A cheatsheet for Fourier transform conventions.

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1 A cheatsheet for different Fourier integral notations.

Damn. There's too many different notations for the Fourier transform. Examples are:

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) \exp(-2\pi i k x) \, dx \\
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) \, dx \\
\tilde{f}(p) &= \sqrt{\frac{1}{2\pi\hbar}} \int_{-\infty}^{\infty} f(x) \exp\left(\frac{-ipx}{\hbar}\right) dx
\end{aligned}$$

There's probably many more, with other variations such as using hats over things instead of twiddles, and so forth.

Unfortunately each of these have different numeric factors for the inverse transform. Having just been bitten by rogue factors of 2π after inoccently switching notations, it seems worthwhile to express the Fourier transform with a general fudge factor in the exponential. Then it can be seen at a glance what constants are required in the inverse transform given anybody's particular choice of the transform definition.

Where to put all the factors can actually be seen from the QM formulation since one is free to treat $\hbar$ as an arbitrary constant, but let's do it from scratch in a mechanical fashion without having to think back to QM as a fundamental.

Suppose we define the Fourier transform as

$$\tilde{f}(s) = \kappa \int_{-\infty}^{\infty} f(x) \exp\left(-i\alpha s x\right) dx$$
$$f(x) = \kappa' \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i\alpha x s\right) ds$$

Now, what factor do we need in the inverse transform to make things work out right? With the Rigor Police on holiday, let's expand the inverse transform integral in terms of the original transform and see what these numeric factors must then be to make this work out.

Omitting temporarily the κ factors to be determined we have

$$\begin{aligned}
f(x) &\propto \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) \exp\left(-i\alpha s u\right) du \right) \exp\left(i\alpha x s\right) ds \\
&= \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} \exp\left(i\alpha s (x-u)\right) ds \\
&= \int_{-\infty}^{\infty} f(u) du \lim_{R\to\infty} 2\pi \frac{1}{\pi\alpha(x-u)} \sin\left(\alpha R(x-u)\right) \\
&= \int_{-\infty}^{\infty} f(u) du 2\pi\delta\left(\alpha(x-u)\right) \\
&= \frac{1}{\alpha} \int_{-\infty}^{\infty} f(v/\alpha) dv 2\pi\delta\left(\alpha x - v\right) \\
&= \frac{2\pi}{\alpha} f((\alpha x)/\alpha) \\
&= \frac{2\pi}{\alpha} f(x)
\end{aligned}$$

Note that to get the result above, after switching order of integration, and assuming that we can take the principle value of the integrals, the usual ad-hoc sinc and exponential integral identification of the delta function was made

$$\begin{aligned}
\mathrm{PV} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(isx\right) ds &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \exp\left(isx\right) ds \\
&= \lim_{R\to\infty} \frac{\sin\left(Rx\right)}{\pi x} \\
&\equiv \delta\left(x\right)
\end{aligned}$$

The end result is that we will need to fix

$$\kappa\kappa' = \frac{\alpha}{2\pi}$$

to have the transform pair produce the desired result. Our transform pair is therefore

$$\tilde{f}(s) = \kappa \int_{-\infty}^{\infty} f(x) \exp(-i\alpha s x)\, dx \Leftrightarrow f(x) = \frac{\alpha}{2\pi\kappa} \int_{-\infty}^{\infty} \tilde{f}(s) \exp(i\alpha s x)\, ds \quad (1)$$

2 A survey of notations.

From 1 we can express the required numeric factors that accompany all the various forward transforms conventions. Let's do a quick survey of the bookshelf, ignoring differences in the i's and j's, differences in the transform variables, and so forth.

From my old systems and signals course, with the book [Haykin(1994)] we have, $\kappa = 1$, and $\alpha = 2\pi$

$$\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) \exp(-2\pi i s x)\, dx \\
f(x) &= \int_{-\infty}^{\infty} \tilde{f}(s) \exp(2\pi i s x)\, ds
\end{aligned}$$

The mathematician's preference, and that of [Bohm(1989)], and [Byron and Fuller(1992)] appears to be the nicely symmetrical version, with $\kappa = 1/\sqrt{2\pi}$, and $\alpha = 1$

$$\begin{aligned}
\tilde{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-isx)\, dx \\
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(s) \exp(isx)\, ds
\end{aligned}$$

From the old circuits course using [Irwin(1993)], and also in the excellent text [Le Page and LePage(1980)], we have $\kappa = 1$, and $\alpha = 1$

$$\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) \exp(-isx)\, dx \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(s) \exp(isx)\, ds
\end{aligned}$$

and finally, the QM specific version from [McMahon(2005)], with $\alpha = p/\hbar$, and $\kappa = 1/\sqrt{2\pi\hbar}$ we have

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{ipx}{\hbar}\right) dx$$
$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{f}(p) \exp\left(\frac{ipx}{\hbar}\right) dp$$

References

[Bohm(1989)] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989.

[Byron and Fuller(1992)] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992.

[Haykin(1994)] S.S. Haykin. Communication systems. 1994.

[Irwin(1993)] J.D. Irwin. *Basic Engineering Circuit Analysis*. MacMillian, 1993.

[Le Page and LePage(1980)] W.R. Le Page and W.R. LePage. *Complex Variables and the Laplace Transform for Engineers*. Courier Dover Publications, 1980.

[McMahon(2005)] D. McMahon. *Quantum Mechanics Demystified*. McGraw-Hill Professional, 2005.