\nu{}^\mu = \mathcal{L}^{-1}(\gamma_\nu) \cdot \gamma^\mu \tag{10}$$

$$x^\mu = y^\nu \Pi_\nu{}^\mu \tag{11}$$

I've seen this expressed using primed indexes and the same symbol Λ used for both the forward and inverse transformation ... lacking skill in tricky index manipulation I've avoided such a notation because I'll probably get it wrong. Instead different symbols for the two different matrixes will be used here and Π was picked for the inverse rather arbitrarily.

With substitution

$$
\begin{aligned}
y^\mu &= x^\nu {\Lambda_\nu}^\mu = (y^\alpha {\Pi_\alpha}^\nu) {\Lambda_\nu}^\mu \\
x^\mu &= y^\nu {\Pi_\nu}^\mu = (x^\alpha {\Lambda_\alpha}^\nu) {\Pi_\nu}^\mu
\end{aligned}
$$

the pair of explicit inverse relationships between the two matrixes can be read off as

$$
{\delta_\alpha}^\mu = {\Pi_\alpha}^\nu {\Lambda_\nu}^\mu = {\Lambda_\alpha}^\nu {\Pi_\nu}^\mu \tag{12}
$$

3.2 Lab frame velocity of particle in tensor form.

In tensor form we want to express the worldline of the particle in the lab frame coordinates. That is

$$
\begin{aligned}
v &= \mathcal{L}(ct\gamma_0) \\
&= \mathcal{L}(x^0\gamma_0) \\
&= x^0\mathcal{L}(\gamma_0)
\end{aligned}
$$

Or

$$
\begin{aligned}
v^\mu &= x^0\mathcal{L}(\gamma_0) \cdot \gamma^\mu \\
&= x^0 {\Lambda_0}^\mu
\end{aligned}
$$

3.3 Lorentz force in tensor form.

The Lorentz force equation 4 in tensor form will also be needed. The bivector F is

$$
F = \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu
$$

So we can write

$$
\begin{aligned}
F \cdot v &= \frac{1}{2} F_{\mu\nu} (\gamma^\mu \wedge \gamma^\nu) \cdot \gamma_\alpha v^\alpha \\
&= \frac{1}{2} F_{\mu\nu} (\gamma^\mu {\delta^\nu}_\alpha - \gamma^\nu {\delta^\mu}_\alpha) v^\alpha \\
&= \frac{1}{2} (v^\alpha F_{\mu\alpha} \gamma^\mu - v^\alpha F_{\alpha\nu} \gamma^\nu)
\end{aligned}
$$

And

$$\begin{aligned}
\dot{v}_\sigma &= \frac{e}{mc}(F \cdot v) \cdot \gamma_\sigma \\
&= \frac{e}{2mc}(v^\alpha F_{\mu\alpha}\gamma^\mu - v^\alpha F_{\alpha\nu}\gamma^\nu) \cdot \gamma_\sigma \\
&= \frac{e}{2mc} v^\alpha (F_{\sigma\alpha} - F_{\alpha\sigma}) \\
&= \frac{e}{mc} v^\alpha F_{\sigma\alpha}
\end{aligned}$$

Or

$$\dot{v}^\sigma = \frac{e}{mc} v^\alpha {F^\sigma}_\alpha \tag{13}$$

3.4 Evolution of Lab frame vector.

Given a lab frame vector with all the (proper) time evolution expressed via the Lorentz transformation

$$y^\mu = x^\nu {\Lambda_\nu}^\mu$$

we want to calculate the derivatives as in the GA procedure

$$\begin{aligned}
\dot{y}^\mu &= x^\nu \dot{\Lambda}_\nu^\mu \\
&= x^\alpha {\delta_\alpha}^\nu \dot{\Lambda}_\nu^\mu \\
&= x^\alpha {\Lambda_\alpha}^\beta {\Pi_\beta}^\nu \dot{\Lambda}_\nu^\mu
\end{aligned}$$

With $y = v$, this is

$$\begin{aligned}
\dot{v}^\sigma &= v^\alpha {\Pi_\alpha}^\nu \dot{\Lambda}_\nu^\sigma \\
&= v^\alpha \frac{e}{mc} {F^\sigma}_\alpha
\end{aligned}$$

So we can make the identification of the bivector field with the Lorentz transformation matrix

$${\Pi_\alpha}^\nu \dot{\Lambda}_\nu^\sigma = \frac{e}{mc} {F^\sigma}_\alpha$$

With an additional summation to invert we have

$$\Lambda_\beta{}^\alpha \Pi_\alpha{}^\nu \dot{\Lambda}_\nu^\sigma = \Lambda_\beta{}^\alpha \frac{e}{mc} F^\sigma{}_\alpha$$

This leaves a tensor differential equation that will provide the complete time evolution of the lab frame worldline for the particle in the field

$$\dot{\Lambda}_\mu^\nu = \frac{e}{mc} \Lambda_\mu{}^\alpha F^\nu{}_\alpha \tag{14}$$

This is the equivalent of the GA equation 5. However, while the GA equation is directly integrable for constant F, how to do this in the equivalent tensor formulation is not so clear.

Want to revisit this, and try to perform this integral in both forms, ideally for both the simpler constant field case, as well as for a more general field. Even better would be to be able to express F in terms of the current density vector, and then treat the proper interaction of two charged particles.

4 Gauge transformation for spin.

In the Baylis article 5 is transformed as $\Lambda \rightarrow \Lambda_{\omega_0} \exp(-i\mathbf{e}_3 \omega_0 \tau)$.

Using this we have

$$\begin{aligned}
\dot{\Lambda} &\rightarrow \frac{d}{d\tau} \left(\Lambda_{\omega_0} \exp(-i\mathbf{e}_3 \omega_0 \tau) \right) \\
&= \dot{\Lambda}_{\omega_0} \exp(-i\mathbf{e}_3 \omega_0 \tau) - \Lambda_{\omega_0} (i\mathbf{e}_3 \omega_0) \exp(-i\mathbf{e}_3 \omega_0 \tau)
\end{aligned}$$

For the transformed 5 this gives

$$\dot{\Lambda}_{\omega_0} \exp(-i\mathbf{e}_3 \omega_0 \tau) - \Lambda_{\omega_0} (i\mathbf{e}_3 \omega_0) \exp(-i\mathbf{e}_3 \omega_0 \tau) = \frac{e}{2mc} F \Lambda_{\omega_0} \exp(-i\mathbf{e}_3 \omega_0 \tau)$$

Canceling the exponentials, and shuffling

$$\dot{\Lambda}_{\omega_0} = \frac{e}{2mc} F \Lambda_{\omega_0} + \Lambda_{\omega_0} (i\mathbf{e}_3 \omega_0) \tag{15}$$

How does he commute the $i\mathbf{e}_3$ term with the Lorentz transform? How about instead transforming as $\Lambda \rightarrow \exp(-i\mathbf{e}_3 \omega_0 \tau) \Lambda_{\omega_0}$.

Using this we have

$$\begin{aligned}
\dot{\Lambda} &\rightarrow \frac{d}{d\tau} \left(\exp(-i\mathbf{e}_3 \omega_0 \tau) \Lambda_{\omega_0} \right) \\
&= \exp(-i\mathbf{e}_3 \omega_0 \tau) \dot{\Lambda}_{\omega_0} - (i\mathbf{e}_3 \omega_0) \exp(-i\mathbf{e}_3 \omega_0 \tau) \Lambda_{\omega_0}
\end{aligned}$$

then, the transformed 5 gives

$$\exp(-i\mathbf{e}_3\omega_0\tau)\dot{\Lambda}_{\omega_0} - (i\mathbf{e}_3\omega_0)\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0} = \frac{e}{2mc}F\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0}$$

Multiplying by the inverse exponential, and shuffling, noting that $\exp(i\mathbf{e}_3\alpha)$ commutes with $i\mathbf{e}_3$, we have

$$\begin{aligned}
\dot{\Lambda}_{\omega_0} &= (i\mathbf{e}_3\omega_0)\Lambda_{\omega_0} + \frac{e}{2mc}\exp(i\mathbf{e}_3\omega_0\tau)F\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0} \\
&= \frac{e}{2mc}\left(\frac{2mc}{e}(i\mathbf{e}_3\omega_0) + \exp(i\mathbf{e}_3\omega_0\tau)F\exp(-i\mathbf{e}_3\omega_0\tau)\right)\Lambda_{\omega_0}
\end{aligned}$$

So, if one writes $F_{\omega_0} = \exp(i\mathbf{e}_3\omega_0\tau)F\exp(-i\mathbf{e}_3\omega_0\tau)$, then the transformed differential equation for the Lorentz transformation takes the form

$$\dot{\Lambda}_{\omega_0} = \frac{e}{2mc}\left(\frac{2mc}{e}(i\mathbf{e}_3\omega_0) + F_{\omega_0}\right)\Lambda_{\omega_0}$$

This is closer to Baylis's equation 31. Dropping ω_0 subscripts this is

$$\dot{\Lambda} = \frac{e}{2mc}\left(\frac{2mc}{e}(i\mathbf{e}_3\omega_0) + F\right)\Lambda$$

A phase change in the Lorentz transformation rotor has introduced an additional term, one that Baylis appears to identify with the spin vector $\mathbf{S}$. My way of getting there seems fishy, so I think that I'm missing something.

Ah, I see. If we go back to 15, then with $\mathbf{S} = \Lambda_{\omega_0}(i\mathbf{e}_3)\tilde{\Lambda}_{\omega_0}$ (an application of a Lorentz transform to the unit bivector for the $\mathbf{e}_2\mathbf{e}_3$ plane), one has

$$\dot{\Lambda}_{\omega_0} = \frac{1}{2}\left(\frac{e}{mc}F + 2\omega_0\mathbf{S}\right)\Lambda_{\omega_0}$$

References

[Baylis et al.(2007)Baylis, Cabrera, and Keselica] W. E. Baylis, R. Cabrera, and D. Keselica. Quantum/classical interface: Fermion spin, 2007. URL `http://www.citebase.org/abstract?id=oai:arXiv.org:0710.3144`.

[Doran and Lasenby(2003)] C. Doran and A.N. Lasenby. *Geometric algebra for physicists*. Cambridge University Press New York, 2003.

[Tong()] Dr. David Tong. Classical mechanics. "`http://www.damtp.cam.ac.uk/user/tong/dynamics.htm`".