Fourier series Vacuum Maxwell's equations.

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Contents

1 Motivation.

In [Bohm(1989)], after finding a formulation of Maxwell's equations that he likes, his next step is to assume the electric and magnetic fields can be expressed in a 3D Fourier series form, with periodicity in some repeated volume of space, and then procedes to evaluate the energy of the field.

1.1 Notation.

A notational table 6 is included below for reference.

2 Setup.

Let's try this. Instead of using the sine and cosine fourier series which looks more complex than it ought to be, use of a complex exponential ought to be cleaner.

2.1 3D Fourier series in complex exponential form.

For a multivector function $f(\mathbf{x}, t)$, periodic in some rectangular spatial volume, let's assume that we have a 3D Fourier series representation.

Define the element of volume for our fundamental wavelengths to be the region bounded by three intervals in the x^1, x^2, x^3 directions respectively

$$
\begin{aligned}
I_1 &= [a^1, a^1 + \lambda_1] \\
I_2 &= [a^2, a^2 + \lambda_2] \\
I_3 &= [a^3, a^3 + \lambda_3]
\end{aligned}
$$

Our assumed Fourier representation is then

$$
f(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}}(t) \exp\left(-\sum_j \frac{2\pi i k_j x^j}{\lambda_j} \right)
$$

Here $\hat{f}_{\mathbf{k}} = \hat{f}_{\{k_1, k_2, k_3\}}$ is indexed over a triplet of integer values, and the k_1, k_2, k_3 indexes take on all integer values in the $[-\infty, \infty]$ range.

Note that we also wish to allow i to not just be a generic complex number, but allow for the use of either the Euclidian or Minkowski pseudoscalar

$$
i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3
$$

Because of this we should not assume that we can commute i, or our exponentials with the functions $f(\mathbf{x}, t)$, or $\hat{f}_{\mathbf{k}}(t)$.

$$
\begin{aligned}
&\int_{x^1 = \partial I_1} \int_{x^2 = \partial I_2} \int_{x^3 = \partial I_3} f(\mathbf{x}, t) e^{2\pi i m_j x^j / \lambda_j} dx^1 dx^2 dx^3 \\
&\qquad = \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}}(t) \int_{x^1 = \partial I_1} \int_{x^2 = \partial I_2} \int_{x^3 = \partial I_3} dx^1 dx^2 dx^3 e^{2\pi i (m_j - k_j) x^j / \lambda_j} dx^1 dx^2 dx^3
\end{aligned}
$$

But each of these integrals is just $\delta_{\mathbf{k},\mathbf{m}} \lambda_1 \lambda_2 \lambda_3$, giving us

$$
\hat{f}_{\mathbf{k}}(t) = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \int_{x^1 = \partial I_1} \int_{x^2 = \partial I_2} \int_{x^3 = \partial I_3} f(\mathbf{x}, t) \exp\left(\sum_j \frac{2\pi i k_j x^j}{\lambda_j} \right) dx^1 dx^2 dx^3
$$

To tidy things up lets invent (or perhaps abuse) some notation to tidy things up. As a subscript on our Fourier coefficients we've used $\mathbf{k}$ as an index. Let's also use it as a vector, and define

$$\mathbf{k} \equiv 2\pi \sum_m \frac{\sigma^m k_m}{\lambda_m} \tag{1}$$

With our spatial vector $\mathbf{x}$ written

$$\mathbf{x} = \sum_m \sigma_m x^m$$

We now have a $\mathbf{k} \cdot \mathbf{x}$ term in the exponential, and can remove when desirable the coordinate summation. If we write $V = \lambda_1 \lambda_2 \lambda_3$ it leaves a nice tidy notation for the 3D fourier series over the volume

$$f(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{2}$$

$$\hat{f}_{\mathbf{k}}(t) = \frac{1}{V} \int f(\mathbf{x}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3 x \tag{3}$$

This allows us to procede without caring about the specifics of the lengths of the sides of the rectangular prism that defines the periodicity of the signal in question.

2.2 Vacuum equation.

Now that we have a desirable seeming Fourier series representation, we want to apply this to Maxwell's equation for the vacuum. We will use the STA formulation of Maxwell's equation, but use the unit convention of Bohm's book.

In [Joot(f)] the STA equivalent to Bohm's notation for Maxwell's equations was found to be

$$F = \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}} \tag{4}$$

$$J = (\rho + \mathbf{j})\gamma_0 \tag{5}$$

$$\nabla F = 4\pi J \tag{6}$$

This is the cgs form of Maxwell's equation, but with the old style $\boldsymbol{\mathcal{H}}$ for $c\mathbf{B}$, and $\boldsymbol{\mathcal{E}}$ for $\mathbf{E}$. In more recent texts $\boldsymbol{\mathcal{E}}$ (as a non-vector) is reserved for electromotive flux. In this set of notes I use Bohm's notation, since the aim is to clarify for myself aspects of his treatment.

For the vacuum equation, we make an explicit spacetime split by premultiplying with γ_0

$$\begin{aligned}
\gamma_0 \nabla &= \gamma_0(\gamma^0 \partial_0 + \gamma^k \partial_k) \\
&= \partial_0 - \gamma^k \gamma_0 \partial_k \\
&= \partial_0 + \gamma_k \gamma_0 \partial_k \\
&= \partial_0 + \sigma_k \partial_k \\
&= \partial_0 + \boldsymbol{\nabla}
\end{aligned}$$

So our vacuum equation is just

$$(\partial_0 + \boldsymbol{\nabla})F = 0 \tag{7}$$

3 First order vacuum solution with Fourier series.

3.1 Basic solution in terms of undetermined coefficients.

Now that a notation for the 3D Fourier series has been established, we can assume a series solution for our field of the form

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{F}_{\mathbf{k}}(t) e^{-2\pi i k_j x^j / \lambda_j} \tag{8}$$

can now apply this to the vacuum Maxwell equation 7. This gives us

$$\begin{aligned}
\sum_{\mathbf{k}} \left(\partial_t \hat{F}_{\mathbf{k}}(t)\right) e^{-2\pi i k_j x^j / \lambda_j} &= -c \sum_{\mathbf{k},m} \sigma^m \hat{F}_{\mathbf{k}}(t) \frac{\partial}{\partial x^m} e^{-2\pi i k_j x^j / \lambda_j} \\
&= -c \sum_{\mathbf{k},m} \sigma^m \hat{F}_{\mathbf{k}}(t) \left(-2\pi \frac{k_m}{\lambda_m}\right) e^{-2\pi i k_j x^j / \lambda_j} \\
&= 2\pi c \sum_{\mathbf{k}} \sum_{m} \frac{\sigma^m k_m}{\lambda_m} \hat{F}_{\mathbf{k}}(t) i e^{-2\pi i k_j x^j / \lambda_j}
\end{aligned}$$

Note that i commutes with $\mathbf{k}$ and since F is also an STA bivector i commutes with F. Putting all this together we have

$$\sum_{\mathbf{k}} \left(\partial_t \hat{F}_{\mathbf{k}}(t)\right) e^{-i\mathbf{k}\cdot\mathbf{x}} = ic \sum_{\mathbf{k}} \mathbf{k} \hat{F}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

Term by term we now have a (big ass, triple infinite) set of very simple first order differential equations, one for each $\mathbf{k}$ triplet of indexes. Specifically this is

$$\hat{F}'_{\mathbf{k}} = ic\mathbf{k}\hat{F}_{\mathbf{k}}$$

With solutions

$$\begin{aligned}
\hat{F}_0 &= C_0 \\
\hat{F}_{\mathbf{k}} &= \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}}
\end{aligned}$$

Here $C_{\mathbf{k}}$ is an undetermined STA bivector. For now we keep this undetermined coefficient on the right hand side of the exponential since no demonstration that it commutes with a factor of the form $\exp(i\mathbf{k}\phi)$. Substitution back into our assumed solution sum we have a solution to Maxwell's equation, in terms of a set of as yet undetermined (bivector) coefficients

$$F(\mathbf{x}, t) = C_0 + \sum_{\mathbf{k}\neq 0} \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x})$$

The special case of $\mathbf{k} = 0$ is now seen to be not so special and can be brought into the sum.

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x})$$

We can also take advantage of the bivector nature of $C_{\mathbf{k}}$, which implies the complex exponential can commute to the left, since the two fold commutation with the pseudoscalar with change sign twice.

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) C_{\mathbf{k}} \tag{9}$$

3.2 Solution as time evolution of initial field.

Now, observe the form of this sum for $t = 0$. This is

$$F(\mathbf{x}, 0) = \sum_{\mathbf{k}} C_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x})$$

So, the C_k coefficients are precisely the Fourier coefficients of $F(\mathbf{x}, 0)$. This is to be expected having repeatedly seen similar results in the Fourier transform treatments of [Joot(c)], [Joot(b)], and [Joot(a)]. We then have an equation for the complete time evolution of any spatially periodic electrodynamic field in terms of the field value at all points in the region at some initial time. Summarizing so far this is

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{x}) \tag{10}$$

$$C_{\mathbf{k}} = \frac{1}{V} \int F(\mathbf{x}', 0) \exp\left(i\mathbf{k} \cdot \mathbf{x}'\right) d^3x' \tag{11}$$

Regrouping slightly we can write this as a convolution with a Fourier kernel (a Green's function). That is

$$F(\mathbf{x}, t) = \frac{1}{V} \int \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})\right) F(\mathbf{x}', 0) d^3x' \tag{12}$$

Or

$$F(\mathbf{x}, t) = \int G(\mathbf{x} - \mathbf{x}', t) F(\mathbf{x}', 0) d^3x' \tag{13}$$

$$G(\mathbf{x}, t) = \frac{1}{V} \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) \tag{14}$$

Okay, that's cool. We've now got the basic periodicity result directly from Maxwell's equation in one shot. No need to drop down to potentials, or even the separate electric or magnetic components of our field $F = \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}$.

3.3 Prettying it up? Questions of commutation.

Now, it is tempting here to write 9 as a single exponential

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct - i\mathbf{k} \cdot \mathbf{x}\right) C_{\mathbf{k}} \qquad \text{VALID?} \tag{15}$$

This would probably allow for a prettier four vector form in terms of $x = x^\mu \gamma_\mu$ replacing the separate $\mathbf{x}$ and $x^0 = ct$ terms. However, such a grouping is not allowable unless one first demonstrates that $e^{i\mathbf{u}}$, and $e^{i\alpha}$, for spatial vector $\mathbf{u}$ and scalar α commute!

To demonstrate that this is in fact the case note that exponential of this dual spatial vector can be written

$$\exp(i\mathbf{u}) = \cos(\mathbf{u}) + i \sin(\mathbf{u})$$

This spatial vector cosine, $\cos(\mathbf{u})$, is a scalar (even powers only), and our sine, $\sin(\mathbf{u}) \propto \mathbf{u}$, is a spatial vector in the direction of $\mathbf{u}$ (odd powers leaves a vector times a scalar). Spatial vectors commute with i (toggles sign twice percolating its way through), therefore pseudoscalar exponentials also commute with i.

This will simplify a lot, and it shows that 15 is in fact a valid representation.

Now, there's one more question of commutation here. Namely, does a dual spatial vector exponential commute with the field itself (or equivalently, one of the Fourier cofficients).

Expanding such a product and attempting term by term commutation should show

$$
\begin{aligned}
e^{i\mathbf{u}}F &= (\cos\mathbf{u} + i\sin\mathbf{u})(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) \\
&= i\sin\mathbf{u}(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) + (\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}})\cos\mathbf{u} \\
&= i(\sin\mathbf{u})\boldsymbol{\mathcal{E}} - (\sin\mathbf{u})\boldsymbol{\mathcal{H}} + F\cos\mathbf{u} \\
&= i(-\boldsymbol{\mathcal{E}}\sin\mathbf{u} + 2\boldsymbol{\mathcal{E}}\cdot\sin\mathbf{u}) + (\boldsymbol{\mathcal{H}}\sin\mathbf{u} - 2\boldsymbol{\mathcal{H}}\cdot\sin\mathbf{u}) + F\cos\mathbf{u} \\
&= 2\sin\mathbf{u}\cdot(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{H}}) + F(\cos\mathbf{u} - i\sin\mathbf{u})
\end{aligned}
$$

That is

$$
e^{i\mathbf{u}}F = 2\sin\mathbf{u}\cdot(\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{H}}) + Fe^{-i\mathbf{u}} \tag{16}
$$

This exponential has one anticommuting term, but also has a scalar component introduced by the portions of the electric and magnetic fields that are colinear with the spatial vector $\mathbf{u}$.

4 Field Energy and momentum.

Given that we have the same structure for our four vector potential solutions as the complete bivector field, it doesn't appear that there is much reason to work in the second order quantities. Following Bohm we should now be prepared to express the field energy density and momentum density in terms of the Fourier coefficients, however unlike Bohm, let's try this using the first order solutions found above.

In cgs units (see [Joot(f)] for verification) these field energy and momentum densities (Poynting vector $\mathbf{P}$) are, respectively

$$
\begin{aligned}
E &= \frac{1}{8\pi}(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2) \\
\mathbf{P} &= \frac{1}{4\pi}(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}})
\end{aligned}
$$

Given that we have a complete field equation without an explicit separation of electric and magnetic components, perhaps this is easier to calculate from the stress energy four vector for energy/momentum. In cgs units this must be

$$T(\gamma_0) = \frac{1}{8\pi} F \gamma_0 \tilde{F} \tag{17}$$

An expansion of this to verify the cgs conversion seems worthwhile.

$$\begin{aligned}
T(\gamma_0) &= \frac{1}{8\pi} F \gamma_0 \tilde{F} \\
&= \frac{-1}{8\pi} (\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) \gamma_0 (\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) \\
&= \frac{1}{8\pi} (\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}})(\boldsymbol{\mathcal{E}} - i\boldsymbol{\mathcal{H}}) \gamma_0 \\
&= \frac{1}{8\pi} \left(\boldsymbol{\mathcal{E}}^2 - (i\boldsymbol{\mathcal{H}})^2 + i(\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{H}}) \right) \gamma_0 \\
&= \frac{1}{8\pi} \left(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2 + 2i^2 \boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{E}} \right) \gamma_0 \\
&= \frac{1}{8\pi} \left(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2 \right) \gamma_0 + \frac{1}{4\pi} \left(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}} \right) \gamma_0
\end{aligned}$$

Good, as expected we have

$$E = T(\gamma_0) \cdot \gamma_0 \tag{18}$$

$$\mathbf{P} = T(\gamma_0) \wedge \gamma_0 \tag{19}$$

FIXME: units here for $\mathbf{P}$ are off by a factor of c. This doesn't matter so much in four vector form $T(\gamma_0)$ where the units naturally take care of themselves.

Okay, let's apply this to our field equation 12, and try to percolate the γ_0 through all the terms of $\tilde{F}(\mathbf{x}, t)$

$$\begin{aligned}
\gamma_0 \tilde{F}(\mathbf{x}, t) &= -\gamma_0 F(\mathbf{x}, t) \\
&= -\gamma_0 \frac{1}{V} \int \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})\right) F(\mathbf{x}', 0) d^3x'
\end{aligned}$$

Taking one factor at a time

$$\begin{aligned}
\gamma_0 \exp\left(i\mathbf{k}ct\right) &= \gamma_0(\cos\left(\mathbf{k}ct\right) + i\sin\left(\mathbf{k}ct\right)) \\
&= \cos\left(\mathbf{k}ct\right) \gamma_0 - i\gamma_0 \sin\left(\mathbf{k}ct\right)) \\
&= \cos\left(\mathbf{k}ct\right) \gamma_0 - i\sin\left(\mathbf{k}ct\right))\gamma_0 \\
&= \exp\left(-i\mathbf{k}ct\right) \gamma_0
\end{aligned}$$

Next, percolate γ_0 through the pseudoscalar exponential.

$$
\begin{aligned}
\gamma_0 e^{i\phi} &= \gamma_0(\cos\phi + i\sin\phi) \\
&= \cos\phi\gamma_0 - i\gamma_0\sin\phi \\
&= e^{-i\phi}\gamma_0
\end{aligned}
$$

Again, the percolation produces a conjugate effect. Lastly, as noted previously F commutes with i. We have therefore

$$
\begin{aligned}
\tilde{F}(\mathbf{x},t)\gamma_0 F(\mathbf{x},t)\gamma_0 &= \frac{1}{V^2}\int\sum_{\mathbf{k},\mathbf{m}} F(\mathbf{a},0)e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x})}e^{i\mathbf{k}ct}e^{-i\mathbf{m}ct}e^{-i\mathbf{m}\cdot(\mathbf{b}-\mathbf{x})}F(\mathbf{b},0)d^3a d^3b \\
&= \frac{1}{V^2}\int\sum_{\mathbf{k},\mathbf{m}} F(\mathbf{a},0)e^{i\mathbf{k}\cdot\mathbf{a}-i\mathbf{m}\cdot\mathbf{b}+i(\mathbf{k}-\mathbf{m})ct-i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}}F(\mathbf{b},0)d^3a d^3b \\
&= \frac{1}{V^2}\int\sum_{\mathbf{k}} F(\mathbf{a},0)F(\mathbf{b},0)e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})}d^3a d^3b \\
&\quad + \frac{1}{V^2}\int\sum_{\mathbf{k}\neq\mathbf{m}} F(\mathbf{a},0)e^{i\mathbf{k}\cdot\mathbf{a}-i\mathbf{m}\cdot\mathbf{b}+i(\mathbf{k}-\mathbf{m})ct-i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}}F(\mathbf{b},0)d^3a d^3b \\
&= \frac{1}{V^2}\int\sum_{\mathbf{k}} F(\mathbf{a},0)F(\mathbf{b},0)e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})}d^3a d^3b \\
&\quad + \frac{1}{V^2}\int\sum_{\mathbf{m},\mathbf{k}\neq 0} F(\mathbf{a},0)e^{i\mathbf{m}\cdot(\mathbf{a}-\mathbf{b})+i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x})+i\mathbf{k}ct}F(\mathbf{b},0)d^3a d^3b
\end{aligned}
$$

Hmm. Messy. The scalar bits of the above are our energy. We have a F^2 like term in the first integral (like the Lagrangian density), but it is at different points, and we have to integrate those with a sort of vector convolution. Given the reciprocal relationships between convolution and multiplication moving between the frequency and time domains in Fourier transforms I'd expect that this first integral can somehow be turned into the sum of the squares of all the Fourier coefficients

$$
\sum_{\mathbf{k}}(C_{\mathbf{k}})^2
$$

which is very much like a discrete version of the Rayleigh energy theorem as derived in [Joot(e)], and is in this case a constant (not a function of time or space) and is dependent on only the initial field. That would mean that the remainder is the Poynting vector, which looks reasonable since it has the appearance of being somewhat antisymmetric.

Hmm, having mostly figured it out without doing the math in this case, the answer pops out. This first integral can be separated cleanly since the pseudoscalar exponentials commute with the bivector field. We then have

$$
\begin{aligned}
&\frac{1}{V^2} \int \sum_{\mathbf{k}} F(\mathbf{a}, 0) F(\mathbf{b}, 0) e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})} d^3 a d^3 b \\
&\quad = \frac{1}{V} \int \sum_{\mathbf{k}} F(\mathbf{a}, 0) e^{i\mathbf{k}\cdot\mathbf{a}} d^3 a \int F(\mathbf{b}, 0) e^{-i\mathbf{k}\cdot\mathbf{b}} d^3 b \\
&\quad = \sum_{\mathbf{k}} \hat{F}_{-\mathbf{k}} \hat{F}_{\mathbf{k}}
\end{aligned}
$$

A side note on subtle notational sneakiness here. In the assumed series solution of 8 $\hat{F}_{\mathbf{k}}(t)$ was the $\mathbf{k}$ Fourier coefficient of $F(\mathbf{x}, t)$, whereas here the use of $\hat{F}_{\mathbf{k}}$ has been used to denote the $\mathbf{k}$ Fourier coefficient of $F(\mathbf{x}, 0)$. An alternative considered and rejected was something messier like $\widehat{F(t=0)}_{\mathbf{k}}$, or the use of the original, less physically significant, $C_{\mathbf{k}}$ coefficients.

The second term could also use a simplification, and it looks like we can separate these $\mathbf{a}$ and $\mathbf{b}$ integrals too

$$
\begin{aligned}
&\frac{1}{V^2} \int \sum_{\mathbf{m},\mathbf{k}\neq 0} F(\mathbf{a}, 0) e^{i\mathbf{m}\cdot(\mathbf{a}-\mathbf{b}) + i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x}) + i\mathbf{k}ct} F(\mathbf{b}, 0) d^3 a d^3 b \\
&\quad = \frac{1}{V} \int \sum_{\mathbf{m},\mathbf{k}\neq 0} F(\mathbf{a}, 0) e^{i(\mathbf{m}+\mathbf{k})\cdot\mathbf{a}} d^3 a e^{i\mathbf{k}ct - i\mathbf{k}\cdot\mathbf{x}} \frac{1}{V} \int F(\mathbf{b}, 0) e^{-i\mathbf{m}\cdot\mathbf{b}} d^3 b \\
&\quad = \sum_{\mathbf{m}} \sum_{\mathbf{k}\neq 0} \hat{F}_{-\mathbf{m}-\mathbf{k}} e^{i\mathbf{k}ct - i\mathbf{k}\cdot\mathbf{x}} \hat{F}_{\mathbf{m}}
\end{aligned}
$$

Making an informed guess that the first integral is a scalar, and the second is a spatial vector, our energy and momentum densities (Poynting vector) respectively are

$$
U \stackrel{?}{=} \frac{1}{8\pi} \sum_{\mathbf{k}} \hat{F}_{-\mathbf{k}} \hat{F}_{\mathbf{k}} \tag{20}
$$

$$
\mathbf{P} \stackrel{?}{=} \frac{1}{8\pi} \sum_{\mathbf{m}} \sum_{\mathbf{k}\neq 0} \hat{F}_{-\mathbf{m}-\mathbf{k}} e^{i\mathbf{k}ct - i\mathbf{k}\cdot\mathbf{x}} \hat{F}_{\mathbf{m}} \tag{21}
$$

Now that much of the math is taken care of, more consideration about the physics implications is required. In particular, relating these abstract quantities to the freqencies and the harmonic oscillator model as Bohm did is desirable (that was the whole point of the exersize).

On the validity of 20, it isn't unreasonable to expect that $\partial U/\partial t = 0$, and $\boldsymbol{\nabla} \cdot \mathbf{P} = 0$ separately in these current free conditions from the energy momentum conservation relation

$$\frac{\partial}{\partial t}\frac{1}{8\pi}\left(\boldsymbol{\mathcal{E}}^2+\boldsymbol{\mathcal{H}}^2\right)+\frac{1}{4\pi}\boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{H}})=-\boldsymbol{\mathcal{E}}\cdot\mathbf{j} \tag{22}$$

Note that an SI derivation of this relation can be found in [Joot(d)]. So it therefore makes some sense that all the time dependence ends up in what has been labelled as the Poynting vector. A proof that the spatial divergence of this quantity is zero would help validate the guess made (or perhaps invalidate it).

Hmm. Again on the validity of identifing the first sum with the energy. It doesn't appear to work for the $\mathbf{k}=0$ case, since that gives you

$$\frac{1}{8\pi V^2}\int F(\mathbf{a},0)F(\mathbf{b},0)d^3ad^3b$$

That is only a scalar if the somehow all the non-scalar parts of that product somehow magically cancel out. Perhaps it's true that the second sum has no scalar part, and if that is the case one would have

$$U \stackrel{?}{=} \frac{1}{8\pi}\sum_{\mathbf{k}}\left\langle \hat{F}_{-\mathbf{k}}\hat{F}_{\mathbf{k}}\right\rangle$$

An explicit calculation of $T(\gamma_0)\cdot\gamma_0$ is probably justified to discarding all other grades, and get just the energy.

So, instead of optimistically hoping that the scalar and spatial vector terms will automatically fall out, it appears that we have to explicitly calculate the dot and wedge products, as in

$$U=-\frac{1}{16\pi}(F\gamma_0F\gamma_0+\gamma_0F\gamma_0F) \tag{23}$$

$$\mathbf{P}=-\frac{1}{16\pi}(F\gamma_0F\gamma_0-\gamma_0F\gamma_0F) \tag{24}$$

and then substitute our Fourier series solution for F to get the desired result. This appears to be getting more complex instead of less so unfortunately, but hopefully following this to a logical conclusion will show in retrospect a faster way to the desired result. A first attempt to do so shows that we have to return to our assumed Fourier solution and revisit some of the assumptions made.

5 Return to the assumed solutions to Maxwell's equation.

An initial attempt to expand 20 properly given the Fourier specification of the Maxwell solution gets into trouble. Consideration of some special cases for specific values of $\mathbf{k}$ shows that there is a problem with the grades of the solution.

Let's reexamine the assumed solution of 12 with respect to grade

$$F(\mathbf{x},t) = \frac{1}{V}\int\sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right)\exp\left(i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\right) F(\mathbf{x}',0)d^3x'$$

For scalar Fourier approximations we are used to the ability to select a subset of the Fourier terms to approximate the field, but except for the $\mathbf{k}=0$ term it appears that a term by term approximation actually introduces noise in the form of non-bivector grades.

Consider first the $\mathbf{k}=0$ term. This gives us a first order approximation of the field which is

$$F(\mathbf{x},t) \approx \frac{1}{V}\int F(\mathbf{x}',0)d^3x'$$

As summation is grade preserving this spatial average of the initial field conditions does have the required grade as desired. Next consider a non-zero fourier term such as $\mathbf{k}=\{1,0,0\}$. For this single term approximation of the field let's write out the field term as

$$F_{\mathbf{k}}(\mathbf{x},t) = \frac{1}{V}\int e^{i\hat{\mathbf{k}}|\mathbf{k}|ct + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}(\boldsymbol{\mathcal{E}}(\mathbf{x}',0) + i\boldsymbol{\mathcal{H}}(\mathbf{x}',0))d^3x'$$

Now, let's expand the exponential. This was shorthand for the product of the exponentials, which seemed to be a reasonable shorthand since we showed they commute. Expanded out this is

$$\begin{aligned}
&\exp(i\hat{\mathbf{k}}|\mathbf{k}|ct + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})) \\
&\quad = (\cos(\mathbf{k}ct) + i\hat{\mathbf{k}}\sin(|\mathbf{k}|ct))(\cos(\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})) + i\sin(\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})))
\end{aligned}$$

For ease of manipulation write $\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) = k\Delta x$, and $\mathbf{k}ct = \boldsymbol{\omega}t$, we have

$$\begin{aligned}
\exp(i\boldsymbol{\omega}t + ik\Delta x) &= \cos(\boldsymbol{\omega}t)\cos(k\Delta x) + i\cos(\boldsymbol{\omega}t)\sin(k\Delta x) \\
&\quad + i\sin(\boldsymbol{\omega}t)\cos(k\Delta x) - \sin(\boldsymbol{\omega}t)\sin(k\Delta x)
\end{aligned}$$

Note that $\cos(\boldsymbol{\omega}t)$ is a scalar, whereas $\sin(\boldsymbol{\omega}t)$ is a (spatial) vector in the direction of $\mathbf{k}$. Multiplying this out with the initial time field $F(\mathbf{x}',0) = \boldsymbol{\mathcal{E}}(\mathbf{x}',0) + i\boldsymbol{\mathcal{H}}(\mathbf{x}',0) = \boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}'$ we can separate into grades.

$$
\begin{aligned}
&\exp(i\omega t + ik\Delta x)(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}') \\
&\quad = \cos(\boldsymbol{\omega} t)(\boldsymbol{\mathcal{E}}' \cos(k\Delta x) - \boldsymbol{\mathcal{H}}' \sin(k\Delta x)) + \sin(\boldsymbol{\omega} t) \times (\boldsymbol{\mathcal{H}}' \sin(k\Delta x) - \boldsymbol{\mathcal{E}}' \cos(k\Delta x)) \\
&\quad + i\cos(\boldsymbol{\omega} t)(\boldsymbol{\mathcal{E}}' \sin(k\Delta x) + \boldsymbol{\mathcal{H}}' \cos(k\Delta x)) - i\sin(\boldsymbol{\omega} t) \times (\boldsymbol{\mathcal{E}}' \sin(k\Delta x) + \boldsymbol{\mathcal{H}}' \cos(k\Delta x)) \\
&\quad - \sin(\boldsymbol{\omega} t) \cdot (\boldsymbol{\mathcal{E}}' \sin(k\Delta x) + \boldsymbol{\mathcal{H}}' \cos(k\Delta x)) \\
&\quad + i(\sin(\boldsymbol{\omega} t) \cdot (\boldsymbol{\mathcal{E}}' \cos(k\Delta x) - \boldsymbol{\mathcal{H}}' \sin(k\Delta x))
\end{aligned}
$$

The first two lines, once integrated, produce the electric and magnetic fields, but the last two are rogue scalar and pseudoscalar terms. These are allowed in so far as they are still solutions to the differential equation, but do not have the desired physical meaning.

If one explicitly sums over pairs of $\{\mathbf{k}, -\mathbf{k}\}$ of index triplets then some cancellation occurs. The cosine cosine products and sine sine products double and the sine cosine terms cancel. We therefore have

$$
\begin{aligned}
&\frac{1}{2}\exp(i\omega t + ik\Delta x)(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}') \\
&\quad = \cos(\boldsymbol{\omega} t)\boldsymbol{\mathcal{E}}' \cos(k\Delta x) + \sin(\boldsymbol{\omega} t) \times \boldsymbol{\mathcal{H}}' \sin(k\Delta x) \\
&\quad + i\cos(\boldsymbol{\omega} t)\boldsymbol{\mathcal{H}}' \cos(k\Delta x) - i\sin(\boldsymbol{\omega} t) \times \boldsymbol{\mathcal{E}}' \sin(k\Delta x) \\
&\quad - \sin(\boldsymbol{\omega} t) \cdot \boldsymbol{\mathcal{E}}' \sin(k\Delta x) \\
&\quad - i\sin(\boldsymbol{\omega} t) \cdot \boldsymbol{\mathcal{H}}' \sin(k\Delta x) \\
&\quad = (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\cos(\boldsymbol{\omega} t)\cos(k\Delta x) - i\sin(\boldsymbol{\omega} t) \times (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\sin(k\Delta x) \\
&\quad - \sin(\boldsymbol{\omega} t) \cdot (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}})\sin(k\Delta x)
\end{aligned}
$$

Here for grouping purposes i is treated as a scalar, which should be justifyable in this specific case. A final grouping produces

$$
\begin{aligned}
\frac{1}{2}\exp(i\omega t + ik\Delta x)(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}') &= (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\cos(\boldsymbol{\omega} t)\cos(k\Delta x) \\
&- i\hat{\mathbf{k}} \times (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\sin(|\boldsymbol{\omega}|t)\sin(k\Delta x) \\
&- \sin(\boldsymbol{\omega} t) \cdot (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\sin(k\Delta x)
\end{aligned}
$$

Observe that despite the grouping of the summation over the pairs of complementary sign index triplets we still have a pure scalar and pure pseudoscalar term above. Allowable by the math since the differential equation had no way of encoding the grade of the desired solution. That only came from the initial time specification of $F(\mathbf{x}', 0)$, but that isn't enough.

Now, from above, we can see that one way to reconsile this grade requirement is to require both $\hat{\mathbf{k}} \cdot \boldsymbol{\mathcal{E}}' = 0$, and $\hat{\mathbf{k}} \cdot \boldsymbol{\mathcal{H}}' = 0$. How can such a requirement make sense given that $\mathbf{k}$ ranges over all directions in space, and that both $\boldsymbol{\mathcal{E}}'$ and $\boldsymbol{\mathcal{H}}'$ could concievably range over many different directions in the volume of periodicity.

With no other way out, it seems that we have to impose two requirements, one on the allowable wavenumber vector directions (which in turn means we can only pick specific orientations of the Fourier volume), and another on the field directions themselves. The electric and magnetic fields must therefore be directed only perpendicular to the wave number vector direction. Wow, that's a pretty severe implication following strictly from a grade requirement!

Thinking back to equation 16, it appears that an implication of this is that we have

$$e^{i\omega t} F(\mathbf{x}', 0) = F(\mathbf{x}', 0) e^{-i\omega t}$$

Knowing this is a required condition should considerably simplify the energy and momentum questions.

6 Appendix. Summary of Notation used.

Here is a summary of the notation, following largely the conventions from [Doran and Lasenby(2003)], but modified here for cgs units as used in [Bohm(1989)] Greek letters range over all indexes and english indexes range over $1, 2, 3$. Bold vectors are spatial enties and non-bold is used for four vectors and scalars. Summation convention is in effect unless otherwise noted, with implied summation over all sets of matched upper and lower indexes.

γ_μ	$\gamma_\mu \cdot \gamma_\nu = \pm\delta^\mu{}_\nu$	Four vector basis vector
$(\gamma_0)^2(\gamma_k)^2$	$= -1$	Minkowski metric
$\sigma_k = \sigma^k$	$= \gamma_k \wedge \gamma_0$	Spatial basis bivector. ($\sigma_k \cdot \sigma_j = \delta_{kj}$)
	$= \gamma_k \gamma_0$	
i	$= \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$	Four-vector pseudoscalar
	$= \gamma_0 \gamma_1 \gamma_2 \gamma_3$	
$\gamma^\mu \cdot \gamma_\nu$	$= \delta^\mu{}_\nu$	Reciprocal basis vectors
x^μ	$= x \cdot \gamma^\mu$	Vector coordinate
x_μ	$= x \cdot \gamma_\mu$	Coordinate for reciprocal basis
x	$= \gamma_\mu x^\mu$	Four vector in terms of coordinates
	$= \gamma^\mu x_\mu$	
$\boldsymbol{\mathcal{E}}$	$= E^k \sigma_k$	Electric field spatial vector
$\boldsymbol{\mathcal{H}}$	$= H^k \sigma_k$	Magnetic field spatial vector
J	$= \gamma_\mu J^\mu$	Current density four vector.
	$= \gamma^\mu J_\mu$	
F	$= \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}$	Faraday bivector
	$= F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$	in terms of Faraday tensor
x^0	$= x \cdot \gamma^0$	Time coordinate (length dim.)
	$= ct$	
$\mathbf{x}$	$= x \wedge \gamma_0$	Spatial vector
	$= x^k \sigma_k$	
J^0	$= J \cdot \gamma^0$	Charge density.
	$= \rho$	(current density dimensions.)
$\mathbf{j}$	$= J \wedge \gamma_0$	Current density spatial vector
	$= J^k \sigma_k$	
∂_μ	$= \partial / \partial x^\mu$	Index up partial.
∂^μ	$= \partial / \partial x_\mu$	Index down partial.
$\partial_{\mu\nu}$	$= \partial / \partial x^\mu \partial / \partial x^\nu$	Index up partial.
∇	$= \sum \gamma^\mu \partial / \partial x^\mu$	Spacetime gradient
	$= \gamma^\mu \partial_\mu$	
	$= \sum \gamma_\mu \partial / \partial x_\mu$	
	$= \gamma_\mu \partial^\mu$	
$\boldsymbol{\nabla}$	$= \sigma^k \partial_k$	Spatial gradient
$\hat{A}_\mathbf{k}$	$= \hat{A}_{k_1,k_2,k_3}$	Fourier coefficient, integer indexes.
$\nabla^2 A$	$= (\nabla \cdot \nabla) A$	Four Laplacian.
	$= (\partial_{00} - \sum_k \partial_{kk}) A$	
x^2	$= x \cdot x$	Four vector square.
	$= x^\mu x_\mu$	
$\mathbf{x}^2$	$= \mathbf{x} \cdot \mathbf{x}$	Spatial vector square.
	$= \sum_{k=1}^3 (x^k)^2$	
	$= \lvert\mathbf{x}\rvert^2$	
$d^3 x$	$= dx^1 dx^2 dx^3$	Spatial volume element.
$\int_{\partial I}$	$= \int_a^b$	Integration range $I = [a, b]$
STA		Space Time Algebra
$(xyz)\tilde{}$	$= \widetilde{xyz} = zyx$	Reverse of a vector product.

While many things could be formulated in a metric signature independent fashion, no effort to do so here has been made. Assume a time positive $(+,-,-,-)$ metric signature. Specifically, that is $(\gamma_0)^2 = 1$, and $(\gamma_k)^2 = -1$.

References

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