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EXPLORING PHYSICS WITH GEOMETRIC ALGEBRA

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DOCUMENT VERSION

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Dedicated to Aurora and Lance, my awesome kids.

PREFACE

This is a somewhat hodge podge, and very exploratory, compilation of Geometric (or Clifford) Algebra related notes on mathematics and Physics.

Most of what appear here as chapters were originally disjoint standalone notes. I eventually accumulated enough of these individual notes that assembling them into a bookish form made some sense, even if only for personal organizational purposes. Since my original notes were disconnected, this assembled form is not necessarily in a logical sequence, so in some cases reading in a chronological sequence (H) may be helpful.

Because of the journaling nature of many of these notes, a reader will find that I do not always know where I am going or what the final result will be ahead of time. This is much different than what you will find in a polished textbook where the author knows the subject like the back of his hand. I sometimes hit dead ends, mistakes, or unproductive paths. You will find repetition and rework of topics that were not initially covered satisfactorily, and unlike a carefully crafted text, these false starts have not all been purged.

The use of this algebra in Physics could be said to be still in its infancy. There is a fair amount Geometric Algebra in advanced treatments like the work of the Cambridge group [10]. There is much less that is easily accessible to someone with undergrad level education. Even a text like Hestenes's New Foundations [19], which has a more elementary target audience is fairly difficult to read. These notes attempt to bridge some of that gap.

I can not promise that I have explained things in a way that is good for anybody else. My audience was essentially myself as I existed at the time of writing, so the prerequisites, both for the mathematics and the Physics, have evolved continually.

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Part I

BASICS AND GEOMETRY

1 INTRODUCTORY CONCEPTS

1.1 MOTIVATION

As an exercise work out axiomatically some of the key vector identities of Geometric Algebra.

Want to at least derive the vector bivector dot product distribution identity

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \tag{1.1}$$

At the same time attempt here to provide a naturally sequenced introduction to the algebra.

1.2 THE AXIOMS

Two basic axioms are required, contraction and associative multiplication respectively

$$\begin{aligned} a^2 &= \text{scalar} \\ a(bc) &= (ab)c = abc \end{aligned} \tag{1.2}$$

Linearity and scalar multiplication should probably also be included for completeness, but even with those this is a surprisingly small set of rules. The choice to impose these as the rules for vector multiplication will be seen to have a rich set of consequences once explored. It will take a fair amount of work to extract all the consequences of this decision, and some of that will be done here.

1.3 CONTRACTION AND THE METRIC

Defining a^2 itself requires introduction of a metric, the specification of the multiplication rules for a particular basis for the vector space. For Euclidean spaces, a requirement that

$$a^2 = |a|^2 \tag{1.3}$$

is sufficient to implicitly define this metric. However, for the Minkowski spaces of special relativity one wants the squares of time and spatial basis vectors to be opposing in sign. Deferring the discussion of metric temporarily one can work with the axioms above to discover their implications, and in particular how these relate to the coordinate vector space constructions that are so familiar.

1.4 SYMMETRIC SUM OF VECTOR PRODUCTS

Squaring a vector sum provides the first interesting feature of the general vector product

$$(a+b)^2 = a^2 + b^2 + ab + ba \tag{1.4}$$

Observe that the LHS is a scalar by the contraction identity, and on the RHS we have scalars a^2 and b^2 by the same. This implies that the symmetric sum of products

$$ab + ba \tag{1.5}$$

is also a scalar, independent of any choice of metric. Symmetric sums of this form have a place in physics over the space of operators, often instantiated in matrix form. There one writes this as the commutator and denotes it as

$$\{a, b\} \equiv ab + ba \tag{1.6}$$

In an Euclidean space one can observe that equation 1.4 has the same structure as the law of cosines so it should not be surprising that this symmetric sum is also related to the dot product. For a Euclidean space where one the notion of perpendicularity can be expressed as

$$|a+b|^2 = |a|^2 + |b|^2 \tag{1.7}$$

we can then see that an implication of the vector product is the fact that perpendicular vectors have the property

$$ab + ba = 0 \tag{1.8}$$

or

$$ba = -ab \tag{1.9}$$

This notion of perpendicularity will also be seen to make sense for non-Euclidean spaces.

Although it retracts from a purist Geometric Algebra approach where things can be done in a coordinate free fashion, the connection between the symmetric product and the standard vector dot product can be most easily shown by considering an expansion with respect to an orthonormal basis.

Lets write two vectors in an orthonormal basis as

$$\begin{aligned} a &= \sum_{\mu} a^{\mu} e_{\mu} \\ b &= \sum_{\mu} b^{\mu} e_{\mu} \end{aligned} \tag{1.10}$$

Here the choice to utilize raised indices rather than lower for the coordinates is taken from physics where summation is typically implied when upper and lower indices are matched as above.

Forming the symmetric product we have

$$\begin{aligned} ab + ba &= \sum_{\mu,\nu} a^{\mu} e_{\mu} b^{\nu} e_{\nu} + b^{\mu} e_{\mu} a^{\nu} e_{\nu} \\ &= \sum_{\mu,\nu} a^{\mu} b^{\nu} \left(e_{\mu} e_{\nu} + e_{\nu} e_{\mu} \right) \\ &= 2 \sum_{\mu} a^{\mu} b^{\mu} {e_{\mu}}^2 + \sum_{\mu \neq \nu} a^{\mu} b^{\nu} \left(e_{\mu} e_{\nu} + e_{\nu} e_{\mu} \right) \end{aligned} \tag{1.11}$$

For an Euclidean space we have ${e_{\mu}}^2 = 1$, and $e_{\nu} e_{\mu} = -e_{\mu} e_{\nu}$, so we are left with

$$\sum_{\mu} a^{\mu} b^{\mu} = \frac{1}{2}(ab + ba) \tag{1.12}$$

This shows that we can make an identification between the symmetric product, and the anti-commutator of physics with the dot product, and then define

$$a \cdot b \equiv \frac{1}{2}\{a, b\} = \frac{1}{2}(ab + ba) \tag{1.13}$$

1.5 ANTISYMMETRIC PRODUCT OF TWO VECTORS (WEDGE PRODUCT)

Having identified or defined the symmetric product with the dot product we are now prepared to examine a general product of two vectors. Employing a symmetric + antisymmetric decomposition we can write such a general product as

$$ab = \overbrace{\boxed{\frac{1}{2}(ab + ba)}}^{a \cdot b} + \overbrace{\boxed{\frac{1}{2}(ab - ba)}}^{a \text{ something } b} \tag{1.14}$$

What is this remaining vector operation between the two vectors

$$a \text{ something } b = \frac{1}{2}(ab - ba) \tag{1.15}$$

One can continue the comparison with the quantum mechanics, and like the anticommutator operator that expressed our symmetric sum in equation eq. (1.6) one can introduce a commutator operator

$$[a, b] \equiv ab - ba \tag{1.16}$$

The commutator however, does not naturally extend to more than two vectors, so as with the scalar part of the vector product (the dot product part), it is desirable to make a different identification for this part of the vector product.

One observation that we can make is that this vector operation changes sign when the operations are reversed. We have

$$b \text{ something } a = \frac{1}{2}(ba - ab) = -a \text{ something } b \tag{1.17}$$

Similarly, if a and b are colinear, say $b = \alpha a$, this product is zero

$$\begin{aligned} a \text{ something}(\alpha a) &= \frac{1}{2}(a(\alpha a) - (\alpha a)a) \\ &= 0 \end{aligned} \tag{1.18}$$

This complete antisymmetry, aside from a potential difference in sign, are precisely the properties of the wedge product used in the mathematics of differential forms. In this differential geometry the wedge product of m one-forms (vectors in this context) can be defined as

$$a_1 \wedge a_2 \cdots \wedge a_m = \frac{1}{m!} \sum a_{i_1} a_{i_2} \cdots a_{i_m} \operatorname{sgn}(\pi(i_1 i_2 \cdots i_m)) \tag{1.19}$$

Here $\operatorname{sgn}(\pi(\cdots))$ is the sign of the permutation of the indices. While we have not gotten yet to products of more than two vectors it is helpful to know that the wedge product will have a place in such a general product. An equation like eq. (1.19) makes a lot more sense after writing it out in full for a few specific cases. For two vectors a_1 and a_2 this is

$$a_1 \wedge a_2 = \frac{1}{2}\left(a_1 a_2 (1) + a_2 a_1 (-1)\right) \tag{1.20}$$

and for three vectors this is

$$\begin{aligned} a_1 \wedge a_2 \wedge a_3 = \frac{1}{6}(&a_1 a_2 a_3 (1) + a_1 a_3 a_2 (-1) \\ &+ a_2 a_1 a_3 (-1) + a_3 a_1 a_2 (1) \\ &+ a_2 a_3 a_1 (1) + a_3 a_2 a_1 (-1)) \end{aligned} \tag{1.21}$$

We will see later that this completely antisymmetrized sum, the wedge product of differential forms will have an important place in this algebra, but like the dot product it is a specific construction of the more general vector product. The choice to identify the antisymmetric sum with the wedge product is an action that amounts to a definition of the wedge product. Explicitly, and complementing the dot product definition of eq. (1.13) for the dot product of two vectors, we say

$$a \wedge b \equiv \frac{1}{2}\left[a, b\right] = \frac{1}{2}(ab - ba) \tag{1.22}$$

Having made this definition, the symmetric and antisymmetric decomposition of two vectors leaves us with a peculiar looking hybrid construction:

$$ab = a \cdot b + a \wedge b \tag{1.23}$$

We had already seen that part of this vector product was not a vector, but was in fact a scalar. We now see that the remainder is also not a vector but is instead something that resides in a different space. In differential geometry this object is called a two form, or a simple element in $\bigwedge^2$. Various labels are available for this object are available in Geometric (or Clifford) algebra, one of which is a 2-blade. 2-vector or bivector is also used in some circumstances, but in dimensions greater than three there are reasons to reserve these labels for a slightly more general construction.

The definition of eq. (1.23) is often used as the starting point in Geometric Algebra introductions. While there is value to this approach I have personally found that the non-axiomatic approach becomes confusing if one attempts to sort out which of the many identities in the algebra are the fundamental ones. That is why my preference is to treat this as a consequence rather than the starting point.

1.6 EXPANSION OF THE WEDGE PRODUCT OF TWO VECTORS

Many introductory geometric algebra treatments try very hard to avoid explicit coordinate treatment. It is true that GA provides infrastructure for coordinate free treatment, however, this avoidance perhaps contributes to making the subject less accessible. Since we are so used to coordinate geometry in vector and tensor algebra, let us take advantage of this comfort, and express the wedge product explicitly in coordinate form to help get some comfort for it.

Employing the definition of eq. (1.22), and an orthonormal basis expansion in coordinates for two vectors a, and b, we have

$$\begin{aligned}
2(a \wedge b) &= (ab - ba) \\
&= \sum_{\mu,\nu} a^\mu b^\nu e_\mu e_\nu - \sum_{\alpha,\beta} a^\alpha b^\beta e_\alpha e_\beta \\
&= \underbrace{\boxed{\sum_{\mu} a^\mu b^\mu - \sum_{\alpha} a^\alpha b^\alpha}}^{=0} + \sum_{\mu \neq \nu} a^\mu b^\nu e_\mu e_\nu - \sum_{\alpha \neq \beta} a^\alpha b^\beta e_\alpha e_\beta \\
&= \sum_{\mu<\nu} (a^\mu b^\nu e_\mu e_\nu + a^\nu b^\mu e_\nu e_\mu) - \sum_{\alpha<\beta} (a^\alpha b^\beta e_\alpha e_\beta + a^\beta b^\alpha e_\beta e_\alpha) \\
&= 2 \sum_{\mu<\nu} (a^\mu b^\nu - a^\nu b^\mu) e_\mu e_\nu
\end{aligned} \tag{1.24}$$

So we have

$$a \wedge b = \sum_{\mu<\nu} \begin{vmatrix} a^\mu & a^\nu \\ b^\mu & b^\nu \end{vmatrix} e_\mu e_\nu \tag{1.25}$$

The similarity to the $\mathbb{R}^3$ vector cross product is not accidental. This similarity can be made explicit by observing the following

$$\begin{aligned}
e_1 e_2 &= e_1 e_2 (e_3 e_3) = (e_1 e_2 e_3) e_3 \\
e_2 e_3 &= e_2 e_3 (e_1 e_1) = (e_1 e_2 e_3) e_1 \\
e_1 e_3 &= e_1 e_3 (e_2 e_2) = -(e_1 e_2 e_3) e_2
\end{aligned} \tag{1.26}$$

This common factor, a product of three normal vectors, or grade three blade, is called the pseudoscalar for $\mathbb{R}^3$. We write $i = e_1e_2e_3$, and can then express the $\mathbb{R}^3$ wedge product in terms of the cross product

$$
\begin{aligned}
a \wedge b &= \begin{vmatrix} a^2 & a^3 \\ b^2 & b^3 \end{vmatrix} e_2e_3 + \begin{vmatrix} a^1 & a^3 \\ b^1 & b^3 \end{vmatrix} e_1e_3 + \begin{vmatrix} a^1 & a^2 \\ b^1 & b^2 \end{vmatrix} e_1e_2 \\
&= (e_1e_2e_3)\left(\begin{vmatrix} a^2 & a^3 \\ b^2 & b^3 \end{vmatrix} e_1 - \begin{vmatrix} a^1 & a^3 \\ b^1 & b^3 \end{vmatrix} e_2 + \begin{vmatrix} a^1 & a^2 \\ b^1 & b^2 \end{vmatrix} e_3 \right)
\end{aligned} \tag{1.27}
$$

This is

$$
a \wedge b = i(a \times b) \tag{1.28}
$$

With this identification we now also have a curious integrated relation where the dot and cross products are united into a single structure

$$
ab = a \cdot b + i(a \times b) \tag{1.29}
$$

1.7 VECTOR PRODUCT IN EXPONENTIAL FORM

One naturally expects there is an inherent connection between the dot and cross products, especially when expressed in terms of the angle between the vectors, as in

$$
\begin{aligned}
a \cdot b &= |a||b| \cos \theta_{a,b} \\
a \times b &= |a||b| \sin \theta_{a,b} \hat{\mathbf{n}}_{a,b}
\end{aligned} \tag{1.30}
$$

However, without the structure of the geometric product the specifics of what connection is is not obvious. In particular the use of eq. (1.29) and the angle relations, one can easily blunder upon the natural complex structure of the geometric product

$$
\begin{aligned}
ab &= a \cdot b + i(a \times b) \\
&= |a||b| \left(\cos \theta_{a,b} + i\hat{\mathbf{n}}_{a,b} \sin \theta_{a,b} \right)
\end{aligned} \tag{1.31}
$$

As we have seen pseudoscalar multiplication in $\mathbb{R}^3$ provides a mapping between a grade 2 blade and a vector, so this $i\hat{\mathbf{n}}$ product is a 2-blade.

In $\mathbb{R}^3$ we also have $i\hat{\mathbf{n}} = \hat{\mathbf{n}}i$ (exercise for reader) and also $i^2 = -1$ (again for the reader), so this 2-blade $i\hat{\mathbf{n}}$ has all the properties of the i of complex arithmetic. We can, in fact, write

$$\begin{aligned} ab &= a \cdot b + i(a \times b) \\ &= |a||b| \exp(i\hat{\mathbf{n}}_{a,b}\theta_{a,b}) \end{aligned} \tag{1.32}$$

In particular, for unit vectors a, b one is able to quaternion exponentials of this form to rotate from one vector to the other

$$b = a \exp(i\hat{\mathbf{n}}_{a,b}\theta_{a,b}) \tag{1.33}$$

This natural GA use of multivector exponentials to implement rotations is not restricted to $\mathbb{R}^3$ or even Euclidean space, and is one of the most powerful features of the algebra.

1.8 PSEUDOSCALAR

In general the pseudoscalar for $\mathbb{R}^N$ is a product of N normal vectors and multiplication by such an object maps m-blades to (N-m) blades.

For $\mathbb{R}^2$ the unit pseudoscalar has a negative square

$$\begin{aligned} (e_1e_2)(e_1e_2) &= -(e_2e_1)(e_1e_2) \\ &= -e_2(e_1e_1)e_2 \\ &= -e_2e_2 \\ &= -1 \end{aligned} \tag{1.34}$$

and we have seen an example of such a planar pseudoscalar in the subspace of the span of two vectors above (where $\hat{\mathbf{n}}i$ was a pseudoscalar for that subspace). In general the sign of the square of the pseudoscalar depends on both the dimension and the metric of the space, so the "complex" exponentials that rotate one vector into another may represent hyperbolic rotations.

For example we have for a four dimensional space the pseudoscalar square is

$$\begin{aligned} i^2 &= (e_0e_1e_2e_3)(e_0e_1e_2e_3) \\ &= -e_0e_0e_1e_2e_3e_1e_2e_3 \\ &= -e_0e_0e_1e_2e_3e_1e_2e_3 \\ &= -e_0e_0e_1e_1e_2e_3e_2e_3 \\ &= e_0e_0e_1e_1e_2e_2e_3e_3 \end{aligned} \tag{1.35}$$

For a Euclidean space where each of the ${e_k}^2 = 1$, we have $i^2 = 1$, but for a Minkowski space where one would have for $k \neq 0$, ${e_0}^2{e_k}^2 = -1$, we have $i^2 = -1$

Such a mixed signature metric will allow for implementation of Lorentz transformations as exponentials (hyperbolic) rotations in a fashion very much like the quaternionic spatial rotations for Euclidean spaces.

It is also worth pointing out that the pseudoscalar multiplication naturally provides a mapping operator into a dual space, as we have seen in the cross product example, mapping vectors to bivectors, or bivectors to vectors. Pseudoscalar multiplication in fact provides an implementation of the Hodge duality operation of differential geometry.

In higher than three dimensions, such as four, this duality operation can in fact map 2-blades to orthogonal 2-blades (orthogonal in the sense of having no common factors). Take for example the typical example of a non-simple element from differential geometry

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4 \tag{1.36}$$

The two blades that compose this sum have no common factors and thus cannot be formed as the wedge product of two vectors. These two blades are orthogonal in a sense that can be made more exact later. As this time we just wish to make the observation that the pseudoscalar provides a natural duality operation between these two subspaces of $\bigwedge^2$. Take for example

$$\begin{aligned}
ie_1 \wedge e_2 &= e_1e_2e_3e_4e_1e_2 \\
&= -e_1e_1e_2e_3e_4e_2 \\
&= -e_1e_1e_2e_2e_3e_4 \\
&\propto e_3e_4
\end{aligned} \tag{1.37}$$

1.9 HIGHER ORDER PRODUCTS

2 GEOMETRIC ALGEBRA. THE VERY QUICKEST INTRODUCTION

2.1 MOTIVATION

An attempt to make a relatively concise introduction to Geometric (or Clifford) Algebra. Much more complete introductions to the subject can be found in [11], [10], and [19].

2.2 AXIOMS

We have a couple basic principles upon which the algebra is based

- Vectors can be multiplied.
- The square of a vector is the (squared) length of that vector (with appropriate generalizations for non-Euclidean metrics).
- Vector products are associative (but not necessarily commutative).

That is really all there is to it, and the rest, paraphrasing Feynman, can be figured out by anybody sufficiently clever.

2.3 BY EXAMPLE. THE 2D CASE

Consider a 2D Euclidean space, and the product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in that space. Utilizing a standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ we can write

$$\begin{aligned}
\mathbf{a} &= \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 \\
\mathbf{b} &= \mathbf{e}_1 y_1 + \mathbf{e}_2 y_2,
\end{aligned} \tag{2.1}$$

and let us write out the product of these two vectors $\mathbf{ab}$, not yet knowing what we will end up with. That is

$$\begin{aligned}
\mathbf{ab} &= (\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2)(\mathbf{e}_1 y_1 + \mathbf{e}_2 y_2) \\
&= \mathbf{e}_1^2 x_1 y_1 + \mathbf{e}_2^2 x_2 y_2 + \mathbf{e}_1 \mathbf{e}_2 x_1 y_2 + \mathbf{e}_2 \mathbf{e}_1 x_2 y_1
\end{aligned} \tag{2.2}$$

From axiom 2 we have $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$, so we have

$$\mathbf{ab} = x_1 y_1 + x_2 y_2 + \mathbf{e}_1\mathbf{e}_2 x_1 y_2 + \mathbf{e}_2\mathbf{e}_1 x_2 y_1. \tag{2.3}$$

We have multiplied two vectors and ended up with a scalar component (and recognize that this part of the vector product is the dot product), and a component that is a "something else". We will call this something else a bivector, and see that it is characterized by a product of non-colinear vectors. These products $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_1$ are in fact related, and we can see that by looking at the case of $\mathbf{b} = \mathbf{a}$. For that we have

$$\begin{aligned} \mathbf{a}^2 &= x_1 x_1 + x_2 x_2 + \mathbf{e}_1\mathbf{e}_2 x_1 x_2 + \mathbf{e}_2\mathbf{e}_1 x_2 x_1 \\ &= |\mathbf{a}|^2 + x_1 x_2(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) \end{aligned} \tag{2.4}$$

Since axiom (2) requires our vectors square to equal its (squared) length, we must then have

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0, \tag{2.5}$$

or

$$\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2. \tag{2.6}$$

We see that Euclidean orthonormal vectors anticommute. What we can see with some additional study is that any colinear vectors commute, and in Euclidean spaces (of any dimension) vectors that are normal to each other anticommute (this can also be taken as a definition of normal).

We can now return to our product of two vectors eq. (2.3) and simplify it slightly

$$\mathbf{ab} = x_1 y_1 + x_2 y_2 + \mathbf{e}_1\mathbf{e}_2(x_1 y_2 - x_2 y_1). \tag{2.7}$$

The product of two vectors in 2D is seen here to have one scalar component, and one bivector component (an irreducible product of two normal vectors). Observe the symmetric and antisymmetric split of the scalar and bivector components above. This symmetry and antisymmetry can be made explicit, introducing dot and wedge product notation respectively

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = x_1 y_1 + x_2 y_2 \\ \mathbf{a}\wedge\mathbf{b} &= \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = \mathbf{e}_1\mathbf{e}_2(x_1 y_y - x_2 y_1). \end{aligned} \tag{2.8}$$

so that the vector product can be written as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \tag{2.9}$$

2.4 PSEUDOSCALAR

In many contexts it is useful to introduce an ordered product of all the unit vectors for the space is called the pseudoscalar. In our 2D case this is

$$i = \mathbf{e}_1\mathbf{e}_2, \tag{2.10}$$

a quantity that we find behaves like the complex imaginary. That can be shown by considering its square

$$\begin{aligned}
(\mathbf{e}_1\mathbf{e}_2)^2 &= (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_2) \\
&= \mathbf{e}_1(\mathbf{e}_2\mathbf{e}_1)\mathbf{e}_2 \\
&= -\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2 \\
&= -(\mathbf{e}_1\mathbf{e}_1)(\mathbf{e}_2\mathbf{e}_2) \\
&= -1^2 \\
&= -1
\end{aligned} \tag{2.11}$$

Here the anticommutation of normal vectors property has been used, as well as (for the first time) the associative multiplication axiom.

In a 3D context, you will see the pseudoscalar in many places (expressing the normals to planes for example). It also shows up in a number of fundamental relationships. For example, if one writes

$$I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \tag{2.12}$$

for the 3D pseudoscalar, then it is also possible to show

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + I(\mathbf{a} \times \mathbf{b}) \tag{2.13}$$

something that will be familiar to the student of QM, where we see this in the context of Pauli matrices. The Pauli matrices also encode a Clifford algebraic structure, but we do not need an explicit matrix representation to do so.

2.5 ROTATIONS

Very much like complex numbers we can utilize exponentials to perform rotations. Rotating in a sense from $\mathbf{e}_1$ to $\mathbf{e}_2$, can be expressed as

$$\begin{aligned}\mathbf{a}e^{i\theta} &= (\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2)(\cos\theta + \mathbf{e}_1\mathbf{e}_2 \sin\theta) \\ &= \mathbf{e}_1(x_1\cos\theta - x_2\sin\theta) + \mathbf{e}_2(x_2\cos\theta + x_1\sin\theta)\end{aligned} \tag{2.14}$$

More generally, even in N dimensional Euclidean spaces, if $\mathbf{a}$ is a vector in a plane, and $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are perpendicular unit vectors in that plane, then the rotation through angle θ is given by

$$\mathbf{a} \rightarrow \mathbf{a}e^{\hat{\mathbf{u}}\hat{\mathbf{v}}\theta}. \tag{2.15}$$

This is illustrated in fig. 2.1

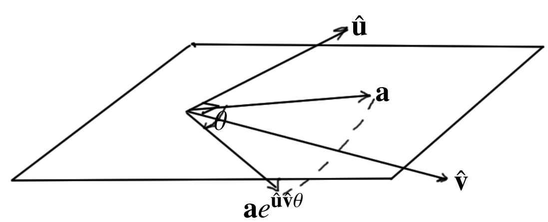

Figure 2.1: Plane rotation

Notice that we have expressed the rotation here without utilizing a normal direction for the plane. The sense of the rotation is encoded by the bivector $\hat{\mathbf{u}}\hat{\mathbf{v}}$ that describes the plane and the orientation of the rotation (or by duality the direction of the normal in a 3D space). By avoiding a requirement to encode the rotation using a normal to the plane we have an method of expressing the rotation that works not only in 3D spaces, but also in 2D and greater than 3D spaces, something that is not possible when we restrict ourselves to traditional vector algebra (where quantities like the cross product can not be defined in a 2D or 4D space, despite the fact that things they may represent, like torque are planar phenomena that do not have any intrinsic requirement for a normal that falls out of the plane.).

When $\mathbf{a}$ does not lie in the plane spanned by the vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, as in fig. 2.2, we must express the rotations differently. A rotation then takes the form

$$\mathbf{a} \rightarrow e^{-\hat{\mathbf{u}}\hat{\mathbf{v}}\theta/2}\mathbf{a}e^{\hat{\mathbf{u}}\hat{\mathbf{v}}\theta/2}. \tag{2.16}$$

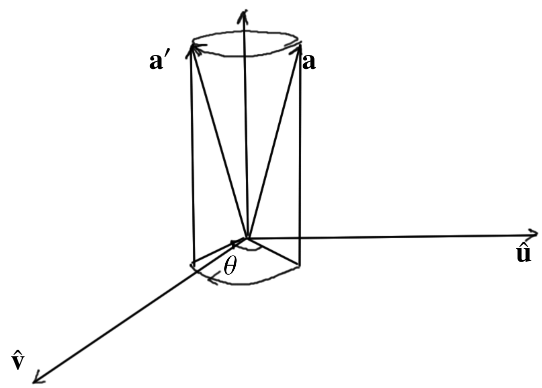

Figure 2.2: 3D rotation

In the 2D case, and when the vector lies in the plane this reduces to the one sided complex exponential operator used above. We see these types of paired half angle rotations in QM, and they are also used extensively in computer graphics under the guise of quaternions.

3 AN (EARLIER) ATTEMPT TO INTUITIVELY INTRODUCE THE DOT, WEDGE, CROSS, AND GEOMETRIC PRODUCTS

3.1 MOTIVATION

Both the NFCM and GAFP books have axiomatic introductions of the generalized (vector, blade) dot and wedge products, but there are elements of both that I was unsatisfied with. Perhaps the biggest issue with both is that they are not presented in a dumb enough fashion.

NFCM presents but does not prove the generalized dot and wedge product operations in terms of symmetric and antisymmetric sums, but it is really the grade operation that is fundamental. You need that to define the dot product of two bivectors for example.

GAFP axiomatic presentation is much clearer, but the definition of generalized wedge product as the totally antisymmetric sum is a bit strange when all the differential forms book give such a different definition.

Here I collect some of my notes on how one starts with the geometric product action on colinear and perpendicular vectors and gets the familiar results for two and three vector products. I may not try to generalize this, but just want to see things presented in a fashion that makes sense to me.

3.2 INTRODUCTION

The aim of this document is to introduce a "new" powerful vector multiplication operation, the geometric product, to a student with some traditional vector algebra background.

The geometric product, also called the Clifford product [1], has remained a relatively obscure mathematical subject. This operation actually makes a great deal of vector manipulation simpler than possible with the traditional methods, and provides a way to naturally expresses many geometric concepts. There is a great deal of information available on the subject, however most of it is targeted for those with a university graduate school background in physics or mathematics. That level of mathematical sophistication should not required to understand the subject.

It is the author's opinion that this could be dumbed down even further, so that it would be palatable for somebody without any traditional vector algebra background.

1 After William Clifford (1845-1879).

3.3 WHAT IS MULTIPLICATION?

The operations of vector addition, subtraction and numeric multiplication have the usual definitions (addition defined in terms of addition of coordinates, and numeric multiplication as a scaling of the vector retaining its direction). Multiplication and division of vectors is often described as "undefined". It is possible however, to define a multiplication, or division operation for vectors, in a natural geometric fashion.

What meaning should be given to multiplication or division of vectors?

3.3.1 *Rules for multiplication of numbers*

Assuming no prior knowledge of how to multiply two vectors (such as the dot, cross, or wedge products to be introduced later) consider instead the rules for multiplication of numbers.

1. Product of two positive numbers is positive. Any consideration of countable sets of objects justifies this rule.
2. Product of a positive and negative number is negative. Example: multiplying a debt (negative number) increases the amount of the debt.
3. Product of a negative and negative number is positive.
4. Multiplication is distributive. Product of a sum is the sum of the products. [2]

$$a(b + c) = ab + ac \tag{3.1}$$

$$(a + b)c = ac + bc \tag{3.2}$$

5. Multiplication is associative. Changing the order that multiplication is grouped by does not change the result.

$$(ab)c = a(bc) \tag{3.3}$$

2 The name of this property is not important and no student should ever be tested on it. It is a word like dividand which countless countless school kids are forced to memorize. Like dividand it is perfectly acceptable to forget it after the test because nobody has to know it to perform division. Since most useful sorts of multiplications have this property this is the least important of the named multiplication properties. This word exists mostly so that authors of math books can impress themselves writing phrases like "a mathematical entity that behaves this way is left and right distributive with respect to addition".

6. Multiplication is commutative. Switching the order of multiplication does not change the result.

$$ab = ba \tag{3.4}$$

Unless the reader had an exceptionally gifted grade three teacher it is likely that rule three was presented without any sort of justification or analogy. This can be considered as a special case of the previous rule. Geometrically, a multiplication by -1 results in an inversion on the number line. If one considers the number line to be a line in space, then this is a 180 degree rotation. Two negative multiplications results in a 360 degree rotation, and thus takes the number back to its original positive or negative segment on its "number line".

3.3.2 *Rules for multiplication of vectors with the same direction*

Having identified the rules for multiplication of numbers, one can use these to define multiplication rules for a simple case, one dimensional vectors. Conceptually a one dimensional vector space can be thought of like a number line, or the set of all numbers as the set of all scalar multiples of a unit vector of a particular direction in space.

It is reasonable to expect the rules for multiplication of two vectors with the same direction to have some of the same characteristics as multiplication of numbers. Lets state this algebraically writing the directed distance from the origin to the points a and b in a vector notation

$$\begin{aligned} \mathbf{a} &= a\mathbf{e} \\ \mathbf{b} &= b\mathbf{e} \end{aligned} \tag{3.5}$$

where **e** is the unit vector alone the line in question.

The product of these two vectors is

$$\mathbf{ab} = ab\mathbf{ee} \tag{3.6}$$

Although no specific meaning has yet been given to the **ee** term yet, one can make a few observations about a product of this form.

1. It is commutative, since $\mathbf{ab} = \mathbf{ba} = ab\mathbf{ee}$.
2. It is distributive since numeric multiplication is.
3. The product of three such vectors is distributive (no matter the grouping of the multiplications there will be a numeric factor and a **eee** factor.

These properties are consistent with half the properties of numeric multiplication. If the other half of the numeric multiplication rules are assumed to also apply we have

1. Product of two vectors in the same direction is positive (rules 1 and 3 above).
2. Product of two vectors pointing in opposite directions is negative (rule 2 above).

This can only be possible by giving the following meaning to the square of a unit vector

$$\mathbf{e}\mathbf{e} = 1 \tag{3.7}$$

Alternately, one can state that the square of a vector is that vectors squared length.

$$\mathbf{a}\mathbf{a} = a^2 \tag{3.8}$$

This property, as well as the associative and distributive properties are the defining properties of the geometric product.

It will be shown shortly that in order to retain this squared vector length property for vectors with components in different directions it will be required to drop the commutative property of numeric multiplication:

$$\mathbf{a}\mathbf{b} \neq \mathbf{b}\mathbf{a} \tag{3.9}$$

This is a choice that will later be observed to have important consequences. There are many types of multiplications that do not have the commutative property. Matrix multiplication is not even necessarily defined when the order is switched. Other multiplication operations (wedge and cross products) change sign when the order is switched.

Another important choice has been made to require the product of two vectors not be a vector itself. This also breaks from the number line analogy since the product of two numbers is still a number. However, it is notable that in order to take roots of a negative number one has to introduce a second number line (the i, or imaginary axis), and so even for numbers, products can be “different” than their factors. Interestingly enough, it will later be possible to show that the choice to not require a vector product to be a vector allow complex numbers to be defined directly in terms of the geometric product of two vectors in a plane.

3.4 AXIOMS

The previous discussion attempts to justify the choice of the following set of axioms for multiplication of vectors

1. linearity
2. associativity
3. contraction

   Square of a vector is its squared length.

This last property is weakened in some circumstances (for example, an alternate definition of vector length is desirable for relativistic calculations.)

3.5 DOT PRODUCT

One can express the dot product in terms of these axioms. This follows by calculating the length of a sum or difference of vectors, starting with the requirement that the vector square is the squared length of that vector.

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, their sum $\mathbf{c} = \mathbf{a} + \mathbf{b}$ has squared length:

$$\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}) = \mathbf{a}^2 + \mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} + \mathbf{b}^2. \tag{3.10}$$

We do not have any specific meaning for the product of vectors, but eq. (3.10) shows that the symmetric sum of such a product:

$$\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} = \text{scalar} \tag{3.11}$$

since the RHS is also a scalar.

Additionally, if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then we must also have:

$$\mathbf{a}^2 + \mathbf{b}^2 = a^2 + b^2. \tag{3.12}$$

This implies a rule for vector multiplication of perpendicular vectors

$$\mathbf{b}\mathbf{a} + \mathbf{a}\mathbf{b} = 0 \tag{3.13}$$

Or,

$$\mathbf{b}\mathbf{a} = -\mathbf{a}\mathbf{b}. \tag{3.14}$$

Note that eq. (3.14) does not assign any meaning to this product of vectors when they perpendicular. Whatever that meaning is, the entity such a perpendicular vector product produces changes sign with commutation.

Performing the same length calculation using standard vector algebra shows that we can identify the symmetric sum of vector products with the dot product:

$$\|\mathbf{c}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2. \tag{3.15}$$

Thus we can make the identity:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{3.16}$$

3.6 COORDINATE EXPANSION OF THE GEOMETRIC PRODUCT

A powerful feature of geometric algebra is that it allows for coordinate free results, and the avoidance of basis selection that coordinates require. While this is true, explicit coordinate expansion, especially initially while making the transition from coordinate based vector algebra, is believed to add clarity to the subject.

Writing a pair of vectors in coordinate vector notation:

$$\mathbf{a} = \sum_i a_i \mathbf{e}_i \tag{3.17}$$

$$\mathbf{b} = \sum_j b_j \mathbf{e}_j \tag{3.18}$$

Despite not yet knowing what meaning to give to the geometric product of two general (non-colinear) vectors, given the axioms above and their consequences we actually have enough information to completely expand the geometric product of two vectors in terms of these coordinates:

$$\begin{aligned}\mathbf{ab} &= \sum_{ij} a_i b_j \mathbf{e}_i \mathbf{e}_j \\ &= \sum_{i=j} a_i b_j \mathbf{e}_i \mathbf{e}_j + \sum_{i \neq j} a_i b_j \mathbf{e}_i \mathbf{e}_j \\ &= \sum_{i} a_i b_i \mathbf{e}_i \mathbf{e}_i + \sum_{i<j} a_i b_j \mathbf{e}_i \mathbf{e}_j + \sum_{j<i} a_i b_j \mathbf{e}_i \mathbf{e}_j \\ &= \sum_{i} a_i b_i + \sum_{i<j} a_i b_j \mathbf{e}_i \mathbf{e}_j + a_j b_i \mathbf{e}_j \mathbf{e}_i \\ &= \sum_{i} a_i b_i + \sum_{i<j} (a_i b_j - b_i a_j) \mathbf{e}_i \mathbf{e}_j\end{aligned} \tag{3.19}$$

This can be summarized nicely in terms of determinants:

$$\mathbf{ab} = \sum_{i} a_i b_i + \sum_{i<j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \mathbf{e}_i \mathbf{e}_j \tag{3.20}$$

This shows, without requiring the "triangle law" expansion of eq. (3.15), that the geometric product has a scalar component that we recognize as the Euclidean vector dot product. It also shows that the remaining bit is a "something else" component. This "something else" is called a bivector. We do not yet know what this bivector is or what to do with it, but will come back to that.

Observe that an interchange of **a** and **b** leaves the scalar part of equation eq. (3.20) unaltered (ie: it is symmetric), whereas an interchange inverts the bivector (ie: it is the antisymmetric part).

3.7 SOME SPECIFIC EXAMPLES TO GET A FEEL FOR THINGS

Moving from the abstract, consider a few specific geometric product example.

- Product of two non-colinear non-orthogonal vectors.

$$(\mathbf{e}_1 + 2\mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_1\mathbf{e}_1 - 2\mathbf{e}_2\mathbf{e}_2 + 2\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2 = -1 + 3\mathbf{e}_2\mathbf{e}_1 \tag{3.21}$$

  Such a product produces both scalar and bivector parts.

- Squaring a bivector

$$(\mathbf{e}_1\mathbf{e}_2)^2 = (\mathbf{e}_1\mathbf{e}_2)(-\mathbf{e}_2\mathbf{e}_1) = -\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_1 = -1 \tag{3.22}$$

  This particular bivector squares to minus one very much like the imaginary number i.

- Product of two perpendicular vectors.

$$(\mathbf{e}_1 + \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2) = 2\mathbf{e}_2\mathbf{e}_1 \tag{3.23}$$

  Such a product generates just a bivector term.

- Product of a bivector and a vector in the plane.

$$(x\mathbf{e}_1 + y\mathbf{e}_2)\mathbf{e}_1\mathbf{e}_2 = x\mathbf{e}_2 - y\mathbf{e}_1 \tag{3.24}$$

  This rotates the vector counterclockwise by 90 degrees.

- General $\mathbb{R}^3$ geometric product of two vectors.

$$\mathbf{x}\mathbf{y} = (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)(y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3) \tag{3.25}$$

$$= \mathbf{x}\cdot\mathbf{y} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}\mathbf{e}_2\mathbf{e}_3 + \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}\mathbf{e}_1\mathbf{e}_3 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\mathbf{e}_1\mathbf{e}_2 \tag{3.26}$$

  Or,

$$\mathbf{x}\mathbf{y} = \mathbf{x}\cdot\mathbf{y} + \begin{vmatrix} \mathbf{e}_2\mathbf{e}_3 & \mathbf{e}_3\mathbf{e}_1 & \mathbf{e}_1\mathbf{e}_2 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{3.27}$$

  Observe that if one identifies $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, and $\mathbf{e}_1\mathbf{e}_2$ with vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ respectively, this second term is the cross product. A precise way to perform this identification will be described later.

  The key thing to observe here is that the structure of the cross product is naturally associated with the geometric product. One can think of the geometric product as a complete product including elements of both the dot and cross product. Unlike the cross product the geometric product is also well defined in two dimensions and greater than three.

These examples are all somewhat random, but give a couple hints of results to come.

3.8 ANTISYMMETRIC PART OF THE GEOMETRIC PRODUCT

Having identified the symmetric sum of vector products with the dot product we can write the geometric product of two arbitrary vectors in terms of this and its difference

$$\begin{aligned} \mathbf{ab} &= \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \\ &= \mathbf{a} \cdot \mathbf{b} + f(\mathbf{a}, \mathbf{b}) \end{aligned} \tag{3.28}$$

Let us examine this second term, the bivector, a mapping of a pair of vectors into a different sort of object of yet unknown properties.

$$f(\mathbf{a}, k\mathbf{a}) = \frac{1}{2}(\mathbf{a}k\mathbf{a} - k\mathbf{aa}) = 0 \tag{3.29}$$

Property: Zero when the vectors are colinear.

$$f(\mathbf{a}, k\mathbf{a} + \mathbf{b}) = \frac{1}{2}(\mathbf{a}(k\mathbf{a} + \mathbf{b}) - (k\mathbf{a} + m\mathbf{b})\mathbf{a}) = f(\mathbf{a}, \mathbf{b}) \tag{3.30}$$

Property: colinear contributions are rejected.

$$f(\alpha\mathbf{a}, \beta\mathbf{b}) = \frac{1}{2}(\alpha\mathbf{a}\beta\mathbf{b} - \beta\mathbf{b}\alpha\mathbf{a}) = \alpha\beta f(\mathbf{a}, \mathbf{b}) \tag{3.31}$$

Property: bilinearity.

$$f(\mathbf{b}, \mathbf{a}) = \frac{1}{2}(\mathbf{ba} - \mathbf{ab}) = -\frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = -f(\mathbf{a}, \mathbf{b}) \tag{3.32}$$

Property: Interchange inverts.

Operationally, these are in fact the properties of what in the calculus of differential forms is called the wedge product (uncoincidentally, these are also all properties of the cross product as well.)

Because the properties are identical the notation from differential forms is stolen, and we write

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \tag{3.33}$$

And as mentioned, the object that this wedge product produces from two vectors is called a bivector.

Strictly speaking the wedge product of differential calculus is defined as an alternating, associative, multilinear form. We have here bilinear, not multilinear and associativity is not meaningful until more than two factors are introduced, however when we get to the product of more than three vectors, we will find that the geometric vector product produces an entity with all of these properties.

Returning to the product of two vectors we can now write

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{3.34}$$

This is often used as the initial definition of the geometric product.

3.9 YES, BUT WHAT IS THAT WEDGE PRODUCT THING

Combination of the symmetric and antisymmetric decomposition in eq. (3.34) shows that the product of two vectors according to the axioms has a scalar part and a bivector part. What is this bivector part geometrically?

One can show that the equation of a plane can be written in terms of bivectors. One can also show that the area of the parallelogram spanned by two vectors can be expressed in terms of the "magnitude" of a bivector. Both of these show that a bivector characterizes a plane and can be thought of loosely as a "plane vector".

Neither the plane equation or the area result are hard to show, but we will get to those later. A more direct way to get an intuitive feel for the geometric properties of the bivector can be obtained by first examining the square of a bivector.

By subtracting the projection of one vector **a** from another **b**, one can form the rejection of **a** from **b**:

$$\mathbf{b}' = \mathbf{b} - (\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} \tag{3.35}$$

With respect to the dot product, this vector is orthogonal to **a**. Since $\mathbf{a} \wedge \hat{\mathbf{a}} = 0$, this allows us to write the wedge product of vectors **a** and **b** as the direct product of two orthogonal vectors

$$\begin{aligned}
\mathbf{a} \wedge \mathbf{b} &= \mathbf{a} \wedge (\mathbf{b} - (\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}) \\
&= \mathbf{a} \wedge \mathbf{b}' \\
&= \mathbf{ab}' \\
&= -\mathbf{b}'\mathbf{a}
\end{aligned} \tag{3.36}$$

The square of the bivector can then be written

$$\begin{aligned}(\mathbf{a} \wedge \mathbf{b})^2 &= (\mathbf{a}\mathbf{b}')(-\mathbf{b}'\mathbf{a}) \\ &= -\mathbf{a}^2(\mathbf{b}')^2.\end{aligned} \tag{3.37}$$

Thus the square of a bivector is negative. It is natural to define a bivector norm:

$$|\mathbf{a} \wedge \mathbf{b}| = \sqrt{-(\mathbf{a} \wedge \mathbf{b})^2} = \sqrt{(\mathbf{a} \wedge \mathbf{b})(\mathbf{b} \wedge \mathbf{a})} \tag{3.38}$$

Dividing by this norm we have an entity that acts precisely like the imaginary number i.

Looking back to eq. (3.34) one can now assign additional meaning to the two parts. The first, the dot product, is a scalar (ie: a real number), and a second part, the wedge product, is a pure imaginary term. Provided $\mathbf{a} \wedge \mathbf{b} \neq 0$, we can write $i = \frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}\wedge\mathbf{b}|}$ and express the geometric product in complex number form:

$$\mathbf{a}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i|\mathbf{a} \wedge \mathbf{b}| \tag{3.39}$$

The complex number system is the algebra of the plane, and the geometric product of two vectors can be used to completely characterize the algebra of an arbitrarily oriented plane in a higher order vector space.

It actually will be very natural to define complex numbers in terms of the geometric product, and we will see later that the geometric product allows for the ad-hoc definition of “complex number” systems according to convenience in many ways.

We will also see that generalizations of complex numbers such as quaternion algebras also find their natural place as specific instances of geometric products.

Concepts familiar from complex numbers such as conjugation, inversion, exponentials as rotations, and even things like the residue theory of complex contour integration, will all have a natural geometric algebra analogue.

We will return to this, but first some more detailed initial examination of the wedge product properties is in order, as is a look at the product of greater than two vectors.

4 COMPARISON OF MANY TRADITIONAL VECTOR AND GA IDENTITIES

4.1 THREE DIMENSIONAL VECTOR RELATIONSHIPS VS N DIMENSIONAL EQUIVALENTS

Here are some comparisons between standard $\mathbb{R}^3$ vector relations and their corresponding wedge and geometric product equivalents. All the wedge and geometric product equivalents here are good for more than three dimensions, and some also for two. In two dimensions the cross product is undefined even if what it describes (like torque) is a perfectly well defined in a plane without introducing an arbitrary normal vector outside of the space.

Many of these relationships only require the introduction of the wedge product to generalize, but since that may not be familiar to somebody with only a traditional background in vector algebra and calculus, some examples are given.

4.1.1 *wedge and cross products are antisymmetric*

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) \tag{4.1}$$

$$\mathbf{v} \wedge \mathbf{u} = -(\mathbf{u} \wedge \mathbf{v}) \tag{4.2}$$

4.1.2 *wedge and cross products are zero when identical*

$$\mathbf{u} \times \mathbf{u} = 0 \tag{4.3}$$

$$\mathbf{u} \wedge \mathbf{u} = 0 \tag{4.4}$$

4.1.3 *wedge and cross products are linear*

These are both linear in the first variable

$$(\mathbf{v} + \mathbf{w}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w} \tag{4.5}$$

$$(\mathbf{v}+\mathbf{w}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w} \tag{4.6}$$

and are linear in the second variable

$$\mathbf{u} \times (\mathbf{v}+\mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \tag{4.7}$$

$$\mathbf{u} \wedge (\mathbf{v}+\mathbf{w}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w} \tag{4.8}$$

4.1.4 *In general, cross product is not associative, but the wedge product is*

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \tag{4.9}$$

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) \tag{4.10}$$

4.1.5 *Wedge and cross product relationship to a plane*

$\mathbf{u} \times \mathbf{v}$ is perpendicular to plane containing $\mathbf{u}$ and $\mathbf{v}$. $\mathbf{u} \wedge \mathbf{v}$ is an oriented representation of the plane containing $\mathbf{u}$ and $\mathbf{v}$.

4.1.6 *norm of a vector*

The norm (length) of a vector is defined in terms of the dot product

$$||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u} \tag{4.11}$$

Using the geometric product this is also true, but this can be also be expressed more compactly as

$$||\mathbf{u}||^2 = \mathbf{u}^2 \tag{4.12}$$

This follows from the definition of the geometric product and the fact that a vector wedge product with itself is zero

$$\mathbf{u}\,\mathbf{u} = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \wedge \mathbf{u} = \mathbf{u} \cdot \mathbf{u} \tag{4.13}$$

4.1.7 *Lagrange identity*

In three dimensions the product of two vector lengths can be expressed in terms of the dot and cross products

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 = (\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2 \tag{4.14}$$

The corresponding generalization expressed using the geometric product is

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 = (\mathbf{u}\cdot\mathbf{v})^2 - (\mathbf{u}\wedge\mathbf{v})^2 \tag{4.15}$$

This follows from by expanding the geometric product of a pair of vectors with its reverse

$$(\mathbf{u}\mathbf{v})(\mathbf{v}\mathbf{u}) = (\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v})(\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\wedge\mathbf{v}) \tag{4.16}$$

4.1.8 *determinant expansion of cross and wedge products*

$$\mathbf{u}\times\mathbf{v} = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \mathbf{e}_i \times \mathbf{e}_j \tag{4.17}$$

$$\mathbf{u}\wedge\mathbf{v} = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \mathbf{e}_i \wedge \mathbf{e}_j \tag{4.18}$$

Without justification or historical context, traditional linear algebra texts will often define the determinant as the first step of an elaborate sequence of definitions and theorems leading up to the solution of linear systems, Cramer's rule and matrix inversion.

An alternative treatment is to axiomatically introduce the wedge product, and then demonstrate that this can be used directly to solve linear systems. This is shown below, and does not require sophisticated math skills to understand.

It is then possible to define determinants as nothing more than the coefficients of the wedge product in terms of "unit k-vectors" ($\mathbf{e}_i \wedge \mathbf{e}_j$ terms) expansions as above.

A one by one determinant is the coefficient of $\mathbf{e}_1$ for an $\mathbb{R}^1$ 1-vector.

A two-by-two determinant is the coefficient of $\mathbf{e}_1 \wedge \mathbf{e}_2$ for an $\mathbb{R}^2$ bivector

A three-by-three determinant is the coefficient of $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ for an $\mathbb{R}^3$ trivector

When linear system solution is introduced via the wedge product, Cramer's rule follows as a side effect, and there is no need to lead up to the end results with definitions of minors, matrices, matrix invertablity, adjoints, cofactors, Laplace expansions, theorems on determinant multiplication and row column exchanges, and so forth.

4.1.9 *Equation of a plane*

For the plane of all points $\mathbf{r}$ through the plane passing through three independent points $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$, the normal form of the equation is

$$((\mathbf{r}_2 - \mathbf{r}_0) \times (\mathbf{r}_1 - \mathbf{r}_0)) \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \tag{4.19}$$

The equivalent wedge product equation is

$$(\mathbf{r}_2 - \mathbf{r}_0) \wedge (\mathbf{r}_1 - \mathbf{r}_0) \wedge (\mathbf{r} - \mathbf{r}_0) = 0 \tag{4.20}$$

4.1.10 *Projective and rejective components of a vector*

For three dimensions the projective and rejective components of a vector with respect to an arbitrary non-zero unit vector, can be expressed in terms of the dot and cross product

$$\mathbf{v} = (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \hat{\mathbf{u}} \times (\mathbf{v} \times \hat{\mathbf{u}}) \tag{4.21}$$

For the general case the same result can be written in terms of the dot and wedge product and the geometric product of that and the unit vector

$$\mathbf{v} = (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{v} \wedge \hat{\mathbf{u}})\hat{\mathbf{u}} \tag{4.22}$$

It is also worthwhile to point out that this result can also be expressed using right or left vector division as defined by the geometric product

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{u})\frac{1}{\mathbf{u}} + (\mathbf{v} \wedge \mathbf{u})\frac{1}{\mathbf{u}} \tag{4.23}$$

$$\mathbf{v} = \frac{1}{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \frac{1}{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}) \tag{4.24}$$

4.1.11 *Area (squared) of a parallelogram is norm of cross product*

$$A^2 = \|\mathbf{u} \times \mathbf{v}\|^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2 \tag{4.25}$$

and is the negated square of a wedge product

$$A^2 = -(\mathbf{u} \wedge \mathbf{v})^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2 \tag{4.26}$$

Note that this squared bivector is a geometric product.

4.1.12 *Angle between two vectors*

$$(\sin\theta)^2 = \frac{||\mathbf{u} \times \mathbf{v}||^2}{||\mathbf{u}||^2 ||\mathbf{v}||^2} \tag{4.27}$$

$$(\sin\theta)^2 = -\frac{(\mathbf{u} \wedge \mathbf{v})^2}{\mathbf{u}^2 \mathbf{v}^2} \tag{4.28}$$

4.1.13 *Volume of the parallelepiped formed by three vectors*

$$V^2 = ||(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}||^2 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}^2 \tag{4.29}$$

$$V^2 = -(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w})^2 = -\left(\sum_{i<j<k} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k \right)^2 = \sum_{i<j<k} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix}^2 \tag{4.30}$$

4.2 SOME PROPERTIES AND EXAMPLES

Some fundamental geometric algebra manipulations will be provided below, showing how this vector product can be used in calculation of projections, area, and rotations. How some of these tie together and correlate concepts from other branches of mathematics, such as complex numbers, will also be shown.

In some cases these examples provide details used above in the cross product and geometric product comparisons.

4.2.1 *Inversion of a vector*

One of the powerful properties of the Geometric product is that it provides the capability to express the inverse of a non-zero vector. This is expressed by:

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{aa}} = \frac{\mathbf{a}}{||\mathbf{a}||^2}. \tag{4.31}$$

4.2.2 *dot and wedge products defined in terms of the geometric product*

Given a definition of the geometric product in terms of the dot and wedge products, adding and subtracting **ab** and **ba** demonstrates that the dot and wedge product of two vectors can also be defined in terms of the geometric product

4.2.3 *The dot product*

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{4.32}$$

This is the symmetric component of the geometric product. When two vectors are colinear the geometric and dot products of those vectors are equal.

As a motivation for the dot product it is normal to show that this quantity occurs in the solution of the length of a general triangle where the third side is the vector sum of the first and second sides $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

$$||\mathbf{c}||^2 = \sum_i (a_i + b_i)^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2\sum_i a_i b_i \tag{4.33}$$

The last sum is then given the name the dot product and other properties of this quantity are then shown (projection, angle between vectors, ...).

This can also be expressed using the geometric product

$$\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}) = \mathbf{a}^2 + \mathbf{b}^2 + (\mathbf{ab} + \mathbf{ba}) \tag{4.34}$$

By comparison, the following equality exists

$$\sum_i a_i b_i = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \tag{4.35}$$

Without requiring expansion by components one can define the dot product exclusively in terms of the geometric product due to its properties of contraction, distribution and associativity. This is arguably a more natural way to define the geometric product. Addition of two similar terms is not immediately required, especially since one of those terms is the wedge product which may also be unfamiliar.

4.2.4 *The wedge product*

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \tag{4.36}$$

This is the antisymmetric component of the geometric product. When two vectors are orthogonal the geometric and wedge products of those vectors are equal.

Switching the order of the vectors negates this antisymmetric geometric product component, and contraction property shows that this is zero if the vectors are equal. These are the defining properties of the wedge product.

4.2.5 *Note on symmetric and antisymmetric dot and wedge product formulas*

A generalization of the dot product that allows computation of the component of a vector "in the direction" of a plane (bivector), or other k-vectors can be found below. Since the signs change depending on the grades of the terms being multiplied, care is required with the formulas above to ensure that they are only used for a pair of vectors.

4.2.6 *Reversing multiplication order. Dot and wedge products compared to the real and imaginary parts of a complex number*

Reversing the order of multiplication of two vectors, has the effect of the inverting the sign of just the wedge product term of the product.

It is not a coincidence that this is a similar operation to the conjugate operation of complex numbers.

The reverse of a product is written in the following fashion

$$\mathbf{ba} = (\mathbf{ab})^{\dagger} \tag{4.37}$$

$$\mathbf{cba} = (\mathbf{abc})^{\dagger} \tag{4.38}$$

Expressed this way the dot and wedge products are

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + (\mathbf{ab})^\dagger) \tag{4.39}$$

This is the symmetric component of the geometric product. When two vectors are colinear the geometric and dot products of those vectors are equal.

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - (\mathbf{ab})^\dagger) \tag{4.40}$$

These symmetric and antisymmetric pairs, the dot and wedge products extract the scalar and bivector components of a geometric product in the same fashion as the real and imaginary components of a complex number are also extracted by its symmetric and antisymmetric components

$$Re(z) = \frac{1}{2}(z + \bar{z}) \tag{4.41}$$

$$Im(z) = \frac{1}{2}(z - \bar{z}) \tag{4.42}$$

This extraction of components also applies to higher order geometric product terms. For example

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \frac{1}{2}(\mathbf{abc} - (\mathbf{abc})^\dagger) = \frac{1}{2}(\mathbf{bca} - (\mathbf{bca})^\dagger) = \frac{1}{2}(\mathbf{cab} - (\mathbf{cab})^\dagger) \tag{4.43}$$

4.2.7 *Orthogonal decomposition of a vector*

Using the Gram-Schmidt process a single vector can be decomposed into two components with respect to a reference vector, namely the projection onto a unit vector in a reference direction, and the difference between the vector and that projection.

With, $\hat{\mathbf{u}} = \mathbf{u}/||\mathbf{u}||$, the projection of $\mathbf{v}$ onto $\hat{\mathbf{u}}$ is

$$\mathrm{Proj}_{\hat{\mathbf{u}}}\, \mathbf{v} = \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v}) \tag{4.44}$$

Orthogonal to that vector is the difference, designated the rejection,

$$\mathbf{v} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v}) = \frac{1}{||\mathbf{u}||^2}(||\mathbf{u}||^2\mathbf{v} - \mathbf{u}(\mathbf{u} \cdot \mathbf{v})) \tag{4.45}$$

The rejection can be expressed as a single geometric algebraic product in a few different ways

$$\frac{\mathbf{u}}{\mathbf{u}^2}(\mathbf{u}\mathbf{v} - \mathbf{u} \cdot \mathbf{v}) = \frac{1}{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}) = \hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v}) = (\mathbf{v} \wedge \hat{\mathbf{u}})\hat{\mathbf{u}} \tag{4.46}$$

The similarity in form between between the projection and the rejection is notable. The sum of these recovers the original vector

$$\mathbf{v} = \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v}) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v}) \tag{4.47}$$

Here the projection is in its customary vector form. An alternate formulation is possible that puts the projection in a form that differs from the usual vector formulation

$$\mathbf{v} = \frac{1}{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v}) + \frac{1}{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}) = (\mathbf{v} \cdot \mathbf{u})\frac{1}{\mathbf{u}} + (\mathbf{v} \wedge \mathbf{u})\frac{1}{\mathbf{u}} \tag{4.48}$$

4.2.8 *A quicker way to the end result*

Working backwards from the end result, it can be observed that this orthogonal decomposition result can in fact follow more directly from the definition of the geometric product itself.

$$\mathbf{v} = \hat{\mathbf{u}}\hat{\mathbf{u}}\mathbf{v} = \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v} + \hat{\mathbf{u}} \wedge \mathbf{v}) \tag{4.49}$$

With this approach, the original geometrical consideration is not necessarily obvious, but it is a much quicker way to get at the same algebraic result.

However, the hint that one can work backwards, coupled with the knowledge that the wedge product can be used to solve sets of linear equations, [1] the problem of orthogonal decomposition can be posed directly,

Let $\mathbf{v} = a\mathbf{u} + \mathbf{x}$, where $\mathbf{u} \cdot \mathbf{x} = 0$. To discard the portions of $\mathbf{v}$ that are colinear with $\mathbf{u}$, take the wedge product

$$\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \wedge (a\mathbf{u} + \mathbf{x}) = \mathbf{u} \wedge \mathbf{x} \tag{4.50}$$

Here the geometric product can be employed

$$\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \wedge \mathbf{x} = \mathbf{u}\mathbf{x} - \mathbf{u} \cdot \mathbf{x} = \mathbf{u}\mathbf{x} \tag{4.51}$$

1 http://www.grassmannalgebra.info/grassmannalgebra/book/bookpdf/TheExteriorProduct.pdf

Because the geometric product is invertible, this can be solved for x

$$\mathbf{x} = \frac{1}{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}) \tag{4.52}$$

The same techniques can be applied to similar problems, such as calculation of the component of a vector in a plane and perpendicular to the plane.

4.2.9 *Area of parallelogram spanned by two vectors*

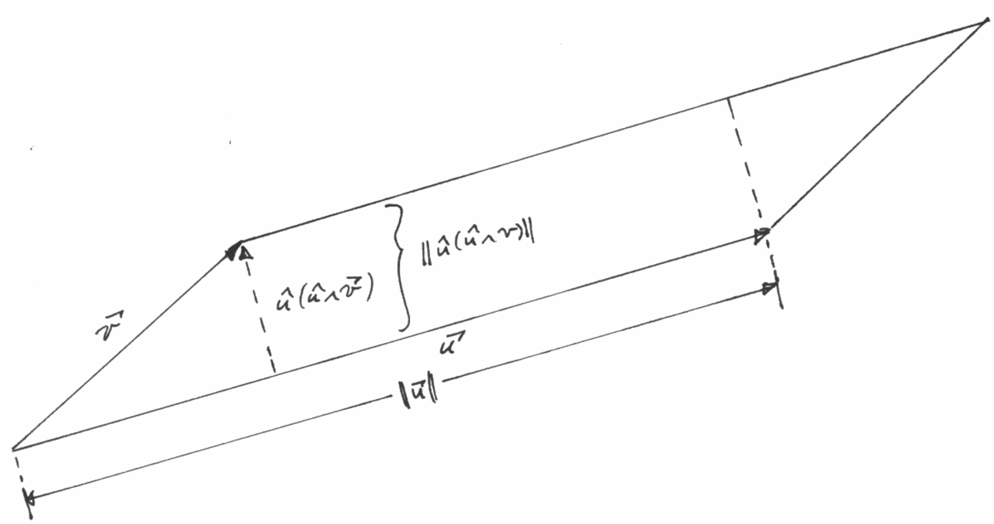

Figure 4.1: parallelogramArea

As depicted in fig. 4.1, one can see that the area of a parallelogram spanned by two vectors is computed from the base times height. In the figure $\mathbf{u}$ was picked as the base, with length $\|\mathbf{u}\|$. Designating the second vector $\mathbf{v}$, we want the component of $\mathbf{v}$ perpendicular to $\hat{\mathbf{u}}$ for the height. An orthogonal decomposition of $\mathbf{v}$ into directions parallel and perpendicular to $\hat{\mathbf{u}}$ can be performed in two ways.

$$\begin{aligned} \mathbf{v} &= \mathbf{v}\hat{\mathbf{u}}\hat{\mathbf{u}} = (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + (\mathbf{v} \wedge \hat{\mathbf{u}})\hat{\mathbf{u}} \\ &= \hat{\mathbf{u}}\hat{\mathbf{u}}\mathbf{v} = \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v}) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v}) \end{aligned} \tag{4.53}$$

The height is the length of the perpendicular component expressed using the wedge as either $\hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v})$ or $(\mathbf{v} \wedge \hat{\mathbf{u}})\hat{\mathbf{u}}$.

Multiplying base times height we have the parallelogram area

$$
\begin{aligned}
A(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u}\| \|\hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v})\| \\
&= \|\hat{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v})\|
\end{aligned}
\tag{4.54}
$$

Since the squared length of an Euclidean vector is the geometric square of that vector, we can compute the squared area of this parallogram by squaring this single scaled vector

$$
A^2 = (\hat{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}))^2 \tag{4.55}
$$

Utilizing both encodings of the perpendicular to $\hat{\mathbf{u}}$ component of $\mathbf{v}$ computed above we have for the squared area

$$
\begin{aligned}
A^2 &= (\hat{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}))^2 \\
&= ((\mathbf{v} \wedge \mathbf{u})\hat{\mathbf{u}})(\hat{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v})) \\
&= (\mathbf{v} \wedge \mathbf{u})(\mathbf{u} \wedge \mathbf{v})
\end{aligned}
\tag{4.56}
$$

Since $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$, we have finally

$$
A^2 = -(\mathbf{u} \wedge \mathbf{v})^2 \tag{4.57}
$$

There are a few things of note here. One is that the parallelogram area can easily be expressed in terms of the square of a bivector. Another is that the square of a bivector has the same property as a purely imaginary number, a negative square.

It can also be noted that a vector lying completely within a plane anticommutes with the bivector for that plane. More generally components of vectors that lie within a plane commute with the bivector for that plane while the perpendicular components of that vector commute. These commutation or anticommutation properties depend both on the vector and the grade of the object that one attempts to commute it with (these properties lie behind the generalized definitions of the dot and wedge product to be seen later).

4.2.10 *Expansion of a bivector and a vector rejection in terms of the standard basis*

If a vector is factored directly into projective and rejective terms using the geometric product $\mathbf{v} = \frac{1}{\mathbf{u}}(\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v})$, then it is not necessarily obvious that the rejection term, a product of vector and bivector is even a vector. Expansion of the vector bivector product in terms of the standard basis vectors has the following form

Let

$$\mathbf{r} = \frac{1}{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v}) = \frac{\mathbf{u}}{\mathbf{u}^2}(\mathbf{u} \wedge \mathbf{v}) = \frac{1}{\|\mathbf{u}\|^2}\mathbf{u}(\mathbf{u} \wedge \mathbf{v}) \tag{4.58}$$

It can be shown that

$$\mathbf{r} = \frac{1}{\|\mathbf{u}\|^2} \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix} \tag{4.59}$$

(a result that can be shown more easily straight from $\mathbf{r} = \mathbf{v} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{v})$).

The rejective term is perpendicular to $\mathbf{u}$, since $\begin{vmatrix} u_i & u_j \\ u_i & u_j \end{vmatrix} = 0$ implies $\mathbf{r} \cdot \mathbf{u} = \mathbf{0}$.

The magnitude of $\mathbf{r}$, is

$$\|\mathbf{r}\|^2 = \mathbf{r} \cdot \mathbf{v} = \frac{1}{\|\mathbf{u}\|^2} \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2 \tag{4.60}$$

.

So, the quantity

$$\|\mathbf{r}\|^2 \|\mathbf{u}\|^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2 \tag{4.61}$$

is the squared area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

It is also noteworthy that the bivector can be expressed as

$$\mathbf{u} \wedge \mathbf{v} = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \mathbf{e}_i \wedge \mathbf{e}_j \tag{4.62}$$

.

Thus is it natural, if one considers each term $\mathbf{e}_i \wedge \mathbf{e}_j$ as a basis vector of the bivector space, to define the (squared) "length" of that bivector as the (squared) area.

Going back to the geometric product expression for the length of the rejection $\frac{1}{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v})$ we see that the length of the quotient, a vector, is in this case is the "length" of the bivector divided by the length of the divisor.

This may not be a general result for the length of the product of two k-vectors, however it is a result that may help build some intuition about the significance of the algebraic operations. Namely,

When a vector is divided out of the plane (parallelogram span) formed from it and another vector, what remains is the perpendicular component of the remaining vector, and its length is the planar area divided by the length of the vector that was divided out.

4.2.11 *Projection and rejection of a vector onto and perpendicular to a plane*

Like vector projection and rejection, higher dimensional analogs of that calculation are also possible using the geometric product.

As an example, one can calculate the component of a vector perpendicular to a plane and the projection of that vector onto the plane.

Let $\mathbf{w} = a\mathbf{u} + b\mathbf{v} + \mathbf{x}$, where $\mathbf{u} \cdot \mathbf{x} = \mathbf{v} \cdot \mathbf{x} = 0$. As above, to discard the portions of $\mathbf{w}$ that are colinear with $\mathbf{u}$ or $\mathbf{u}$, take the wedge product

$$\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} = (a\mathbf{u} + b\mathbf{v} + \mathbf{x}) \wedge \mathbf{u} \wedge \mathbf{v} = \mathbf{x} \wedge \mathbf{u} \wedge \mathbf{v} \tag{4.63}$$

Having done this calculation with a vector projection, one can guess that this quantity equals $\mathbf{x}(\mathbf{u} \wedge \mathbf{v})$. One can also guess there is a vector and bivector dot product like quantity such that the allows the calculation of the component of a vector that is in the "direction of a plane". Both of these guesses are correct, and the validating these facts is worthwhile. However, skipping ahead slightly, this to be proved fact allows for a nice closed form solution of the vector component outside of the plane:

$$\mathbf{x} = (\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v})\frac{1}{\mathbf{u} \wedge \mathbf{v}} = \frac{1}{\mathbf{u} \wedge \mathbf{v}}(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) \tag{4.64}$$

Notice the similarities between this planar rejection result a the vector rejection result. To calculation the component of a vector outside of a plane we take the volume spanned by three vectors (trivector) and "divide out" the plane.

Independent of any use of the geometric product it can be shown that this rejection in terms of the standard basis is

$$\mathbf{x} = \frac{1}{(A_{u,v})^2} \sum_{i<j<k} \begin{vmatrix} w_i & w_j & w_k \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ \mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k \end{vmatrix} \tag{4.65}$$

Where

$$(A_{u,v})^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} = -(\mathbf{u} \wedge \mathbf{v})^2 \tag{4.66}$$

is the squared area of the parallelogram formed by $\mathbf{u}$, and $\mathbf{v}$.

The (squared) magnitude of $\mathbf{x}$ is

$$\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{w} = \frac{1}{(A_{u,v})^2} \sum_{i<j<k} \begin{vmatrix} w_i & w_j & w_k \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{vmatrix}^2 \tag{4.67}$$

Thus, the (squared) volume of the parallelepiped (base area times perpendicular height) is

$$\sum_{i<j<k} \begin{vmatrix} w_i & w_j & w_k \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{vmatrix}^2 \tag{4.68}$$

Note the similarity in form to the w,u,v trivector itself

$$\sum_{i<j<k} \begin{vmatrix} w_i & w_j & w_k \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{vmatrix} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k \tag{4.69}$$

which, if you take the set of $\mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k$ as a basis for the trivector space, suggests this is the natural way to define the length of a trivector. Loosely speaking the length of a vector is a length, length of a bivector is area, and the length of a trivector is volume.

4.2.12 *Product of a vector and bivector. Defining the "dot product" of a plane and a vector*

In order to justify the normal to a plane result above, a general examination of the product of a vector and bivector is required. Namely,

$$\mathbf{w}(\mathbf{u} \wedge \mathbf{v}) = \sum_{i,j<k} w_i \mathbf{e}_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \mathbf{e}_j \wedge \mathbf{e}_k \tag{4.70}$$

This has two parts, the vector part where $i = j$ or $i = k$, and the trivector parts where no indices equal. After some index summation trickery, and grouping terms and so forth, this is

$$\mathbf{w}(\mathbf{u} \wedge \mathbf{v}) = \sum_{i<j} (w_i \mathbf{e}_j - w_j \mathbf{e}_i) \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} + \sum_{i<j<k} \begin{vmatrix} w_i & w_j & w_k \\ u_i & u_j & u_k \\ v_i & v_j & v_k \end{vmatrix} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k \tag{4.71}$$

The trivector term is $\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v}$. Expansion of $(\mathbf{u} \wedge \mathbf{v})\mathbf{w}$ yields the same trivector term. This is the completely symmetric part, and the vector term is negated. Like the geometric product of two vectors, this geometric product can be grouped into symmetric and antisymmetric parts, one of which is a pure k-vector. In analogy the antisymmetric part of this product can be called a generalized dot product, and is roughly speaking the dot product of a "plane" (bivector), and a vector.

The properties of this generalized dot product remain to be explored, but first here is a summary of the notation

$$\mathbf{w}(\mathbf{u} \wedge \mathbf{v}) = \mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) + \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} \tag{4.72}$$

$$(\mathbf{u} \wedge \mathbf{v})\mathbf{w} = -\mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) + \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} \tag{4.73}$$

$$\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{w}(\mathbf{u} \wedge \mathbf{v}) + (\mathbf{u} \wedge \mathbf{v})\mathbf{w}) \tag{4.74}$$

$$\mathbf{w} \cdot (\mathbf{u} \wedge \mathbf{v}) = \frac{1}{2}(\mathbf{w}(\mathbf{u} \wedge \mathbf{v}) - (\mathbf{u} \wedge \mathbf{v})\mathbf{w}) \tag{4.75}$$

Let $\mathbf{w} = \mathbf{x} + \mathbf{y}$, where $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$, and $\mathbf{y} \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{v} = \mathbf{0}$. Expressing $\mathbf{w}$ and the $\mathbf{u} \wedge \mathbf{v}$, products in terms of these components is

$$\mathbf{w}(\mathbf{u} \wedge \mathbf{v}) = \mathbf{x}(\mathbf{u} \wedge \mathbf{v}) + \mathbf{y}(\mathbf{u} \wedge \mathbf{v}) = \mathbf{x} \cdot (\mathbf{u} \wedge \mathbf{v}) + \mathbf{y} \cdot (\mathbf{u} \wedge \mathbf{v}) + \mathbf{y} \wedge \mathbf{u} \wedge \mathbf{v} \tag{4.76}$$

With the conditions and definitions above, and some manipulation, it can be shown that the term $\mathbf{y} \cdot (\mathbf{u} \wedge \mathbf{v}) = \mathbf{0}$, which then justifies the previous solution of the normal to a plane problem. Since the vector term of the vector bivector product the name dot product is zero when the vector is perpendicular to the plane (bivector), and this vector, bivector "dot product" selects only the components that are in the plane, so in analogy to the vector-vector dot product this name itself is justified by more than the fact this is the non-wedge product term of the geometric vector-bivector product.

4.2.13 *Complex numbers*

There is a one to one correspondence between the geometric product of two $\mathbb{R}^2$ vectors and the field of complex numbers.

Writing, a vector in terms of its components, and left multiplying by the unit vector $\mathbf{e}_1$ yields

$$Z = \mathbf{e}_1\mathbf{P} = \mathbf{e}_1(x\mathbf{e}_1 + y\mathbf{e}_2) = x(1) + y(\mathbf{e}_1\mathbf{e}_2) = x(1) + y(\mathbf{e}_1 \wedge \mathbf{e}_2) \tag{4.77}$$

The unit scalar and unit bivector pair $1, \mathbf{e}_1 \wedge \mathbf{e}_2$ can be considered an alternate basis for a two dimensional vector space. This alternate vector representation is closed with respect to the geometric product

$$\begin{aligned} Z_1Z_2 &= \mathbf{e}_1(x_1\mathbf{e}_1 + y_1\mathbf{e}_2)\mathbf{e}_1(x_2\mathbf{e}_1 + y_2\mathbf{e}_2) \\ &= (x_1 + y_1\mathbf{e}_1\mathbf{e}_2)(x_2 + y_2\mathbf{e}_1\mathbf{e}_2) \\ &= x_1x_2 + y_1y_2(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_1\mathbf{e}_2) \end{aligned} \tag{4.78}$$

$$+(x_1y_2 + x_2y_1)\mathbf{e}_1\mathbf{e}_2$$

This closure can be observed after calculation of the square of the unit bivector above, a quantity

$$(\mathbf{e}_1 \wedge \mathbf{e}_2)^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -1 \tag{4.79}$$

that has the characteristics of the complex number $i^2 = -1$.

This fact allows the simplification of the product above to

$$Z_1Z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) \tag{4.80}$$

Thus what is traditionally the defining, and arguably arbitrary seeming, rule of complex number multiplication, is found to follow naturally from the higher order structure of the geometric product, once that is applied to a two dimensional vector space.

It is also informative to examine how the length of a vector can be represented in terms of a complex number. Taking the square of the length

$$\begin{aligned} \mathbf{P} \cdot \mathbf{P} &= (x\mathbf{e}_1 + y\mathbf{e}_2) \cdot (x\mathbf{e}_1 + y\mathbf{e}_2) \\ &= (\mathbf{e}_1 Z)\mathbf{e}_1 Z \\ &= ((x - y\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_1)\mathbf{e}_1 Z \\ &= (x - y(\mathbf{e}_1 \wedge \mathbf{e}_2))Z \end{aligned} \tag{4.81}$$

This right multiplication of a vector with $\mathbf{e}_1$, is named the conjugate

$$\overline{Z} = x - y(\mathbf{e}_1 \wedge \mathbf{e}_2) \tag{4.82}$$

And with that definition, the length of the original vector can be expressed as

$$\mathbf{P} \cdot \mathbf{P} = \overline{Z}Z \tag{4.83}$$

This is also a natural definition of the length of a complex number, given the fact that the complex numbers can be considered an isomorphism with the two dimensional Euclidean vector space.

4.2.14 *Rotation in an arbitrarily oriented plane*

A point $\mathbf{P}$, of radius $\mathbf{r}$, located at an angle θ from the vector $\hat{\mathbf{u}}$ in the direction from $\mathbf{u}$ to $\mathbf{v}$, can be expressed as

$$\mathbf{P} = r\left(\hat{\mathbf{u}}\cos\theta + \frac{\hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v})}{\|\hat{\mathbf{u}}(\hat{\mathbf{u}} \wedge \mathbf{v})\|}\sin\theta\right) = r\hat{\mathbf{u}}\left(\cos\theta + \frac{(\mathbf{u} \wedge \mathbf{v})}{\|\hat{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v})\|}\sin\theta\right) \tag{4.84}$$

Writing $\mathbf{I}_{\mathbf{u},\mathbf{v}} = \frac{\mathbf{u} \wedge \mathbf{v}}{\|\hat{\mathbf{u}}(\mathbf{u} \wedge \mathbf{v})\|}$, the square of this bivector has the property $\mathbf{I}_{\mathbf{u},\mathbf{v}}{}^2 = -1$ of the imaginary unit complex number.

This allows the point to be specified as a complex exponential

$$= \hat{\mathbf{u}}r(\cos\theta + \mathbf{I}_{\mathbf{u},\mathbf{v}}\sin\theta) = \hat{\mathbf{u}}r\exp(\mathbf{I}_{\mathbf{u},\mathbf{v}}\theta) \tag{4.85}$$

Complex numbers could be expressed in terms of the $\mathbb{R}^2$unit bivector $\mathbf{e}_1 \wedge \mathbf{e}_2$. However this isomorphism really only requires a pair of linearly independent vectors in a plane (of arbitrary dimension).

4.2.15 *Quaternions*

Similar to complex numbers the geometric product of two $\mathbb{R}^3$ vectors can be used to define quaternions. Pre and Post multiplication with $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ can be used to express a vector in terms of the quaternion unit numbers i, j, k, as well as describe all the properties of those numbers.

4.2.16 *Cross product as outer product*

Cross product can be written as a scaled outer product

$$\mathbf{a} \times \mathbf{b} = -i(\mathbf{a} \wedge \mathbf{b}) \tag{4.86}$$

$$
\begin{aligned}
i^2 &= (\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 \\
&= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \\
&= -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 \\
&= \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 \\
&= -\mathbf{e}_3\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 \\
&= -1
\end{aligned}
\tag{4.87}
$$

The equivalence of the $\mathbb{R}^3$ cross product and the wedge product expression above can be confirmed by direct multiplication of $-i = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ with a determinant expansion of the wedge product

$$
\mathbf{u} \wedge \mathbf{v} = \sum_{1<=i<j<=3} (u_i v_j - v_i u_j)\mathbf{e}_i \wedge \mathbf{e}_j = \sum_{1<=i<j<=3} (u_i v_j - v_i u_j)\mathbf{e}_i\mathbf{e}_j \tag{4.88}
$$

5 CRAMER'S RULE

5.1 Cramer's rule, determinants, and matrix inversion can be naturally expressed in terms of the wedge product

The use of the wedge product in the solution of linear equations can be quite useful.

This does not require any notion of geometric algebra, only an exterior product and the concept of similar elements, and a nice example of such a treatment can be found in Solution of Linear equations section of [4].

Traditionally, instead of using the wedge product, Cramer's rule is usually presented as a generic algorithm that can be used to solve linear equations of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$ (or equivalently to invert a matrix). Namely

$$\mathbf{x} = \frac{1}{|\mathbf{A}|}\,\mathrm{adj}(\mathbf{A}) \tag{5.1}$$

This is a useful theoretic result. For numerical problems row reduction with pivots and other methods are more stable and efficient.

When the wedge product is coupled with the Clifford product and put into a natural geometric context, the fact that the determinants are used in the expression of $\mathbb{R}^N$ parallelogram area and parallelepiped volumes (and higher dimensional generalizations of these) also comes as a nice side effect.

As is also shown below, results such as Cramer's rule also follow directly from the property of the wedge product that it selects non identical elements. The end result is then simple enough that it could be derived easily if required instead of having to remember or look up a rule.

5.1.1 *Two variables example*

$$\begin{bmatrix}\mathbf{a} & \mathbf{b}\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \mathbf{a}x + \mathbf{b}y = \mathbf{c} \tag{5.2}$$

Pre and post multiplying by **a** and **b**.

$$(\mathbf{a}x + \mathbf{b}y) \wedge \mathbf{b} = (\mathbf{a} \wedge \mathbf{b})x = \mathbf{c} \wedge \mathbf{b} \tag{5.3}$$

$$\mathbf{a} \wedge (\mathbf{a}x + \mathbf{b}y) = (\mathbf{a} \wedge \mathbf{b})y = \mathbf{a} \wedge \mathbf{c} \tag{5.4}$$

Provided $\mathbf{a} \wedge \mathbf{b} \neq 0$ the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\mathbf{a} \wedge \mathbf{b}} \begin{bmatrix} \mathbf{c} \wedge \mathbf{b} \\ \mathbf{a} \wedge \mathbf{c} \end{bmatrix} \tag{5.5}$$

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$, this is Cramer's rule since the $\mathbf{e}_1 \wedge \mathbf{e}_2$ factors of the wedge products

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{5.6}$$

divide out.

Similarly, for three, or N variables, the same ideas hold

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{d} \tag{5.7}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} \begin{bmatrix} \mathbf{d} \wedge \mathbf{b} \wedge \mathbf{c} \\ \mathbf{a} \wedge \mathbf{d} \wedge \mathbf{c} \\ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{d} \end{bmatrix} \tag{5.8}$$

Again, for the three variable three equation case this is Cramer's rule since the $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ factors of all the wedge products divide out, leaving the familiar determinants.

5.1.2 *A numeric example*

When there are more equations than variables case, if the equations have a solution, each of the k-vector quotients will be scalars

To illustrate here is the solution of a simple example with three equations and two unknowns.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \tag{5.9}$$

The right wedge product with $(1,1,1)$ solves for x

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \wedge \begin{bmatrix}1\\1\\1\end{bmatrix} x = \begin{bmatrix}1\\1\\2\end{bmatrix} \wedge \begin{bmatrix}1\\1\\1\end{bmatrix} \tag{5.10}$$

and a left wedge product with $(1,1,0)$ solves for y

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \wedge \begin{bmatrix}1\\1\\1\end{bmatrix} y = \begin{bmatrix}1\\1\\0\end{bmatrix} \wedge \begin{bmatrix}1\\1\\2\end{bmatrix} \tag{5.11}$$

Observe that both of these equations have the same factor, so one can compute this only once (if this was zero it would indicate the system of equations has no solution).

Collection of results for x and y yields a Cramer's rule like form (writing $\mathbf{e}_i \wedge \mathbf{e}_j = \mathbf{e}_{ij}$):

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{(1,1,0)\wedge(1,1,1)} \begin{bmatrix}(1,1,2)\wedge(1,1,1)\\(1,1,0)\wedge(1,1,2)\end{bmatrix} = \frac{1}{\mathbf{e}_{13}+\mathbf{e}_{23}} \begin{bmatrix}-\mathbf{e}_{13}-\mathbf{e}_{23}\\2\mathbf{e}_{13}+2\mathbf{e}_{23}\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix} \tag{5.12}$$

6 TORQUE

Torque is generally defined as the magnitude of the perpendicular force component times distance, or work per unit angle.

Suppose a circular path in an arbitrary plane containing orthonormal vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is parametrized by angle.

$$\mathbf{r} = r(\hat{\mathbf{u}}\cos\theta + \hat{\mathbf{v}}\sin\theta) = r\hat{\mathbf{u}}(\cos\theta + \hat{\mathbf{u}}\hat{\mathbf{v}}\sin\theta) \tag{6.1}$$

By designating the unit bivector of this plane as the imaginary number

$$\mathbf{i} = \hat{\mathbf{u}}\hat{\mathbf{v}} = \hat{\mathbf{u}} \wedge \hat{\mathbf{v}} \tag{6.2}$$

$$\mathbf{i}^2 = -1 \tag{6.3}$$

this path vector can be conveniently written in complex exponential form

$$\mathbf{r} = r\hat{\mathbf{u}}e^{\mathbf{i}\theta} \tag{6.4}$$

and the derivative with respect to angle is

$$\frac{d\mathbf{r}}{d\theta} = r\hat{\mathbf{u}}\mathbf{i}e^{\mathbf{i}\theta} = \mathbf{r}\mathbf{i} \tag{6.5}$$

So the torque, the rate of change of work W, due to a force F, is

$$\tau = \frac{dW}{d\theta} = \mathbf{F}\cdot\frac{d\mathbf{r}}{d\theta} = \mathbf{F}\cdot(\mathbf{r}\mathbf{i}) \tag{6.6}$$

Unlike the cross product description of torque, $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$ no vector in a normal direction had to be introduced, a normal that does not exist in two dimensions or in greater than three dimensions. The unit bivector describes the plane and the orientation of the rotation, and the sense of the rotation is relative to the angle between the vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$.

6.1 EXPANDING THE RESULT IN TERMS OF COMPONENTS

At a glance this does not appear much like the familiar torque as a determinant or cross product, but this can be expanded to demonstrate its equivalence. Note that the cross product is hiding there in the bivector $\mathbf{i} = \hat{\mathbf{u}} \wedge \hat{\mathbf{v}}$. Expanding the position vector in terms of the planar unit vectors

$$\mathbf{ri} = \left(r_u \hat{\mathbf{u}} + r_v \hat{\mathbf{v}}\right) \hat{\mathbf{u}}\hat{\mathbf{v}} = r_u \hat{\mathbf{v}} - r_v \hat{\mathbf{u}} \tag{6.7}$$

and expanding the force by components in the same direction plus the possible perpendicular remainder term

$$\mathbf{F} = F_u \hat{\mathbf{u}} + F_v \hat{\mathbf{v}} + \mathbf{F}_{\perp \hat{\mathbf{u}},\hat{\mathbf{v}}} \tag{6.8}$$

and then taking dot products yields is the torque

$$\tau = \mathbf{F} \cdot (\mathbf{ri}) = r_u F_v - r_v F_u \tag{6.9}$$

This determinant may be familiar from derivations with $\hat{\mathbf{u}} = \mathbf{e}_1$, and $\hat{\mathbf{v}} = \mathbf{e}_2$ (See the Feynman lectures Volume I for example).

6.2 GEOMETRICAL DESCRIPTION

When the magnitude of the "rotational arm" is factored out, the torque can be written as

$$\tau = \mathbf{F} \cdot (\mathbf{ri}) = |\mathbf{r}|(\mathbf{F} \cdot (\hat{\mathbf{r}}\mathbf{i})) \tag{6.10}$$

The vector $\hat{\mathbf{r}}\mathbf{i}$ is the unit vector perpendicular to the $\mathbf{r}$. Thus the torque can also be described as the product of the magnitude of the rotational arm times the component of the force that is in the direction of the rotation (ie: the work done rotating something depends on length of the lever, and the size of the useful part of the force pushing on it).

6.3 SLIGHT GENERALIZATION. APPLICATION OF THE FORCE TO A LEVER NOT IN THE PLANE

If the rotational arm that the force is applied to is not in the plane of rotation then only the components of the lever arm direction and the component of the force that are in the plane will contribute to the work done. The calculation above was general with respect to the direction of the force, so to generalize it for an arbitrarily oriented lever arm, the quantity $\mathbf{r}$ needs to be replaced by the projection of $\mathbf{r}$ onto the plane of rotation.

That component in the plane (bivector) $\mathbf{i}$ can be described with the geometric product nicely

$$\mathbf{r}_{\mathbf{i}} = (\mathbf{r} \cdot \mathbf{i})\frac{1}{\mathbf{i}} = -(\mathbf{r} \cdot \mathbf{i})\mathbf{i} \tag{6.11}$$

Thus, the vector with this magnitude that is perpendicular to this in the plane of the rotation is

$$\mathbf{r}_{\mathbf{i}}\mathbf{i} = -(\mathbf{r} \cdot \mathbf{i})\mathbf{i}^2 = (\mathbf{r} \cdot \mathbf{i}) \tag{6.12}$$

So, the most general for torque for rotation constrained to the plane i is:

$$\tau = \mathbf{F} \cdot (\mathbf{r} \cdot \mathbf{i}) \tag{6.13}$$

This makes sense when once considers that only the dot product part of $\mathbf{ri} = \mathbf{r} \cdot \mathbf{i} + \mathbf{r} \wedge \mathbf{i}$ contributes to the component of $\mathbf{r}$ in the plane, and when the lever is in the rotational plane this wedge product component of $\mathbf{ri}$ is zero.

6.4 EXPRESSING TORQUE AS A BIVECTOR

The general expression for torque for a rotation constrained to a plane has been found to be:

$$\tau = \mathbf{F} \cdot (\mathbf{r} \cdot \mathbf{i}) \tag{6.14}$$

We have an expectation that torque should have a form similar to the traditional vector torque

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = -\mathbf{i}_3(\mathbf{r} \wedge \mathbf{F}) \tag{6.15}$$

Note that here $\mathbf{i}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit pseudoscalar for $\mathbb{R}^3$, not the unit bivector for the rotational plane. We should be able to express torque in a form related to $\mathbf{r} \wedge \mathbf{F}$, but modified in a fashion that results in a scalar value.

When the rotation is not constrained to a specific plane the motion will be in

$$\mathbf{i} = \frac{\hat{\mathbf{r}} \wedge \mathbf{r}'}{\|\hat{\mathbf{r}} \wedge \mathbf{r}'\|} \tag{6.16}$$

The lever arm component in this plane is

$$
\begin{aligned}
\mathbf{r} \cdot \mathbf{i} &= \frac{1}{2}(\mathbf{r}\mathbf{i} - \mathbf{i}\mathbf{r}) \\
&= \frac{1}{2\|\hat{\mathbf{r}} \wedge \mathbf{r}'\|}(\mathbf{r}(\hat{\mathbf{r}} \wedge \mathbf{r}') - (\hat{\mathbf{r}} \wedge \mathbf{r}')\mathbf{r}) \\
&= \frac{1}{\|\hat{\mathbf{r}} \wedge \mathbf{r}'\|}\mathbf{r}(\hat{\mathbf{r}} \wedge \mathbf{r}')
\end{aligned}
\tag{6.17}
$$

So the torque in this natural plane of rotation is

$$
\begin{aligned}
\tau &= \mathbf{F} \cdot (\mathbf{r} \cdot \mathbf{i}) \\
&= \frac{1}{\|\hat{\mathbf{r}} \wedge \mathbf{r}'\|}\mathbf{F} \cdot (\mathbf{r}(\hat{\mathbf{r}} \wedge \mathbf{r}')) \\
&= \frac{1}{2\|\hat{\mathbf{r}} \wedge \mathbf{r}'\|}\left(\mathbf{F}\mathbf{r}(\hat{\mathbf{r}} \wedge \mathbf{r}') + (\mathbf{r}' \wedge \hat{\mathbf{r}})\mathbf{r}\mathbf{F}\right) \\
&= \frac{1}{2}(\mathbf{F}\mathbf{r}\mathbf{i} + (\mathbf{F}\mathbf{r}\mathbf{i})^\dagger) = \frac{1}{2}(\mathbf{i}\mathbf{r}\mathbf{F} + (\mathbf{i}\mathbf{r}\mathbf{F})^\dagger) \\
&= \langle \mathbf{i}\mathbf{r}\mathbf{F} \rangle_0
\end{aligned}
\tag{6.18}
$$

The torque is the scalar part of $\mathbf{i}(\mathbf{r}\mathbf{F})$.

$$
\tau = \langle \mathbf{i}(\mathbf{r} \cdot \mathbf{F} + \mathbf{r} \wedge \mathbf{F}) \rangle_0
\tag{6.19}
$$

Since the bivector scalar product $\mathbf{i}(\mathbf{r} \cdot \mathbf{F})$ here contributes only a bivector part the scalar part comes only from the $\mathbf{i}(\mathbf{r} \wedge \mathbf{F})$ component, and one can write the torque in a fashion that is very similar to the vector cross product torque. Here is both for comparison

$$
\begin{aligned}
\tau &= \langle \mathbf{i}(\mathbf{r} \wedge \mathbf{F}) \rangle_0 \\
\boldsymbol{\tau} &= -\mathbf{i}_3(\mathbf{r} \wedge \mathbf{F})
\end{aligned}
\tag{6.20}
$$

Note again that $\mathbf{i}$ here is the unit bivector for the plane of rotation and not the unit 3D pseudoscalar $\mathbf{i}_3$.

6.5 PLANE COMMON TO FORCE AND VECTOR

Physical intuition provides one further way to express this. Namely, the unit bivector for the rotational plane should also be in the plane common to $\mathbf{F}$ and $\mathbf{r}$

$$
\mathbf{i} = \frac{\mathbf{F} \wedge \mathbf{r}}{\sqrt{-(\mathbf{F} \wedge \mathbf{r})^2}}
\tag{6.21}
$$

So the torque is

$$\begin{aligned}
\tau &= \frac{1}{\sqrt{-(\mathbf{F} \wedge \mathbf{r})^2}} \left\langle (\mathbf{F} \wedge \mathbf{r})(\mathbf{r} \wedge \mathbf{F}) \right\rangle_0 \\
&= \frac{1}{\sqrt{-(\mathbf{F} \wedge \mathbf{r})^2}} (\mathbf{F} \wedge \mathbf{r})(\mathbf{r} \wedge \mathbf{F}) \\
&= \frac{-(\mathbf{r} \wedge \mathbf{F})^2}{\sqrt{-(\mathbf{r} \wedge \mathbf{F})^2}} \\
&= \sqrt{-(\mathbf{r} \wedge \mathbf{F})^2} \\
&= \|\mathbf{r} \wedge \mathbf{F}\|
\end{aligned} \tag{6.22}$$

Above the $\langle \cdots \rangle_0$ could be dropped because the quantity has only a scalar part. The fact that the sign of the square root can be either plus or minus follows from the fact that the orientation of the unit bivector in the $\mathbf{r}$, $\mathbf{F}$ plane has two possibilities. The positive root selection here is due to the orientation picked for $\mathbf{i}$.

For comparison, this can also be expressed with the cross product:

$$\begin{aligned}
\tau &= \sqrt{-(\mathbf{r} \wedge \mathbf{F})^2} \\
&= \sqrt{-(\mathbf{r} \wedge \mathbf{F})(\mathbf{r} \wedge \mathbf{F})} \\
&= \sqrt{-((\mathbf{r} \times \mathbf{F})\mathbf{i}_3)(\mathbf{i}_3(\mathbf{r} \times \mathbf{F}))} \\
&= \sqrt{(\mathbf{r} \times \mathbf{F})^2} \\
&= \|\mathbf{r} \times \mathbf{F}\| \\
&= \|\boldsymbol{\tau}\|
\end{aligned} \tag{6.23}$$

6.6 TORQUE AS A BIVECTOR

It is natural to drop the magnitude in the torque expression and name the bivector quantity

$$\mathbf{r} \wedge \mathbf{F} \tag{6.24}$$

This defines both the plane of rotation (when that rotation is unconstrained) and the orientation of the rotation, since inverting either the force or the arm position will invert the rotational direction.

When examining the general equations for motion of a particle of fixed mass we will see this quantity again related to the non-radial component of that particles acceleration. Thus we define a torque bivector

$$\boldsymbol{\tau} = \mathbf{r} \wedge \mathbf{F} \tag{6.25}$$

The magnitude of this bivector is our scalar torque, the rate of change of work on the object with respect to the angle of rotation.

7 DERIVATIVES OF A UNIT VECTOR

7.1 FIRST DERIVATIVE OF A UNIT VECTOR

7.1.1 *Expressed with the cross product*

It can be shown that a unit vector derivative can be expressed using the cross product. Two cross product operations are required to get the result back into the plane of the rotation, since a unit vector is constrained to circular (really perpendicular to itself) motion.

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) = \frac{1}{\|\mathbf{r}\|^3}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) \times \mathbf{r} = \left(\hat{\mathbf{r}} \times \frac{1}{\|\mathbf{r}\|}\frac{d\mathbf{r}}{dt}\right) \times \hat{\mathbf{r}} \tag{7.1}$$

This derivative is the rejective component of $\frac{d\mathbf{r}}{dt}$ with respect to $\hat{\mathbf{r}}$, but is scaled by $1/\|\mathbf{r}\|$. How to calculate this result can be found in other places, such as [38].

7.2 EQUIVALENT RESULT UTILIZING THE GEOMETRIC PRODUCT

The equivalent geometric product result can be obtained by calculating the derivative of a vector $\mathbf{r} = r\hat{\mathbf{r}}$.

$$\frac{d\mathbf{r}}{dt} = r\frac{d\hat{\mathbf{r}}}{dt} + \hat{\mathbf{r}}\frac{dr}{dt} \tag{7.2}$$

7.2.1 *Taking dot products*

One trick is required first (as was also the case in the Salus and Hille derivation), which is expressing $\frac{dr}{dt}$ via the dot product.

$$\begin{aligned}\frac{d(r^2)}{dt} &= 2r\frac{dr}{dt}\\ \frac{d(\mathbf{r}\cdot\mathbf{r})}{dt} &= 2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\end{aligned} \tag{7.3}$$

Thus,

$$\frac{dr}{dt} = \hat{\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} \tag{7.4}$$

Taking dot products of the derivative above yields

$$\begin{aligned}
\hat{\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} &= \hat{\mathbf{r}} \cdot r\frac{d\hat{\mathbf{r}}}{dt} + \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\frac{dr}{dt} \\
&= \mathbf{r} \cdot \frac{d\hat{\mathbf{r}}}{dt} + \frac{dr}{dt} \\
&= \mathbf{r} \cdot \frac{d\hat{\mathbf{r}}}{dt} + \hat{\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt}
\end{aligned} \tag{7.5}$$

$$\implies \mathbf{r} \cdot \frac{d\hat{\mathbf{r}}}{dt} = \mathbf{0} \tag{7.6}$$

One could alternatively prove this with a diagram.

7.2.2 *Taking wedge products*

As in linear equation solution, the $\hat{\mathbf{r}}$ component can be eliminated by taking a wedge product

$$\begin{aligned}
\hat{\mathbf{r}} \wedge \frac{d\mathbf{r}}{dt} &= \hat{\mathbf{r}} \wedge r\frac{d\hat{\mathbf{r}}}{dt} + \hat{\mathbf{r}} \wedge \hat{\mathbf{r}}\frac{dr}{dt} \\
&= r\hat{\mathbf{r}} \wedge \frac{d\hat{\mathbf{r}}}{dt} \\
&= \mathbf{r} \wedge \frac{d\hat{\mathbf{r}}}{dt} \\
&= \mathbf{r} \wedge \frac{d\hat{\mathbf{r}}}{dt} + \mathbf{r} \cdot \frac{d\hat{\mathbf{r}}}{dt} \\
&= \mathbf{r}\frac{d\hat{\mathbf{r}}}{dt}
\end{aligned} \tag{7.7}$$

This allows expression of $\frac{d\hat{\mathbf{r}}}{dt}$ in terms of $\frac{d\mathbf{r}}{dt}$ in various ways (compare to the cross product results above)

$$\begin{aligned}
\frac{d\hat{\mathbf{r}}}{dt} &= \frac{1}{\mathbf{r}}\left(\hat{\mathbf{r}} \wedge \frac{d\mathbf{r}}{dt}\right) \\
&= \frac{1}{\|\mathbf{r}\|}\hat{\mathbf{r}}\left(\hat{\mathbf{r}} \wedge \frac{d\mathbf{r}}{dt}\right) \\
&= \frac{1}{\|\mathbf{r}\|}\left(\frac{d\mathbf{r}}{dt} - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt})\right)
\end{aligned} \tag{7.8}$$

Thus this derivative is the component of $\frac{1}{\|\mathbf{r}\|}\frac{d\mathbf{r}}{dt}$ in the direction perpendicular to $\mathbf{r}$.

7.2.3 *Another view*

When the objective is not comparing to the cross product, it is also notable that this unit vector derivative can be written

$$\mathbf{r}\frac{d\hat{\mathbf{r}}}{dt} = \hat{\mathbf{r}} \wedge \frac{d\mathbf{r}}{dt} \tag{7.9}$$

8 RADIAL COMPONENTS OF VECTOR DERIVATIVES

8.1 FIRST DERIVATIVE OF A RADIALLY EXPRESSED VECTOR

Having calculated the derivative of a unit vector, the total derivative of a radially expressed vector can be calculated

$$\begin{aligned}(r\hat{\mathbf{r}})' &= r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}' \\ &= r'\hat{\mathbf{r}} + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')\end{aligned} \tag{8.1}$$

There are two components. One is in the $\hat{\mathbf{r}}$ direction (linear component) and the other perpendicular to that (a rotational component) in the direction of the rejection of $\hat{\mathbf{r}}$ from $\mathbf{r}'$.

8.2 SECOND DERIVATIVE OF A VECTOR

Taking second derivatives of a radially expressed vector, we have

$$\begin{aligned}(r\hat{\mathbf{r}})'' &= (r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}')' \\ &= r''\hat{\mathbf{r}} + r'\hat{\mathbf{r}}' + (r\hat{\mathbf{r}}')' \\ &= r''\hat{\mathbf{r}} + (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + (r\hat{\mathbf{r}}')'\end{aligned} \tag{8.2}$$

Expanding the last term takes a bit more work

$$\begin{aligned}(r\hat{\mathbf{r}}')' &= (\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'))' \\ &= \hat{\mathbf{r}}'(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}}' \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\ &= (1/r)(\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'))(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}}' \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\ &= (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 + \hat{\mathbf{r}}(\hat{\mathbf{r}}' \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'')\end{aligned} \tag{8.3}$$

There are three terms to this. One a scalar (negative) multiple of $\hat{\mathbf{r}}$, and another, the rejection of $\hat{\mathbf{r}}$ from $\mathbf{r}''$. The middle term here remains to be expanded. In particular,

$$\begin{aligned}
\hat{\mathbf{r}}' \wedge \mathbf{r}' &= \hat{\mathbf{r}}' \wedge (r\hat{\mathbf{r}}' + r'\hat{\mathbf{r}}) \\
&= r'\hat{\mathbf{r}}' \wedge \hat{\mathbf{r}} \\
&= r'/2(\hat{\mathbf{r}}'\hat{\mathbf{r}} - \hat{\mathbf{r}}\hat{\mathbf{r}}') \\
&= r'/2r((\mathbf{r}' \wedge \hat{\mathbf{r}})\hat{\mathbf{r}}\hat{\mathbf{r}} - \hat{\mathbf{r}}\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')) \\
&= r'/2r(\mathbf{r}' \wedge \hat{\mathbf{r}} - \hat{\mathbf{r}} \wedge \mathbf{r}') \\
&= -(r'/r)\hat{\mathbf{r}} \wedge \mathbf{r}'
\end{aligned} \tag{8.4}$$

$$\implies (r\hat{\mathbf{r}}')' = (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 - (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \tag{8.5}$$

$$\begin{aligned}
\implies (r\hat{\mathbf{r}})'' &= r''\hat{\mathbf{r}} + (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 - (r'/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\
&= r''\hat{\mathbf{r}} + (1/r)\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}')^2 + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'') \\
&= \hat{\mathbf{r}}\left(r'' + (1/r)(\hat{\mathbf{r}} \wedge \mathbf{r}')^2\right) + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}'')
\end{aligned} \tag{8.6}$$

There are two terms here that are in the $\hat{\mathbf{r}}$ direction (the bivector square is a negative scalar), and one rejective term in the direction of the component perpendicular to $\hat{\mathbf{r}}$ relative to $\mathbf{r}''$.

9 ROTATIONAL DYNAMICS

9.1 GA INTRODUCTION OF ANGULAR VELOCITY

By taking the first derivative of a radially expressed vector we have the velocity

$$\mathbf{v} = r'\hat{\mathbf{r}} + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{r}') = \hat{\mathbf{r}}(v_r + \hat{\mathbf{r}} \wedge \mathbf{v}) \tag{9.1}$$

Or,

$$\hat{\mathbf{r}}\mathbf{v} = v_r + \hat{\mathbf{r}} \wedge \mathbf{v} \tag{9.2}$$

$$\hat{\mathbf{r}}\mathbf{v} = v_r + (1/r)\mathbf{r} \wedge \mathbf{v} \tag{9.3}$$

Put this way, the earlier calculus exercise to derive this seems a bit silly, since it is probably clear that $v_r = \hat{\mathbf{r}} \cdot \mathbf{v}$.

Anyways, let us work with velocity expressed this way in a few ways.

9.1.1 *Speed in terms of linear and rotational components*

$$|\mathbf{v}|^2 = v_r^2 + (\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{v}))^2 \tag{9.4}$$

And,

$$\begin{aligned}
(\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{v}))^2 &= (\mathbf{v} \wedge \hat{\mathbf{r}})\hat{\mathbf{r}}\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{v}) \\
&= (\mathbf{v} \wedge \hat{\mathbf{r}})(\hat{\mathbf{r}} \wedge \mathbf{v}) \\
&= -(\hat{\mathbf{r}} \wedge \mathbf{v})^2 \\
&= |\hat{\mathbf{r}} \wedge \mathbf{v}|^2
\end{aligned} \tag{9.5}$$

$$\begin{aligned}
\implies |\mathbf{v}|^2 &= v_r^2 + |\hat{\mathbf{r}} \wedge \mathbf{v}|^2 \\
&= v_r^2 + |\hat{\mathbf{r}} \wedge \mathbf{v}|^2
\end{aligned} \tag{9.6}$$

So, we can assign a physical significance to the bivector.

$$|\hat{\mathbf{r}} \wedge \mathbf{v}| = |v_\perp| \tag{9.7}$$

The bivector $|\hat{\mathbf{r}} \wedge \mathbf{v}|$ has the magnitude of the non-radial component of the velocity. This equals the magnitude of the component of the velocity perpendicular to its radial component (ie: the angular component of the velocity).

9.1.2 *angular velocity. Prep*

Because $|\hat{\mathbf{r}} \wedge \mathbf{v}|$ is the non-radial velocity component, for small angles $v_\perp/r$ will equal the angle between the vector and its displacement.

This allows for the calculation of the rate of change of that angle with time, what it called the scalar angular velocity (dimensions are $1/t$ not x/t). This can be done by taking the sin as the ratio of the length of the non-radial component of the delta to the length of the displaced vector.

$$\sin d\theta = \frac{|\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge d\mathbf{r})|}{|\mathbf{r} + d\mathbf{r}|} \tag{9.8}$$

With $d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = \mathbf{v}dt$, the angular velocity is

$$\begin{aligned}
\sin d\theta &= \frac{1}{|\mathbf{r} + \mathbf{v}dt|}|\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{v})dt| \\
&= \frac{1}{|\mathbf{r} + \mathbf{v}dt|}|(\hat{\mathbf{r}} \wedge \mathbf{v})dt| \\
\frac{\sin d\theta}{|dt|} &= \frac{1}{|\mathbf{r} + \mathbf{v}dt|}|\hat{\mathbf{r}} \wedge \mathbf{v}| \\
&= \frac{1}{|\mathbf{r}||\mathbf{r} + \mathbf{v}dt|}|\mathbf{r} \wedge \mathbf{v}|
\end{aligned} \tag{9.9}$$

In the limit, taking $dt > 0$, this is

$$\omega = \frac{d\theta}{dt} = \frac{1}{\mathbf{r}^2}|\mathbf{r} \wedge \mathbf{v}| \tag{9.10}$$

9.1.3 angular velocity. Summarizing

Here is a summary of calculations so far involving the $\mathbf{r} \wedge \mathbf{v}$ bivector

$$\begin{aligned}
\mathbf{v} &= \hat{\mathbf{r}} v_r + \frac{\hat{\mathbf{r}}}{|\mathbf{r}|}(\mathbf{r} \wedge \mathbf{v}) \\
\frac{d\hat{\mathbf{r}}}{dt} &= \frac{\hat{\mathbf{r}}}{\mathbf{r}^2}(\mathbf{r} \wedge \mathbf{v}) \\
|v_\perp| &= \frac{1}{|\mathbf{r}|}|\mathbf{r} \wedge \mathbf{v}| \\
\omega = \frac{d\theta}{dt} &= \frac{1}{\mathbf{r}^2}|\mathbf{r} \wedge \mathbf{v}|
\end{aligned} \tag{9.11}$$

It makes sense to give the bivector a name. Given its magnitude the angular velocity bivector $\boldsymbol{\omega}$ is designated

$$\boldsymbol{\omega} = \frac{\mathbf{r} \wedge \mathbf{v}}{\mathbf{r}^2} \tag{9.12}$$

So the linear and rotational components of the velocity can thus be expressed in terms of this, as can our unit vector derivative, scalar angular velocity, and perpendicular velocity magnitude:

$$\begin{aligned}
\omega = \frac{d\theta}{dt} &= |\boldsymbol{\omega}| \\
\mathbf{v} &= \hat{\mathbf{r}} v_r + \mathbf{r}\boldsymbol{\omega} \\
&= \hat{\mathbf{r}}(v_r + r\boldsymbol{\omega}) \\
\frac{d\hat{\mathbf{r}}}{dt} &= \hat{\mathbf{r}}\boldsymbol{\omega} \\
|v_\perp| &= r|\boldsymbol{\omega}|
\end{aligned} \tag{9.13}$$

This is similar to the vector angular velocity ($\boldsymbol{\omega} = (\mathbf{r} \times \mathbf{v})/r^2$), but instead of lying perpendicular to the plane of rotation, it defines the plane of rotation (for a vector $\mathbf{a}$, $\mathbf{a} \wedge \boldsymbol{\omega}$ is zero if the vector is in the plane and non-zero if the vector has a component outside of the plane).

9.1.4 Explicit perpendicular unit vector

If one introduces a unit vector $\hat{\boldsymbol{\theta}}$ in the direction of rejection of $\mathbf{r}$ from $d\mathbf{r}$, the total velocity takes the symmetrical form

$$\begin{aligned}
\mathbf{v} &= v_r \hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}} \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}
\end{aligned} \tag{9.14}$$

9.1.5 *acceleration in terms of angular velocity bivector*

Taking derivatives of velocity, one can with a bit of work, express acceleration in terms of radial and non-radial components

$$
\begin{aligned}
\mathbf{a} &= (\hat{\mathbf{r}} v_r + \mathbf{r}\omega)' \\
&= \hat{\mathbf{r}}' v_r + \hat{\mathbf{r}} v_r' + \mathbf{r}'\omega + \mathbf{r}\omega' \\
&= \hat{\mathbf{r}}\omega v_r + \hat{\mathbf{r}} v_r' + \mathbf{r}'\omega + \mathbf{r}\omega' \\
&= \hat{\mathbf{r}}\omega v_r + \hat{\mathbf{r}} a_r + \mathbf{v}\omega + \mathbf{r}\omega'
\end{aligned}
\tag{9.15}
$$

But,

$$
\begin{aligned}
\omega' &= ((1/r^2)(\mathbf{r} \wedge \mathbf{v}))' \\
&= (-2/r^3) r' (\mathbf{r} \wedge \mathbf{v}) + (1/r^2)(\mathbf{v} \wedge \mathbf{v} + \mathbf{r} \wedge \mathbf{a}) \\
&= -(2/r) v_r \omega + (1/r^2)(\mathbf{r} \wedge \mathbf{a})
\end{aligned}
\tag{9.16}
$$

So,

$$
\begin{aligned}
\mathbf{a} &= \hat{\mathbf{r}} a_r - \hat{\mathbf{r}}\omega v_r + \mathbf{v}\omega + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a}) \\
&= \hat{\mathbf{r}} a_r - (\mathbf{v} - \mathbf{r}\omega)\omega + \mathbf{v}\omega + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a}) \\
\\
&= \hat{\mathbf{r}} a_r + \mathbf{r}\omega^2 + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a}) \\
&= \hat{\mathbf{r}}(a_r + r\omega^2) + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a})
\end{aligned}
\tag{9.17}
$$

Note that ω^2 is a negative scalar, so as normal writing $\|\omega\|^2 = -\omega^2$, we have acceleration in a fashion similar to the traditional cross product form:

$$
\begin{aligned}
\mathbf{a} &= \hat{\mathbf{r}}(a_r - r\|\omega\|^2) + \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a}) \\
&= \hat{\mathbf{r}}(a_r - r\|\omega\|^2 + \hat{\mathbf{r}} \wedge \mathbf{a})
\end{aligned}
\tag{9.18}
$$

In the traditional representation, this last term, the non-radial acceleration component, is often expressed as a derivative.

In terms of the wedge product, this can be done by noting that

$$
(\mathbf{r} \wedge \mathbf{v})' = \mathbf{v} \wedge \mathbf{v} + \mathbf{r} \wedge \mathbf{a} = \mathbf{r} \wedge \mathbf{a}
\tag{9.19}
$$

$$
\begin{aligned}
\mathbf{a} &= \hat{\mathbf{r}}(a_r - r\|\omega\|^2) + \frac{\mathbf{r}}{r^2}(\mathbf{r} \wedge \mathbf{v})' \\
&= \hat{\mathbf{r}}(a_r - r\|\omega\|^2) + \frac{1}{\mathbf{r}}\frac{d(\mathbf{r}^2\omega)}{dt}
\end{aligned}
\tag{9.20}
$$

Expressed in terms of force (for constant mass) this is

$$
\begin{aligned}
\mathbf{F} &= m\mathbf{a} \\
&= \hat{\mathbf{r}}(ma_r) + (m\mathbf{r})\omega^2 + \frac{1}{\mathbf{r}}\frac{d(m\mathbf{r}^2\omega)}{dt} \\
&= \mathbf{F}_r + (m\mathbf{r})\omega^2 + \frac{1}{\mathbf{r}}\frac{d(m\mathbf{r}^2\omega)}{dt}
\end{aligned}
\tag{9.21}
$$

Alternately, the non-radial term can be expressed in terms of torque

$$
\begin{aligned}
\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a}) = \hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge m\mathbf{a}) \\
&= \frac{\mathbf{r}}{r^2}(\mathbf{r} \wedge \mathbf{F}) \\
&= \frac{1}{\mathbf{r}}(\mathbf{r} \wedge \mathbf{F}) \\
&= \frac{1}{\mathbf{r}}\boldsymbol{\tau}
\end{aligned}
\tag{9.22}
$$

Thus the torque bivector, which in magnitude was the angular derivative of the work done by the force $\|\boldsymbol{\tau}\| = \tau = \frac{dW}{d\theta} = \mathbf{F} \cdot \frac{d\mathbf{r}}{d\theta}$ is also expressible as a time derivative

$$
\begin{aligned}
\boldsymbol{\tau} &= \frac{d(m\mathbf{r}^2\omega)}{dt} \\
&= \frac{d(m\mathbf{r} \wedge \mathbf{v})}{dt} \\
&= \frac{d(\mathbf{r} \wedge m\mathbf{v})}{dt} \\
&= \frac{d(\mathbf{r} \wedge \mathbf{p})}{dt}
\end{aligned}
\tag{9.23}
$$

This bivector $m\mathbf{r}^2\omega = \mathbf{r} \wedge \mathbf{p}$ is called the angular momentum, designated $\mathbf{J}$. It is related to the total momentum as follows

$$
\mathbf{p} = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) + \frac{1}{\mathbf{r}}\mathbf{J}
\tag{9.24}
$$

So the total force is

$$\mathbf{F} = \mathbf{F}_r + m\mathbf{r}\omega^2 + \frac{1}{\mathbf{r}}\frac{d\mathbf{J}}{dt} \tag{9.25}$$

Observe that for a purely radial (ie: central) force, we must have $\frac{d\mathbf{J}}{dt} = 0$ so, the angular momentum must be constant.

9.1.6 *Kepler's laws example*

This follows the [38] treatment, modified for the GA notation.

Consider the gravitational force

$$\begin{aligned} m\mathbf{a} &= -G\frac{mM}{r^2}\hat{\mathbf{r}} \\ \mathbf{a} &= -GM\frac{\hat{\mathbf{r}}}{r^2} = -\rho\frac{\hat{\mathbf{r}}}{r^2} \end{aligned} \tag{9.26}$$

Or,

$$\frac{\hat{\mathbf{r}}}{r^2} = -\frac{1}{\rho}\frac{d\mathbf{v}}{dt} \tag{9.27}$$

The unit vector derivative is

$$\begin{aligned} \frac{d\hat{\mathbf{r}}}{dt} &= \frac{\hat{\mathbf{r}}}{r}(\hat{\mathbf{r}} \wedge \mathbf{v}) \\ &= \frac{\hat{\mathbf{r}}}{r^2}\frac{\mathbf{J}}{m} \\ &= -\frac{1}{m\rho}\frac{d\mathbf{v}}{dt}\mathbf{J} \\ &= \frac{d(-\frac{1}{m\rho}\mathbf{v}\mathbf{J})}{dt} \end{aligned} \tag{9.28}$$

The last because $\mathbf{J}$, m, and ρ are all constant.

Before continuing, let us examine this funny vector bivector product term. In general a vector bivector product will have vector and trivector parts, but the differential equation implies that this is a vector. Let us confirm this

$$\begin{aligned} \mathbf{v}\mathbf{J} &= \mathbf{v}(\mathbf{r} \wedge m\mathbf{v}) \\ &= (m\mathbf{v}^2)\hat{\mathbf{v}}(\mathbf{r} \wedge \hat{\mathbf{v}}) \\ &= -(m\mathbf{v}^2)\hat{\mathbf{v}}(\hat{\mathbf{v}} \wedge \mathbf{r}) \end{aligned} \tag{9.29}$$

So, this is in fact a vector, it is the rejective component of $\mathbf{r}$ from the direction of $\hat{\mathbf{v}}$ scaled by $-m\mathbf{v}^2$. We can also calculate the product $\mathbf{J}\mathbf{v}$ from this:

$$\begin{aligned}
\mathbf{v}\mathbf{J} &= -(m\mathbf{v}^2)\hat{\mathbf{v}}(\hat{\mathbf{v}} \wedge \mathbf{r}) \\
&= -(m\mathbf{v}^2)(\mathbf{r} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}} \\
&= -(\mathbf{r} \wedge m\mathbf{v})\mathbf{v} \\
&= -\mathbf{J}\mathbf{v}
\end{aligned} \tag{9.30}$$

This antisymetrical result $\mathbf{v}\mathbf{J} = -\mathbf{J}\mathbf{v}$ is actually the defining property of the vector bivector "dot product" (unlike the vector dot product which is the symmetrical parts). This vector bivector dot product selects the vector component, leaving the trivector part. Since $\mathbf{v}$ lies completely in the plane of the angular velocity bivector $\mathbf{v} \wedge \mathbf{J} = 0$ in this case.

Anyways, back to the problem, integrating with respect to time, and introducing a vector integration constant $\mathbf{e}$ we have

$$\hat{\mathbf{r}} + \frac{1}{m\rho}\mathbf{v}\mathbf{J} = \mathbf{e} \tag{9.31}$$

Multiplying by $\mathbf{r}$

$$\begin{aligned}
r + \frac{1}{m\rho}\mathbf{r}\mathbf{v}\mathbf{J} &= \mathbf{r}\mathbf{e} \\
r + \frac{1}{m^2\rho}(\mathbf{r} \cdot \mathbf{p} + \mathbf{J})\mathbf{J} &= \mathbf{r} \cdot \mathbf{e} + \mathbf{r} \wedge \mathbf{e}
\end{aligned} \tag{9.32}$$

This results in three equations, one for each of the scalar, vector, and bivector parts

$$\begin{aligned}
r + \frac{\mathbf{J}^2}{m^2\rho} &= \mathbf{r} \cdot \mathbf{e} \\
\frac{1}{m\rho}(\mathbf{r} \cdot \mathbf{v})\mathbf{J} &= 0 \\
\mathbf{r} \wedge \mathbf{e} &= 0
\end{aligned} \tag{9.33}$$

The first of these equations is the result from Salas and Hille (integration constant differs in sign though).

$$r - \frac{J^2}{m^2\rho} = \mathbf{r} \cdot \mathbf{e} \tag{9.34}$$

9.1.7 *Circular motion*

For circular motion $v_r = a_r = 0$, so:

$$\mathbf{v} = \mathbf{r}\omega \tag{9.35}$$

$$\mathbf{a} = \hat{\mathbf{r}}\left(-\frac{\mathbf{v}^2}{r} + \hat{\mathbf{r}} \wedge \mathbf{a}\right) \tag{9.36}$$

For constant circular motion:

$$\begin{aligned}
\mathbf{a} &= \mathbf{v}\omega + \mathbf{r}\omega' \\
&= \mathbf{v}\omega + \mathbf{r}(\mathbf{0}) \\
&= \mathbf{r}(\omega)^2 \\
&= -\mathbf{r}|\omega|^2
\end{aligned} \tag{9.37}$$

ie: the $\hat{\mathbf{r}}(\hat{\mathbf{r}} \wedge \mathbf{a})$ term is zero... all acceleration is inwards.
Can also expand this in terms of $\mathbf{r}$ and $\mathbf{v}$:

$$\begin{aligned}
\mathbf{a} &= \mathbf{r}\left(\omega\right)^2 \\
&= \mathbf{r}\left(\frac{1}{\mathbf{r}}\mathbf{v}\right)^2 \\
&= -\mathbf{r}\left(\mathbf{v}\frac{1}{\mathbf{r}}\frac{1}{\mathbf{r}}\mathbf{v}\right) \\
&= -\mathbf{r}\left(\frac{\mathbf{v}^2}{\mathbf{r}^2}\right) \\
&= -\frac{1}{\mathbf{r}}\mathbf{v}^2
\end{aligned} \tag{9.38}$$

10 BIVECTOR GEOMETRY

10.1 MOTIVATION

Consider the derivative of a vector parametrized bivector square such as

$$\frac{d}{d\lambda}(\mathbf{x} \wedge \mathbf{k})^2 = \left(\frac{d\mathbf{x}}{d\lambda} \wedge \mathbf{k}\right)(\mathbf{x} \wedge \mathbf{k}) + (\mathbf{x} \wedge \mathbf{k})\left(\frac{d\mathbf{x}}{d\lambda} \wedge \mathbf{k}\right) \tag{10.1}$$

where $\mathbf{k}$ is constant. In this case, the left hand side is a scalar so the right hand side, this symmetric product of bivectors must also be a scalar. In the more general case, do we have any reason to assume a symmetric bivector product is a scalar as is the case for the symmetric vector product?

Here this question is considered, and examination of products of intersecting bivectors is examined. We take intersecting bivectors to mean that there a common vector ($\mathbf{k}$ above) can be factored from both of the two bivectors, leaving a vector remainder. Since all non coplanar bivectors in $\mathbb{R}^3$ intersect this examination will cover the important special case of three dimensional plane geometry.

A result of this examination is that many of the concepts familiar from vector geometry such as orthogonality, projection, and rejection will have direct bivector equivalents.

General bivector geometry, in spaces where non-coplanar bivectors do not necessarily intersect (such as in $\mathbb{R}^4$) is also considered. Some of the results require plane intersection, or become simpler in such circumstances. This will be pointed out when appropriate.

10.2 COMPONENTS OF GRADE TWO MULTIVECTOR PRODUCT

The geometric product of two bivectors can be written:

$$\mathbf{AB} = \langle \mathbf{AB} \rangle_0 + \langle \mathbf{AB} \rangle_2 + \langle \mathbf{AB} \rangle_4 = \mathbf{A} \cdot \mathbf{B} + \langle \mathbf{AB} \rangle_2 + \mathbf{A} \wedge \mathbf{B} \tag{10.2}$$

$$\mathbf{BA} = \langle \mathbf{BA} \rangle_0 + \langle \mathbf{BA} \rangle_2 + \langle \mathbf{BA} \rangle_4 = \mathbf{B} \cdot \mathbf{A} + \langle \mathbf{BA} \rangle_2 + \mathbf{B} \wedge \mathbf{A} \tag{10.3}$$

Because we have three terms involved, unlike the vector dot and wedge product we cannot generally separate these terms by symmetric and antisymmetric parts. However forming those sums will still worthwhile, especially for the case of intersecting bivectors since the last term will be zero in that case.

10.2.1 Sign change of each grade term with commutation

Starting with the last term we can first observe that

$$\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A} \tag{10.4}$$

To show this let $\mathbf{A} = \mathbf{a} \wedge \mathbf{b}$, and $\mathbf{B} = \mathbf{c} \wedge \mathbf{d}$. When $\mathbf{A} \wedge \mathbf{B} \neq 0$, one can write:

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B} &= \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \\ &= -\mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \wedge \mathbf{a} \\ &= \mathbf{c} \wedge \mathbf{d} \wedge \mathbf{a} \wedge \mathbf{b} \\ &= \mathbf{B} \wedge \mathbf{A} \end{aligned} \tag{10.5}$$

To see how the signs of the remaining two terms vary with commutation form:

$$\begin{aligned} (\mathbf{A} + \mathbf{B})^2 &= (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B}) \\ &= \mathbf{A}^2 + \mathbf{B}^2 + \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} \end{aligned} \tag{10.6}$$

When $\mathbf{A}$ and $\mathbf{B}$ intersect we can write $\mathbf{A} = \mathbf{a} \wedge \mathbf{x}$, and $\mathbf{B} = \mathbf{b} \wedge \mathbf{x}$, thus the sum is a bivector

$$(\mathbf{A} + \mathbf{B}) = (\mathbf{a} + \mathbf{b}) \wedge \mathbf{x} \tag{10.7}$$

And so, the square of the two is a scalar. When $\mathbf{A}$ and $\mathbf{B}$ have only non intersecting components, such as the grade two $\mathbb{R}^4$ multivector $\mathbf{e}_{12} + \mathbf{e}_{34}$, the square of this sum will have both grade four and scalar parts.

Since the LHS = RHS, and the grades of the two also must be the same. This implies that the quantity

$$\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} = \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A} + \langle \mathbf{A}\mathbf{B} \rangle_2 + \langle \mathbf{B}\mathbf{A} \rangle_2 + \mathbf{A} \wedge \mathbf{B} + \mathbf{B} \wedge \mathbf{A} \tag{10.8}$$

is a scalar $\iff$ $\mathbf{A} + \mathbf{B}$ is a bivector, and in general has scalar and grade four terms. Because this symmetric sum has no grade two terms, regardless of whether $\mathbf{A}$, and $\mathbf{B}$ intersect, we have:

$$\langle \mathbf{A}\mathbf{B} \rangle_2 + \langle \mathbf{B}\mathbf{A} \rangle_2 = 0 \tag{10.9}$$

$$\implies \langle \mathbf{AB} \rangle_2 = -\langle \mathbf{BA} \rangle_2 \tag{10.10}$$

One would intuitively expect $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. This can be demonstrated by forming the complete symmetric sum

$$\begin{aligned}
\mathbf{AB} + \mathbf{BA} &= \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A} + \langle \mathbf{AB} \rangle_2 + \langle \mathbf{BA} \rangle_2 + \mathbf{A} \wedge \mathbf{B} + \mathbf{B} \wedge \mathbf{A} \\
&= \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A} + \langle \mathbf{AB} \rangle_2 - \langle \mathbf{AB} \rangle_2 + \mathbf{A} \wedge \mathbf{B} + \mathbf{A} \wedge \mathbf{B} \\
&= \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A} + 2\mathrm{A} \wedge \mathbf{B}
\end{aligned} \tag{10.11}$$

The LHS commutes with interchange of $\mathbf{A}$ and $\mathbf{B}$, as does $\mathbf{A} \wedge \mathbf{B}$. So for the RHS to also commute, the remaining grade 0 term must also:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{10.12}$$

10.2.2 *Dot, wedge and grade two terms of bivector product*

Collecting the results of the previous section and substituting back into eq. (10.2) we have:

$$\mathbf{A} \cdot \mathbf{B} = \left\langle \frac{\mathbf{AB} + \mathbf{BA}}{2} \right\rangle_0 \tag{10.13}$$

$$\langle \mathbf{AB} \rangle_2 = \frac{\mathbf{AB} - \mathbf{BA}}{2} \tag{10.14}$$

$$\mathbf{A} \wedge \mathbf{B} = \left\langle \frac{\mathbf{AB} + \mathbf{BA}}{2} \right\rangle_4 \tag{10.15}$$

When these intersect in a line the wedge term is zero, so for that special case we can write:

$$\mathbf{A} \cdot \mathbf{B} = \frac{\mathbf{AB} + \mathbf{BA}}{2}$$

$$\langle \mathbf{AB} \rangle_2 = \frac{\mathbf{AB} - \mathbf{BA}}{2}$$

$$\mathbf{A} \wedge \mathbf{B} = 0$$

(note that this is always the case for $\mathbb{R}^3$).

10.3 INTERSECTION OF PLANES

Starting with two planes specified parametrically, each in terms of two direction vectors and a point on the plane:

$$
\begin{aligned}
\mathbf{x} &= \mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v} \\
\mathbf{y} &= \mathbf{q} + a\mathbf{w} + b\mathbf{z}
\end{aligned}
\tag{10.16}
$$

If these intersect then all points on the line must satisfy $\mathbf{x} = \mathbf{y}$, so the solution requires:

$$
\mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{q} + a\mathbf{w} + b\mathbf{z}
\tag{10.17}
$$

$$
\implies (\mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v}) \wedge \mathbf{w} \wedge \mathbf{z} = (\mathbf{q} + a\mathbf{w} + b\mathbf{z}) \wedge \mathbf{w} \wedge \mathbf{z} = \mathbf{q} \wedge \mathbf{w} \wedge \mathbf{z}
\tag{10.18}
$$

Rearranging for β, and writing $\mathbf{B} = \mathbf{w} \wedge \mathbf{z}$:

$$
\beta = \frac{\mathbf{q} \wedge \mathbf{B} - (\mathbf{p} + \alpha\mathbf{u}) \wedge \mathbf{B}}{\mathbf{v} \wedge \mathbf{B}}
\tag{10.19}
$$

Note that when the solution exists the left vs right order of the division by $\mathbf{v} \wedge \mathbf{B}$ should not matter since the numerator will be proportional to this bivector (or else the β would not be a scalar).

Substitution of β back into $\mathbf{x} = \mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v}$ (all points in the first plane) gives you a parametric equation for a line:

$$
\mathbf{x} = \mathbf{p} + \frac{(\mathbf{q} - \mathbf{p}) \wedge \mathbf{B}}{\mathbf{v} \wedge \mathbf{B}}\mathbf{v} + \alpha\frac{1}{\mathbf{v} \wedge \mathbf{B}}((\mathbf{v} \wedge \mathbf{B})\mathbf{u} - (\mathbf{u} \wedge \mathbf{B})\mathbf{v})
\tag{10.20}
$$

Where a point on the line is:

$$
\mathbf{p} + \frac{(\mathbf{q} - \mathbf{p}) \wedge \mathbf{B}}{\mathbf{v} \wedge \mathbf{B}}\mathbf{v}
\tag{10.21}
$$

And a direction vector for the line is:

$$
\frac{1}{\mathbf{v} \wedge \mathbf{B}}((\mathbf{v} \wedge \mathbf{B})\mathbf{u} - (\mathbf{u} \wedge \mathbf{B})\mathbf{v})
\tag{10.22}
$$

$$\propto (\mathbf{v} \wedge \mathbf{B})^2 \mathbf{u} - (\mathbf{v} \wedge \mathbf{B})(\mathbf{u} \wedge \mathbf{B})\mathbf{v} \tag{10.23}$$

Now, this result is only valid if $\mathbf{v} \wedge \mathbf{B} \neq 0$ (ie: line of intersection is not directed along $\mathbf{v}$), but if that is the case the second form will be zero. Thus we can add the results (or any non-zero linear combination of) allowing for either of $\mathbf{u}$, or $\mathbf{v}$ to be directed along the line of intersection:

$$a\left((\mathbf{v} \wedge \mathbf{B})^2 \mathbf{u} - (\mathbf{v} \wedge \mathbf{B})(\mathbf{u} \wedge \mathbf{B})\mathbf{v}\right) + b\left((\mathbf{u} \wedge \mathbf{B})^2 \mathbf{v} - (\mathbf{u} \wedge \mathbf{B})(\mathbf{v} \wedge \mathbf{B})\mathbf{u}\right) \tag{10.24}$$

Alternately, one could formulate this in terms of $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$. Is there a more symmetrical form for this direction vector?

10.3.1 *Vector along line of intersection in $\mathbb{R}^3$*

For $\mathbb{R}^3$ one can solve the intersection problem using the normals to the planes. For simplicity put the origin on the line of intersection (and all planes through a common point in $\mathbb{R}^3$ have at least a line of intersection). In this case, for bivectors $\mathbf{A}$ and $\mathbf{B}$, normals to those planes are $i\mathbf{A}$, and $i\mathbf{B}$ respectively. The plane through both of those normals is:

$$(i\mathbf{A}) \wedge (i\mathbf{B}) = \frac{(i\mathbf{A})(i\mathbf{B}) - (i\mathbf{B})(i\mathbf{A})}{2} = \frac{\mathbf{BA} - \mathbf{AB}}{2} = \langle \mathbf{BA} \rangle_2 \tag{10.25}$$

The normal to this plane

$$i\langle \mathbf{BA} \rangle_2 \tag{10.26}$$

is directed along the line of intersection. This result is more appealing than the general $\mathbb{R}^N$ result of eq. (10.24), not just because it is simpler, but also because it is a function of only the bivectors for the planes, without a requirement to find or calculate two specific independent direction vectors in one of the planes.

10.3.2 *Applying this result to $\mathbb{R}^N$*

If you reject the component of $\mathbf{A}$ from $\mathbf{B}$ for two intersecting bivectors:

$$\mathrm{Rej}_{\mathbf{A}}(\mathbf{B}) = \frac{1}{\mathbf{A}} \langle \mathbf{AB} \rangle_2 \tag{10.27}$$

the line of intersection remains the same ... that operation rotates $\mathbf{B}$ so that the two are mutually perpendicular. This essentially reduces the problem to that of the three dimensional case, so the solution has to be of the same form... you just need to calculate a "pseudoscalar" (what you are calling the join), for the subspace spanned by the two bivectors.

That can be computed by taking any direction vector that is on one plane, but is not in the second. For example, pick a vector $\mathbf{u}$ in the plane $\mathbf{A}$ that is not on the intersection of $\mathbf{A}$ and $\mathbf{B}$. In mathese that is $\mathbf{u} = \frac{1}{\mathbf{A}}(\mathbf{A} \cdot \mathbf{u})$ (or $\mathbf{u} \wedge \mathbf{A} = 0$), where $\mathbf{u} \wedge \mathbf{B} \neq 0$. Thus a pseudoscalar for this subspace is:

$$\mathbf{i} = \frac{\mathbf{u} \wedge \mathbf{B}}{|\mathbf{u} \wedge \mathbf{B}|} \tag{10.28}$$

To calculate the direction vector along the intersection we do not care about the scaling above. Also note that provided $\mathbf{u}$ has a component in the plane $\mathbf{A}$, $\mathbf{u} \cdot \mathbf{A}$ is also in the plane (it is rotated $\pi/2$ from $\frac{1}{\mathbf{A}}(\mathbf{A} \cdot \mathbf{u})$.

Thus, provided that $\mathbf{u} \cdot \mathbf{A}$ is not on the intersection, a scaled "pseudoscalar" for the subspace can be calculated by taking from any vector $\mathbf{u}$ with a component in the plane $\mathbf{A}$:

$$\mathbf{i} \propto (\mathbf{u} \cdot \mathbf{A}) \wedge \mathbf{B} \tag{10.29}$$

Thus a vector along the intersection is:

$$\mathbf{d} = ((\mathbf{u} \cdot \mathbf{A}) \wedge \mathbf{B})\langle \mathbf{A}\mathbf{B} \rangle_2 \tag{10.30}$$

Interchange of $\mathbf{A}$ and $\mathbf{B}$ in either the trivector or bivector terms above would also work.

Without showing the steps one can write the complete parametric solution of the line through the planes of equations eq. (10.16) in terms of this direction vector:

$$\mathbf{x} = \mathbf{p} + \left(\frac{(\mathbf{q} - \mathbf{p}) \wedge \mathbf{B}}{(\mathbf{d} \cdot \mathbf{A}) \wedge \mathbf{B}} \right) (\mathbf{d} \cdot \mathbf{A}) + \alpha \mathbf{d} \tag{10.31}$$

Since $(\mathbf{d} \cdot \mathbf{A}) \neq 0$ and $(\mathbf{d} \cdot \mathbf{A}) \wedge \mathbf{B} \neq 0$ (unless $\mathbf{A}$ and $\mathbf{B}$ are coplanar), observe that this is a natural generator of the pseudoscalar for the subspace, and as such shows up in the expression above.

Also observe the non-coincidental similarity of the $\mathbf{q} - \mathbf{p}$ term to Cramer's rule (a ration of determinants).

10.4 COMPONENTS OF A GRADE TWO MULTIVECTOR

The procedure to calculate projections and rejections of planes onto planes is similar to a vector projection onto a space.

To arrive at that result we can consider the product of a grade two multivector $\mathbf{A}$ with a bivector $\mathbf{B}$ and its inverse (the restriction that $\mathbf{B}$ be a bivector, a grade two multivector that can be written as a wedge product of two vectors, is required for general invertability).

$$\begin{aligned}
\mathbf{A}\frac{1}{\mathbf{B}}\mathbf{B} &= \left(\mathbf{A}\cdot\frac{1}{\mathbf{B}} + \left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2 + \mathbf{A}\wedge\frac{1}{\mathbf{B}}\right)\mathbf{B} \\
&= \mathbf{A}\cdot\frac{1}{\mathbf{B}}\mathbf{B} \\
&+ \left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2\cdot\mathbf{B} + \left\langle\left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2\mathbf{B}\right\rangle_2 + \left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2\wedge\mathbf{B} \\
&+ \left(\mathbf{A}\wedge\frac{1}{\mathbf{B}}\right)\cdot\mathbf{B} + \left\langle\mathbf{A}\wedge\frac{1}{\mathbf{B}}\mathbf{B}\right\rangle_4 + \mathbf{A}\wedge\frac{1}{\mathbf{B}}\wedge\mathbf{B}
\end{aligned} \tag{10.32}$$

Since $\frac{1}{\mathbf{B}} = -\frac{\mathbf{B}}{|\mathbf{B}|^2}$, this implies that the 6-grade term $\mathbf{A}\wedge\frac{1}{\mathbf{B}}\wedge\mathbf{B}$ is zero. Since the LHS has grade 2, this implies that the 0-grade and 4-grade terms are zero (also independently implies that the 6-grade term is zero). This leaves:

$$\mathbf{A} = \mathbf{A}\cdot\frac{1}{\mathbf{B}}\mathbf{B} \qquad + \left\langle\left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2\mathbf{B}\right\rangle_2 + \left(\mathbf{A}\wedge\frac{1}{\mathbf{B}}\right)\cdot\mathbf{B} \tag{10.33}$$

This could be written somewhat more symmetrically as

$$\begin{aligned}
\mathbf{A} &= \sum_{i=0,2,4}\left\langle\left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_i\mathbf{B}\right\rangle_2 \\
&= \left\langle\left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle\mathbf{B} + \left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2\mathbf{B} + \left\langle\mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_4\mathbf{B}\right\rangle_2
\end{aligned} \tag{10.34}$$

This is also a more direct way to derive the result in retrospect.

Looking at eq. (10.33) we have three terms. The first is

$$\mathbf{A}\cdot\frac{1}{\mathbf{B}}\mathbf{B} \tag{10.35}$$

This is the component of $\mathbf{A}$ that lies in the plane $\mathbf{B}$ (the projection of $\mathbf{A}$ onto $\mathbf{B}$).

The next is

$$\left\langle \left\langle \mathbf{A} \frac{1}{\mathbf{B}} \right\rangle_2 \mathbf{B} \right\rangle_2 \tag{10.36}$$

If $\mathbf{B}$ and $\mathbf{A}$ have any intersecting components, this is the components of $\mathbf{A}$ from the intersection that are perpendicular to $\mathbf{B}$ with respect to the bivector dot product. ie: This is the rejective term.

And finally,

$$\left(\mathbf{A} \wedge \frac{1}{\mathbf{B}} \right) \cdot \mathbf{B} \tag{10.37}$$

This is the remainder, the non-projective and non-coplanar terms. Greater than three dimensions is required to generate such a term. Example:

$$\begin{aligned} \mathbf{A} &= \mathbf{e}_{12} + \mathbf{e}_{23} + \mathbf{e}_{43} \\ \mathbf{B} &= \mathbf{e}_{34} \end{aligned} \tag{10.38}$$

Product terms for these are:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= 1 \\ \langle \mathbf{A}\mathbf{B} \rangle_2 &= \mathbf{e}_{24} \\ \mathbf{A} \wedge \mathbf{B} &= \mathbf{e}_{1234} \end{aligned} \tag{10.39}$$

The decomposition is thus:

$$\mathbf{A} = \left(\mathbf{A} \cdot \mathbf{B} + \langle \mathbf{A}\mathbf{B} \rangle_2 + \mathbf{A} \wedge \mathbf{B} \right) \frac{1}{\mathbf{B}} = (1 + \mathbf{e}_{24} + \mathbf{e}_{1234})\mathbf{e}_{43} \tag{10.40}$$

10.4.1 *Closer look at the grade two term*

The grade two term of eq. (10.36) can be expanded using its antisymmetric bivector product representation

$$\begin{aligned} \left\langle \mathbf{A} \frac{1}{\mathbf{B}} \right\rangle_2 \mathbf{B} &= \frac{1}{2} \left(\mathbf{A} \frac{1}{\mathbf{B}} - \frac{1}{\mathbf{B}} \mathbf{A} \right) \mathbf{B} \\ &= \frac{1}{2} \left(\mathbf{A} - \frac{1}{\mathbf{B}} \mathbf{A} \mathbf{B} \right) \\ &= \frac{1}{2} \left(\mathbf{A} - \frac{1}{\hat{\mathbf{B}}} \mathbf{A} \hat{\mathbf{B}} \right) \end{aligned} \tag{10.41}$$

Observe here one can restrict the examination to the case where $\mathbf{B}$ is a unit bivector without loss of generality.

$$\begin{aligned}\left\langle \mathbf{A}\frac{1}{\mathbf{i}}\right\rangle_2 \mathbf{i} &= \frac{1}{2}\left(\mathbf{A} + \mathbf{i}\mathbf{A}\mathbf{i}\right) \\ &= \frac{1}{2}\left(\mathbf{A} - \mathbf{i}^\dagger\mathbf{A}\mathbf{i}\right)\end{aligned} \tag{10.42}$$

The second term is a rotation in the plane $\mathbf{i}$, by 180 degrees:

$$\mathbf{i}^\dagger\mathbf{A}\mathbf{i} = e^{-\mathbf{i}\pi/2}\mathbf{A}e^{\mathbf{i}\pi/2} \tag{10.43}$$

So, any components of $\mathbf{A}$ that are completely in the plane cancel out (ie: the $\mathbf{A} \cdot \frac{1}{\mathbf{i}}\mathbf{i}$ component).

Also, if $\langle \mathbf{A}\mathbf{i}\rangle_4 \neq 0$ then those components of $\mathbf{A}\mathbf{i}$ commute so

$$\begin{aligned}\left\langle \mathbf{A} - \mathbf{i}^\dagger\mathbf{A}\mathbf{i}\right\rangle_4 &= \langle \mathbf{A}\rangle_4 - \left\langle \mathbf{i}^\dagger\mathbf{A}\mathbf{i}\right\rangle_4 \\ &= \langle \mathbf{A}\rangle_4 - \left\langle \mathbf{i}^\dagger\mathbf{i}\mathbf{A}\right\rangle_4 \\ &= \langle \mathbf{A}\rangle_4 - \langle \mathbf{A}\rangle_4 \\ &= 0\end{aligned} \tag{10.44}$$

This implies that we have only grade two terms, and the final grade selection in eq. (10.36) can be dropped:

$$\left\langle \left\langle \mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2 \mathbf{B}\right\rangle_2 = \left\langle \mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2 \mathbf{B} \tag{10.45}$$

It is also possible to write this in a few alternate variations which are useful to list explicitly so that one can recognize them in other contexts:

$$\begin{aligned}\left\langle \mathbf{A}\frac{1}{\mathbf{B}}\right\rangle_2 \mathbf{B} &= \frac{1}{2}\left(\mathbf{A} - \frac{1}{\mathbf{B}}\mathbf{A}\mathbf{B}\right) \\ &= \frac{1}{2}\left(\mathbf{A} + \hat{\mathbf{B}}\mathbf{A}\hat{\mathbf{B}}\right) \\ &= \frac{1}{2}\left(\hat{\mathbf{B}}\mathbf{A} - \mathbf{A}\hat{\mathbf{B}}\right)\hat{\mathbf{B}} \\ &= \left\langle \hat{\mathbf{B}}\mathbf{A}\right\rangle_2 \hat{\mathbf{B}} \\ &= \hat{\mathbf{B}}\left\langle \mathbf{A}\hat{\mathbf{B}}\right\rangle_2\end{aligned} \tag{10.46}$$

10.4.2 *Projection and Rejection*

Equation (10.45) can be substituted back into eq. (10.33) yielding:

$$\mathbf{A} = \mathbf{A} \cdot \frac{1}{\mathbf{B}}\mathbf{B} \qquad + \left\langle \mathbf{A}\frac{1}{\mathbf{B}} \right\rangle_2 \mathbf{B} + \left(\mathbf{A} \wedge \frac{1}{\mathbf{B}} \right) \cdot \mathbf{B} \tag{10.47}$$

Now, for the special case where $\mathbf{A} \wedge \mathbf{B} = 0$ (all bivector components of the grade two multivector $\mathbf{A}$ have a common vector with bivector $\mathbf{B}$) we can write

$$\begin{aligned} \mathbf{A} &= \mathbf{A} \cdot \frac{1}{\mathbf{B}}\mathbf{B} + \left\langle \mathbf{A}\frac{1}{\mathbf{B}} \right\rangle_2 \mathbf{B} \\ &= \mathbf{B}\frac{1}{\mathbf{B}} \cdot \mathbf{A} + \mathbf{B}\left\langle \frac{1}{\mathbf{B}}\mathbf{A} \right\rangle_2 \end{aligned} \tag{10.48}$$

This is

$$\mathbf{A} = \mathrm{Proj}_{\mathbf{B}}(\mathbf{A}) + \mathrm{Rej}_{\mathbf{B}}(\mathbf{A}) \tag{10.49}$$

It is worth verifying that these two terms are orthogonal (with respect to the grade two vector dot product)

$$\begin{aligned} \mathrm{Proj}_{\mathbf{B}}(\mathbf{A}) \cdot \mathrm{Rej}_{\mathbf{B}}(\mathbf{A}) &= \langle \mathrm{Proj}_{\mathbf{B}}(\mathbf{A})\, \mathrm{Rej}_{\mathbf{B}}(\mathbf{A}) \rangle \\ &= \left\langle \mathbf{A} \cdot \frac{1}{\mathbf{B}}\mathbf{B}\mathbf{B}\left\langle \frac{1}{\mathbf{B}}\mathbf{A} \right\rangle_2 \right\rangle \\ &= \frac{1}{4\mathbf{B}^2}\langle (\mathbf{AB} + \mathbf{BA})(\mathbf{BA} - \mathbf{AB}) \rangle \\ &= \frac{1}{4\mathbf{B}^2}\langle \mathbf{ABBA} - \mathbf{ABAB} + \mathbf{BABA} - \mathbf{BAAB} \rangle \\ &= \frac{1}{4\mathbf{B}^2}\langle -\mathbf{ABAB} + \mathbf{BABA} \rangle \end{aligned} \tag{10.50}$$

Since we have introduced the restriction $\mathbf{A} \wedge \mathbf{B} \neq 0$, we can use the dot product to reorder product terms:

$$\mathbf{AB} = -\mathbf{BA} + 2\mathbf{A} \cdot \mathbf{B} \tag{10.51}$$

This can be used to reduce the grade zero term above:

$$\begin{aligned} \langle \mathbf{BABA} - \mathbf{ABAB} \rangle &= \langle \mathbf{BA}(-\mathbf{AB} + 2\mathbf{A} \cdot \mathbf{B}) - (-\mathbf{BA} + 2\mathbf{A} \cdot \mathbf{B})\mathbf{AB} \rangle \\ &= +2(\mathbf{A} \cdot \mathbf{B})\langle \mathbf{BA} - \mathbf{AB} \rangle \\ &= +4(\mathbf{A} \cdot \mathbf{B})\langle \langle \mathbf{BA} \rangle_2 \rangle \\ &= 0 \end{aligned} \tag{10.52}$$

This proves orthogonality as expected.

10.4.3 *Grade two term as a generator of rotations*

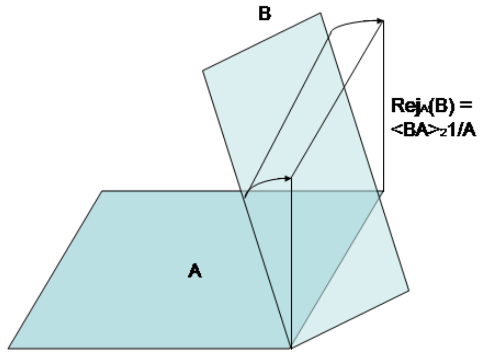

Figure 10.1: Bivector rejection. Perpendicular component of plane

Figure 10.1 illustrates how the grade 2 component of the bivector product acts as a rotation in the rejection operation.

Provided that $\mathbf{A}$ and $\mathbf{B}$ are not coplanar, $\langle\mathbf{AB}\rangle_2$ is a plane mutually perpendicular to both.

Given two mutually perpendicular unit bivectors $\mathbf{A}$ and $\mathbf{B}$, we can in fact write:

$$\mathbf{B} = \mathbf{A}\langle\mathbf{BA}\rangle_2 \tag{10.53}$$

$$\mathbf{B} = \langle\mathbf{AB}\rangle_2\mathbf{A} \tag{10.54}$$

Compare this to a unit bivector for two mutually perpendicular vectors:

$$\mathbf{b} = \mathbf{a}(\mathbf{a} \wedge \mathbf{b}) \tag{10.55}$$

$$\mathbf{b} = (\mathbf{b} \wedge \mathbf{a})\mathbf{a} \tag{10.56}$$

In both cases, the unit bivector functions as an imaginary number, applying a rotation of $\pi/2$ rotating one of the perpendicular entities onto the other.

As with vectors one can split the rotation of the unit bivector into half angle left and right rotations. For example, for the same mutually perpendicular pair of bivectors one can write

$$\begin{aligned}
\mathbf{B} &= \mathbf{A}\langle\mathbf{BA}\rangle_2 \\
&= \mathbf{A}e^{\langle\mathbf{BA}\rangle_2\pi/2} \\
&= e^{-\langle\mathbf{BA}\rangle_2\pi/4}\mathbf{A}e^{\langle\mathbf{BA}\rangle_2\pi/4} \\
&= \left(\frac{1}{\sqrt{2}}(1-\mathbf{BA})\right)\mathbf{A}\left(\frac{1}{\sqrt{2}}(1+\mathbf{BA})\right)
\end{aligned} \tag{10.57}$$

Direct multiplication can be used to verify that this does in fact produce the desired result.

In general, writing

$$\mathbf{i} = \frac{\langle\mathbf{BA}\rangle_2}{|\langle\mathbf{BA}\rangle_2|} \tag{10.58}$$

the rotation of plane **B** towards **A** by angle θ can be expressed with either a single sided full angle

$$\begin{aligned}
\mathrm{R}_{\theta:\mathbf{A}\rightarrow\mathbf{B}}(\mathbf{A}) &= \mathbf{A}e^{\mathbf{i}\theta} \\
&= e^{-\mathbf{i}\theta}\mathbf{A}
\end{aligned} \tag{10.59}$$

or double sided the half angle rotor formulas:

$$\mathrm{R}_{\theta:\mathbf{A}\rightarrow\mathbf{B}}(\mathbf{A}) = e^{-\mathbf{i}\theta/2}\mathbf{A}e^{\mathbf{i}\theta/2} = \mathbf{R}^\dagger\mathbf{A}\mathbf{R} \tag{10.60}$$

Where:

$$\begin{aligned}
\mathbf{R} &= e^{\mathbf{i}\theta/2} \\
&= \cos(\theta/2) + \frac{\langle\mathbf{BA}\rangle_2}{|\langle\mathbf{BA}\rangle_2|}\sin(\theta/2)
\end{aligned} \tag{10.61}$$

As with half angle rotors applied to vectors, there are two possible orientations to rotate. Here the orientation of the rotation is such that the angle is measured along the minimal arc between the two, where the angle between the two is in the range $(0, \pi)$ as opposed to the $(\pi, 2\pi)$ rotational direction.

10.4.4 *Angle between two intersecting planes*

Worth pointing out for comparison to the vector result, one can use the bivector dot product to calculate the angle between two intersecting planes. This angle of separation θ between the two can be expressed using the exponential:

$$\hat{\mathbf{B}} = \hat{\mathbf{A}} e^{\frac{\langle \mathbf{BA} \rangle_2}{|\langle \mathbf{BA} \rangle_2|}\theta} \tag{10.62}$$

$$\implies -\hat{\mathbf{A}}\hat{\mathbf{B}} = e^{\frac{\langle \mathbf{BA} \rangle_2}{|\langle \mathbf{BA} \rangle_2|}\theta} \tag{10.63}$$

Taking the grade zero terms of both sides we have:

$$-\left\langle \hat{\mathbf{A}}\hat{\mathbf{B}} \right\rangle = \left\langle e^{\frac{\langle \mathbf{BA} \rangle_2}{|\langle \mathbf{BA} \rangle_2|}\theta} \right\rangle \tag{10.64}$$

$$\implies \cos(\theta) = -\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \tag{10.65}$$

The sine can be obtained by selecting the grade two terms

$$-\left\langle \hat{\mathbf{A}}\hat{\mathbf{B}} \right\rangle_2 = \frac{\langle \mathbf{BA} \rangle_2}{|\langle \mathbf{BA} \rangle_2|} \sin(\theta) \tag{10.66}$$

$$\implies \sin(\theta) = \frac{|\langle \mathbf{BA} \rangle_2|}{|\mathbf{A}||\mathbf{B}|} \tag{10.67}$$

Note that the strictly positive sine result here is consistent with the fact that the angle is being measured such that it is in the $(0, \pi)$ range.

10.4.5 *Rotation of an arbitrarily oriented plane*

As stated in a few of the GA books the rotor equation is a rotation representation that works for all grade vectors. Let us verify this for the bivector case. Given a plane through the origin spanned by two direction vectors and rotated about the origin in a plane specified by unit magnitude rotor $\mathbf{R}$, the rotated plane will be specified by the wedge of the rotations applied to the two direction vectors. Let

$$\mathbf{A} = \mathbf{u} \wedge \mathbf{v} \tag{10.68}$$

Then,

$$\begin{aligned}
R(\mathbf{A}) &= R(\mathbf{u}) \wedge R(\mathbf{v}) \\
&= (\mathbf{R}^\dagger \mathbf{u} \mathbf{R}) \wedge (\mathbf{R}^\dagger \mathbf{v} \mathbf{R}) \\
&= \frac{1}{2}(\mathbf{R}^\dagger \mathbf{u} \mathbf{R} \mathbf{R}^\dagger \mathbf{v} \mathbf{R} - \mathbf{R}^\dagger \mathbf{v} \mathbf{R} \mathbf{R}^\dagger \mathbf{u} \mathbf{R}) \\
&= \frac{1}{2}(\mathbf{R}^\dagger \mathbf{u} \mathbf{v} \mathbf{R} - \mathbf{R}^\dagger \mathbf{v} \mathbf{u} \mathbf{R}) \\
&= \mathbf{R}^\dagger \frac{\mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}}{2} \mathbf{R} \\
&= \mathbf{R}^\dagger \mathbf{u} \wedge \mathbf{v} \mathbf{R} \\
&= \mathbf{R}^\dagger \mathbf{A} \mathbf{R}
\end{aligned} \tag{10.69}$$

Observe that with this half angle double sided rotation equation, any component of $\mathbf{A}$ in the plane of rotation, or any component that does not intersect the plane of rotation, will be unchanged by the rotor since it will commute with it. In those cases the opposing sign half angle rotations will cancel out. Only the components of the plane that are perpendicular to the rotational plane will be changed by this rotation operation.

10.5 A COUPLE OF REDUCTION FORMULA EQUIVALENTS FROM $\mathbb{R}^3$ VECTOR GEOMETRY

The reduction of the $\mathbb{R}^3$ dot of cross products to dot products can be naturally derived using GA arguments. Writing i as the $\mathbb{R}^3$ pseudoscalar we have:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \frac{\mathbf{a} \wedge \mathbf{b}}{i} \cdot \frac{\mathbf{c} \wedge \mathbf{d}}{i} \\
&= \frac{1}{2}\left(\frac{\mathbf{a} \wedge \mathbf{b}}{i} \frac{\mathbf{c} \wedge \mathbf{d}}{i} + \frac{\mathbf{c} \wedge \mathbf{d}}{i} \frac{\mathbf{a} \wedge \mathbf{b}}{i} \right) \\
&= -\frac{1}{2}\left((\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{c} \wedge \mathbf{d})(\mathbf{a} \wedge \mathbf{b}) \right) \\
&= -(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) - (\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d})
\end{aligned} \tag{10.70}$$

In $\mathbb{R}^3$ this last term must be zero, thus one can write

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = -(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) \tag{10.71}$$

This is now in a form where it can be reduced to products of vector dot products.

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) &= \frac{1}{2}\langle(\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{c} \wedge \mathbf{d})(\mathbf{a} \wedge \mathbf{b})\rangle \\
&= \frac{1}{2}\langle(\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{d} \wedge \mathbf{c})(\mathbf{b} \wedge \mathbf{a})\rangle \\
&= \frac{1}{2}\langle(\mathbf{a}\mathbf{b} - \mathbf{a} \cdot \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{d} \wedge \mathbf{c})(\mathbf{b}\mathbf{a} - \mathbf{b} \cdot \mathbf{a})\rangle \\
&= \frac{1}{2}\langle\mathbf{a}\mathbf{b}(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{d} \wedge \mathbf{c})\mathbf{b}\mathbf{a}\rangle \\
&= \frac{1}{2}\langle\mathbf{a}(\mathbf{b} \cdot (\mathbf{c} \wedge \mathbf{d}) + \mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{d}))((\mathbf{d} \wedge \mathbf{c}) \cdot \mathbf{b} + (\mathbf{d} \wedge \mathbf{c}) \wedge \mathbf{b})\mathbf{a}\rangle \\
&= \frac{1}{2}\langle\mathbf{a}(\mathbf{b} \cdot (\mathbf{c} \wedge \mathbf{d})) + ((\mathbf{d} \wedge \mathbf{c}) \cdot \mathbf{b})\mathbf{a}\rangle \\
&= \frac{1}{2}\langle\mathbf{a}((\mathbf{b} \cdot \mathbf{c})\mathbf{d} - (\mathbf{b} \cdot \mathbf{d})\mathbf{c}) + (\mathbf{d}(\mathbf{c} \cdot \mathbf{b}) - \mathbf{c}(\mathbf{d} \cdot \mathbf{b}))\mathbf{a}\rangle \\
&= \frac{1}{2}((\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) + (\mathbf{d} \cdot \mathbf{a})(\mathbf{c} \cdot \mathbf{b}) - (\mathbf{c} \cdot \mathbf{a})(\mathbf{d} \cdot \mathbf{b})) \\
&= (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})
\end{aligned}
\tag{10.72}
$$

Summarizing with a comparison to the $\mathbb{R}^3$ relations we have:

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = -(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \tag{10.73}$$

$$(\mathbf{a} \wedge \mathbf{c}) \cdot (\mathbf{b} \wedge \mathbf{c}) = -(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}^2(\mathbf{a} \cdot \mathbf{b}) \tag{10.74}$$

The bivector relations hold for all of $\mathbb{R}^N$.

11 TRIVECTOR GEOMETRY

11.1 MOTIVATION

The direction vector for two intersecting planes can be found to have the form:

$$a\left((\mathbf{v} \wedge \mathbf{B})^2\mathbf{u} - (\mathbf{v} \wedge \mathbf{B})(\mathbf{u} \wedge \mathbf{B})\mathbf{v}\right) + b\left((\mathbf{u} \wedge \mathbf{B})^2\mathbf{v} - (\mathbf{u} \wedge \mathbf{B})(\mathbf{v} \wedge \mathbf{B})\mathbf{u}\right) \tag{11.1}$$

While trying to put eq. (11.1) into a form that eliminated $\mathbf{u}$, and $\mathbf{v}$ in favor of $\mathbf{A} = \mathbf{u} \wedge \mathbf{v}$ symmetric and antisymmetric formulations for the various grade terms of a trivector product looked like they could be handy. Here is a summary of those results.

11.2 GRADE COMPONENTS OF A TRIVECTOR PRODUCT

11.2.1 *Grade 6 term*

Writing two trivectors in terms of mutually orthogonal components

$$\mathbf{A} = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \mathbf{xyz} \tag{11.2}$$

and

$$\mathbf{B} = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{uvw} \tag{11.3}$$

Assuming that there is no common vector between the two, the wedge of these is

$$
\begin{aligned}
\mathbf{A} \wedge \mathbf{B} &= \langle \mathbf{AB} \rangle_6 \\
&= \langle \mathbf{xyzuvw} \rangle_6 \\
&= \langle \mathbf{yz}(\mathbf{xu})\mathbf{vw} \rangle_6 \\
&= \langle \mathbf{yz}(-\mathbf{ux} + 2\mathbf{u} \cdot \mathbf{x})\mathbf{vw} \rangle_6 \\
&= -\langle \mathbf{yzu}(\mathbf{xv})\mathbf{w} \rangle_6 \\
&= -\langle \mathbf{yzu}(-\mathbf{vx} + 2\mathbf{v} \cdot \mathbf{x})\mathbf{w} \rangle_6 \\
&= \langle \mathbf{yzuv}(\mathbf{xw}) \rangle_6 \\
&= \cdots \\
&= -\langle \mathbf{uvwxyz} \rangle_6 \\
&= -\langle \mathbf{BA} \rangle_6 \\
&= -\mathbf{B} \wedge \mathbf{A}
\end{aligned}
\tag{11.4}
$$

Note above that any interchange of terms inverts the sign (demonstrated explicitly for all the $\mathbf{x}$ interchanges).

As an aside, this sign change on interchange is taken as the defining property of the wedge product in differential forms. That property also implies also that the wedge product is zero when a vector is wedged with itself since zero is the only value that is the negation of itself. Thus we see explicitly how the notation of using the wedge for the highest grade term of two blades is consistent with the traditional wedge product definition.

The end result here is that the grade 6 term of a trivector trivector product changes sign on interchange of the trivectors:

$$
\langle \mathbf{AB} \rangle_6 = -\langle \mathbf{BA} \rangle_6 \tag{11.5}
$$

11.2.2 *Grade 4 term*

For a trivector product to have a grade 4 term there must be a common vector between the two

$$
\mathbf{A} = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \mathbf{xyz} \tag{11.6}
$$

and

$$
\mathbf{B} = \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{z} = \mathbf{uvz} \tag{11.7}
$$

The grade four term of the product is

$$
\begin{aligned}
\langle \mathbf{BA} \rangle_4 &= \langle \mathbf{uvzxyz} \rangle_4 \\
&= \langle \mathbf{uvzzxy} \rangle_4 \\
&= \mathbf{z}^2 \langle \mathbf{uvxy} \rangle_4 \\
&= \mathbf{z}^2 \langle \mathbf{u}(\mathbf{vx})\mathbf{y} \rangle_4 \\
&= \mathbf{z}^2 \langle \mathbf{u}(-\mathbf{xv} + 2\mathbf{x} \cdot \mathbf{v})\mathbf{y} \rangle_4 \\
&= -\mathbf{z}^2 \langle \mathbf{uxvy} \rangle_4 \\
&= \cdots \\
&= \mathbf{z}^2 \langle \mathbf{xyuv} \rangle_4 \\
&= \langle \mathbf{xyzzuv} \rangle_4 \\
&= \langle \mathbf{xyzuvz} \rangle_4 \\
&= \langle \mathbf{xyzuvz} \rangle_4 \\
&= \langle \mathbf{AB} \rangle_4
\end{aligned}
\tag{11.8}
$$

Thus the grade 4 term commutes on interchange:

$$
\langle \mathbf{AB} \rangle_4 = \langle \mathbf{BA} \rangle_4 \tag{11.9}
$$

11.2.3 *Grade 2 term*

Similar to above, for a trivector product to have a grade 2 term there must be two common vectors between the two

$$
\mathbf{A} = \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \mathbf{xyz} \tag{11.10}
$$

and

$$
\mathbf{B} = \mathbf{u} \wedge \mathbf{y} \wedge \mathbf{z} = \mathbf{uyz} \tag{11.11}
$$

The grade two term of the product is

$$
\begin{aligned}
\langle \mathbf{AB} \rangle_2 &= \langle \mathbf{xyzuyz} \rangle_2 \\
&= \langle \mathbf{xyzyzu} \rangle_2 \\
&= (\mathbf{yz})^2 \langle \mathbf{xu} \rangle_2 \\
&= -(\mathbf{yz})^2 \langle \mathbf{ux} \rangle_2 \\
&= -\langle \mathbf{BA} \rangle_2
\end{aligned}
\tag{11.12}
$$

The grade 2 term anticommutes on interchange:

$$
\langle \mathbf{AB} \rangle_2 = -\langle \mathbf{BA} \rangle_2 \tag{11.13}
$$

11.2.4 *Grade 0 term*

Any grade 0 terms are due to products of the form $\mathbf{A} = k\mathbf{B}$

$$
\begin{aligned}
\langle \mathbf{AB} \rangle_0 &= \langle k\mathbf{BB} \rangle_0 \\
&= \langle \mathbf{B}k\mathbf{B} \rangle_0 \\
&= \langle \mathbf{BA} \rangle_0
\end{aligned}
\tag{11.14}
$$

The grade 2 term commutes on interchange:

$$
\langle \mathbf{AB} \rangle_0 = \langle \mathbf{BA} \rangle_0 \tag{11.15}
$$

11.2.5 *combining results*

$$
\mathbf{AB} = \langle \mathbf{AB} \rangle_0 + \langle \mathbf{AB} \rangle_2 + \langle \mathbf{AB} \rangle_4 + \langle \mathbf{AB} \rangle_6
$$

$$
\begin{aligned}
\mathbf{BA} &= \langle \mathbf{BA} \rangle_0 + \langle \mathbf{BA} \rangle_2 + \langle \mathbf{BA} \rangle_4 + \langle \mathbf{BA} \rangle_6 \\
&= \langle \mathbf{AB} \rangle_0 - \langle \mathbf{AB} \rangle_2 + \langle \mathbf{AB} \rangle_4 - \langle \mathbf{AB} \rangle_6
\end{aligned}
\tag{11.16}
$$

These can be combined to express each of the grade terms as subsets of the symmetric and antisymmetric parts:

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} = \langle \mathbf{AB} \rangle_0 &= \left\langle \frac{\mathbf{AB} + \mathbf{BA}}{2} \right\rangle_0 \\
\langle \mathbf{AB} \rangle_2 &= \left\langle \frac{\mathbf{AB} - \mathbf{BA}}{2} \right\rangle_2 \\
\langle \mathbf{AB} \rangle_4 &= \left\langle \frac{\mathbf{AB} + \mathbf{BA}}{2} \right\rangle_4 \\
\mathbf{A} \wedge \mathbf{B} = \langle \mathbf{AB} \rangle_6 &= \left\langle \frac{\mathbf{AB} - \mathbf{BA}}{2} \right\rangle_6
\end{aligned} \tag{11.17}$$

Note that above I have been somewhat loose with the argument above. A grade three vector will have the following form:

$$\sum_{i<j<k} D_{ijk} \mathbf{e}_{ijk} \tag{11.18}$$

Where D_{ijk} is the determinant of ijk components of the vectors being wedged. Thus the product of two trivectors will be of the following form:

$$\sum_{i<j<k} \sum_{i'<j'<k'} D_{ijk} D'_{i'j'k'} (\mathbf{e}_{ijk} \mathbf{e}_{i'j'k'}) \tag{11.19}$$

It is really each of these $\mathbf{e}_{ijk}\mathbf{e}_{i'j'k'}$ products that have to be considered in the grade and sign arguments above. The end result will be the same though... one would just have to present it a bit more carefully for a true proof.

11.2.6 *Intersecting trivector cases*

As with the intersecting bivector case, when there is a line of intersection between the two volumes one can write:

$$\begin{aligned}
\mathbf{A} \cdot \mathbf{B} = \langle \mathbf{AB} \rangle_0 &= \left\langle \frac{\mathbf{AB} + \mathbf{BA}}{2} \right\rangle_0 \\
\langle \mathbf{AB} \rangle_2 &= \frac{\mathbf{AB} - \mathbf{BA}}{2} \\
\langle \mathbf{AB} \rangle_4 &= \left\langle \frac{\mathbf{AB} + \mathbf{BA}}{2} \right\rangle_4 \\
\mathbf{A} \wedge \mathbf{B} = \langle \mathbf{AB} \rangle_6 &= 0
\end{aligned} \tag{11.20}$$

And if these volumes intersect in a plane a further simplification is possible:

$$
\begin{aligned}
\mathbf{A} \cdot \mathbf{B} = \langle \mathbf{AB} \rangle_0 &= \frac{\mathbf{AB} + \mathbf{BA}}{2} \\
\langle \mathbf{AB} \rangle_2 &= \frac{\mathbf{AB} - \mathbf{BA}}{2} \\
\langle \mathbf{AB} \rangle_4 &= 0 \\
\mathbf{A} \wedge \mathbf{B} = \langle \mathbf{AB} \rangle_6 &= 0
\end{aligned}
\tag{11.21}
$$

12 MULTIVECTOR PRODUCT GRADE ZERO TERMS

One can show that the grade zero component of a multivector product is independent of the order of the terms:

$$\langle \mathbf{AB} \rangle = \langle \mathbf{BA} \rangle \tag{12.1}$$

Doran/Lasenby has an elegant proof of this, but a dumber proof using an explicit expansion by basis also works and highlights the similarities with the standard component definition of the vector dot product.

Writing:

$$\mathbf{A} = \sum_i \langle \mathbf{A} \rangle_i \tag{12.2}$$

$$\mathbf{B} = \sum_i \langle \mathbf{B} \rangle_i \tag{12.3}$$

The product of $\mathbf{A}$ and $\mathbf{B}$ is:

$$\begin{aligned}\mathbf{AB} &= \sum_{ij} \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \\ &= \sum_{ij} \sum_{k=0}^{\min(i,j)} \Big\langle \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \Big\rangle_{2k+|i-j|}\end{aligned} \tag{12.4}$$

$$\mathbf{AB} = \sum_{ij} \sum_{k=0}^{\min(i,j)} \Big\langle \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \Big\rangle_{2k+|i-j|} \tag{12.5}$$

To get a better feel for this, consider an example

$$\mathbf{A} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{12} + \mathbf{e}_{13} + \mathbf{e}_{34} + \mathbf{e}_{345} \tag{12.6}$$

$$\mathbf{B} = \mathbf{e}_2 + \mathbf{e}_{21} + \mathbf{e}_{23} \tag{12.7}$$

$$\mathbf{AB} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{12} + \mathbf{e}_{13} + \mathbf{e}_{34} + \mathbf{e}_{345})(\mathbf{e}_2 + \mathbf{e}_{21} + \mathbf{e}_{23}) \tag{12.8}$$

Here are multivectors with grades ranging from zero to three. This multiplication will include vector/vector, vector/bivector, vector/trivector, bivector/bivector, and bivector/trivector. Some of these will be grade lowering, some grade preserving and some grade raising.

Only the like grade terms can potentially generate grade zero terms, so the grade zero terms of the product in eq. (12.5) are:

$$\mathbf{AB} = \sum_{i=j} \left\langle \langle \mathbf{A} \rangle_i \langle \mathbf{B} \rangle_j \right\rangle \tag{12.9}$$

Using the example above we have

$$\langle \mathbf{AB} \rangle = \langle (\mathbf{e}_1 + \mathbf{e}_2)\mathbf{e}_2 \rangle + \langle (\mathbf{e}_{12} + \mathbf{e}_{13} + \mathbf{e}_{34})\mathbf{e}_{21} \rangle \tag{12.10}$$

In general one can introduce an orthonormal basis $\sigma^k = \{\boldsymbol{\sigma}_i^k\}_i$ for each of the $\langle\rangle_k$ spaces. Here orthonormal is with respect to the k-vector dot product

$$\boldsymbol{\sigma}_i^k \cdot \boldsymbol{\sigma}_j^k = (-1)^{k(k-1)/2} \delta_{ij} \tag{12.11}$$

then one can decompose each of the k-vectors with respect to that basis:

$$\langle \mathbf{A} \rangle_k = \sum_i \left(\langle \mathbf{A} \rangle_k \cdot \boldsymbol{\sigma}_i^k \right) \frac{1}{\boldsymbol{\sigma}_i^k} \tag{12.12}$$

$$\langle \mathbf{B} \rangle_k = \sum_j \left(\langle \mathbf{B} \rangle_k \cdot \boldsymbol{\sigma}_j^k \right) \frac{1}{\boldsymbol{\sigma}_j^k} \tag{12.13}$$

Thus the scalar part of the product is

$$\begin{aligned} \langle \mathbf{AB} \rangle &= \sum_{k,i,j} \left\langle \left(\langle \mathbf{A} \rangle_k \cdot \boldsymbol{\sigma}_i^k \right) \frac{1}{\boldsymbol{\sigma}_i^k} \left(\langle \mathbf{B} \rangle_k \cdot \boldsymbol{\sigma}_j^k \right) \frac{1}{\boldsymbol{\sigma}_j^k} \right\rangle \\ &= \sum_{k,i,j} \left\langle \boldsymbol{\sigma}_i^k \boldsymbol{\sigma}_j^k \right\rangle \left(\langle \mathbf{A} \rangle_k \cdot \boldsymbol{\sigma}_i^k \right) \left(\langle \mathbf{B} \rangle_k \cdot \boldsymbol{\sigma}_j^k \right) \\ &= \sum_{k,i,j} (-1)^{k(k-1)/2} \, \delta_{ij} \left(\langle \mathbf{A} \rangle_k \cdot \boldsymbol{\sigma}_i^k \right) \left(\langle \mathbf{B} \rangle_k \cdot \boldsymbol{\sigma}_j^k \right) \end{aligned} \tag{12.14}$$

Thus the complete scalar product can be written

$$\langle \mathbf{AB} \rangle = \sum_{k,i} (-1)^{k(k-1)/2} \left(\langle \mathbf{A} \rangle_k \cdot \boldsymbol{\sigma}_i^k \right) \left(\langle \mathbf{B} \rangle_k \cdot \boldsymbol{\sigma}_i^k \right) \tag{12.15}$$

Note, compared to the vector dot product, the alternation in sign, which is dependent on the grades involved.

Also note that this now trivially proves that the scalar product is commutative.

Perhaps more importantly we see how similar this generalized dot product is to the standard component formulation of the vector dot product we are so used to. At a glance the component-less geometric algebra formulation seems so much different than the standard vector dot product expressed in terms of components, but we see here that this is in fact not the case.

13 BLADE GRADE REDUCTION

13.1 GENERAL TRIPLE PRODUCT REDUCTION FORMULA

Consideration of the reciprocal frame bivector decomposition required the following identity

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \mathbf{A}_a \cdot (\mathbf{A}_b \cdot \mathbf{A}_c) \tag{13.1}$$

This holds when $a + b \leq c$, and $a <= b$. Similar equations for vector wedge blade dot blade reduction can be found in NFCM, but intuition let me to believe the above generalization was valid.

To prove this use the definition of the generalized dot product of two blades:

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \langle (\mathbf{A}_a \wedge \mathbf{A}_b) \mathbf{A}_c \rangle_{|c-(a+b)|} \tag{13.2}$$

The subsequent discussion is restricted to the $b \geq a$ case. Would have to think whether this restriction is required.

$$\begin{aligned}
\mathbf{A}_a \wedge \mathbf{A}_b &= \mathbf{A}_a \mathbf{A}_b - \sum_{i=|b-a|, i+=2}^{a+b} \langle \mathbf{A}_a \mathbf{A}_b \rangle_i \\
&= \mathbf{A}_a \mathbf{A}_b - \sum_{k=0}^{a-1} \langle \mathbf{A}_a \mathbf{A}_b \rangle_{2k+b-a}
\end{aligned} \tag{13.3}$$

Back substitution gives:

$$\langle (\mathbf{A}_a \wedge \mathbf{A}_b) \mathbf{A}_c \rangle_{|c-(a+b)|} = \langle \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \rangle_{|c-(a+b)|} - \sum_{k=0}^{a-1} \langle \langle \mathbf{A}_a \mathbf{A}_b \rangle_{2k+b-a} \mathbf{A}_c \rangle_{c-a-b} \tag{13.4}$$

Temporarily writing $\langle \mathbf{A}_a \mathbf{A}_b \rangle_{2k+b-a} = \mathbf{C}_i$,

$$\begin{aligned}
\langle \mathbf{A}_a \mathbf{A}_b \rangle_{2k+b-a} \mathbf{A}_c &= \sum_{j=c-i, j+=2}^{c+i} \langle \mathbf{C}_i \mathbf{A}_c \rangle_j \\
&= \sum_{r=0}^{i} \langle \mathbf{C}_i \mathbf{A}_c \rangle_{c-i+2r} \\
&= \sum_{r=0}^{2k+b-a} \langle \mathbf{C}_i \mathbf{A}_c \rangle_{c-2k-b+a+2r} \\
&= \sum_{r=0}^{2k+b-a} \langle \mathbf{C}_i \mathbf{A}_c \rangle_{c-b+a+2(r-k)}
\end{aligned} \tag{13.5}$$

We want the only the following grade terms:

$$c - b + a + 2(r - k) = c - b - a \implies r = k - a \tag{13.6}$$

There are many such k, r combinations, but we have a $k \in [0, a - 1]$ constraint, which implies $r \in [-a, -1]$. This contradicts with r strictly positive, so there are no such grade elements.

This gives an intermediate result, the reduction of the triple product to a direct product, removing the explicit wedge:

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \langle \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \rangle_{c-a-b} \tag{13.7}$$

$$\begin{aligned}
\langle \mathbf{A}_a \mathbf{A}_b \mathbf{A}_c \rangle_{c-a-b} &= \langle \mathbf{A}_a (\mathbf{A}_b \mathbf{A}_c) \rangle_{c-a-b} \\
&= \left\langle \mathbf{A}_a \sum_i \langle \mathbf{A}_b \mathbf{A}_c \rangle_i \right\rangle_{c-a-b} \\
&= \left\langle \sum_j \left\langle \mathbf{A}_a \sum_i \langle \mathbf{A}_b \mathbf{A}_c \rangle_i \right\rangle_j \right\rangle_{c-a-b}
\end{aligned} \tag{13.8}$$

Explicitly specifying the grades here is omitted for simplicity. The lowest grade of these is $(c - b) - a$, and all others are higher, so grade selection excludes them.

By definition

$$\langle \mathbf{A}_b \mathbf{A}_c \rangle_{c-b} = \mathbf{A}_b \cdot \mathbf{A}_c \tag{13.9}$$

so that lowest grade term is thus

$$\langle \mathbf{A}_a \langle \mathbf{A}_b \mathbf{A}_c \rangle_{c-b} \rangle_{c-a-b} = \langle \mathbf{A}_a (\mathbf{A}_b \cdot \mathbf{A}_c) \rangle_{c-a-b} = \mathbf{A}_a \cdot (\mathbf{A}_b \cdot \mathbf{A}_c) \tag{13.10}$$

This completes the proof.

13.2 REDUCTION OF GRADE OF DOT PRODUCT OF TWO BLADES

The result above can be applied to reducing the dot product of two blades. For $k <= s$:

$$(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3 \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s) \tag{13.11}$$

$$\begin{aligned}
&= (\mathbf{a}_1 \wedge (\mathbf{a}_2 \wedge \mathbf{a}_3 \cdots \wedge \mathbf{a}_k)) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s) \\
&= (\mathbf{a}_1 \cdot ((\mathbf{a}_2 \wedge \mathbf{a}_3 \cdots \wedge \mathbf{a}_k)) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s)) \\
&= (\mathbf{a}_1 \cdot (\mathbf{a}_2 \cdot (\mathbf{a}_3 \cdots \wedge \mathbf{a}_k)) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s)) \\
&= \cdots \\
&= \mathbf{a}_1 \cdot (\mathbf{a}_2 \cdot (\mathbf{a}_3 \cdot (\cdots (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_s)))))
\end{aligned} \tag{13.12}$$

This can be reduced to a single determinant, as is done in the Flanders' differential forms book definition of the $\bigwedge^k$ inner product (which is then used to define the Hodge dual).

The first such product is:

$$\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) = \sum (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u) \mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \wedge \mathbf{b}_k \tag{13.13}$$

Next, take dot product with $\mathbf{a}_{k-1}$:

1. $k = 2$

$$\begin{aligned}
&\mathbf{a}_{k-1} \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k)) \\
&= \sum_{v \neq u} (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\
&= \sum_{u<v} (-1)^{v-1} (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_u) + \sum_{u<v} (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\
&= + \sum_{u<v} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) - \sum_{u<v} (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_u) \\
&= + \sum_{u<v} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_1 \cdot \mathbf{b}_v) - (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_u)
\end{aligned} \tag{13.14}$$

$$
- \sum_{u<v} \begin{vmatrix} \mathbf{a}_{k-1} \cdot \mathbf{b}_u & \mathbf{a}_{k-1} \cdot \mathbf{b}_v \\ \mathbf{a}_k \cdot \mathbf{b}_u & \mathbf{a}_k \cdot \mathbf{b}_v \end{vmatrix} \tag{13.15}
$$

2. $k > 2$

$$
\mathbf{a}_{k-1} \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k)) \tag{13.16}
$$

$$
\begin{aligned}
&= \sum (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u) \mathbf{a}_{k-1} \cdot (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \wedge \mathbf{b}_k) \\
&= \sum_{v<u} (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u)(-1)^{v-1} (\mathbf{a}_{k-1} \cdot \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_u \cdots \wedge \mathbf{b}_k) \\
&+ \sum_{v>u} (-1)^{u-1} (\mathbf{a}_k \cdot \mathbf{b}_u)(-1)^{v} (\mathbf{a}_{k-1} \cdot \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned} \tag{13.17}
$$

Add negation exponents, and use a change of variables for the first sum

$$
\begin{aligned}
&= \sum_{u<v} (-1)^{v+u} (\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_{k-1} \cdot \mathbf{b}_u)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\
&- \sum_{u<v} (-1)^{u+v} (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_{k-1} \cdot \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned} \tag{13.18}
$$

Merge sums:

$$
\begin{aligned}
&= \sum_{u<v} (-1)^{u+v} \big((\mathbf{a}_k \cdot \mathbf{b}_v)(\mathbf{a}_{k-1} \cdot \mathbf{b}_u) - (\mathbf{a}_k \cdot \mathbf{b}_u)(\mathbf{a}_{k-1} \cdot \mathbf{b}_v) \big) \\
&(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
\end{aligned} \tag{13.19}
$$

$$
\mathbf{a}_{k-1} \cdot (\mathbf{a}_k \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k)) = \tag{13.20}
$$

$$
\sum_{u<v} (-1)^{u+v} \begin{vmatrix} \mathbf{a}_{k-1} \cdot \mathbf{b}_u & \mathbf{a}_{k-1} \cdot \mathbf{b}_v \\ \mathbf{a}_k \cdot \mathbf{b}_u & \mathbf{a}_k \cdot \mathbf{b}_v \end{vmatrix} (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k)
$$

Note that special casing $k = 2$ does not seem to be required because in that case $-1^{u+v} = -1^{1+2} = -1$, so this is identical to eq. (13.15) after all.

13.2.1 *Pause to reflect*

Although my initial aim was to show that $\mathbf{A}_k \cdot \mathbf{B}_k$ could be expressed as a determinant as in the differential forms book (different sign though), and to determine exactly what that determinant is, there are some useful identities that fall out of this even just for this bivector kvector dot product expansion.

Here is a summary of some of the things figured out so far

1. Dot product of grade one blades.

   Here we have a result that can be expressed as a one by one determinant. Worth mentioning to explicitly show the sign.

$$\mathbf{a} \cdot \mathbf{b} = \det[\mathbf{a} \cdot \mathbf{b}] \tag{13.21}$$

2. Dot product of grade two blades.

$$(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2) = -\begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{vmatrix} = -\det[\mathbf{a}_i \cdot \mathbf{b}_j] \tag{13.22}$$

3. Dot product of grade two blade with grade > 2 blade.

$$\begin{aligned} &(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) \\ &= \sum_{u<v} (-1)^{u+v-1} (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \end{aligned} \tag{13.23}$$

Observe how similar this is to the vector blade dot product expansion:

$$\mathbf{a} \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) = \sum (-1)^{i-1} (\mathbf{a} \cdot \mathbf{b}_i)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_i \cdots \wedge \mathbf{b}_k) \tag{13.24}$$

13.2.1.1 *Expand it for $k = 3$*

Explicit expansion of eq. (13.23) for the $k = 3$ case, is also helpful to get a feel for the equation:

$$\begin{aligned}(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3) = (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2)\mathbf{b}_3 \\ + (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_3 \wedge \mathbf{b}_1)\mathbf{b}_2 \\ + (\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_2 \wedge \mathbf{b}_3)\mathbf{b}_1\end{aligned} \tag{13.25}$$

Observe the cross product like alternation in sign and indices. This suggests that a more natural way to express the sign coefficient may be via a $\operatorname{sgn}(\pi)$ expression for the sign of the permutation of indices.

13.3 TRIVECTOR DOT PRODUCT

With the result of eq. (13.23), or the earlier equivalent determinant expression in equation eq. (13.20) we are now in a position to evaluate the dot product of a trivector and a greater or equal grade blade.

$$\begin{aligned}&\mathbf{a}_1 \cdot ((\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k)) \\ &= \sum_{u<v} (-1)^{u+v-1} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v) \mathbf{a}_1 \cdot (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\ &= \sum_{w<u<v} (-1)^{u+v+w} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_w \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\ &+ \sum_{u<w<v} (-1)^{u+v+w-1} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_w \cdots \check{\mathbf{b}}_v \cdots \wedge \mathbf{b}_k) \\ &+ \sum_{u<v<w} (-1)^{u+v+w} (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)\end{aligned} \tag{13.26}$$

Change the indices of summation and grouping like terms we have:

$$\begin{aligned}\sum_{u<v<w} (-1)^{u+v+w} ((\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_v \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_u) \\ - (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\ + (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w) \\ )(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)\end{aligned} \tag{13.27}$$

Now, each of the embedded dot products were in fact determinants:

$$(\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_x \wedge \mathbf{b}_y) = -\begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_x & \mathbf{a}_2 \cdot \mathbf{b}_y \\ \mathbf{a}_3 \cdot \mathbf{b}_x & \mathbf{a}_3 \cdot \mathbf{b}_y \end{vmatrix} \tag{13.28}$$

Thus, we can expand these triple dot products like so (factor of -1 omitted):

$$\begin{aligned}
&(\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_v \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_u) \\
&- (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_w)(\mathbf{a}_1 \cdot \mathbf{b}_v) \\
&+ (\mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v)(\mathbf{a}_1 \cdot \mathbf{b}_w) \\
&= (\mathbf{a}_1 \cdot \mathbf{b}_u) \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_v & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_v & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix} \\
&- (\mathbf{a}_1 \cdot \mathbf{b}_v) \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix} \\
&+ (\mathbf{a}_1 \cdot \mathbf{b}_w) \begin{vmatrix} \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_v \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_v \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_u & \mathbf{a}_1 \cdot \mathbf{b}_v & \mathbf{a}_1 \cdot \mathbf{b}_w \\ \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_v & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_v & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix}
\end{aligned} \tag{13.29}$$

Final back substitution gives:

$$\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) \\
&= \sum_{u<v<w} (-1)^{u+v+w-1} \begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_u & \mathbf{a}_1 \cdot \mathbf{b}_v & \mathbf{a}_1 \cdot \mathbf{b}_w \\ \mathbf{a}_2 \cdot \mathbf{b}_u & \mathbf{a}_2 \cdot \mathbf{b}_v & \mathbf{a}_2 \cdot \mathbf{b}_w \\ \mathbf{a}_3 \cdot \mathbf{b}_u & \mathbf{a}_3 \cdot \mathbf{b}_v & \mathbf{a}_3 \cdot \mathbf{b}_w \end{vmatrix} (\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)
\end{aligned} \tag{13.30}$$

In particular for $k = 3$ we have

$$\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3) \\
&= -\begin{vmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \mathbf{a}_1 \cdot \mathbf{b}_3 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \mathbf{a}_2 \cdot \mathbf{b}_3 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 & \mathbf{a}_3 \cdot \mathbf{b}_3 \end{vmatrix} = -\det[\mathbf{a}_i \cdot \mathbf{b}_j]
\end{aligned} \tag{13.31}$$

This can be substituted back into eq. (13.30) to put it in a non determinant form.

$$
\begin{aligned}
&(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \cdots \wedge \mathbf{b}_k) \\
&= \sum_{u<v<w} (-1)^{u+v+w} (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_u \wedge \mathbf{b}_v \wedge \mathbf{b}_w)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_u \cdots \check{\mathbf{b}}_v \cdots \check{\mathbf{b}}_w \cdots \wedge \mathbf{b}_k)
\end{aligned}
\tag{13.32}
$$

13.4 INDUCTION ON THE RESULT

It is pretty clear that recursively performing these calculations will yield similar determinant and inner dot product reduction results.

13.4.1 *dot product of like grade terms as determinant*

Let us consider the equal grade case first, summarizing the results so far

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \det[\mathbf{a} \cdot \mathbf{b}] \\
(\mathbf{a}_1 \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2) &= -\det[\mathbf{a}_i \cdot \mathbf{b}_j] \\
(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \mathbf{b}_3) &= -\det[\mathbf{a}_i \cdot \mathbf{b}_j]
\end{aligned}
\tag{13.33}
$$

What will the sign be for the higher grade equivalents? It has the appearance of being related to the sign associated with blade reversion. To verify this calculate the dot product of a blade formed from a set of perpendicular unit vectors with itself.

$$
\begin{aligned}
&(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_k) \\
&= (-1)^{k(k-1)/2} (\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) \cdot (\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_2 \wedge \mathbf{e}_1) \\
&= (-1)^{k(k-1)/2} \mathbf{e}_1 \cdot (\mathbf{e}_2 \cdots (\mathbf{e}_k \cdot (\mathbf{e}_k \wedge \cdots \wedge \mathbf{e}_2 \wedge \mathbf{e}_1))) \\
&= (-1)^{k(k-1)/2} \mathbf{e}_1 \cdot (\mathbf{e}_2 \cdots (\mathbf{e}_{k-1} \cdot (\mathbf{e}_{k-1} \wedge \cdots \wedge \mathbf{e}_2 \wedge \mathbf{e}_1))) \\
&= \cdots \\
&= (-1)^{k(k-1)/2}
\end{aligned}
\tag{13.34}
$$

This fixes the sign, and provides the induction hypothesis for the general case:

$$
(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) = (-1)^{k(k-1)/2} \det[\mathbf{a}_i \cdot \mathbf{b}_j]
\tag{13.35}
$$

Alternately, one can remove the sign change coefficient with reversion of one of the blades:

$$(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \cdot (\mathbf{b}_k \wedge \mathbf{b}_{k-1} \wedge \cdots \wedge \mathbf{b}_1) = \det[\mathbf{a}_i \cdot \mathbf{b}_j] \tag{13.36}$$

13.4.2 *Unlike grades*

Let us summarize the results for unlike grades at the same time reformulating the previous results in terms of index permutation, also writing for brevity $\mathbf{A}_s = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_s$, and $\mathbf{B}_k = \mathbf{b}_1 \wedge \cdots \wedge \mathbf{b}_k$:

$$\mathbf{A}_1 \cdot \mathbf{B}_k = \sum_i \operatorname{sgn}(\pi(i, 1, 2, \cdots \check{i} \cdots, k))(\mathbf{A}_1 \cdot \mathbf{b}_i)(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_i \cdots \wedge \mathbf{b}_k) \tag{13.37}$$

$$\mathbf{A}_2 \cdot \mathbf{B}_k = \sum_{i_1 < i_2} \operatorname{sgn}(\pi(i_1, i_2, 1, 2, \cdots \check{i}_1 \cdots \check{i}_2 \cdots, k)) \tag{13.38}$$

$$\mathbf{A}_2 \cdot (\mathbf{b}_{i_1} \wedge \mathbf{b}_{i_2})(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_{i_1} \cdots \check{\mathbf{b}}_{i_2} \cdots \wedge \mathbf{b}_k) \tag{13.39}$$

$$\mathbf{A}_3 \cdot \mathbf{B}_k = \sum_{i_1 < i_2 < i_3} \operatorname{sgn}(\pi(i_1, i_2, i_3, 1, 2, \cdots \check{i}_1 \cdots \check{i}_2 \cdots \check{i}_3 \cdots, k)) \tag{13.40}$$

$$\mathbf{A}_3 \cdot (\mathbf{b}_{i_1} \wedge \mathbf{b}_{i_2} \wedge \mathbf{b}_{i_3})(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_{i_1} \cdots \check{\mathbf{b}}_{i_2} \cdots \check{\mathbf{b}}_{i_3} \cdots \wedge \mathbf{b}_k) \tag{13.41}$$

We see that the dot product consumes any of the excess sign variation not described by the sign of the permutation of indices.

The induction hypothesis is basically described above (change 3 to s, and add extra dots):

$$\mathbf{A}_s \cdot \mathbf{B}_k = \sum_{i_1 < i_2 \cdots < i_s} \operatorname{sgn}(\pi(i_1, i_2 \cdots, i_s, 1, 2, \cdots \check{i}_1 \cdots \check{i}_2 \cdots \check{i}_s \cdots, k))$$

$$\mathbf{A}_s \cdot (\mathbf{b}_{i_1} \wedge \mathbf{b}_{i_2} \cdots \wedge \mathbf{b}_{i_s})(\mathbf{b}_1 \wedge \cdots \check{\mathbf{b}}_{i_1} \cdots \check{\mathbf{b}}_{i_2} \cdots \check{\mathbf{b}}_{i_s} \cdots \wedge \mathbf{b}_k) \tag{13.42}$$

13.4.3 *Perform the induction*

In a sense this has already been done. The steps will be pretty much the same as the logic that produced the bivector and trivector results. Thinking about typing this up in latex is not fun, so this will be left for a paper proof.

14 MORE DETAILS ON NFCM PLANE FORMULATION

14.1 WEDGE PRODUCT FORMULA FOR A PLANE

The equation of the plane with bivector **U** through point **a** is given by

$$(\mathbf{x} - \mathbf{a}) \wedge \mathbf{U} = 0 \tag{14.1}$$

or

$$\mathbf{x} \wedge \mathbf{U} = \mathbf{a} \wedge \mathbf{U} = \mathbf{T} \tag{14.2}$$

14.1.1 *Examining this equation in more details*

Without any loss of generality one can express this plane equation in terms of a unit bivector **i**

$$\mathbf{x} \wedge \mathbf{i} = \mathbf{a} \wedge \mathbf{i} \tag{14.3}$$

As with the line equation, to express this in the "standard" parametric form, right multiplication with $1/\mathbf{i}$ is required.

$$(\mathbf{x} \wedge \mathbf{i})\frac{1}{\mathbf{i}} = (\mathbf{a} \wedge \mathbf{i})\frac{1}{\mathbf{i}} \tag{14.4}$$

We have a trivector bivector product here, which in general has a vector, trivector, and 5-vector component. Since $\mathbf{i} \wedge \mathbf{i} = 0$, the 5-vector component is zero:

$$\mathbf{x} \wedge \mathbf{i} \wedge -\mathbf{i} = 0 \tag{14.5}$$

and intuition says that the trivector component will also be zero. However, as well as providing verification of this, expansion of this product will also demonstrate how to find the projective and rejective components of a vector with respect to a plane (ie: components in and out of the plane).

14.1.2 *Rejection from a plane product expansion*

Here is an explicit expansion of the rejective term above

$$
\begin{aligned}
(\mathbf{x} \wedge \mathbf{i})\frac{1}{\mathbf{i}} &= -(\mathbf{x} \wedge \mathbf{i})\mathbf{i} \\
&= -\frac{1}{2}(\mathbf{x}\mathbf{i} + \mathbf{i}\mathbf{x})\mathbf{i} \\
&= \frac{1}{2}(\mathbf{x} - \mathbf{i}\mathbf{x}\mathbf{i}) \\
&= \frac{1}{2}(\mathbf{x} - (\mathbf{x}\mathbf{i} + 2\mathbf{i} \cdot \mathbf{x})\mathbf{i}) \\
&= \mathbf{x} - (\mathbf{i} \cdot \mathbf{x})\mathbf{i}
\end{aligned}
\tag{14.6}
$$

In this last term the quantity $\mathbf{i} \cdot \mathbf{x}$ is a vector in the plane. This can be demonstrated by writing $\mathbf{i}$ in terms of a pair of orthonormal vectors $\mathbf{i} = \hat{\mathbf{u}}\hat{\mathbf{v}} = \hat{\mathbf{u}} \wedge \hat{\mathbf{v}}$.

$$
\begin{aligned}
\mathbf{i} \cdot \mathbf{x} &= (\hat{\mathbf{u}} \wedge \hat{\mathbf{v}}) \cdot \mathbf{x} \\
&= \hat{\mathbf{u}}(\hat{\mathbf{v}} \cdot \mathbf{x}) - \hat{\mathbf{v}}(\hat{\mathbf{u}} \cdot \mathbf{x})
\end{aligned}
\tag{14.7}
$$

Thus, $(\mathbf{i} \cdot \mathbf{x}) \wedge \mathbf{i} = 0$, and $(\mathbf{i} \cdot \mathbf{x})\mathbf{i} = (\mathbf{i} \cdot \mathbf{x}) \cdot \mathbf{i}$. Inserting this above we have the end result

$$
\begin{aligned}
(\mathbf{x} \wedge \mathbf{i})\frac{1}{\mathbf{i}} &= \mathbf{x} - (\mathbf{i} \cdot \mathbf{x}) \cdot \mathbf{i} \\
&= \mathbf{a} - (\mathbf{i} \cdot \mathbf{a}) \cdot \mathbf{i}
\end{aligned}
\tag{14.8}
$$

Or

$$
\mathbf{x} - \mathbf{a} = (\mathbf{i} \cdot (\mathbf{x} - \mathbf{a})) \cdot \mathbf{i}
\tag{14.9}
$$

This is actually the standard parametric equation of a plane, but expressed in terms of a unit bivector that describes the plane instead of in terms of a pair of vectors in the plane.

To demonstrate this expansion of the right hand side is required

$$
\begin{aligned}
(\mathbf{i} \cdot \mathbf{x}) \cdot \mathbf{i} &= (\hat{\mathbf{u}}(\hat{\mathbf{v}} \cdot \mathbf{x}) - \hat{\mathbf{v}}(\hat{\mathbf{u}} \cdot \mathbf{x}))\hat{\mathbf{u}}\hat{\mathbf{v}} \\
&= \hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{x}) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \mathbf{x})
\end{aligned}
\tag{14.10}
$$

Substituting this back yields:

$$
\begin{aligned}
\mathbf{x} &= \mathbf{a} + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot (\mathbf{x} - \mathbf{a})) + \hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot (\mathbf{x} - \mathbf{a})) \\
&= \mathbf{a} + s\hat{\mathbf{u}} + t\hat{\mathbf{v}} \\
&= \mathbf{a} + s'\mathbf{y} + t'\mathbf{w}
\end{aligned}
\tag{14.11}
$$

Where $\mathbf{y}$ and $\mathbf{w}$ are two arbitrary, but non-colinear vectors in the plane.

In words this says that the plane is specified by a point in the plane, and the span of any pair of linearly independent vectors directed in that plane.

An expression of this form, or a normal form in terms of the cross product is often how the plane is defined, and the analysis above demonstrates that the bivector wedge product formula,

$$\mathbf{x} \wedge \mathbf{U} = \mathbf{a} \wedge \mathbf{U} \tag{14.12}$$

where specific direction vectors in the plane need not be explicitly specified, also implicitly contains this parametric representation.

14.1.3 *Orthonormal decomposition of a vector with respect to a plane*

With the expansion above we have a separation of a vector into two components, and these can be demonstrated to be the components that are directed entirely within and out of the plane.

Rearranging terms from above we have:

$$\begin{aligned} \mathbf{x} &= (\mathbf{x} \cdot \mathbf{i}) \cdot \frac{1}{\mathbf{i}} + (\mathbf{x} \wedge \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \\ &= (\mathbf{x} \cdot \mathbf{i}) \frac{1}{\mathbf{i}} + (\mathbf{x} \wedge \mathbf{i}) \frac{1}{\mathbf{i}} \end{aligned} \tag{14.13}$$

Writing the vector $\mathbf{x}$ in terms of components parallel and perpendicular to the plane

$$\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{x}_{\|} \tag{14.14}$$

Only the $\mathbf{x}_{\|}$ component contributes to the dot product and only the $\mathbf{x}_{\perp}$ component contributes to the wedge product:

$$\begin{aligned} \mathbf{x} &= (\mathbf{x}_{\|} \cdot \mathbf{i}) \cdot \frac{1}{\mathbf{i}} + (\mathbf{x}_{\perp} \wedge \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \\ \mathbf{x}_{\|} &= (\mathbf{x} \cdot \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \\ \mathbf{x}_{\perp} &= (\mathbf{x} \wedge \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \end{aligned} \tag{14.15}$$

So, just as in the orthonormal decomposition of a vector with respect to a unit vector, this gives us a way to calculate components of a vector in and rejected from any plane, a very useful result in its own right.

Returning to back to the equation of a plane we have

$$-(\mathbf{x} \wedge \mathbf{i})\mathbf{i} = -(\mathbf{a} \wedge \mathbf{i})\mathbf{i} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \tag{14.16}$$

Thus, for the fixed point in the plane, the quantity

$$\mathbf{d} = (\mathbf{a} \wedge \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \tag{14.17}$$

is the component of that vector perpendicular to the plane or the minimal length directed vector from the origin to the plane (directrix). In terms of the unit bivector for the plane and its directrix the equation of a plane becomes

$$\mathbf{x} \wedge \mathbf{i} = \mathbf{d}\mathbf{i} = \mathbf{d} \wedge \mathbf{i} \tag{14.18}$$

Note that the directrix is a normal to the plane.

14.1.4 *Alternate derivation of orthonormal planar decomposition*

This could alternately be derived by expanding the vector unit bivector product directly

$$\begin{aligned}
\mathbf{x}\mathbf{i}\frac{1}{\mathbf{i}} &= (\mathbf{x} \cdot \mathbf{i} + \mathbf{x} \wedge \mathbf{i})\frac{1}{\mathbf{i}} \\
&= -(\mathbf{x} \cdot \mathbf{i}) \cdot \mathbf{i} - (\mathbf{x} \cdot \mathbf{i}) \wedge \mathbf{i} - (\mathbf{x} \wedge \mathbf{i})\mathbf{i} \\
&= -(\mathbf{x} \cdot \mathbf{i}) \cdot \mathbf{i} - (\mathbf{x} \wedge \mathbf{i}) \cdot \mathbf{i} - \langle(\mathbf{x} \wedge \mathbf{i})\mathbf{i}\rangle_3 - (\mathbf{x} \wedge \mathbf{i}) \wedge \mathbf{i} \\
&= (\mathbf{x} \cdot \mathbf{i}) \cdot \frac{1}{\mathbf{i}} + (\mathbf{x} \wedge \mathbf{i}) \cdot \frac{1}{\mathbf{i}} - \langle(\mathbf{x} \wedge \mathbf{i})\mathbf{i}\rangle_3
\end{aligned} \tag{14.19}$$

Since the LHS of this equation is the vector $\mathbf{x}$, the RHS must also be a vector, which demonstrates that the term

$$\langle(\mathbf{x} \wedge \mathbf{i})\mathbf{i}\rangle_3 = 0 \tag{14.20}$$

So, one has

$$\begin{aligned}
\mathbf{x} &= (\mathbf{x} \cdot \mathbf{i}) \cdot \frac{1}{\mathbf{i}} + (\mathbf{x} \wedge \mathbf{i}) \cdot \frac{1}{\mathbf{i}} \\
&= (\mathbf{x} \cdot \mathbf{i})\frac{1}{\mathbf{i}} + (\mathbf{x} \wedge \mathbf{i})\frac{1}{\mathbf{i}}
\end{aligned} \tag{14.21}$$

14.2 GENERALIZATION OF ORTHOGONAL DECOMPOSITION TO COMPONENTS WITH RESPECT TO A HYPERVOLUME

Having observed how to directly calculate the components of a vector in and out of a plane, we can now do the same thing for a rth degree volume element spanned by an r-blade hypervolume element $\mathbf{U}$.

14.2.1 *Hypervolume element and its inverse written in terms of a spanning orthonormal set*

We take $\mathbf{U}$ to be a simple element, not an arbitrary multivector of grade r. Such an element can always be written in the form

$$\mathbf{U} = k\mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_r \tag{14.22}$$

Where $\mathbf{u}_k$ are unit vectors that span the volume element.
The inverse of $\mathbf{U}$ is thus

$$\begin{aligned}
\mathbf{U}^{-1} &= \frac{\mathbf{U}^\dagger}{\mathbf{U}\mathbf{U}^\dagger} \\
&= \frac{k\mathbf{u}_r\cdots\mathbf{u}_1}{(k\mathbf{u}_1\cdots\mathbf{u}_r)(k\mathbf{u}_r\mathbf{u}_{r-1}\cdots\mathbf{u}_1)} \\
&= \frac{\mathbf{u}_r\cdots\mathbf{u}_1}{k}
\end{aligned} \tag{14.23}$$

14.2.2 *Expanding the product*

Having gathered the required introductory steps we are now in a position to express the vector $\mathbf{x}$ in terms of components projected into and rejected from this hypervolume

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}\mathbf{U}\frac{1}{\mathbf{U}} \\
&= (\mathbf{x}\cdot\mathbf{U} + \mathbf{x}\wedge\mathbf{U})\frac{1}{\mathbf{U}}
\end{aligned} \tag{14.24}$$

The dot product term can be expanded to

$$
\begin{aligned}
&(\mathbf{x} \cdot \mathbf{U})\frac{1}{\mathbf{U}} \\
&= k((\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_2\mathbf{u}_3 \cdots \mathbf{u}_r - (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_1\mathbf{u}_3\mathbf{u}_4 \cdots \mathbf{u}_r + \cdots)\frac{1}{\mathbf{U}} \\
&= (\mathbf{x} \cdot \mathbf{u}_1)(\mathbf{u}_2\mathbf{u}_3 \cdots \mathbf{u}_r)(\mathbf{u}_r\mathbf{u}_{r-1} \cdots \mathbf{u}_1) - (\mathbf{x} \cdot \mathbf{u}_2)(\mathbf{u}_1\mathbf{u}_3\mathbf{u}_4 \cdots \mathbf{u}_r)(\mathbf{u}_{r-1} \cdots \mathbf{u}_1) + \cdots \\
&= (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots
\end{aligned}
\tag{14.25}
$$

This demonstrates that $(\mathbf{x} \cdot \mathbf{U})\frac{1}{\mathbf{U}}$ is a vector. Because all the potential $3, 5, ...2r - 1$ grade terms of this product are zero one can write

$$
(\mathbf{x} \cdot \mathbf{U})\frac{1}{\mathbf{U}} = \left\langle (\mathbf{x} \cdot \mathbf{U})\frac{1}{\mathbf{U}} \right\rangle_1 = (\mathbf{x} \cdot \mathbf{U}) \cdot \frac{1}{\mathbf{U}}
\tag{14.26}
$$

In general the product of a $r - 1$-blade and an r-blade such as $(\mathbf{x} \cdot \mathbf{A}_r)\mathbf{B}_r)$ could potentially have any of these higher order terms.

Summarizing the results so far we have

$$
\begin{aligned}
\mathbf{x} &= (\mathbf{x} \cdot \mathbf{U})\frac{1}{\mathbf{U}} + (\mathbf{x} \wedge \mathbf{U})\frac{1}{\mathbf{U}} \\
&= (\mathbf{x} \cdot \mathbf{U}) \cdot \frac{1}{\mathbf{U}} + (\mathbf{x} \wedge \mathbf{U})\frac{1}{\mathbf{U}}
\end{aligned}
\tag{14.27}
$$

Since the RHS of this equation is a vector, this implies that the LHS is also a vector and thus

$$
\begin{aligned}
(\mathbf{x} \wedge \mathbf{U})\frac{1}{\mathbf{U}} &= \left\langle (\mathbf{x} \wedge \mathbf{U})\frac{1}{\mathbf{U}} \right\rangle_1 \\
&= (\mathbf{x} \wedge \mathbf{U}) \cdot \frac{1}{\mathbf{U}}
\end{aligned}
\tag{14.28}
$$

Thus we have an explicit formula for the projective and rejective terms of a vector with respect to a hypervolume element $\mathbf{U}$:

$$
\begin{aligned}
\mathbf{x} &= (\mathbf{x} \cdot \mathbf{U})\frac{1}{\mathbf{U}} + (\mathbf{x} \wedge \mathbf{U})\frac{1}{\mathbf{U}} \\
&= (\mathbf{x} \cdot \mathbf{U}) \cdot \frac{1}{\mathbf{U}} + (\mathbf{x} \wedge \mathbf{U}) \cdot \frac{1}{\mathbf{U}} \\
&= \frac{-1^{r(r-1)/2}}{|\mathbf{U}|^2} \left((\mathbf{x} \cdot \mathbf{U}) \cdot \mathbf{U} + (\mathbf{x} \wedge \mathbf{U}) \cdot \mathbf{U} \right)
\end{aligned}
\tag{14.29}
$$

14.2.3 *Special note. Interpretation for projection and rejective components of a line*

The proof above utilized the general definition of the dot product of two blades, the selection of the lowest grade element of the product:

$$\mathbf{A}_k \cdot \mathbf{B}_j = \left\langle \mathbf{A}_k \mathbf{B}_j \right\rangle_{|k-j|} \tag{14.30}$$

Because of this, the scalar-vector dot product is perfectly well defined

$$a \cdot \mathbf{b} = \langle a\mathbf{b} \rangle_{1-0} = a\mathbf{b} \tag{14.31}$$

So, when $\mathbf{U}$ is a vector, the equations above also hold.

14.2.4 *Explicit expansion of projective and rejective components*

Having calculated the explicit vector expansion of the projective term to prove that the all the higher grade product terms were zero, this can be used to explicitly expand the projective and rejective components in terms of a set of unit vectors that span the hypervolume

$$\begin{aligned}
\mathbf{x}_\parallel &= (\mathbf{x} \cdot \mathbf{U}) \cdot \frac{1}{\mathbf{U}} \\
&= (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots \\
\mathbf{x}_\perp &= (\mathbf{x} \wedge \mathbf{U}) \cdot \frac{1}{\mathbf{U}} \\
&= \mathbf{x} - (\mathbf{x} \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{x} \cdot \mathbf{u}_2)\mathbf{u}_2 - \cdots
\end{aligned} \tag{14.32}$$

Recall here that the unit vectors $\mathbf{u}_k$ are not the standard basis vectors. They are instead an arbitrary set of orthonormal vectors that span the hypervolume element $\mathbf{U}$.

14.3 VERIFICATION THAT PROJECTIVE AND REJECTIVE COMPONENTS ARE ORTHOGONAL

In NFCM, for the equation of a line, it is demonstrated that the two vector components (directrix and parametrization) are orthogonal, and that the directrix is the minimal distance to the line from the origin. That can be done here too for the hypervolume result.

$$
\begin{aligned}
\mathbf{x} \wedge \mathbf{U} &= \mathbf{a} \wedge \mathbf{U} \\
(\mathbf{x} \wedge \mathbf{U})\mathbf{U}^{-1} &= (\mathbf{a} \wedge \mathbf{U})\mathbf{U}^{-1} \\
(\mathbf{x}\mathbf{U} - \mathbf{x} \cdot \mathbf{U})\mathbf{U}^{-1} &= (\mathbf{a} \wedge \mathbf{U})\mathbf{U}^{-1} \\
\mathbf{x} &= (\mathbf{x} \cdot \mathbf{U})\mathbf{U}^{-1} + (\mathbf{a} \wedge \mathbf{U})\mathbf{U}^{-1} \\
&= \boldsymbol{\alpha}\mathbf{U}^{-1} + \mathbf{d}
\end{aligned}
\tag{14.33}
$$

This first component, the projective term $\boldsymbol{\alpha}\mathbf{U}^{-1} = (\mathbf{x} \cdot \mathbf{U})\mathbf{U}^{-1}$, can be interpreted as a parametrization term. The last component, the rejective term $\mathbf{d} = (\mathbf{a} \wedge \mathbf{U})\mathbf{U}^{-1}$ is identified as the directrix. Calculation of $|\mathbf{x}|$ allows us to verify the physical interpretation of this vector.

Expansion of the projective term has previously shown that given

$$
\mathbf{U} = k\mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \wedge \mathbf{u}_r
\tag{14.34}
$$

then the expansion of this parametrization term has the form

$$
\boldsymbol{\alpha} = \left(\sum_{i=1}^{r} \alpha_i \mathbf{u}_i \right) \mathbf{U}
\tag{14.35}
$$

This is a very specific parametrization, a $r - 1$ grade parametrization $\boldsymbol{\alpha}$ with r free variables, producing a vector directed strictly in hypervolume spanned by $\mathbf{U}$.

We can calculate the length of the projective component of $\mathbf{x}$ expressed in terms of this parametrization:

$$
\begin{aligned}
{\mathbf{x}_{\|}}^2 &= \left((\mathbf{x} \cdot \mathbf{U})\mathbf{U}^{-1} \right)^2 \\
&= \boldsymbol{\alpha} \frac{\mathbf{U}^\dagger}{\mathbf{U}\mathbf{U}^\dagger} \left(\frac{\mathbf{U}^\dagger}{\mathbf{U}\mathbf{U}^\dagger} \right)^\dagger \boldsymbol{\alpha}^\dagger \\
&= \boldsymbol{\alpha} \frac{\mathbf{U}^\dagger}{|\mathbf{U}|^2} \frac{\mathbf{U}}{|\mathbf{U}|^2} \boldsymbol{\alpha}^\dagger \\
&= \frac{|\boldsymbol{\alpha}|^2}{|\mathbf{U}|^2}
\end{aligned}
\tag{14.36}
$$

$$
\begin{aligned}
\mathbf{x}^2 &= (\alpha\mathbf{U}^{-1})^2 + \mathbf{d}^2 + 2(\alpha\mathbf{U}^{-1}) \cdot \mathbf{d} \\
&= \frac{|\alpha|^2}{|\mathbf{U}|^2} + \mathbf{d}^2 + 2(\alpha\mathbf{U}^{-1}) \cdot \mathbf{d}
\end{aligned}
\tag{14.37}
$$

Direct computation shows that this last dot product term is zero

$$
\begin{aligned}
(\alpha\mathbf{U}^{-1}) \cdot \mathbf{d} &= (\alpha\mathbf{U}^{-1}) \cdot ((\mathbf{a} \wedge \mathbf{U})\mathbf{U}^{-1}) \\
&= (\alpha\mathbf{U}^{-1}) \cdot (\mathbf{U}^{-1}(\mathbf{U} \wedge \mathbf{a})) \\
&= \frac{(-1)^{r(r-1)/2}}{|\mathbf{U}|^2}(\alpha\mathbf{U}^{-1}) \cdot (\mathbf{U}(\mathbf{U} \wedge \mathbf{a})) \\
&= \frac{(-1)^{r(r-1)/2}}{|\mathbf{U}|^2}\left\langle \alpha\mathbf{U}^{-1}\mathbf{U}(\mathbf{U} \wedge \mathbf{a})\right\rangle_0 \\
&= \frac{(-1)^{r(r-1)/2}}{|\mathbf{U}|^2}\left\langle \alpha(\mathbf{U} \wedge \mathbf{a})\right\rangle_0
\end{aligned}
\tag{14.38}
$$

This last term is a product of an $r-1$ grade blade and a $r+1$ grade blade. The lowest order term of this product has grade $r+1-(r-1) = 2$, which implies that $\langle \alpha(\mathbf{U} \wedge \mathbf{a})\rangle_0 = 0$. This demonstrates explicitly that the parametrization term is perpendicular to the rejective term as expected.

The length from the origin to the volume is thus

$$
\mathbf{x}^2 = \frac{|\alpha|^2}{|\mathbf{U}|^2} + \mathbf{d}^2
\tag{14.39}
$$

This is minimized when $\alpha = 0$. Thus $\mathbf{d}$ is a vector directed from the origin to the hypervolume, perpendicular to that hypervolume, and also has the minimal distance to that space.

15 QUATERNIONS

Like complex numbers, quaternions may be written as a multivector with scalar and bivector components (a 0,2-multivector).

$$q = \alpha + \mathbf{B} \tag{15.1}$$

Where the complex number has one bivector component, and the quaternions have three.

One can describe quaternions as 0,2-multivectors where the basis for the bivector part is left handed. There is not really anything special about quaternion multiplication, or complex number multiplication, for that matter. Both are just a specific examples of a 0,2-multivector multiplication. Other quaternion operations can also be found to have natural multivector equivalents. The most important of which is likely the quaternion conjugate, since it implies the norm and the inverse. As a multivector, like complex numbers, the conjugate operation is reversal:

$$\overline{q} = q^\dagger = \alpha - \mathbf{B} \tag{15.2}$$

Thus $|q|^2 = q\overline{q} = \alpha^2 - \mathbf{B}^2$. Note that this norm is a positive definite as expected since a bivector square is negative.

To be more specific about the left handed basis property of quaternions one can note that the quaternion bivector basis is usually defined in terms of the following properties

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \tag{15.3}$$

$$\mathbf{ij} = -\mathbf{ji}, \mathbf{ik} = -\mathbf{ki}, \mathbf{jk} = -\mathbf{kj} \tag{15.4}$$

$$\mathbf{ij} = \mathbf{k} \tag{15.5}$$

The first two properties are satisfied by any set of orthogonal unit bivectors for the space. The last property, which could also be written $\mathbf{ijk} = -1$, amounts to a choice for the orientation of this bivector basis of the 2-vector part of the quaternion.

As an example suppose one picks

$$\mathbf{i} = \mathbf{e}_2\mathbf{e}_3 \tag{15.6}$$

$$\mathbf{j} = \mathbf{e}_3\mathbf{e}_1 \tag{15.7}$$

Then the third bivector required to complete the basis set subject to the properties above is

$$\mathbf{ij} = \mathbf{e}_2\mathbf{e}_1 = \mathbf{k} \tag{15.8}$$

.

Suppose that, instead of the above, one picked a slightly more natural bivector basis, the duals of the unit vectors obtained by multiplication with the pseudoscalar ($\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_i$). These bivectors are

$$\mathbf{i} = \mathbf{e}_2\mathbf{e}_3, \mathbf{j} = \mathbf{e}_3\mathbf{e}_1, \mathbf{k} = \mathbf{e}_1\mathbf{e}_2 \tag{15.9}$$

.

A 0,2-multivector with this as the basis for the bivector part would have properties similar to the standard quaternions (anti-commutative unit quaternions, negation for unit quaternion square, same conjugate, norm and inversion operations, ...), however the triple product would have the value $\mathbf{ijk} = 1$, instead of -1.

15.1 QUATERNION AS GENERATOR OF DOT AND CROSS PRODUCT

The product of pure quaternions is noted as being a generator of dot and cross products. This is also true of a vector bivector product.

Writing a vector $\mathbf{x}$ as

$$\mathbf{x} = \sum_i x_i\mathbf{e}_i = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{15.10}$$

And a bivector $\mathbf{B}$ (where for short, $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$) as:

$$\mathbf{B} = \sum_i b_i\mathbf{e}_i I = b_1\mathbf{e}_{23} + b_2\mathbf{e}_{31} + b_3\mathbf{e}_{12} \tag{15.11}$$

The product of these two is

$$
\begin{aligned}
\mathbf{x}\mathbf{B} &= (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)(b_1\mathbf{e}_{23} + b_2\mathbf{e}_{31} + b_3\mathbf{e}_{12}) \\
&= (x_3b_2 - x_2b_3)\mathbf{e}_1 + (x_1b_3 - x_3b_1)\mathbf{e}_2 + (x_2b_1 - x_1b_2)\mathbf{e}_3 \\
&+ (x_1b_1 + x_2b_2 + x_3b_3)\mathbf{e}_{123}
\end{aligned}
\tag{15.12}
$$

Looking at the vector and trivector components of this we recognize the dot product and negated cross product immediately (as with multiplication of pure quaternions).

Those products are, in fact, $\mathbf{x} \cdot \mathbf{B}$ and $\mathbf{x} \wedge \mathbf{B}$ respectively.

Introducing a vector and bivector basis $\alpha = \{\mathbf{e}_i\}$, and $\beta = \{\mathbf{e}_i I\}$, we can express the dot product and cross product of the associated coordinate vectors in terms of vector bivectors products as follows:

$$
[\mathbf{x}]_\alpha \cdot [\mathbf{B}]_\beta = \frac{\mathbf{B} \wedge \mathbf{x}}{I}
\tag{15.13}
$$

$$
[\mathbf{x}]_\alpha \times [\mathbf{B}]_\beta = [\mathbf{B} \cdot \mathbf{x}]_\alpha
\tag{15.14}
$$

16

CAUCHY EQUATIONS EXPRESSED AS A GRADIENT

The complex number derivative, when it exists, is defined as:

$$\frac{\delta f}{\delta z} = \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$f'(z) = \text{limit}_{|\delta z| \rightarrow 0} \quad \frac{\delta f}{\delta z}$$

Like any two variable function, this limit requires that all limiting paths produce the same result, thus it is minimally necessary that the limits for the particular cases of $\delta z = \delta x + i\delta y$ exist for both $\delta x = 0$, and $\delta y = 0$ independently. Of course there are other possible ways for $\delta z \rightarrow 0$, such as spiraling inwards paths. Apparently it can be shown that if the specific cases are satisfied, then this limit exists for any path (I am not sure how to show that, nor will try, at least now).

Examining each of these cases separately, we have for $\delta x = 0$, and $f(z) = u(x, y) + iv(x, y)$:

$$\begin{aligned}
\frac{\delta f}{\delta z} &= \frac{u(x, y+\delta y) + iv(x, y+\delta y)}{i\delta y} \\
&\rightarrow -i\frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial y}
\end{aligned} \tag{16.1}$$

and for $\delta y = 0$

$$\begin{aligned}
\frac{\delta f}{\delta z} &= \frac{u(x+\delta x, y) + iv(x+\delta x, y)}{\delta x} \\
&\rightarrow \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}
\end{aligned} \tag{16.2}$$

If these are equal regardless of the path, then equating real and imaginary parts of these respective equations we have:

$$\begin{aligned}
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0 \\
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0
\end{aligned} \tag{16.3}$$

Now, these are strikingly similar to the gradient, and we make this similarly explicit using the planar pseudoscalar $i = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$ as the unit imaginary. For the first equation, pre multiplying by $1 = \mathbf{e}_{11}$, and post multiplying by $\mathbf{e}_2$ we have:

$$\mathbf{e}_1 \frac{\partial \mathbf{e}_{12} v}{\partial x} + \mathbf{e}_2 \frac{\partial u}{\partial y} = 0,$$

and for the second, pre multiply by $\mathbf{e}_1$, and post multiply the ∂_y term by $1 = \mathbf{e}_{22}$, and rearrange:

$$\mathbf{e}_1 \frac{\partial u}{\partial x} + \mathbf{e}_2 \frac{\partial \mathbf{e}_{12} v}{\partial y} = 0.$$

Adding these we have:

$$\mathbf{e}_1 \frac{\partial u + \mathbf{e}_{12}}{\partial x} + \mathbf{e}_2 \frac{\partial u + \mathbf{e}_{12} v}{\partial y} = 0.$$

Since $f = u + iv$, this is just

$$\mathbf{e}_1 \frac{\partial f}{\partial x} + \mathbf{e}_2 \frac{\partial f}{\partial y} = 0. \tag{16.4}$$

Or,

$$\nabla f = 0 \tag{16.5}$$

By taking second partial derivatives and equating mixed partials we are used to seeing these Cauchy-Riemann equations take this form as second order equations:

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{16.6}$$

$$\nabla^2 v = v_{xx} + v_{yy} = 0 \tag{16.7}$$

Given this, eq. (16.5) is something that we could have perhaps guessed, since the square root of the Laplacian operator, is in fact the gradient (there are an infinite number of such square roots, since any rotation of the coordinate system that expresses the gradient also works). However, a guess of this is not required since we see this explicitly through some logical composition of relationships.

The end result is that we can make a statement that in regions where the complex function is analytic (has a derivative), the gradient of that function is zero in that region.

This is a kind of interesting result and I expect that this will relevant when figuring out how the geometric calculus all fits together.

16.1 VERIFY WE STILL HAVE THE CAUCHY EQUATIONS HIDING IN THE GRADIENT

We have:

$$\nabla f \mathbf{e}_1 = \nabla(\mathbf{e}_1 u - \mathbf{e}_2 v) = 0$$

If this is to be zero, both the scalar and bivector parts of this equation must also be zero.

$$\begin{aligned}(\nabla \cdot f)\mathbf{e}_1 &= \nabla \cdot (\mathbf{e}_1 u - \mathbf{e}_2 v) \\ &= (\mathbf{e}_1 \partial_x + \mathbf{e}_2 \partial_y) \cdot (\mathbf{e}_1 u - \mathbf{e}_2 v) \\ &= (\partial_x u - \partial_y v) = 0\end{aligned} \tag{16.8}$$

$$\begin{aligned}(\nabla \wedge f)\mathbf{e}_1 &= \nabla \wedge (\mathbf{e}_1 u - \mathbf{e}_2 v) \\ &= (\mathbf{e}_1 \partial_x + \mathbf{e}_2 \partial_y) \wedge (\mathbf{e}_1 u - \mathbf{e}_2 v) \\ &= -\mathbf{e}_1 \wedge \mathbf{e}_2 (\partial_x v + \partial_y u) = 0\end{aligned} \tag{16.9}$$

We therefore see that this recovers the expected pair of Cauchy equations:

$$\begin{aligned}\partial_x u - \partial_y v &= 0 \\ \partial_x v + \partial_y u &= 0\end{aligned} \tag{16.10}$$

17

LEGENDRE POLYNOMIALS

Exercise 8.4, from [19].

Find the first couple terms of the Legendre polynomial expansion of

$$\frac{1}{|\mathbf{x} - \mathbf{a}|} \tag{17.1}$$

Write

$$f(x) = \frac{1}{|\mathbf{x}|} \tag{17.2}$$

Expanding $f(\mathbf{x} - \mathbf{a})$ about $\mathbf{x}$ we have

$$\frac{1}{|\mathbf{x} - \mathbf{a}|} = \sum_{k=0} \frac{1}{k!}(-\mathbf{a} \cdot \nabla)^k \frac{1}{|\mathbf{x}|} \tag{17.3}$$

Expanding the first term we have

$$\begin{aligned}
-\mathbf{a} \cdot \nabla \frac{1}{|\mathbf{x}|} &= \frac{1}{|\mathbf{x}|^2} \mathbf{a} \cdot \nabla |\mathbf{x}| \\
&= \frac{1}{|\mathbf{x}|^2} \mathbf{a} \cdot \nabla (\mathbf{x}^2)^{1/2} \\
&= \frac{1}{|\mathbf{x}|^2} \frac{(1/2)}{(|\mathbf{x}|^2)^{1/2}} \mathbf{a} \cdot \nabla \mathbf{x}^2 \\
&= \frac{\mathbf{a} \cdot \mathbf{x}}{|\mathbf{x}|^3}
\end{aligned} \tag{17.4}$$

Expansion of the second derivative term is

$$\begin{aligned}
\frac{(-\mathbf{a} \cdot \nabla)}{2} \frac{(-\mathbf{a} \cdot \nabla)}{1} \frac{1}{|\mathbf{x}|} &= \frac{\mathbf{a} \cdot \nabla}{2} \left(\frac{-\mathbf{a} \cdot \mathbf{x}}{|\mathbf{x}|^3} \right) \\
&= \frac{-1}{2} \left(\frac{\mathbf{a} \cdot \nabla (\mathbf{a} \cdot \mathbf{x})}{|\mathbf{x}|^3} + (\mathbf{a} \cdot \mathbf{x}) \mathbf{a} \cdot \nabla \frac{1}{|\mathbf{x}|^3} \right)
\end{aligned} \tag{17.5}$$

For this we need

$$\mathbf{a} \cdot \nabla(\mathbf{a} \cdot \mathbf{x}) = \mathbf{a} \cdot (\mathbf{a} \cdot \nabla \mathbf{x}) = \mathbf{a}^2 \tag{17.6}$$

And

$$\begin{aligned} \mathbf{a} \cdot \nabla \frac{1}{|\mathbf{x}|^k} &= k \frac{1}{|\mathbf{x}|^{k-1}} \mathbf{a} \cdot \nabla \frac{1}{|\mathbf{x}|} \\ &= k \frac{1}{|\mathbf{x}|^{k-1}} \frac{-\mathbf{a} \cdot \mathbf{x}}{|\mathbf{x}|^3} \\ &= -k \frac{\mathbf{a} \cdot \mathbf{x}}{|\mathbf{x}|^{k+2}} \end{aligned} \tag{17.7}$$

Thus the second derivative term is

$$\frac{-1}{2} \left(\frac{\mathbf{a}^2}{|\mathbf{x}|^3} - 3 \frac{(\mathbf{a} \cdot \mathbf{x})^2}{|\mathbf{x}|^5} \right) = \frac{(1/2) \left(3(\mathbf{a} \cdot \mathbf{x})^2 - \mathbf{a}^2 \mathbf{x}^2 \right)}{|\mathbf{x}|^5} \tag{17.8}$$

Summing these terms we have

$$\frac{1}{|\mathbf{x} - \mathbf{a}|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{a} \cdot \mathbf{x}}{|\mathbf{x}|^3} + \frac{(1/2) \left(3(\mathbf{a} \cdot \mathbf{x})^2 - \mathbf{a}^2 \mathbf{x}^2 \right)}{|\mathbf{x}|^5} + \cdots \tag{17.9}$$

NFCM writes this as

$$\frac{1}{|\mathbf{x} - \mathbf{a}|} = \frac{P_0(\mathbf{xa})}{|\mathbf{x}|} + \frac{P_1(\mathbf{xa})}{|\mathbf{x}|^3} + \frac{P_2(\mathbf{xa})}{|\mathbf{x}|^5} + \cdots \tag{17.10}$$

And calls $P_i = P_i(\mathbf{xa})$ terms the Legendre polynomials. This is not terribly clear since one expects a different form for the Legendre polynomials.

Using the Taylor formula one can derive a recurrence relation for these that makes the calculation a bit simpler

$$\begin{aligned} \frac{P_{k+1}}{|\mathbf{x}|^{2(k+1)+1}} &= \frac{-\mathbf{a} \cdot \nabla}{k+1} \left(\frac{P_k}{|\mathbf{x}|^{2k+1}} \right) \\ &= \frac{-1}{k+1} \left(\frac{\mathbf{a} \cdot \nabla(P_k}{|\mathbf{x}|^{2k+1}} + P_k \frac{\mathbf{a} \cdot \nabla}{|\mathbf{x}|^{2k+1}} \right) \\ &= \frac{1}{k+1} \left(P_k(2k+1) \frac{\mathbf{a} \cdot \mathbf{x}}{|\mathbf{x}|^{2k+3}} - \mathbf{x}^2 \frac{\mathbf{a} \cdot \nabla P_k}{|\mathbf{x}|^{2k+3}} \right) \end{aligned} \tag{17.11}$$

Or

$$(k+1) P_{k+1} = P_k(2k+1) \mathbf{a} \cdot \mathbf{x} - \mathbf{x}^2 \mathbf{a} \cdot \nabla P_k \tag{17.12}$$

Some of these have been calculated

$$
\begin{aligned}
P_0 &= 1 \\
P_1 &= \mathbf{a} \cdot \mathbf{x} \\
P_2 &= \frac{1}{2}(3(\mathbf{a} \cdot \mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2)
\end{aligned}
\tag{17.13}
$$

And for the derivatives

$$
\begin{aligned}
\mathbf{a} \cdot \nabla P_0 &= 0 \\
\mathbf{a} \cdot \nabla P_1 &= \mathbf{a}^2 \\
\mathbf{a} \cdot \nabla P_2 &= \frac{1}{2}((3)(2)(\mathbf{a} \cdot \mathbf{x})\mathbf{a}^2 - 2\mathbf{a}^2\mathbf{x} \cdot \mathbf{a}) \\
&= 2\mathbf{a}^2(\mathbf{x} \cdot \mathbf{a})
\end{aligned}
\tag{17.14}
$$

Using the recurrence relation one can calculate P_3 for example.

$$
\begin{aligned}
P_3 &= (1/3)\left(\frac{5}{2}(3(\mathbf{a} \cdot \mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2)(\mathbf{a} \cdot \mathbf{x}) - 2\mathbf{x}^2\mathbf{a}^2(\mathbf{x} \cdot \mathbf{a})\right) \\
&= (1/3)(\mathbf{a} \cdot \mathbf{x})\left(\frac{5}{2}(3(\mathbf{a} \cdot \mathbf{x})^2 - \mathbf{a}^2\mathbf{x}^2) - 2\mathbf{x}^2\mathbf{a}^2\right) \\
&= (\mathbf{a} \cdot \mathbf{x})\left(\frac{5}{2}((\mathbf{a} \cdot \mathbf{x})^2) - 3/2\mathbf{x}^2\mathbf{a}^2\right) \\
&= \frac{1}{2}(\mathbf{a} \cdot \mathbf{x})(5(\mathbf{a} \cdot \mathbf{x})^2 - 3\mathbf{x}^2\mathbf{a}^2)
\end{aligned}
\tag{17.15}
$$

17.1 PUTTING THINGS IN STANDARD LEGENDRE POLYNOMIAL FORM

This is still pretty laborious to calculate, especially because of not having a closed form recurrence relation for $\mathbf{a} \cdot \nabla P_k$. Let us relate these to the standard Legendre polynomial form.

Observe that we can write

$$
\begin{aligned}
P_0(\mathbf{xa}) &= 1 \\
\frac{P_1(\mathbf{xa})}{|\mathbf{x}||\mathbf{a}|} &= \hat{\mathbf{a}} \cdot \hat{\mathbf{x}} \\
\frac{P_2(\mathbf{xa})}{|\mathbf{x}|^2|\mathbf{a}|^2} &= \frac{1}{2}(3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})^2 - 1) \\
\frac{P_3(\mathbf{xa})}{|\mathbf{x}|^3|\mathbf{a}|^3} &= \frac{1}{2}(5(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})^3 - 3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}))
\end{aligned}
\tag{17.16}
$$

With this scaling, we have the standard form for the Legendre polynomials, and can write

$$\frac{1}{\mathbf{x}-\mathbf{a}} = \frac{1}{|\mathbf{x}|}\left(P_0 + \frac{|\mathbf{a}|}{|\mathbf{x}|}P_1(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) + \left(\frac{|\mathbf{a}|}{|\mathbf{x}|}\right)^2 P_2(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) + \left(\frac{|\mathbf{a}|}{|\mathbf{x}|}\right)^3 P_3(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) + \cdots\right) \tag{17.17}$$

17.2 SCALING STANDARD FORM LEGENDRE POLYNOMIALS

Since the odd Legendre polynomials have only odd terms and even have only even terms this allows for the scaled form that NFCM uses.

$$\begin{aligned}
P_0(\mathbf{xa}) &= P_0(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) \\
P_1(\mathbf{xa}) &= |\mathbf{x}||\mathbf{a}|P_1(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) = \mathbf{a}\cdot\mathbf{x} \\
P_2(\mathbf{xa}) &= |\mathbf{x}|^2|\mathbf{a}|^2P_2(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) = \frac{1}{2}(3(\mathbf{a}\cdot\mathbf{x})^2 - \mathbf{x}^2\mathbf{a}^2) \\
P_3(\mathbf{xa}) &= |\mathbf{x}|^3|\mathbf{a}|^3P_3(\hat{\mathbf{a}}\cdot\hat{\mathbf{x}}) = \frac{1}{2}(5(\mathbf{a}\cdot\mathbf{x})^3 - 3(\mathbf{a}\cdot\mathbf{x})\mathbf{x}^2\mathbf{a}^2)
\end{aligned} \tag{17.18}$$

Every term for the k^{th} polynomial is a permutation of the geometric product $\mathbf{x}^k\mathbf{a}^k$.

This allows for writing some of these terms using the wedge product. Using the product expansion:

$$(\mathbf{a}\cdot\mathbf{x})^2 = (\mathbf{a}\wedge\mathbf{x})^2 + \mathbf{a}^2\mathbf{x}^2 \tag{17.19}$$

Thus we have:

$$\begin{aligned}
P_2(\mathbf{xa}) &= (\mathbf{a}\cdot\mathbf{x})^2 + \frac{1}{2}(\mathbf{a}\wedge\mathbf{x})^2 \\
&= (\mathbf{a}\cdot\mathbf{x})^2 - \frac{1}{2}|\mathbf{a}\wedge\mathbf{x}|^2
\end{aligned} \tag{17.20}$$

This is nice geometrically since the directional dependence of this term on the co-linearity and perpendicularity of the vectors **a** and **x** is clear.

Doing the same for the P_3:

$$\begin{aligned}
P_3(\mathbf{xa}) &= (\mathbf{a}\cdot\mathbf{x})\frac{1}{2}(5(\mathbf{a}\cdot\mathbf{x})^2 - 3\mathbf{x}^2\mathbf{a}^2) \\
&= (\mathbf{a}\cdot\mathbf{x})\frac{1}{2}(2(\mathbf{a}\cdot\mathbf{x})^2 + 3(\mathbf{a}\wedge\mathbf{x})^2) \\
&= (\mathbf{a}\cdot\mathbf{x})((\mathbf{a}\cdot\mathbf{x})^2 - \frac{3}{2}|\mathbf{a}\wedge\mathbf{x}|^2)
\end{aligned} \tag{17.21}$$

I suppose that one could get the same geometrical interpretation with a standard Legendre expansion in terms of $\hat{\mathbf{a}} \cdot \hat{\mathbf{x}} = cos(\theta)$ terms, by collect both $sin(\theta)$ and $cos(\theta)$ powers, but one can see the power of writing things explicitly in terms of the original vectors.

17.3 NOTE ON NFCM LEGENDRE POLYNOMIAL NOTATION

In NFCM's slightly abusive notation P_k was used with various meanings. He wrote $P_k(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}}) = \frac{P_k(\mathbf{xa})}{|\mathbf{x}|^k|\mathbf{a}|^k}$.

Note for example that the standard first degree Legendre polynomial $P_1(x) = x$ evaluated with a $\mathbf{xa}$ value:

$$\begin{aligned}\frac{1}{|\mathbf{x}||\mathbf{a}|}P_1(x)|_{x=\mathbf{xa}} &= \hat{\mathbf{x}}\hat{\mathbf{a}} \\ &= \hat{\mathbf{x}} \cdot \hat{\mathbf{a}} + \hat{\mathbf{x}} \wedge \hat{\mathbf{a}}\end{aligned} \tag{17.22}$$

This has a bivector component in addition to the component identical to the standard Legendre polynomial term (the first part).

By luck it happens that the scalar part of this equals $P_1(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})$, but this is not the case for other terms. Example, $P_2(\mathbf{xa})$:

$$\begin{aligned}P_2(x)|_{x=\mathbf{xa}} &= \frac{1}{2}(3(\mathbf{xa})^2 - 1) \\ &= \frac{1}{2}(3(-\mathbf{ax} + 2\mathbf{a} \cdot \mathbf{x})(\mathbf{xa}) - 1) \\ &= \frac{1}{2}(3(-\mathbf{a}^2\mathbf{x}^2 + 2(\mathbf{a} \cdot \mathbf{x})^2 + 2(\mathbf{a} \cdot \mathbf{x})(\mathbf{x} \wedge \mathbf{a})) - 1) \\ &= -(3/2)\mathbf{a}^2\mathbf{x}^2 + 3(\mathbf{a} \cdot \mathbf{x})^2 + 3(\mathbf{a} \cdot \mathbf{x})(\mathbf{x} \wedge \mathbf{a}) - 1/2\end{aligned} \tag{17.23}$$

Scaling this by $1/(\mathbf{a}^2\mathbf{x}^2)$ is

$$-\frac{3}{2} + 3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})^2 + 3(\hat{\mathbf{a}} \cdot \hat{\mathbf{x}})(\hat{\mathbf{x}} \wedge \hat{\mathbf{a}}) - \frac{1}{\mathbf{a}^2\mathbf{x}^2} \tag{17.24}$$

The scalar part of this is not anything recognizable.

18 LEVI-CIVITICA SUMMATION IDENTITY

18.1 MOTIVATION

In [5] it is left to the reader to show

$$\sum_k \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im} \tag{18.1}$$

18.2 A MECHANICAL PROOF

Although it is not mathematical, this is easy to prove, at least for 3D. The following perl code does the trick

```
#!/usr/bin/perl

$e{111} = 0  ; $e{112} = 0  ; $e{113} = 0 ;
$e{121} = 0  ; $e{122} = 0  ; $e{123} = 1 ;
$e{131} = 0  ; $e{132} = -1 ; $e{133} = 0 ;

$e{211} = 0  ; $e{212} = 0  ; $e{213} = -1 ;
$e{221} = 0  ; $e{222} = 0  ; $e{223} = 0 ;
$e{231} = 1  ; $e{232} = 0  ; $e{233} = 0 ;

$e{311} = 0  ; $e{312} = 1  ; $e{313} = 0 ;
$e{321} = -1 ; $e{322} = 0  ; $e{323} = 0 ;
$e{331} = 0  ; $e{332} = 0  ; $e{333} = 0 ;

$d{11} = 1 ; $d{12} = 0 ; $d{13} = 0 ;
$d{21} = 0 ; $d{22} = 1 ; $d{23} = 0 ;
$d{31} = 0 ; $d{32} = 0 ; $d{33} = 1 ;

# prove: \sum_k e_{ijk} e_{klm}
#   = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}
#print "$e{123} $e{113} $e{321}\n" ;

for ( my $i = 1 ; $i <= 3 ; $i++ ) {
 for ( my $j = 1 ; $j <= 3 ; $j++ ) {
```

```
  for ( my $l = 1 ; $l <= 3 ; $l++ ) {
   for ( my $m = 1 ; $m <= 3 ; $m++ ) {
    my $lhs = 0 ;
    my $rhs = $d{"${i}${l}"} * $d{"${j}${m}"}
            - $d{"${j}${l}"} * $d{"${i}${m}"} ;

    for ( my $k = 1 ; $k <= 3 ; $k++ ) {
     $lhs += $e{"${i}${j}${k}"} * $e{"${k}${l}${m}"} ;
    }

    if ( $rhs != $lhs ) {
     print 'ERROR: \\sum_{k=1}^{3} \\epsilon_{' .
           "${i}${j}k" .
           '} \\epsilon_{' .
           "k${l}${m}} = $lhs\n" ;

     print 'ERROR: \\delta_{' .
           "${i}${l}}" .  '\\delta_{' .
           "${j}${m}" .  '} - \\delta_{' .
           "${j}${l}" . '}\\delta_{' .
           "${i}${m}} = $rhs\n\n" ;
     } else {
      print "$lhs &= " .
            '\\sum_{k=1}^{3} \\epsilon_{' .
            "${i}${j}k" . '} \\epsilon_{' .
            "k${l}${m}} = " . '\\delta_{' .
            "${i}${l}}" . '\\delta_{' . "${j}${m}" .
            '} - \\delta_{' . "${j}${l}" .
            '}\\delta_{' . "${i}${m}}" .
            ' \\\\' . " \n" ;
} } } } }
```

The output produced has all the variations of indices, such as

$$
\begin{aligned}
0 &= \sum_{k=1}^{3} \epsilon_{11k}\epsilon_{k11} = \delta_{11}\delta_{11} - \delta_{11}\delta_{11} \\
0 &= \sum_{k=1}^{3} \epsilon_{11k}\epsilon_{k12} = \delta_{11}\delta_{12} - \delta_{11}\delta_{12} \\
&\vdots \\
0 &= \sum_{k=1}^{3} \epsilon_{11k}\epsilon_{k33} = \delta_{13}\delta_{13} - \delta_{13}\delta_{13} \\
0 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k11} = \delta_{11}\delta_{21} - \delta_{21}\delta_{11} \\
1 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k12} = \delta_{11}\delta_{22} - \delta_{21}\delta_{12} \\
0 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k13} = \delta_{11}\delta_{23} - \delta_{21}\delta_{13} \\
-1 &= \sum_{k=1}^{3} \epsilon_{12k}\epsilon_{k21} = \delta_{12}\delta_{21} - \delta_{22}\delta_{11} \\
&\vdots
\end{aligned}
\tag{18.2}
$$

18.3 PROOF USING BIVECTOR DOT PRODUCT

This identity can also be derived from an expansion of the bivector dot product in two different ways.

$$
\begin{aligned}
(\mathbf{e}_i \wedge \mathbf{e}_j) \cdot (\mathbf{e}_m \wedge \mathbf{e}_n) &= ((\mathbf{e}_i \wedge \mathbf{e}_j) \cdot \mathbf{e}_m) \cdot \mathbf{e}_n \\
&= (\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_m) - \mathbf{e}_j(\mathbf{e}_i \cdot \mathbf{e}_m)) \cdot \mathbf{e}_n \\
&= (\mathbf{e}_i\delta_{jm} - \mathbf{e}_j\delta_{im}) \cdot \mathbf{e}_n \\
&= \delta_{in}\delta_{jm} - \delta_{jn}\delta_{im}
\end{aligned}
\tag{18.3}
$$

Expressing the wedge product in terms duality, using the pseudoscalar $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, we have

$$
(\mathbf{e}_i \wedge \mathbf{e}_j)\mathbf{e}_k = I\epsilon_{ijk} \tag{18.4}
$$

Or

$$\mathbf{e}_i \wedge \mathbf{e}_j = I \sum_k \epsilon_{ijk} \mathbf{e}_k \tag{18.5}$$

Then the bivector dot product is

$$\begin{aligned}
(\mathbf{e}_i \wedge \mathbf{e}_j) \cdot (\mathbf{e}_m \wedge \mathbf{e}_n) &= \left\langle I \sum_k \epsilon_{ijk} \mathbf{e}_k I \sum_p \epsilon_{mnp} \mathbf{e}_p \right\rangle \\
&= I^2 \sum_{k,p} \epsilon_{ijk} \epsilon_{mnp} \left\langle \mathbf{e}_k \mathbf{e}_p \right\rangle \\
&= -\sum_{k,p} \epsilon_{ijk} \epsilon_{mnp} \delta_{kp} \\
&= -\sum_k \epsilon_{ijk} \epsilon_{mnk}
\end{aligned} \tag{18.6}$$

Comparing the two expansions we have

$$\sum_k \epsilon_{ijk} \epsilon_{mnk} = \delta_{jn} \delta_{im} - \delta_{in} \delta_{jm} \tag{18.7}$$

Which is equivalent to the original identity (after an index switcheroo). Note both the dimension and metric dependencies in this proof.

19

SOME NFCM EXERCISE SOLUTIONS AND NOTES

Solutions for problems in chapter 2 I recall that some of the problems from this chapter of [19] were fairly tricky. Did I end up doing them all? I intended to revisit these and make sure I understood it all. As I do so, write up solutions, starting with 1.3, a question on the Geometric Algebra group.

Another thing I recall from the text is that I was fairly confused about all the mass of identities by the time I got through it, and it was not clear to me which were the fundamental ones. Eventually I figured out that it is really grade selection that is the fundamental operation, and found better presentations of axiomatic treatment in [10].

For reference the GA axioms are

- vector product is linear

$$
\begin{aligned}
a(\alpha b + \beta c) &= \alpha a b + \beta a c \\
(\alpha a + \beta b) c &= \alpha a c + \beta b c
\end{aligned}
\tag{19.1}
$$

- distribution of vector product

$$
(ab)c = a(bc) = abc \tag{19.2}
$$

- vector contraction

$$
a^2 \in \mathbb{R} \tag{19.3}
$$

  For a Euclidean space, this provides the length $a^2 = |a|^2$, but for relativity and conformal geometry this specific meaning is not required.

The definition of the generalized dot between two blades is

$$
A_r \cdot B_s = \langle AB \rangle_{|r-s|} \tag{19.4}
$$

and the generalized wedge product definition for two blades is

$$A_r \wedge B_s = \langle AB \rangle_{r+s}. \tag{19.5}$$

With these definitions and the GA axioms everything else should logically follow.

I personally found it was really easy to go around in circles attempting the various proofs, and intended to revisit all of these and prove them all for myself making sure I did not invoke any circular arguments and used only things already proven.

19.0.1 *Exercise 1.3*

Solve for x

$$\alpha x + a x \cdot b = c \tag{19.6}$$

where α is a scalar and all the rest are vectors.

19.0.1.1 *Solution*

Can dot or wedge the entire equation with the constant vectors. In particular

$$\begin{aligned} c \cdot b &= (\alpha x + a x \cdot b) \cdot b \\ &= (\alpha + a \cdot b) x \cdot b \end{aligned} \tag{19.7}$$

$$\implies x \cdot b = \frac{c \cdot b}{\alpha + a \cdot b} \tag{19.8}$$

and

$$\begin{aligned} c \wedge a &= (\alpha x + a x \cdot b) \wedge a \\ &= \alpha (x \wedge a) + \overbrace{\boxed{(a \wedge a)}}^{=0} (x \cdot b) \wedge a \end{aligned} \tag{19.9}$$

$$\implies x \wedge a = \frac{1}{\alpha} (c \wedge a) \tag{19.10}$$

This last can be reduced by dotting with b, and then substitute the result for $x \cdot b$ from above

$$\begin{aligned}(x \wedge a) \cdot b &= x(a \cdot b) - (x \cdot b)a \\ &= x(a \cdot b) - \frac{c \cdot b}{\alpha + a \cdot b} a\end{aligned} \tag{19.11}$$

Thus the final solution is

$$x = \frac{1}{a \cdot b}\left(\frac{c \cdot b}{\alpha + a \cdot b} a + \frac{1}{\alpha}(c \wedge a) \cdot b\right) \tag{19.12}$$

Question: was there a geometric or physical motivation for this question. I can not recall one?

19.1 SEQUENTIAL PROOFS OF REQUIRED IDENTITIES

19.1.1 *Split of symmetric and antisymmetric parts of the vector product*

NFCM defines the vector dot and wedge products in terms of the symmetric and antisymmetric parts, and not in terms of grade selection.

The symmetric and antisymmetric split of a vector product takes the form

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) \tag{19.13}$$

Observe that if the two vectors are colinear, say $b = \alpha a$, then this is

$$ab = \frac{\alpha}{2}(a^2 + a^2) + \frac{\alpha}{2}(a^2 - a^2) \tag{19.14}$$

The antisymmetric part is zero for any colinear vectors, while the symmetric part is a scalar by the contraction axiom eq. (19.3).

Now, suppose that one splits the vector b into a part that is explicit colinear with a, as in $b = \alpha a + c$.

Here one can observe that none of the colinear component of this vector contributes to the antisymmetric part of the split

$$\begin{aligned}\frac{1}{2}(ab - ba) &= \frac{1}{2}(a(\alpha a + c) - (\alpha a + c)a) \\ &= \frac{1}{2}(ac - ca)\end{aligned} \tag{19.15}$$

So, in a very loose fashion the symmetric part can be observed to be due to only colinear parts of the vectors whereas colinear components of the vectors do not contribute at all to the antisymmetric part of the product split. One can see that there is a notion of parallelism and perpendicularity built into this construction.

What is of interest here is to show that this symmetric and antisymmetric split also provides the scalar and bivector parts of the product, and thus matches the definitions of generalized dot and wedge products.

While it has been observed that the symmetric product is a scalar for colinear vectors it has not been demonstrated that this is necessarily a scalar in the general case.

Consideration of the square of $a + b$ is enough to do so.

$$\begin{aligned} (a+b)^2 &= a^2 + b^2 + ab + ba \\ &\Longrightarrow \end{aligned} \tag{19.16}$$

$$\frac{1}{2}\left((a+b)^2 - a^2 - b^2\right) = \frac{1}{2}(ab + ba) \tag{19.17}$$

We have only scalar terms on the LHS, which demonstrates that the symmetric product is necessarily a scalar. This is despite the fact that the exact definition of a^2 (ie: the metric for the space) has not been specified, nor even a requirement that this vector square is even satisfies $a^2 >= 0$. Such an omission is valuable since it allows for a natural formulation of relativistic four-vector algebra where both signs are allowed for the vector square.

Observe that eq. (19.17) provides a generalization of the Pythagorean theorem. If one defines, as in Euclidean space, that two vectors are perpendicular by

$$(a+b)^2 = a^2 + b^2 \tag{19.18}$$

Then one necessarily has

$$\frac{1}{2}(ab + ba) = 0 \tag{19.19}$$

So, that we have as a consequence of this perpendicularity definition a sign inversion on reversal

$$ba = -ab \tag{19.20}$$

This equation contains the essence of the concept of grade. The product of a pair of vectors is grade two if reversal of the factors changes the sign, which in turn implies the two factors must be perpendicular.

Given a set of vectors that, according to the symmetric vector product (dot product) are all either mutually perpendicular or colinear, grouping by colinear sets determines the grade

$$a_1 a_2 a_3 ... a_m = (b_{j_1} b_{j_2} ...)(b_{k_1} b_{k_2} ...)...(b_{l_1} b_{l_2} ...) \tag{19.21}$$

after grouping in pairs of colinear vectors (for which the squares are scalars) the count of the remaining elements is the grade. By example, suppose that e_i is a normal basis for $\mathbb{R}^N$ $e_i \cdot e_j \propto \delta_{ij}$, and one wishes to determine the grade of a product. Permuting this product so that it is ordered by index leaves it in a form that the grade can be observed by inspection

$$\begin{aligned}
e_3 e_7 e_1 e_2 e_1 e_7 e_6 e_7 &= -e_3 e_1 e_7 e_2 e_1 e_7 e_6 e_7 \\
&= e_1 e_3 e_7 e_2 e_1 e_7 e_6 e_7 \\
&= ... \\
&\propto e_1 e_1 e_2 e_3 e_6 e_7 e_7 e_7 \\
&= (e_1 e_1) e_2 e_3 e_6 (e_7 e_7) e_7 \\
&\propto e_2 e_3 e_6 e_7
\end{aligned} \tag{19.22}$$

This is an example of a grade four product. Given this implicit definition of grade, one can then see that the antisymmetric product of two vectors is necessarily grade two. An explicit enumeration of a vector product in terms of an explicit normal basis and associated coordinates is helpful here to demonstrate this.

Let

$$\begin{aligned}
a &= \sum_i a_i e_i \\
b &= \sum_j b_j e_j
\end{aligned} \tag{19.23}$$

now, form the product

$$\begin{aligned}
ab &= \sum_i \sum_j a_i b_j e_i e_j \\
&= \sum_{i<j} a_i b_j e_i e_j + \sum_{i=j} a_i b_j e_i e_j + \sum_{i>j} a_i b_j e_i e_j \\
&= \sum_{i<j} a_i b_j e_i e_j + \sum_{i=j} a_i b_j e_i e_j + \sum_{j>i} a_j b_i e_j e_i \\
&= \sum_{i<j} a_i b_j e_i e_j + \sum_{i=j} a_i b_j e_i e_j - \sum_{i<j} a_j b_i e_i e_j \\
&= \sum_i a_i b_i (e_i)^2 + \sum_{i<j} (a_i b_j - a_j b_i) e_i e_j
\end{aligned} \tag{19.24}$$

similarly

$$ba = \sum_i a_i b_i (e_i)^2 - \sum_{i<j} (a_i b_j - a_j b_i) e_i e_j \tag{19.25}$$

Thus the symmetric and antisymmetric products are respectively

$$\begin{aligned}\frac{1}{2}(ab + ba) &= \sum_i a_i b_i (e_i)^2 \\ \frac{1}{2}(ab - ba) &= \sum_{i<j} (a_i b_j - a_j b_i) e_i e_j\end{aligned} \tag{19.26}$$

The first part as shown above with non-coordinate arguments is a scalar. Each term in the antisymmetric product has a grade two term, which as a product of perpendicular vectors cannot be reduced any further, so it is therefore grade two in its entirety.

following the definitions of eq. (19.4) and eq. (19.5) respectively, one can then write

$$\begin{aligned}a \cdot b &= \frac{1}{2}(ab + ba) \\ a \wedge b &= \frac{1}{2}(ab - ba)\end{aligned} \tag{19.27}$$

These can therefore be seen to be a consequence of the definitions and axioms rather than a required a-priori definition in their own right. Establishing these as derived results is important to avoid confusion when one moves on to general higher grade products. The vector dot and wedge products are not sufficient by themselves if taken as a fundamental definition to establish the required results for such higher grade products (in particular the useful formulas for vector times blade dot and wedge products should be observed to be derived results as opposed to definitions).

19.1.2 *bivector dot with vector reduction*

In the 1.3 solution above the identity

$$(a \wedge b) \cdot c = a(b \cdot c) - (a \cdot c)b \tag{19.28}$$

was used. Let us prove this.

$$
\begin{aligned}
(a \wedge b) \cdot c &= \langle (a \wedge b) c \rangle_1 \\
&\implies \\
2(a \wedge b) \cdot c &= \langle abc - bac \rangle_1 \\
&= \langle abc - b(-ca + 2a \cdot c) \rangle_1 \\
&= \langle abc + bca \rangle_1 - 2b(a \cdot c) \\
&= \langle a(b \cdot c + b \wedge c) + (b \cdot c + b \wedge c) a \rangle_1 - 2b(a \cdot c) \\
&= 2a(b \cdot c) + a \cdot (b \wedge c) + (b \wedge c) \cdot a - 2b(a \cdot c)
\end{aligned}
\tag{19.29}
$$

To complete the proof we need $a \cdot B = -B \cdot a$, but once that is demonstrated, one is left with the desired identity after dividing through by 2.

19.1.3 *vector bivector dot product reversion*

Prove $a \cdot B = -B \cdot a$.

20 OUTERMORPHISM QUESTION

20.1

[10] has an example of a linear operator.

$$F(a) = a + \alpha(a \cdot f_1)f_2. \tag{20.1}$$

This is used to compute the determinant without putting the operator in matrix form.

20.1.1 *bivector outermorphism*

Their first step is to compute the wedge of this function applied to two vectors. Doing this myself (not omitting steps), I get:

$$\begin{aligned}
F(a \wedge b) &= F(a) \wedge F(b) \\
&= (a + \alpha(a \cdot f_1)f_2) \wedge (b + \alpha(b \cdot f_1)f_2) \\
&= a \wedge b + \alpha(a \cdot f_1)f_2 \wedge b + \alpha(b \cdot f_1)a \wedge f_2 + \alpha^2(a \cdot f_1)(b \cdot f_1)\overbrace{\boxed{f_2 \wedge f_2}}^{=0} \\
&= a \wedge b + \alpha\left((b \cdot f_1)a - (a \cdot f_1)b\right) \wedge f_2 \\
&= a \wedge b + \alpha\left((a \wedge b) \cdot f_1\right) \wedge f_2
\end{aligned} \tag{20.2}$$

This has a very similar form to the original function F. In particular one can write

$$\begin{aligned}
F(a) &= a + \alpha(a \cdot f_1)f_2 \\
&= a + \langle\alpha(a \cdot f_1)f_2\rangle_1 \\
&= a + \langle\alpha(a \cdot f_1)f_2\rangle_{0+1} \\
&= a + \alpha(a \cdot f_1) \wedge f_2
\end{aligned} \tag{20.3}$$

Here the fundamental definition of the wedge product as the highest grade part of a product of blades has been used to show that the new bivector function defined via outermorphism has the same form as the original, once we put the original in the new form that applies to bivector and vector:

$$F(A) = A + \alpha(A \cdot f_1) \wedge f_2 \tag{20.4}$$

20.1.2 *Induction*

Now, proceeding inductively, assuming that this is true for some grade k blade A, one can calculate $F(A) \wedge F(b)$ for a vector b:

$$\begin{aligned}
&F(A) \wedge F(b) \\
&= (A + \alpha(A \cdot f_1) \wedge f_2) \wedge (b + \alpha(b \cdot f_1)f_2) \\
&= A \wedge b + \alpha(b \cdot f_1)A \wedge f_2 + \alpha((A \cdot f_1) \wedge f_2) \wedge b + \alpha^2(b \cdot f_1)((A \cdot f_1) \wedge f_2) \wedge f_2 \\
&= A \wedge b + \alpha \left((b \cdot f_1)A - (A \cdot f_1) \wedge b\right) \wedge f_2 \\
&= A \wedge b + \alpha\langle(b \cdot f_1)A - (A \cdot f_1)b\rangle_k \wedge f_2
\end{aligned} \tag{20.5}$$

Now, similar to the bivector case, this inner quantity can be reduced, but it is messier to do so:

$$\langle(b \cdot f_1)A - (A \cdot f_1)b\rangle_k = \frac{1}{2}\Big\langle b f_1 A - A f_1 b + f_1(bA + (-1)^k Ab)\Big\rangle_k \tag{20.6}$$

$$\implies \langle(b \cdot f_1)A - (A \cdot f_1)b\rangle_k = \frac{1}{2}\langle b f_1 A - A f_1 b\rangle_k + \langle f_1(b \wedge A)\rangle_k \tag{20.7}$$

Consider first the right hand expression:

$$\begin{aligned}
\langle f_1(b \wedge A)\rangle_k &= f_1 \cdot (b \wedge A) \\
&= (-1)^k f_1 \cdot (A \wedge b) \\
&= (-1)^k(-1)^k(A \wedge b) \cdot f_1 \\
&= (A \wedge b) \cdot f_1
\end{aligned} \tag{20.8}$$

The right hand expression in eq. (20.7) can be shown to equal zero. That is messier still and the calculation can be found at the end.

Using that equals zero result we now have:

$$F(A) \wedge F(b) = A \wedge b + \alpha((A \wedge b) \cdot f_1) \wedge f_2 \tag{20.9}$$

This completes the induction.

20.1.3 *Can the induction be avoided?*

Now, GAFP did not do this induction, nor even claim it was required. The statement is "It follows that", after only calculating the bivector case. Is there a reason that they would be able to make such a statement without proof that is obvious to them perhaps but not to me?

It has been pointed out that this question is answered, “yes, the induction can be avoided”, in [30] page 148.

20.2 APPENDIX. MESSY REDUCTION FOR INDUCTION

Q: Is there an easier way to do this?

Here we want to show that

$$\frac{1}{2}\langle b f_1 A - A f_1 b\rangle_k = 0$$

Expanding the innards of this expression to group A and b parts together:

$$\begin{aligned}
b f_1 A - A f_1 b &= (f_1 b - 2 b \wedge f_1) A - A(b f_1 - 2 f_1 \wedge b) \\
&= f_1 b A - A b f_1 - 2(b \wedge f_1) A + 2A(f_1 \wedge b) \\
&= f_1(b \cdot A + b \wedge A) - (A \cdot b + A \wedge b) f_1 \\
&\quad - 2\left((b \wedge f_1) \cdot A + \langle (b \wedge f_1) A\rangle_k + (b \wedge f_1) \wedge A\right) \\
&\quad + 2\left(A \cdot (f_1 \wedge b) + \langle A(f_1 \wedge b)\rangle_k + A \wedge (f_1 \wedge b)\right)
\end{aligned} \tag{20.10}$$

the grade $k-2$, and grade $k+2$ terms of the bivector product cancel (we are also only interested in the grade-k parts so can discard them). This leaves:

$$f_1 \wedge (b \cdot A) - (A \cdot b) \wedge f_1 + f_1 \cdot (b \wedge A) - (A \wedge b) \cdot f_1 - 2\langle (b \wedge f_1) A\rangle_k + 2\langle A(f_1 \wedge b)\rangle_k$$

The bivector, blade product part of this is the antisymmetric part of that product so those two last terms can be expressed with the commutator relationship for a bivector with blade: $\langle B_2 A\rangle_k = \frac{1}{2}(B_2 A - A B_2)$:

$$\begin{aligned}
2\langle A(f_1 \wedge b)\rangle_k - 2\langle (b \wedge f_1) A\rangle_k &= A(f_1 \wedge b) - (f_1 \wedge b) A - (b \wedge f_1) A + A(b \wedge f_1) \\
&= A(f_1 \wedge b) - (f_1 \wedge b) A + (f_1 \wedge b) A - A(f_1 \wedge b) \\
&= 0
\end{aligned} \tag{20.11}$$

So, we now have to show that we have zero for the remainder:

$$\begin{aligned} 2\langle b f_1 A - A f_1 b \rangle_k &= f_1 \wedge (b \cdot A) - (A \cdot b) \wedge f_1 \\ &\quad + f_1 \cdot (b \wedge A) - (A \wedge b) \cdot f_1 \\ &= (-1)^{k-1} f_1 \wedge (A \cdot b) - (-1)^{k-1} f_1 \wedge (A \cdot b) \\ &\quad + (-1)^k f_1 \cdot (A \wedge b) - (-1)^k f_1 \cdot (A \wedge b) \\ &= 0 \end{aligned} \tag{20.12}$$

20.3 NEW OBSERVATION

Looking again, I think I see one thing that I missed. The text said they were constructing the action on a general multivector. So, perhaps they meant b to be a blade. This is a typesetting subtlety if that is the case. Let us assume that is what they meant, and that b is a grade k blade. This makes the coefficient of the scalar α in equation 4.147 :

$$a \cdot f_1 f_2 \wedge b + b \cdot f_1 a \wedge f_2 = \Big((b \cdot f_1)a + (-1)^k (a \cdot f_1) b\Big) \wedge f_2 \tag{20.13}$$

whereas they have:

$$\big((b \cdot f_1)a - (a \cdot f_1)b\big) \wedge f_2$$

So, no, I think they must have intended b to be a vector, not an arbitrary grade blade. Now, indirectly, it has been proven here that for a vectors x, y, and a grade-k blade B:

$$(A \wedge x) \cdot y = A(x \cdot y) - (A \cdot y) \wedge x \tag{20.14}$$

Or,

$$(A \wedge x) \cdot y = (y \cdot x)A + (-1)^k (y \cdot A) \wedge x \tag{20.15}$$

(changed variable names to disassociate this from the specifics of this particular example), which is a generalization of the wedge product with dot product distribution identity for vectors:

$$(a \wedge b) \cdot c = a(b \cdot c) - (a \cdot c) \wedge b \tag{20.16}$$

I believe I have seen a still more general form of eq. (20.14) in a Hestenes paper, but did not think about using it a-priori. Regardless, it does not really appear the the GAFP text was treating b as anything but a vector, since there would have to be a $(-1)^k$ factor on equation 4.147 for it to be general.

Part II

PROJECTION

21 RECIPROCAL FRAME VECTORS

21.1 APPROACH WITHOUT GEOMETRIC ALGEBRA

Without employing geometric algebra, one can use the projection operation expressed as a dot product and calculate the a vector orthogonal to a set of other vectors, in the direction of a reference vector.

Such a calculation also yields $\mathbb{R}^N$ results in terms of determinants, and as a side effect produces equations for parallelogram area, parallelepiped volume and higher dimensional analogues as a side effect (without having to employ change of basis diagonalization arguments that do not work well for higher dimensional subspaces).

21.1.1 *Orthogonal to one vector*

The simplest case is the vector perpendicular to another. In anything but $\mathbb{R}^2$ there are a whole set of such vectors, so to express this as a non-set result a reference vector is required.

Calculation of the coordinate vector for this case follows directly from the dot product. Borrowing the GA term, we subtract the projection to calculate the rejection.

$$
\begin{aligned}
\text{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right) &= \mathbf{v} - \mathbf{v}\cdot\hat{\mathbf{u}}\hat{\mathbf{u}} \\
&= \frac{1}{\mathbf{u}^2}(\mathbf{v}\mathbf{u}^2 - \mathbf{v}\cdot\mathbf{u}\mathbf{u}) \\
&= \frac{1}{\mathbf{u}^2}\sum v_i\mathbf{e}_i u_j u_j - v_j u_j u_i \mathbf{e}_i \\
&= \frac{1}{\mathbf{u}^2}\sum u_j\mathbf{e}_i \begin{vmatrix} v_i & v_j \\ u_i & u_j \end{vmatrix} \\
&= \frac{1}{\mathbf{u}^2}\sum_{i<j}(u_i\mathbf{e}_j - u_j\mathbf{e}_i)\begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}
\end{aligned}
\tag{21.1}
$$

Thus we can write the rejection of $\mathbf{v}$ from $\hat{\mathbf{u}}$ as:

$$
\text{Rej}_{\hat{\mathbf{u}}}\left(\mathbf{v}\right) = \frac{1}{\mathbf{u}^2}\sum_{i<j}\begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}\begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix}
\tag{21.2}
$$

Or introducing some shorthand:

$$
\begin{aligned}
D^{\mathbf{uv}}_{ij} &= \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \\
D^{\mathbf{ue}}_{ij} &= \begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix}
\end{aligned}
\tag{21.3}
$$

eq. (21.2) can be expressed in a form that will be slightly more convenient for larger sets of vectors:

$$
\mathrm{Rej}_{\hat{\mathbf{u}}}(\mathbf{v}) = \frac{1}{\mathbf{u}^2} \sum_{i<j} D^{\mathbf{uv}}_{ij} D^{\mathbf{ue}}_{ij}
\tag{21.4}
$$

Note that although the GA axiom $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}$ has been used in equations eq. (21.2) and eq. (21.4) above and the derivation, that was not necessary to prove them. This can, for now, be thought of as a notational convenience, to avoid having to write $\mathbf{u} \cdot \mathbf{u}$, or $\|\mathbf{u}\|^2$.

This result can be used to express the $\mathbb{R}^N$ area of a parallelogram since we just have to multiply the length of $\mathrm{Rej}_{\hat{\mathbf{u}}}(\mathbf{v})$:

$$
\|\mathrm{Rej}_{\hat{\mathbf{u}}}(\mathbf{v})\|^2 = \mathrm{Rej}_{\hat{\mathbf{u}}}(\mathbf{v}) \cdot \mathbf{v} = \frac{1}{\mathbf{u}^2} \sum_{i<j} \left(D^{\mathbf{uv}}_{ij}\right)^2
\tag{21.5}
$$

with the length of the base $\|\mathbf{u}\|$. [FIXME: insert figure.]

Thus the area (squared) is:

$$
\mathrm{A}^2_{\mathbf{u},\mathbf{v}} = \sum_{i<j} \left(D^{\mathbf{uv}}_{ij}\right)^2
\tag{21.6}
$$

For the special case of a vector in $\mathbb{R}^2$ this is

$$
\mathrm{A}_{\mathbf{u},\mathbf{v}} = |D^{\mathbf{uv}}_{12}| = \mathrm{abs}\left(\begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} \right)
\tag{21.7}
$$

21.1.2 *Vector orthogonal to two vectors in direction of a third*

The same procedure can be followed for three vectors, but the algebra gets messier. Given three vectors **u**, **v**, and **w** we can calculate the component **w**′ of **w** perpendicular to **u** and **v**. That is:

$$
\begin{aligned}
\mathbf{v}' &= \mathbf{v} - \mathbf{v}\cdot\hat{\mathbf{u}}\hat{\mathbf{u}} \\
\implies & \\
\mathbf{w}' &= \mathbf{w} - \mathbf{w}\cdot\hat{\mathbf{u}}\hat{\mathbf{u}} - \mathbf{w}\cdot\hat{\mathbf{v}}'\hat{\mathbf{v}}'
\end{aligned}
\tag{21.8}
$$

After expanding this out, a number of the terms magically cancel out and one is left with

$$
\begin{aligned}
\mathbf{w}'' = \mathbf{w}'(\mathbf{u}^2\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2) = \mathbf{u}&\left(-\mathbf{u}\cdot\mathbf{w}\mathbf{v}^2 + (\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})\right) \\
+\mathbf{v}&\left(-\mathbf{u}^2(\mathbf{v}\cdot\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})\right) \\
+\mathbf{w}&\left(\mathbf{u}^2\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2\right)
\end{aligned}
\tag{21.9}
$$

And this in turn can be expanded in terms of coordinates and the results collected yielding

$$
\begin{aligned}
\mathbf{w}'' &= \sum \mathbf{e}_i u_j v_k \left(u_i \begin{vmatrix} v_j & v_k \\ w_j & w_k \end{vmatrix} - v_i \begin{vmatrix} u_j & u_k \\ w_j & w_k \end{vmatrix} w_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \right) \\
&= \sum \mathbf{e}_i u_j v_k \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \\
&= \sum_{i,j<k} \mathbf{e}_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \\
&= \left(\sum_{i<j<k} + \sum_{j<i<k} + \sum_{j<k<i} \right) \mathbf{e}_i \begin{vmatrix} u_j & u_k \\ v_j & v_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix}.
\end{aligned}
\tag{21.10}
$$

Expanding the sum of the denominator in terms of coordinates:

$$
\mathbf{u}^2\mathbf{v}^2 - (\mathbf{u}\cdot\mathbf{v})^2 = \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2
\tag{21.11}
$$

and using a change of summation indices, our final result for the vector perpendicular to two others in the direction of a third is:

$$\mathrm{Rej}_{\hat{\mathbf{u}},\hat{\mathbf{v}}}(\mathbf{w}) = \frac{\sum_{i<j<k} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ \mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k \end{vmatrix}}{\sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2} \tag{21.12}$$

As a small aside, it is notable here to observe that span $\left\{ \begin{vmatrix} u_i & u_j \\ \mathbf{e}_i & \mathbf{e}_j \end{vmatrix} \right\}$ is the null space for the vector $\mathbf{u}$, and the set span $\left\{ \begin{vmatrix} u_i & u_j & u_k \\ v_i & v_j & v_k \\ \mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k \end{vmatrix} \right\}$ is the null space for the two vectors $\mathbf{u}$ and $\mathbf{v}$ respectively.

Since the rejection is a normal to the set of vectors it must necessarily include these cross product like determinant terms.

As in eq. (21.4), use of a $D^{\mathbf{uvw}}_{ijk}$ notation allows for a more compact result:

$$\mathrm{Rej}_{\hat{\mathbf{u}}\hat{\mathbf{v}}}(\mathbf{w}) = \left(\sum_{i<j} \left(D^{\mathbf{uv}}_{ij} \right)^2 \right)^{-1} \sum_{i<j<k} D^{\mathbf{uvw}}_{ijk} D^{\mathbf{uve}}_{ijk} \tag{21.13}$$

And, as before this yields the Volume of the parallelepiped by multiplying perpendicular height:

$$\|\mathrm{Rej}_{\hat{\mathbf{u}}\hat{\mathbf{v}}}(\mathbf{w})\| = \mathrm{Rej}_{\hat{\mathbf{u}}\hat{\mathbf{v}}}(\mathbf{w}) \cdot \mathbf{w} = \left(\sum_{i<j} \left(D^{\mathbf{uv}}_{ij} \right)^2 \right)^{-1} \sum_{i<j<k} \left(D^{\mathbf{uvw}}_{ijk} \right)^2 \tag{21.14}$$

by the base area.

Thus the squared volume of a parallelepiped spanned by the three vectors is:

$$\mathrm{V}^2_{\mathbf{u},\mathbf{v},\mathbf{w}} = \sum_{i<j<k} \left(D^{\mathbf{uvw}}_{ijk} \right)^2. \tag{21.15}$$

The simplest case is for $\mathbb{R}^3$ where we have only one summand:

$$\mathrm{V}_{\mathbf{u},\mathbf{v},\mathbf{w}} = |D^{\mathbf{uvw}}_{ijk}| = \mathrm{abs}\left(\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right). \tag{21.16}$$

21.1.3 *Generalization. Inductive Hypothesis*

There are two things to prove

1. hypervolume of parallelepiped spanned by vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$

$$\mathrm{V}^2_{\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_k} = \sum_{i_1<i_2<\cdots<i_k} \left(D^{\mathbf{u}_{i_1}\mathbf{u}_{i_2}\cdots\mathbf{u}_{i_k}}_{i_1 i_2\cdots i_k}\right)^2 \tag{21.17}$$

2. Orthogonal rejection of a set of vectors in direction of another.

$$\mathrm{Rej}_{\hat{\mathbf{u}}_1\cdots\hat{\mathbf{u}}_{k-1}}\left(\mathbf{u}_k\right) = \frac{\sum_{i_1<\cdots<i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_{k-1}}\mathbf{e}}_{i_1\cdots i_k}}{\sum_{i_1<\cdots<i_{k-1}} \left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_{k-1}}}_{i_1\cdots i_{k-1}}\right)^2} \tag{21.18}$$

I cannot recall if I ever did the inductive proof for this. Proving for the initial case is done (since it is proved for both the two and three vector cases). For the limiting case where $k = n$ it can be observed that this is normal to all the others, so the only thing to prove for that case is if the scaling provided by hypervolume eq. (21.17) is correct.

21.1.4 *Scaling required for reciprocal frame vector*

Presuming an inductive proof of the general result of eq. (21.18) is possible, this rejection has the property

$$\mathrm{Rej}_{\hat{\mathbf{u}}_1\cdots\hat{\mathbf{u}}_{k-1}}\left(\mathbf{u}_k\right) \cdot \mathbf{u}_i \propto \delta_{ki}$$

With the scaling factor picked so that this equals δ_{ki}, the resulting “reciprocal frame vector” is

$$\mathbf{u}^k = \frac{\sum_{i_1<\cdots<i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_{k-1}}\mathbf{e}}_{i_1\cdots i_k}}{\sum_{i_1<\cdots<i_k} \left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k}\right)^2} \tag{21.19}$$

The superscript notation is borrowed from Doran/Lasenby, and denotes not a vector raised to a power, but this this special vector satisfying the following orthogonality and scaling criteria:

$$\mathbf{u}^k \cdot \mathbf{u}_i = \delta_{ki}. \tag{21.20}$$

Note that for $k = n - 1$, eq. (21.19) reduces to

$$\mathbf{u}^n = \frac{D^{\mathbf{u}_1 \cdots \mathbf{u}_{n-1} \mathbf{e}}_{1 \cdots (n-1)}}{D^{\mathbf{u}_1 \cdots \mathbf{u}_n}_{1 \cdots n}}. \tag{21.21}$$

This or some other scaled version of this is likely as close as we can come to generalizing the cross product as an operation that takes vectors to vectors.

21.1.5 *Example. $\mathbb{R}^3$ case. Perpendicular to two vectors*

Observe that for $\mathbb{R}^3$, writing $\mathbf{u} = \mathbf{u}_1$, $\mathbf{v} = \mathbf{u}_2$, $\mathbf{w} = \mathbf{u}_3$, and $\mathbf{w}' = {\mathbf{u}_3}^3$ this is:

$$\mathbf{w}' = \frac{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}} = \frac{\mathbf{u} \times \mathbf{v}}{(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}} \tag{21.22}$$

This is the cross product scaled by the (signed) volume for the parallelepiped spanned by the three vectors.

21.2 DERIVATION WITH GA

Regression with respect to a set of vectors can be expressed directly. For vectors $\mathbf{u}_i$ write $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \mathbf{u}_k$. Then for any vector we have:

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}\mathbf{B}\frac{1}{\mathbf{B}} \\
&= \left\langle \mathbf{x}\mathbf{B}\frac{1}{\mathbf{B}} \right\rangle_1 \\
&= \left\langle (\mathbf{x} \cdot \mathbf{B} + \mathbf{x} \wedge \mathbf{B})\frac{1}{\mathbf{B}} \right\rangle_1
\end{aligned} \tag{21.23}$$

All the grade three and grade five terms are selected out by the grade one operation, leaving just

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{B})\cdot\frac{1}{\mathbf{B}} + (\mathbf{x}\wedge\mathbf{B})\cdot\frac{1}{\mathbf{B}}. \tag{21.24}$$

This last term is the rejective component.

$$\mathrm{Rej}_{\mathbf{B}}\left(\mathbf{x}\right) = (\mathbf{x}\wedge\mathbf{B})\cdot\frac{1}{\mathbf{B}} = \frac{(\mathbf{x}\wedge\mathbf{B})\cdot\mathbf{B}^\dagger}{\mathbf{B}\mathbf{B}^\dagger} \tag{21.25}$$

Here we see in the denominator the squared sum of determinants in the denominator of eq. (21.18):

$$\mathbf{B}\mathbf{B}^\dagger = \sum_{i_1<\cdots<i_k}\left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k}\right)^2 \tag{21.26}$$

In the numerator we have the dot product of two wedge products, each expressible as sums of determinants:

$$\mathbf{B}^\dagger = (-1)^{k(k-1)/2}\sum_{i_1<\cdots<i_k} D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k}\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k} \tag{21.27}$$

And

$$\mathbf{x}\wedge\mathbf{B} = \sum_{i_1<\cdots<i_{k+1}} D^{\mathbf{x}\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_{k+1}}\mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_{k+1}} \tag{21.28}$$

Dotting these is all the grade one components of the product. Performing that calculation would likely provide an explicit confirmation of the inductive hypothesis of eq. (21.18). This can be observed directly for the $k+1 = n$ case. That product produces a Laplace expansion sum.

$$(\mathbf{x}\wedge\mathbf{B})\cdot\mathbf{B}^\dagger = D^{\mathbf{x}\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{12\cdots n}\left(\mathbf{e}_1 D^{\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{234\cdots n} - \mathbf{e}_2 D^{\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{134\cdots n} + \mathbf{e}_3 D^{\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{124\cdots n}\right) \tag{21.29}$$

$$(\mathbf{x}\wedge\mathbf{B})\cdot\frac{1}{\mathbf{B}} = \frac{D^{\mathbf{x}\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{12\cdots n} D^{\mathbf{e}\mathbf{u}_1\cdots\mathbf{u}_{n-1}}_{12\cdots n}}{\sum_{i_1<\cdots<i_k}\left(D^{\mathbf{u}_{i_1}\cdots\mathbf{u}_{i_k}}_{i_1\cdots i_k}\right)^2} \tag{21.30}$$

Thus eq. (21.18) for the $k = n - 1$ case is proved without induction. A proof for the $k + 1 < n$ case would be harder. No proof is required if one picks the set of basis vectors $\mathbf{e}_i$ such that $\mathbf{e}_i \wedge \mathbf{B} = 0$ (then the $k + 1 = n$ result applies). I believe that proves the general case too if one observes that a rotation to any other basis in the span of the set of vectors only changes the sign of the each of the determinants, and the product of the two sign changes will then have value one.

Follow through of the details for a proof of original non GA induction hypothesis is probably not worthwhile since this reciprocal frame vector problem can be tackled with a different approach using a subspace pseudovector.

It is notable that although this had no induction in the argument above, note that it is fundamentally required. That is because there is an inductive proof required to prove that the general wedge and dot product vector formulas:

$$\mathbf{x} \cdot \mathbf{B} = \frac{1}{2}(\mathbf{x}\mathbf{B} - (-1)^k \mathbf{B}\mathbf{x}) \tag{21.31}$$

$$\mathbf{x} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{x}\mathbf{B} + (-1)^k \mathbf{B}\mathbf{x}) \tag{21.32}$$

from the GA axioms (that is an easier proof without the mass of indices and determinant products.)

21.3 PSEUDOVECTOR FROM REJECTION

As noted in the previous section the reciprocal frame vector $\mathbf{u}^k$ is the vector in the direction of $\mathbf{u}_k$ that has no component in span $\mathbf{u}_1, \cdots, \mathbf{u}_{k-1}$, normalized such that $\mathbf{u}_k \cdot \mathbf{u}^k = 1$. Explicitly, with $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \wedge \mathbf{u}_{k-1}$ this is:

$$\mathbf{u}^k = \frac{(\mathbf{u}_k \wedge \mathbf{B}) \cdot \mathbf{B}}{\mathbf{u}_k \cdot \big((\mathbf{u}_k \wedge \mathbf{B}) \cdot \mathbf{B}\big)} \tag{21.33}$$

This is derived from eq. (21.25), after noting that $\frac{\mathbf{B}^\dagger}{\mathbf{B}\mathbf{B}^\dagger} \propto \mathbf{B}$, and further scaling to produce the desired orthonormal property of equation eq. (21.20) that defines the reciprocal frame vector.

21.3.1 *back to reciprocal result*

Now, eq. (21.33) looks considerably different from the Doran/Lasenby result. Reduction to a direct pseudovector/blade product is possible since the dot product here can be converted to a direct product.

$$\mathbf{x} = \mathbf{u}_k - (\mathbf{u}_k \cdot \mathbf{B}) \cdot \tfrac{1}{\mathbf{B}}$$

$$\begin{aligned}
(\mathbf{u}_k \wedge \mathbf{B}) \cdot \mathbf{B} &= \boxed{(\mathbf{x}\mathbf{B})} \cdot \mathbf{B} \\
&= \langle \mathbf{x}\mathbf{B}\mathbf{B} \rangle_1 \\
&= \mathbf{x}\mathbf{B}^2 \\
&= \left(\left(\mathbf{u}_k - (\mathbf{u}_k \cdot \mathbf{B}) \cdot \frac{1}{\mathbf{B}} \right) \wedge \mathbf{B} \right) \mathbf{B} \\
&= (\mathbf{u}_k \wedge \mathbf{B})\mathbf{B}
\end{aligned} \tag{21.34}$$

Thus eq. (21.33) is a scaled pseudovector for the subspace defined by span $\mathbf{u}_i$, multiplied by a k-1 blade.

21.4 COMPONENTS OF A VECTOR

The delta property of eq. (21.20) allows one to use the reciprocal frame vectors and the basis that generated them to calculate the coordinates of the a vector with respect to this (not necessarily orthonormal) basis.

That is a pretty powerful result, but somewhat obscured by the Doran/Lasenby super/sub script notation.

Suppose one writes a vector in span $\mathbf{u}_i$ in terms of unknown coefficients

$$\mathbf{a} = \sum a_i \mathbf{u}_i \tag{21.35}$$

Dotting with $\mathbf{u}^j$ gives:

$$\mathbf{a} \cdot \mathbf{u}^j = \sum a_i \mathbf{u}_i \cdot \mathbf{u}^j = \sum a_i \delta_{ij} = a_j \tag{21.36}$$

Thus

$$\mathbf{a} = \sum (\mathbf{a} \cdot \mathbf{u}^i) \mathbf{u}_i \tag{21.37}$$

Similarly, writing this vectors in terms of $\mathbf{u}^i$ we have

$$\mathbf{a} = \sum b_i \mathbf{u}^i \tag{21.38}$$

Dotting with $\mathbf{u}_j$ gives:

$$\mathbf{a} \cdot \mathbf{u}_j = \sum b_i \mathbf{u}^i \cdot \mathbf{u}_j = \sum b_i \delta_{ij} = b_j \tag{21.39}$$

Thus

$$\mathbf{a} = \sum (\mathbf{a} \cdot \mathbf{u}_i) \mathbf{u}^i \tag{21.40}$$

We are used to seeing the equation for components of a vector in terms of a basis in the following form:

$$\mathbf{a} = \sum (\mathbf{a} \cdot \mathbf{u}_i) \mathbf{u}_i \tag{21.41}$$

This is true only when the basis vectors are orthonormal. Equations eq. (21.37) and eq. (21.40) provide the general decomposition of a vector in terms of a general linearly independent set.

21.4.1 *Reciprocal frame vectors by solving coordinate equation*

A more natural way to these results are to take repeated wedge products. Given a vector decomposition in terms of a basis $\mathbf{u}_i$, we want to solve for a_i:

$$\mathbf{a} = \sum_{i=1}^{k} a_i \mathbf{u}_i \tag{21.42}$$

The solution, from the wedge is:

$$\mathbf{a} \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \check{\mathbf{u}}_i \cdots \wedge \mathbf{u}_k = a_i (-1)^{i-1} \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \tag{21.43}$$

$$\implies a_i = (-1)^{i-1} \frac{\mathbf{a} \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \check{\mathbf{u}}_i \cdots \wedge \mathbf{u}_k}{\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k}) \tag{21.44}$$

The complete vector in terms of components is thus:

$$\mathbf{a} = \sum (-1)^{i-1} \frac{\mathbf{a} \wedge \left(\mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \check{\mathbf{u}}_i \cdots \wedge \mathbf{u}_k\right)}{\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k} \mathbf{u}_i \tag{21.45}$$

We are used to seeing the coordinates expressed in terms of dot products instead of wedge products. As in $\mathbb{R}^3$ where the pseudovector allows wedge products to be expressed in terms of the dot product we can do the same for the general case.

Writing $\mathbf{B} \in \bigwedge^{k-1}$ and $\mathbf{I} \in \bigwedge^{k}$ we want to reduce an equation of the following form

$$\frac{\mathbf{a} \wedge \mathbf{B}}{\mathbf{I}} = \frac{1}{\mathbf{I}} \frac{\mathbf{aB} + (-1)^{k-1}\mathbf{Ba}}{2} \tag{21.46}$$

The pseudovector either commutes or anticommutes with a vector in the subspace depending on the grade

$$\begin{aligned}
\mathbf{Ia} &= \mathbf{I} \cdot \mathbf{a} + \overbrace{\boxed{\mathbf{I} \wedge \mathbf{a}}}^{=0} \\
&= (-1)^{k-1} \mathbf{a} \cdot \mathbf{I} \\
&= (-1)^{k-1} \mathbf{aI}
\end{aligned} \tag{21.47}$$

Substituting back into eq. (21.46) we have

$$\begin{aligned}
\frac{\mathbf{a} \wedge \mathbf{B}}{\mathbf{I}} &= (-1)^{k-1} \frac{\mathbf{a}\left(\frac{1}{\mathbf{I}}\mathbf{B}\right) + \left(\frac{1}{\mathbf{I}}\mathbf{B}\right)\mathbf{a}}{2} \\
&= (-1)^{k-1} \mathbf{a} \cdot \left(\frac{1}{\mathbf{I}}\mathbf{B}\right) \\
&= \mathbf{a} \cdot \left(\mathbf{B}\frac{1}{\mathbf{I}}\right)
\end{aligned} \tag{21.48}$$

With $\mathbf{I} = \mathbf{u}_1 \wedge \cdots \mathbf{u}_k$, and $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \check{\mathbf{u}}_i \cdots \wedge \mathbf{u}_k$, back substitution back into eq. (21.45) is thus

$$\mathbf{a} = \sum \mathbf{a} \cdot \left((-1)^{i-1} \mathbf{B} \frac{1}{\mathbf{I}}\right) \mathbf{u}_i \tag{21.49}$$

The final result yields the reciprocal frame vector $\mathbf{u}^k$, and we see how to arrive at this result naturally attempting to answer the question of how to find the coordinates of a vector with respect to a (not necessarily orthonormal) basis.

$$\mathbf{a} = \sum \mathbf{a} \cdot \overbrace{\boxed{\left((\mathbf{u}_1 \wedge \mathbf{u}_2 \cdots \check{\mathbf{u}}_i \cdots \wedge \mathbf{u}_k) \frac{(-1)^{i-1}}{\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k} \right)}}^{\mathbf{u}^k} \mathbf{u}_i \tag{21.50}$$

21.5 COMPONENTS OF A BIVECTOR

To find the coordinates of a bivector with respect to an arbitrary basis we have a similar problem. For a vector basis $\mathbf{a}_i$, introduce a bivector basis $\mathbf{a}_i \wedge \mathbf{a}_j$, and write

$$\mathbf{B} = \sum_{u<v} b_{uv} \mathbf{a}_u \wedge \mathbf{a}_v \tag{21.51}$$

Wedging with $\mathbf{a}_i \wedge \mathbf{a}_j$ will select all but the ij component. Specifically

$$\begin{aligned} \mathbf{B} \wedge (\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_i \cdots \check{\mathbf{a}}_j \cdots \wedge \mathbf{a}_k) &= b_{ij} \mathbf{a}_i \wedge \mathbf{a}_j \wedge (\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_i \cdots \check{\mathbf{a}}_j \cdots \wedge \mathbf{a}_k) \\ &= b_{ij} (-1)^{j-2+i-1} (\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k) \end{aligned} \tag{21.52}$$

Thus

$$b_{ij} = (-1)^{i+j-3} \mathbf{B} \wedge \frac{(\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_i \cdots \check{\mathbf{a}}_j \cdots \wedge \mathbf{a}_k)}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k} \tag{21.53}$$

We want to put this in dot product form like eq. (21.50). To do so we need a generalized grade reduction formula

$$(\mathbf{A}_a \wedge \mathbf{A}_b) \cdot \mathbf{A}_c = \mathbf{A}_a \cdot (\mathbf{A}_b \cdot \mathbf{A}_c) \tag{21.54}$$

This holds when $a + b \leq c$. Writing $\mathbf{A} = \mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_i \cdots \check{\mathbf{a}}_j \cdots \wedge \mathbf{a}_k$, and $\mathbf{I} = \mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k$, we have

$$\begin{aligned} (\mathbf{B} \wedge \mathbf{A}) \frac{1}{\mathbf{I}} &= (\mathbf{B} \wedge \mathbf{A}) \cdot \frac{1}{\mathbf{I}} \\ &= \mathbf{B} \cdot \left(\mathbf{A} \cdot \frac{1}{\mathbf{I}} \right) \\ &= \mathbf{B} \cdot \left(\mathbf{A} \frac{1}{\mathbf{I}} \right) \end{aligned} \tag{21.55}$$

Thus the bivector in terms of its coordinates for this basis is:

$$\sum_{u<v} \mathbf{B} \cdot \left((\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_u \cdots \check{\mathbf{a}}_v \cdots \wedge \mathbf{a}_k) \frac{(-1)^{u+v-2-1}}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k} \right) \mathbf{a}_u \wedge \mathbf{a}_v \tag{21.56}$$

It is easy to see how this generalizes to higher order blades since eq. (21.54) is good for all required grades. In all cases, the form is going to be the same, with only differences in sign and the number of omitted vectors in the $\mathbf{A}$ blade.

For example for a trivector

$$\mathbf{T} = \sum_{u<v<w} t_{uvw} \mathbf{a}_u \wedge \mathbf{a}_v \wedge \mathbf{a}_w \tag{21.57}$$

It is pretty straightforward to show that this can be decomposed as follows

$$\mathbf{T} = \sum_{u<v<w} \mathbf{T} \cdot \left((\mathbf{a}_1 \wedge \cdots \check{\mathbf{a}}_u \cdots \check{\mathbf{a}}_v \cdots \check{\mathbf{a}}_w \cdots \wedge \mathbf{a}_k) \frac{(-1)^{u+v+w-3-2-1}}{\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_k} \right) \mathbf{a}_u \wedge \mathbf{a}_v \wedge \mathbf{a}_w \tag{21.58}$$

21.5.1 *Compare to GAFP*

Doran/Lasenby's GAFP demonstrates eq. (21.50), and with some incomprehensible steps skips to a generalized result of the form [1]

$$\mathbf{B} = \sum_{i<j} \mathbf{B} \cdot \left(\mathbf{a}^j \wedge \mathbf{a}^i \right) \mathbf{a}_i \wedge \mathbf{a}_j \tag{21.59}$$

GAFP states this for general multivectors instead of bivectors, but the idea is the same.

This makes intuitive sense based on the very similar vector result. This does not show that the generalized reciprocal frame k-vectors calculated in eq. (21.56) or eq. (21.58) can be produced simply by wedging the corresponding individual reciprocal frame vectors.

To show that either takes algebraic identities that I do not know, or am not thinking of as applicable. Alternately perhaps it would just take simple brute force.

Easier is to demonstrate the validity of the final result directly. Then assuming my direct calculations are correct implicitly demonstrates equivalence.

1 In retrospect I do not think that the in between steps had anything to do with logical sequence. The authors wanted some of the results for subsequent stuff (like: rotor recovery) and sandwiched it between the vector and reciprocal frame multivector results somewhat out of sequence.

Starting with $\mathbf{B}$ as defined in eq. (21.51), take dot products with $\mathbf{a}^j \wedge \mathbf{a}^i$.

$$\begin{aligned}\mathbf{B} \cdot (\mathbf{a}^j \wedge \mathbf{a}^i) &= \sum_{u<v} b_{uv} (\mathbf{a}_u \wedge \mathbf{a}_v) \cdot (\mathbf{a}^j \wedge \mathbf{a}^i) \\ &= \sum_{u<v} b_{uv} \begin{vmatrix} \mathbf{a}_u \cdot \mathbf{a}^i & \mathbf{a}_u \cdot \mathbf{a}^j \\ \mathbf{a}_v \cdot \mathbf{a}^i & \mathbf{a}_v \cdot \mathbf{a}^j \end{vmatrix} \\ &= \sum_{u<v} b_{uv} \begin{vmatrix} \delta_{ui} & \delta_{uj} \\ \delta_{vi} & \delta_{vj} \end{vmatrix}\end{aligned} \tag{21.60}$$

Consider this determinant when $u = i$ for example

$$\begin{vmatrix} \delta_{ui} & \delta_{uj} \\ \delta_{vi} & \delta_{vj} \end{vmatrix} = \begin{vmatrix} 1 & \delta_{ij} \\ \delta_{vi} & \delta_{vj} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \delta_{vi} & \delta_{vj} \end{vmatrix} = \delta_{vj} \tag{21.61}$$

If any one index is common, then both must be common ($ij = uv$) for this determinant to have a non-zero (ie: one) value. On the other hand, if no index is common then all the δ's are zero.

Like eq. (21.20) this demonstrates an orthonormal selection behavior like the reciprocal frame vector. It has the action:

$$(\mathbf{a}_i \wedge \mathbf{a}_j) \cdot (\mathbf{a}^v \wedge \mathbf{a}^u) = \delta_{ij,uv} \tag{21.62}$$

This means that we can write b_{uv} directly in terms of a bivector dot product

$$b_{uv} = \mathbf{B} \cdot (\mathbf{a}^v \wedge \mathbf{a}^u) \tag{21.63}$$

and thus proves eq. (21.59). Proof of the general result also follows from the determinant expansion of the respective blade dot products.

21.5.2 *Direct expansion of bivector in terms of reciprocal frame vectors*

Looking at linear operators I realized that the result for bivectors above can follow more easily from direct expansion of a bivector written in terms of vector factors:

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} &= \sum (\mathbf{a} \cdot \mathbf{u}_i \mathbf{u}^i) \wedge (\mathbf{b} \cdot \mathbf{u}_j \mathbf{u}^j) \\ &= \sum_{i<j} \left(\mathbf{a} \cdot \mathbf{u}_i \mathbf{b} \cdot \mathbf{u}_j - \mathbf{a} \cdot \mathbf{u}_j \mathbf{b} \cdot \mathbf{u}_i \right) \mathbf{u}^i \wedge \mathbf{u}^j \\ &= \sum_{i<j} \begin{vmatrix} \mathbf{a} \cdot \mathbf{u}_i & \mathbf{a} \cdot \mathbf{u}_j \\ \mathbf{b} \cdot \mathbf{u}_i & \mathbf{b} \cdot \mathbf{u}_j \end{vmatrix} \mathbf{u}^i \wedge \mathbf{u}^j\end{aligned} \tag{21.64}$$

When the set of vectors $\mathbf{u}_i = \mathbf{u}^i$ are orthonormal we have already calculated this result when looking at the wedge product in a differential forms context:

$$\mathbf{a} \wedge \mathbf{b} = \sum_{i<j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \mathbf{u}_i \wedge \mathbf{u}_j \tag{21.65}$$

For this general case for possibly non-orthonormal frames, this determinant of dot products can be recognized as the dot product of two blades

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{u}_j \wedge \mathbf{u}_i) &= \mathbf{a} \cdot (\mathbf{b} \cdot (\mathbf{u}_j \wedge \mathbf{u}_i)) \\ &= \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{u}_j \mathbf{u}_i - \mathbf{b} \cdot \mathbf{u}_i \mathbf{u}_j) \\ &= \mathbf{b} \cdot \mathbf{u}_j \mathbf{a} \cdot \mathbf{u}_i - \mathbf{b} \cdot \mathbf{u}_i \mathbf{a} \cdot \mathbf{u}_j \end{aligned} \tag{21.66}$$

Thus we have a decomposition of the bivector directly into a sum of components for the reciprocal frame bivectors:

$$\mathbf{a} \wedge \mathbf{b} = \sum_{i<j} \left((\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{u}_j \wedge \mathbf{u}_i) \right) \mathbf{u}^i \wedge \mathbf{u}^j \tag{21.67}$$

22

MATRIX REVIEW

22.1 MOTIVATION

My initial intention for subset of notes was to get a feel for the similarities and differences between GA and matrix approaches to solution of projection. Attempting to write up that comparison I found gaps in my understanding of the matrix algebra. In particular the topic of projection as well as the related ideas of pseudoinverses and SVD were not adequately covered in my university courses, nor my texts from those courses. Here is my attempt to write up what I understand of these subjects and explore the gaps in my knowledge.

Particularly helpful was Gilbert Strang's excellent MIT lecture on subspace projection (available on the MIT opencourseware website). Much of the notes below are probably detailed in his course textbook.

There is some GA content here, but the focus in this chapter is not neccessarily GA.

22.2 SUBSPACE PROJECTION IN MATRIX NOTATION

22.2.1 *Projection onto line*

The simplest sort of projection to compute is projection onto a line. Given a direction vector b, and a line with direction vector u as in fig. 22.1.

The projection onto u is some value:

$$p = \alpha u \tag{22.1}$$

and we can write

$$b = p + e \tag{22.2}$$

where e is the component perpendicular to the line u.

Expressed in terms of the dot product this relationship is described by:

$$(b - p) \cdot a = 0 \tag{22.3}$$

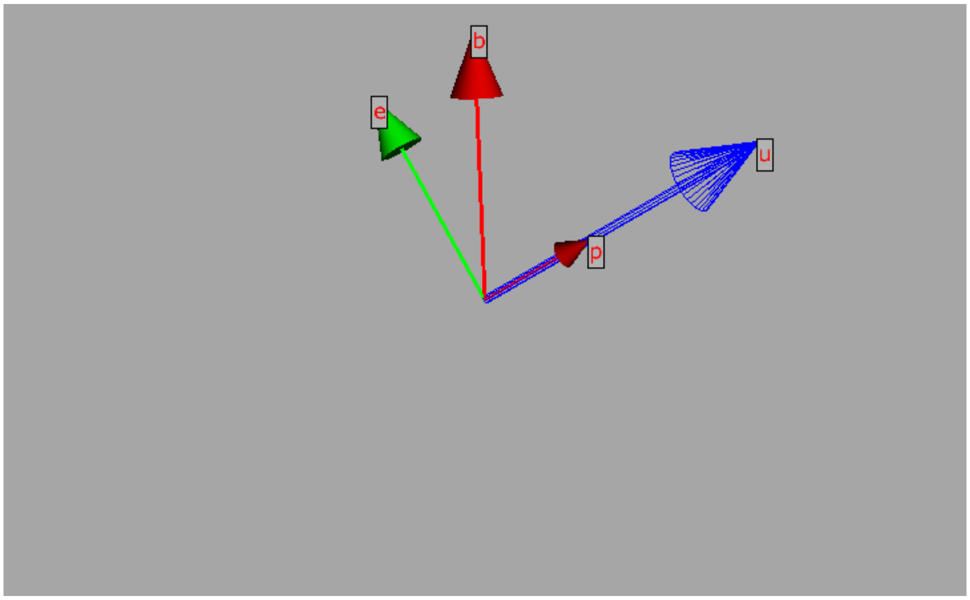

Figure 22.1: Projection onto line

Or,

$$b \cdot a = \alpha a \cdot a \tag{22.4}$$

and solving for α and substituting we have:

$$p = a \frac{a \cdot b}{a \cdot a} \tag{22.5}$$

In matrix notation that is:

$$p = a \left(\frac{a^{\mathrm{T}} b}{a^{\mathrm{T}} a} \right) \tag{22.6}$$

Following Gilbert Strang's MIT lecture on subspace projection, the parenthesis can be moved to directly express this as a projection matrix operating on b.

$$p = \left(\frac{a a^{\mathrm{T}}}{a^{\mathrm{T}} a} \right) b = Pb \tag{22.7}$$

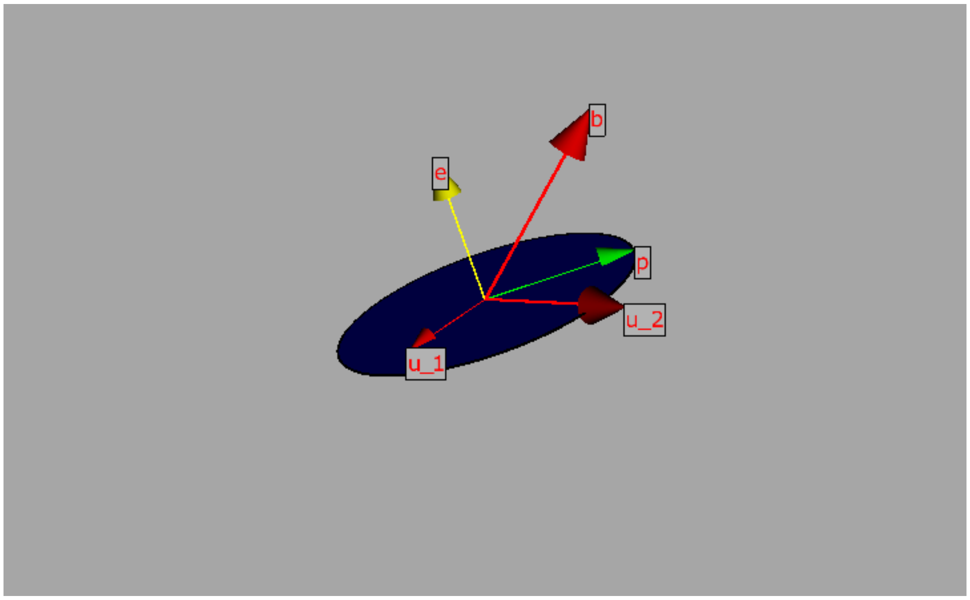

Figure 22.2: Projection onto plane

22.2.2 *Projection onto plane (or subspace)*

Calculation of the projection matrix to project onto a plane is similar. The variables to solve for are p, and e in as fig. 22.2.

For projection onto a plane (or hyperplane) the idea is the same, splitting the vector into a component in the plane and an perpendicular component. Since the idea is the same for any dimensional subspace, explicit specification of the summation range is omitted here so the result is good for higher dimensional subspaces as well as the plane:

$$b - p = e \tag{22.8}$$

$$p = \sum \alpha_i u_i \tag{22.9}$$

however, we get a set of equations, one for each direction vector in the plane

$$(b - p) \cdot u_i = 0 \tag{22.10}$$

Expanding p explicitly and rearranging we have the following set of equations:

$$b \cdot u_i = (\sum_s \alpha_s u_s) \cdot u_i \tag{22.11}$$

putting this in matrix form

$$[b \cdot u_i]_i = \left[(\textstyle\sum_s \alpha u_s) \cdot u_i\right]_i \tag{22.12}$$

Writing $U = \begin{bmatrix} u_1 & u_2 & \cdots \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} u_1^{\mathrm{T}} \\ u_2^{\mathrm{T}} \\ \vdots \end{bmatrix} b &= \left[(\textstyle\sum_s \alpha_s u_s) \cdot u_i\right] \\ &= \left[u_i \cdot u_j\right]_{ij} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} \end{aligned} \tag{22.13}$$

Solving for the vector of unknown coefficients $\alpha = [\alpha_i]_i$ we have

$$\alpha = \left[u_i \cdot u_j\right]_{ij}^{-1} U^{\mathrm{T}} b \tag{22.14}$$

And

$$p = U\alpha = U\left[u_i \cdot u_j\right]_{ij}^{-1} U^{\mathrm{T}} b \tag{22.15}$$

However, this matrix in the middle is just $U^{\mathrm{T}}U$:

$$\begin{aligned} \begin{bmatrix} u_1^{\mathrm{T}} \\ u_2^{\mathrm{T}} \\ \vdots \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \ldots \end{bmatrix} &= \begin{bmatrix} u_1^{\mathrm{T}} u_1 & u_1^{\mathrm{T}} u_2 & \ldots \\ u_2^{\mathrm{T}} u_1 & u_2^{\mathrm{T}} u_2 & \ldots \\ \vdots & & \end{bmatrix} \\ &= \left[u_i^{\mathrm{T}} u_j\right]_{ij} \\ &= \left[u_i \cdot u_j\right]_{ij} \end{aligned} \tag{22.16}$$

This provides the final result:

$$\mathrm{Proj}_U(b) = U(U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}b \tag{22.17}$$

22.2.3 *Simplifying case. Orthonormal basis for column space*

To evaluate eq. (22.17) we need only full column rank for U, but this will be messy in general due to the matrix inversion required for the center product. That can be avoided by picking an orthonormal basis for the vector space that we are projecting on. With an orthonormal column basis that central product term to invert is:

$$U^\mathrm{T} U = [u_i^\mathrm{T} u_j]_{ij} = [\delta_{ij}]_{ij} = I_{r,r} \tag{22.18}$$

Therefore, the projection matrix can be expressed using the two exterior terms alone:

$$\mathrm{Proj}_U = U(U^\mathrm{T} U)^{-1} U^\mathrm{T} = U U^\mathrm{T} \tag{22.19}$$

22.2.4 *Numerical expansion of left pseudoscalar matrix with matrix*

Numerically expanding the projection matrix $A(A^\mathrm{T} A)^{-1} A^\mathrm{T}$ is not something that we want to do, but the simpler projection matrix of equation eq. (22.19) that we get with an orthonormal basis makes this not so daunting.

Let us do this to get a feel for things.

22.2.4.1 $\mathbb{R}^4$ *plane projection example*

Take a simple projection onto the plane spanned by the following two orthonormal vectors

$$u_1 = \frac{\sqrt{2}}{4} \begin{bmatrix} 1 \\ 2 \\ \sqrt{3} \\ 0 \end{bmatrix} \tag{22.20}$$

$$u_2 = \frac{\sqrt{2}}{4} \begin{bmatrix} -\sqrt{3} \\ 0 \\ 1 \\ 2 \end{bmatrix} \tag{22.21}$$

Thus the projection matrix is:

$$P = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} u_1^{\mathrm{T}} \\ u_2^{\mathrm{T}} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & -\sqrt{3} \\ 2 & 0 \\ \sqrt{3} & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & \sqrt{3} & 0 \\ -\sqrt{3} & 0 & 1 & 2 \end{bmatrix} \tag{22.22}$$

$$\implies P = \frac{1}{8} \begin{bmatrix} 4 & 2 & 0 & -2\sqrt{3} \\ 2 & 4 & 2\sqrt{3} & 0 \\ 0 & 2\sqrt{3} & 4 & 2 \\ -2\sqrt{3} & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 0 & -\sqrt{3}/4 \\ 1/4 & 1/2 & \sqrt{3}/4 & 0 \\ 0 & \sqrt{3}/4 & 1/2 & 1/4 \\ -\sqrt{3}/4 & 0 & 1/4 & 1/2 \end{bmatrix} \tag{22.23}$$

What can be said about this just by looking at the matrix itself?

1. One can verify by inspection that $Pu_1 = u_1$ and $Pu_2 = u_2$. This is what we expected so this validates all the math performed so far. Good!

2. It is symmetric. Analytically, we know to expect this, since for a a full column rank matrix A the transpose of the projection matrix is:

$$P^{\mathrm{T}} = \left(A \frac{1}{A^{\mathrm{T}} A} A^{\mathrm{T}} \right)^{\mathrm{T}} = P. \tag{22.24}$$

3. In this particular case columns 2,4 and columns 1,3 are each pairs of perpendicular vectors. Is something like this to be expected in general for projection matrices?

4. We expect this to be a rank two matrix, so the null space has dimension two. This can be verified.

22.2.5 *Return to analytic treatment*

Let us look at the matrix for projection onto an orthonormal basis in a bit more detail. This simpler form allows for some observations that are a bit harder in the general form.

Suppose we have a vector n that is perpendicular to all the orthonormal vectors u_i that span the subspace. We can then write:

$$u_i \cdot n = 0 \tag{22.25}$$

Or,

$$u_i^{\mathrm{T}} n = 0 \tag{22.26}$$

In block matrix form for all u_i that is:

$$[u_i^{\mathrm{T}} n]_i = [u_i^{\mathrm{T}}]_i n = U^{\mathrm{T}} n = 0 \tag{22.27}$$

This is all we need to verify that our projection matrix indeed produces a zero for any vector completely outside of the subspace:

$$\mathrm{Proj}_U(n) = U(U^{\mathrm{T}} n) = U0 = 0 \tag{22.28}$$

Now we have seen numerically that UU^{T} is not an identity matrix despite operating as one on any vector that lies completely in the subspace.

Having seen the action of this matrix on vectors in the null space, we can now directly examine the action of this matrix on any vector that lies in the span of the set $\{u_i\}$. By linearity it is sufficient to do this calculation for a particular u_i:

$$\begin{aligned}
UU^{\mathrm{T}} u_i &= U \begin{bmatrix} u_1^{\mathrm{T}} u_i \\ u_2^{\mathrm{T}} u_i \\ \vdots \\ u_r^{\mathrm{T}} u_i \end{bmatrix} \\
&= \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \delta_{si} \end{bmatrix}_s \\
&= \sum_{k=1}^{r} u_k \delta_{ki} \\
&= u_i
\end{aligned} \tag{22.29}$$

This now completes the validation of the properties of this matrix (in its simpler form with an orthonormal basis for the subspace).

22.3 MATRIX PROJECTION VS. GEOMETRIC ALGEBRA

I found it pretty interesting how similar the projection product is to the projection matrix above from traditional matrix algebra. It is worthwhile to write this out and point out the similarities and differences.

22.3.1 *General projection matrix*

We have shown above, provided a matrix A is of full column rank, a projection onto its columns can be written:

$$\mathrm{Proj}_A(x) = \left(A \frac{1}{A^\mathrm{T} A} A^\mathrm{T}\right) x \tag{22.30}$$

Now contrast this with the projection written in terms of a vector dot product with a non-null blade

$$\mathrm{Proj}_A(x) = A\left(\frac{1}{A} \cdot x\right) = A \frac{1}{A^\dagger A}\left(A^\dagger \cdot x\right) \tag{22.31}$$

This is a curious correspondence and naive comparison could lead one to think that perhaps there the concepts of matrix transposition and blade reversal are equivalent.

This is not actually the case since the matrix transposition actually corresponds to the adjoint operation of a linear transformation for blades. Be that as it may, there appears to be other situations other than projections where matrix operations of the form $B^\mathrm{T} A$ end up with GA equivalents in the form $B^\dagger A$. An example is the rigid body equations where the body angular velocity bivector corresponding to a rotor R is of the form $\mathbf{\Omega} = R' R^\dagger$, whereas the matrix form for a rotation matrix R is of the form $\mathbf{\Omega} = R' R^\dagger$.

22.3.2 *Projection matrix full column rank requirement*

The projection matrix derivations above required full column rank. A reformulation in terms of a generalized matrix (Moore-Penrose) inverse, or SVD can eliminate this full column rank requirement for the formulation of the projection matrix.

We will get to this later, but we never really proved that full column rank implies $A^\mathrm{T} A$ invertability.

If one writes out the matrix $A^\mathrm{T} A$ in full

Now, if $A = [a_i]_i$, the matrix

$$A^\mathrm{T} A = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & \ldots \\ a_2 \cdot a_1 & a_2 \cdot a_2 & \ldots \\ \vdots & & \end{bmatrix}. \tag{22.32}$$

This is an invertible matrix provided $\{a_i\}_i$ is a linearly independent set of vectors. For full column rank to imply invertability, it would be sufficient to prove that the determinant of this matrix was non-zero.

I am not sure how to show that this is true with just matrix algebra, however one can identify the determinant of the matrix of eq. (22.32), after an adjustment of sign for reversion, as the GA dot product of a k-blade:

$$(-1)^{k(k-1)/2}(a_1 \wedge \cdots \wedge a_k) \cdot (a_1 \wedge \cdots \wedge a_k). \tag{22.33}$$

Linear independence means that this wedge product is non-zero, and therefore the dot product, and thus original determinant is also non-zero.

When the $A^\mathrm{T}A$ matrix of eq. (22.32) is invertible that inverse can be written using a cofactor matrix (adjoint expansion):

Let us write this out, where C_{ij} are the cofactor matrices of $A^\mathrm{T}A$, we have:

$$\frac{1}{A^\mathrm{T}A} = \frac{1}{\left|A^\mathrm{T}A\right|}[C_{ij}]^\mathrm{T} \tag{22.34}$$

Observe that the denominator here is exactly the determinant of eq. (22.33). This illustrates the motivation of Hestenes to label the explicit alternating vector-vector dot product expansion of a blade-vector dot product the “generalized Laplace expansion”.

22.3.3 *Projection onto orthonormal columns*

When the columns of the matrix A are orthonormal, the projection matrix is reduced to:

$$\mathrm{Proj}_A(x) = \left(AA^\mathrm{T}\right)x. \tag{22.35}$$

The corresponding GA entity is a projection onto a unit magnitude blade. With that scaling the inverse term also drops out leaving:

$$\mathrm{Proj}_A(x) = A\left(A^\dagger \cdot x\right) \tag{22.36}$$

This helps point out the similarity between the matrix inverse $\frac{1}{A^\mathrm{T}A}$ and the blade product inverse $\frac{1}{A^\dagger A}$ is only on the surface, since this blade product is only a scalar.

22.4 PROOF OF OMITTED DETAILS AND AUXILIARY STUFF

22.4.1 *That we can remove parenthesis to form projection matrix in line projection equation*

Remove the parenthesis in some of these expressions may not always be correct, so it is worth demonstrating that this is okay as done to calculate the projection matrix P in eq. (22.7). We only need to look at the numerator since the denominator is a scalar in this case.

$$\begin{aligned}
(aa^{\mathrm{T}})b &= [a_i a_j]_{ij} [b_i]_i \\
&= \left[\sum_k a_i a_k b_k\right]_i \\
&= \left[a_i \sum_k a_k b_k\right]_i \\
&= [a_i]_i a^{\mathrm{T}} b \\
&= a(a^{\mathrm{T}} b)
\end{aligned} \tag{22.37}$$

22.4.2 *Any invertible scaling of column space basis vectors does not change the projection*

Suppose that one introduces an alternate basis for the column space

$$v_i = \sum \alpha_{ik} u_k \tag{22.38}$$

This can be expressed in matrix form as:

$$V = UE \tag{22.39}$$

or

$$UE^{-1} = V \tag{22.40}$$

We should expect that the projection onto the plane expressed with this alternate basis should be identical to the original. Verification is straightforward:

$$
\begin{aligned}
\mathrm{Proj}_V &= V\left(V^{\mathrm{T}}V\right)^{-1}V^{\mathrm{T}} \\
&= \left(UE^{-1}\right)\left(\left(UE^{-1}\right)^{\mathrm{T}}\left(UE^{-1}\right)\right)^{-1}\left(UE^{-1}\right)^{\mathrm{T}} \\
&= \left(UE^{-1}\right)\left({E^{-1}}^{T}U^{\mathrm{T}}UE^{-1}\right)^{-1}\left(UE^{-1}\right)^{\mathrm{T}} \\
&= \left(UE^{-1}\right)E\left(U^{\mathrm{T}}U\right)^{-1}E^{\mathrm{T}}\left(UE^{-1}\right)^{\mathrm{T}} \\
&= U\left(U^{\mathrm{T}}U\right)^{-1}E^{\mathrm{T}}{E^{-1}}^{\mathrm{T}}U^{\mathrm{T}} \\
&= U\left(U^{\mathrm{T}}U\right)^{-1}U^{\mathrm{T}} \\
&= \mathrm{Proj}_U
\end{aligned}
\tag{22.41}
$$

23

OBLIQUE PROJECTION AND RECIPROCAL FRAME VECTORS

23.1 MOTIVATION

Followup on wikipedia projection article's description of an oblique projection. Calculate this myself.

23.2 USING GA. OBLIQUE PROJECTION ONTO A LINE

INSERT DIAGRAM.

Problem is to project a vector $\mathbf{x}$ onto a line with direction $\hat{\mathbf{p}}$, along a direction vector $\hat{\mathbf{d}}$.

Write:

$$\mathbf{x} + \alpha\hat{\mathbf{d}} = \beta\hat{\mathbf{p}} \tag{23.1}$$

and solve for $\mathbf{p} = \beta\hat{\mathbf{p}}$. Wedging with $\hat{\mathbf{d}}$ provides the solution:

$$\mathbf{x} \wedge \hat{\mathbf{d}} + \alpha \overbrace{\boxed{\hat{\mathbf{d}} \wedge \hat{\mathbf{d}}}}^{=0} = \beta\hat{\mathbf{p}} \wedge \hat{\mathbf{d}} \tag{23.2}$$

$$\implies \beta = \frac{\mathbf{x} \wedge \hat{\mathbf{d}}}{\hat{\mathbf{p}} \wedge \hat{\mathbf{d}}} \tag{23.3}$$

So the "oblique" projection onto this line (using direction $\hat{\mathbf{d}}$) is:

$$\mathrm{Proj}_{\hat{\mathbf{d}}\rightarrow\hat{\mathbf{p}}}(\mathbf{x}) = \frac{\mathbf{x} \wedge \hat{\mathbf{d}}}{\hat{\mathbf{p}} \wedge \hat{\mathbf{d}}}\hat{\mathbf{p}} \tag{23.4}$$

This also shows that we do not need unit vectors for this sort of projection operation, since we can scale these two vectors by any quantity since they are in both the numerator and denominator.

Let $\mathbf{D}$, and $\mathbf{P}$ be vectors in the directions of $\hat{\mathbf{d}}$, and $\hat{\mathbf{p}}$ respectively. Then the projection can also be written:

$$\mathrm{Proj}_{\mathbf{D}\to\mathbf{P}}(\mathbf{x}) = \frac{\mathbf{x} \wedge \mathbf{D}}{\mathbf{P} \wedge \mathbf{D}}\mathbf{P} \tag{23.5}$$

It is interesting to see projection expressed here without any sort of dot product when all our previous projection calculations had intrinsic requirements for a metric.

Now, let us compare this to the matrix forms of projection that we have become familiar with. For the matrix result we need a metric, but because this result is intrinsically non-metric, we can introduce one if convenient and express this result with that too. Such an expansion is:

$$\begin{aligned}
\frac{\mathbf{x} \wedge \mathbf{D}}{\mathbf{P} \wedge \mathbf{D}}\mathbf{P} &= \mathbf{x} \wedge \mathbf{D}\frac{\mathbf{D} \wedge \mathbf{P}}{\mathbf{D} \wedge \mathbf{P}}\frac{1}{\mathbf{P} \wedge \mathbf{D}}\mathbf{P} \\
&= (\mathbf{x} \wedge \mathbf{D}) \cdot (\mathbf{D} \wedge \mathbf{P})\frac{1}{|\mathbf{P} \wedge \mathbf{D}|^2}\mathbf{P} \\
&= ((\mathbf{x} \wedge \mathbf{D}) \cdot \mathbf{D}) \cdot \mathbf{P}\frac{1}{|\mathbf{P} \wedge \mathbf{D}|^2}\mathbf{P} \\
&= (\mathbf{x}\mathbf{D}^2 - \mathbf{x} \cdot \mathbf{D}\mathbf{D}) \cdot \mathbf{P}\frac{1}{|\mathbf{P} \wedge \mathbf{D}|^2}\mathbf{P} \\
&= \frac{\mathbf{x} \cdot \mathbf{P}\mathbf{D}^2 - \mathbf{x} \cdot \mathbf{D}\mathbf{D} \cdot \mathbf{P}}{\mathbf{P}^2\mathbf{D}^2 - (\mathbf{P} \cdot \mathbf{D})^2}\mathbf{P}
\end{aligned} \tag{23.6}$$

This gives us the projection explicitly:

$$\mathrm{Proj}_{\mathbf{D}\to\mathbf{P}}(\mathbf{x}) = \left(\mathbf{x} \cdot \frac{\mathbf{P}\mathbf{D}^2 - \mathbf{D}\mathbf{D} \cdot \mathbf{P}}{\mathbf{P}^2\mathbf{D}^2 - (\mathbf{P} \cdot \mathbf{D})^2}\right)\mathbf{P} \tag{23.7}$$

It sure does not simplify things to expand things out, but we now have things prepared to express in matrix form.

Assuming a euclidean metric, and a bit of playing shows that the denominator can be written more simply as:

$$\mathbf{P}^2\mathbf{D}^2 - (\mathbf{P} \cdot \mathbf{D})^2 = \left|U^{\mathrm{T}}U\right| \tag{23.8}$$

where:

$$U = \begin{bmatrix}\mathbf{P} & \mathbf{D}\end{bmatrix} \tag{23.9}$$

Similarly the numerator can be written:

$$\mathbf{x} \cdot \mathbf{P}\mathbf{D}^2 - \mathbf{x} \cdot \mathbf{D}\mathbf{D} \cdot \mathbf{P} = D^{\mathrm{T}} U \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} U^{\mathrm{T}} \mathbf{x}. \tag{23.10}$$

Combining these yields a projection matrix:

$$\mathrm{Proj}_{\mathbf{D} \to \mathbf{P}}(\mathbf{x}) = \left(\mathbf{P} \frac{1}{\left| U^{\mathrm{T}} U \right|} \mathbf{D}^{\mathrm{T}} U \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} U^{\mathrm{T}} \right) \mathbf{x}. \tag{23.11}$$

The alternation above suggests that this is related to the matrix inverse of something. Let us try to calculate this directly instead.

23.3 OBLIQUE PROJECTION ONTO A LINE USING MATRICES

Let us start at the same place as in eq. (23.1), except that we know we can discard the unit vectors and work with any vectors in the projection directions:

$$\mathbf{x} + \alpha \mathbf{D} = \beta \mathbf{P} \tag{23.12}$$

Assuming an inner product, we have two sets of results:

$$\begin{aligned} \langle \mathbf{P}, \mathbf{x} \rangle + \alpha \langle \mathbf{P}, \mathbf{D} \rangle &= \beta \langle \mathbf{P}, \mathbf{P} \rangle \\ \langle \mathbf{D}, \mathbf{x} \rangle + \alpha \langle \mathbf{D}, \mathbf{D} \rangle &= \beta \langle \mathbf{D}, \mathbf{P} \rangle \end{aligned} \tag{23.13}$$

and can solve this for α, and β.

$$\begin{bmatrix} \langle \mathbf{P}, \mathbf{D} \rangle & \langle \mathbf{P}, \mathbf{P} \rangle \\ \langle \mathbf{D}, \mathbf{D} \rangle & \langle \mathbf{D}, \mathbf{P} \rangle \end{bmatrix} \begin{bmatrix} -\alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \langle \mathbf{P}, \mathbf{x} \rangle \\ \langle \mathbf{D}, \mathbf{x} \rangle \end{bmatrix} \tag{23.14}$$

If our inner product is defined by $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* A \mathbf{v}$, we have:

$$\begin{aligned} \begin{bmatrix} \langle \mathbf{P}, \mathbf{D} \rangle & \langle \mathbf{P}, \mathbf{P} \rangle \\ \langle \mathbf{D}, \mathbf{D} \rangle & \langle \mathbf{D}, \mathbf{P} \rangle \end{bmatrix} &= \begin{bmatrix} \mathbf{P}^* A \mathbf{D} & \mathbf{P}^* A \mathbf{P} \\ \mathbf{D}^* A \mathbf{D} & \mathbf{D}^* A \mathbf{P} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{P} & \mathbf{D} \end{bmatrix}^* A \begin{bmatrix} \mathbf{D} & \mathbf{P} \end{bmatrix} \end{aligned} \tag{23.15}$$

Thus the solution to eq. (23.14) is

$$\begin{bmatrix} -\alpha \\ \beta \end{bmatrix} = \left(\frac{1}{\begin{bmatrix} \mathbf{P} & \mathbf{D} \end{bmatrix}^* A \begin{bmatrix} \mathbf{D} & \mathbf{P} \end{bmatrix}} \begin{bmatrix} \mathbf{P} & \mathbf{D} \end{bmatrix}^* A \right) \mathbf{x} \tag{23.16}$$

Again writing $U = \begin{bmatrix} \mathbf{P} & \mathbf{D} \end{bmatrix}$, this is:

$$\begin{aligned} \begin{bmatrix} -\alpha \\ \beta \end{bmatrix} &= \left(\frac{1}{U^*AU \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} U^* A \right) \mathbf{x} \\ &= \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{U^*AU} U^* A \right) \mathbf{x} \end{aligned} \tag{23.17}$$

Since we only care about solution for β to find the projection, we have to discard half the inversion work, and just select that part of the solution (suggests that a Cramer's rule method is more efficient than matrix inversion in this case) :

$$\beta = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\alpha \\ \beta \end{bmatrix} \tag{23.18}$$

Thus the solution of this oblique projection problem in terms of matrices is:

$$\mathrm{Proj}_{\mathbf{D}\rightarrow\mathbf{P}}(\mathbf{x}) = \left(\mathbf{P} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{U^*AU} U^* A \right) \mathbf{x} \tag{23.19}$$

Which is:

$$\mathrm{Proj}_{\mathbf{D}\rightarrow\mathbf{P}}(\mathbf{x}) = \left(\mathbf{P} \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{U^*AU} U^* A \right) \mathbf{x} \tag{23.20}$$

Explicit expansion can be done easily enough to show that this is identical to eq. (23.7), so the question of what we were implicitly inverting in eq. (23.11) is answered.

23.4 OBLIQUE PROJECTION ONTO HYPERPLANE

Now that we have got this directed projection problem solved for a line in both GA and matrix form, the next logical step is a k-dimensional hyperplane projection. The equation to solve is now:

$$\mathbf{x} + \alpha \mathbf{D} = \sum \beta_i \mathbf{P}_i \tag{23.21}$$

23.4.1 *Non metric solution using wedge products*

For $\mathbf{x}$ with some component not in the hyperplane, we can wedge with $P = \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \cdots \wedge \mathbf{P}_k$

$$\mathbf{x} \wedge P + \alpha \mathbf{D} \wedge P = \sum_{i=1}^{k} \beta_i \overbrace{\boxed{\mathbf{P}_i \wedge P}}^{=0}$$

Thus the projection onto the hyperplane spanned by P is going from $\mathbf{x}$ along $\mathbf{D}$ is $\mathbf{x} + \alpha \mathbf{D}$:

$$\mathrm{Proj}_{\mathbf{D}\to P}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \wedge P}{\mathbf{D} \wedge P}\mathbf{D} \tag{23.22}$$

23.4.1.1 *Q: reduction of this*

When P is a single vector we can reduce this to our previous result:

$$\begin{aligned}
\mathrm{Proj}_{\mathbf{D}\to\mathbf{P}}(\mathbf{x}) &= \mathbf{x} - \frac{\mathbf{x} \wedge \mathbf{P}}{\mathbf{D} \wedge \mathbf{P}}\mathbf{D} \\
&= \frac{1}{\mathbf{D} \wedge \mathbf{P}} \left((\mathbf{D} \wedge \mathbf{P})\mathbf{x} - (\mathbf{x} \wedge \mathbf{P})\mathbf{D} \right) \\
&= \frac{1}{\mathbf{D} \wedge \mathbf{P}} \left((\mathbf{D} \wedge \mathbf{P}) \cdot \mathbf{x} - (\mathbf{x} \wedge \mathbf{P}) \cdot \mathbf{D} \right) \\
&= \frac{1}{\mathbf{D} \wedge \mathbf{P}} \left(\mathbf{D}\mathbf{P} \cdot \mathbf{x} - \mathbf{P}\mathbf{D} \cdot \mathbf{x} - \mathbf{x}\mathbf{P} \cdot \mathbf{D} + \mathbf{P}\mathbf{x} \cdot \mathbf{D} \right) \\
&= \frac{1}{\mathbf{D} \wedge \mathbf{P}} \left(\mathbf{D}\mathbf{P} \cdot \mathbf{x} - \mathbf{x}\mathbf{P} \cdot \mathbf{D} \right)
\end{aligned} \tag{23.23}$$

Which is:

$$\mathrm{Proj}_{\mathbf{D}\to\mathbf{P}}(\mathbf{x}) = \frac{1}{\mathbf{P} \wedge \mathbf{D}} \mathbf{P} \cdot (\mathbf{D} \wedge \mathbf{x}). \tag{23.24}$$

A result that is equivalent to our original eq. (23.5). Can we similarly reduce the general result to something of this form. Initially I wrote:

$$
\begin{aligned}
\mathrm{Proj}_{\mathbf{D}\rightarrow P}(\mathbf{x}) &= \mathbf{x} - \frac{\mathbf{x}\wedge P}{\mathbf{D}\wedge P}\mathbf{D} \\
&= \frac{\mathbf{D}\wedge P}{\mathbf{D}\wedge P}\mathbf{x} - \frac{\mathbf{x}\wedge P}{\mathbf{D}\wedge P}\mathbf{D} \\
&= \frac{1}{\mathbf{D}\wedge P}\left((\mathbf{D}\wedge P)\mathbf{x} - (\mathbf{x}\wedge P)\mathbf{D}\right) \\
&= \frac{1}{\mathbf{D}\wedge P}\left((\mathbf{D}\wedge P)\cdot\mathbf{x} - (\mathbf{x}\wedge P)\cdot\mathbf{D}\right) \\
&= \frac{1}{\mathbf{D}\wedge P}\left(\mathbf{D}P\cdot\mathbf{x} - P\mathbf{D}\cdot\mathbf{x} - \mathbf{x}P\cdot\mathbf{D} + P\mathbf{x}\cdot\mathbf{D}\right) \\
&= \frac{1}{\mathbf{D}\wedge P}\left(\mathbf{D}P\cdot\mathbf{x} - \mathbf{x}P\cdot\mathbf{D}\right) \\
&= -\frac{1}{\mathbf{D}\wedge P}P\cdot(\mathbf{D}\wedge\mathbf{x})
\end{aligned}
\tag{23.25}
$$

However, I am not sure that about the manipulations done on the last few lines where P has grade greater than 1 (ie: the triple product expansion and recollection later).

23.4.2 *hyperplane directed projection using matrices*

To solve eq. (23.21) using matrices, we can take a set of inner products:

$$
\begin{aligned}
\langle\mathbf{D},\mathbf{x}\rangle + \alpha\langle\mathbf{D},\mathbf{D}\rangle &= \sum_{u=1}^{k}\beta_u\langle\mathbf{D},\mathbf{P}_u\rangle \\
\langle\mathbf{P}_i,\mathbf{x}\rangle + \alpha\langle\mathbf{P}_i,\mathbf{D}\rangle &= \sum_{u=1}^{k}\beta_u\langle\mathbf{P}_i,\mathbf{P}_u\rangle
\end{aligned}
\tag{23.26}
$$

Write $\mathbf{D} = \mathbf{P}_{k+1}$, and $\alpha = -\beta_{k+1}$ for symmetry, which reduces this to:

$$
\begin{aligned}
\langle\mathbf{P}_{k+1},\mathbf{x}\rangle &= \sum_{u=1}^{k}\beta_u\langle\mathbf{P}_{k+1},\mathbf{P}_u\rangle + \beta_{k+1}\langle\mathbf{P}_{k+1},\mathbf{P}_{k+1}\rangle \\
\langle\mathbf{P}_i,\mathbf{x}\rangle &= \sum_{u=1}^{k}\beta_u\langle\mathbf{P}_i,\mathbf{P}_u\rangle + \beta_{k+1}\langle\mathbf{P}_i,\mathbf{P}_{k+1}\rangle
\end{aligned}
\tag{23.27}
$$

That is the following set of equations:

$$\langle \mathbf{P}_i, \mathbf{x} \rangle = \sum_{u=1}^{k+1} \beta_u \langle \mathbf{P}_i, \mathbf{P}_u \rangle \tag{23.28}$$

Which we can now express as a single matrix equation (for $i, j \in [1, k+1]$) :

$$\left[\langle \mathbf{P}_i, \mathbf{x} \rangle \right]_i = \left[\langle \mathbf{P}_i, \mathbf{P}_j \rangle \right]_{ij} \left[\beta_i \right]_i \tag{23.29}$$

Solving for $\boldsymbol{\beta} = \left[\beta_i \right]_i$, gives:

$$\boldsymbol{\beta} = \frac{1}{\left[\langle \mathbf{P}_i, \mathbf{P}_j \rangle \right]_{ij}} \left[\langle \mathbf{P}_i, \mathbf{x} \rangle \right]_i \tag{23.30}$$

The projective components of interest are $\sum_{i=1}^{k} \beta_i \mathbf{P}_i$. In matrix form that is:

$$\begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_k \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_k \end{bmatrix} \begin{bmatrix} I_{k,k} & 0_{k,1} \end{bmatrix} \boldsymbol{\beta} \tag{23.31}$$

Therefore the directed projection is:

$$\mathrm{Proj}_{\mathbf{D}\to P}(\mathbf{x}) = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_k \end{bmatrix} \begin{bmatrix} I_{k,k} & 0_{k,1} \end{bmatrix} \frac{1}{\left[\langle \mathbf{P}_i, \mathbf{P}_j \rangle \right]_{ij}} \left[\langle \mathbf{P}_i, \mathbf{x} \rangle \right]_i \tag{23.32}$$

As before writing $U = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_k & \mathbf{D} \end{bmatrix}$, and write $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* A \mathbf{v}$. The directed projection is now:

$$\mathrm{Proj}_{\mathbf{D}\to P}(\mathbf{x}) = \left(U \begin{bmatrix} I_{k,k} \\ 0_{1,k} \end{bmatrix} \begin{bmatrix} I_{k,k} & 0_{k,1} \end{bmatrix} \frac{1}{U^* A U} U^* A \right) \mathbf{x}$$

$$= \left(U \begin{bmatrix} I_{k,k} & 0_{k,1} \\ 0_{1,k} & 0_{1,1} \end{bmatrix} \frac{1}{U^* A U} U^* A \right) \mathbf{x} \tag{23.33}$$

23.5 PROJECTION USING RECIPROCAL FRAME VECTORS

In a sense the projection operation is essentially a calculation of components of vectors that span a given subspace. We can also calculate these components using a reciprocal frame. To start with consider just orthogonal projection, where the equation to solve is:

$$\mathbf{x} = \mathbf{e} + \sum \beta_j \mathbf{P}_j \tag{23.34}$$

and $\mathbf{e} \cdot \mathbf{P}_i = 0$.
Introduce a reciprocal frame $\{\mathbf{P}^j\}$ that also spans the space of $\{\mathbf{P}_j\}$, and is defined by:

$$\mathbf{P}_i \cdot \mathbf{P}^j = \delta_{ij} \tag{23.35}$$

With this we have:

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{P}^i &= \overset{=0}{\boxed{\mathbf{e} \cdot \mathbf{P}^i}} + \sum \beta_j \mathbf{P}_j \cdot \mathbf{P}^i \\
&= \sum \beta_j \delta_{ij} \\
&= \beta_i
\end{aligned} \tag{23.36}$$

$$\mathbf{x} = \mathbf{e} + \sum \mathbf{P}_j (\mathbf{P}^j \cdot \mathbf{x})$$

For a Euclidean metric the projection part of this is:

$$\mathrm{Proj}_P(\mathbf{x}) = \left(\sum \mathbf{P}_j (\mathbf{P}^j)^{\mathrm{T}}\right) \mathbf{x} \tag{23.37}$$

Note that there is a freedom to remove the dot product that was employed to form the matrix representation of eq. (23.37) that may not be obvious. I did not find that this was obvious, when seen in Prof. Gilbert Strang's MIT OCW lectures, and had to prove it for myself. That proof is available at the end of 22 comparing the geometric and matrix projection operations , in the 'That we can remove parenthesis to form projection matrix in line projection equation.' subsection.

Writing $P = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_k \end{bmatrix}$ and for the reciprocal frame vectors: $Q = \begin{bmatrix} \mathbf{P}^1 & \mathbf{P}^2 & \cdots & \mathbf{P}^k \end{bmatrix}$

We now have the following simple calculation for the projection matrix onto a set of linearly independent vectors (columns of P):

$$\text{Proj}_P(\mathbf{x}) = PQ^{\text{T}}\mathbf{x}. \tag{23.38}$$

Compare to the general projection matrix previously calculated when the columns of P weare not orthonormal:

$$\text{Proj}_P(\mathbf{x}) = P\frac{1}{P^{\text{T}}P}P^{\text{T}}\mathbf{x} \tag{23.39}$$

With orthonormal columns the $P^T P$ becomes identity and the inverse term drops out, and we get something similar with reciprocal frames. As a side effect this shows us how to calculate without GA the reciprocal frame vectors. Those vectors are thus the columns of

$$Q = P\frac{1}{P^{\text{T}}P} \tag{23.40}$$

We are thus able to get a specific understanding of some of the interior terms of the general orthogonal projection matrix.

Also note that the orthonormality of these columns is confirmed by observing that $Q^{\text{T}}P = \frac{1}{P^{\text{T}}P}P^{\text{T}}P = I$.

23.5.1 *example/verification*

As an example to see that this works write:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{23.41}$$

$$P^{\text{T}}P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{23.42}$$

$$\frac{1}{P^{\text{T}}P} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{23.43}$$

$$Q = P\frac{1}{P^{\mathrm{T}}P} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 2 \end{bmatrix} \tag{23.44}$$

By inspection these columns have the desired properties.

23.6 DIRECTED PROJECTION IN TERMS OF RECIPROCAL FRAMES

Suppose that one has a set of vectors $\{\mathbf{P}_i\}$ that span the subspace that contains the vector to be projected $\mathbf{x}$. If one wants to project onto a subset of those P_k, say, the first k of l of these vectors, and wants the projection directed along components of the remaining $l-k$ of these vectors, then solution of the following is required:

$$\mathbf{x} = \sum_{j=1}^{l} \beta_j \mathbf{P}_j$$

This (affine?) projection is then just the $\sum_{j=1}^{k} \beta_j \mathbf{P}_j$ components of this vector.
Given a reciprocal frame for the space, the solution follows immediately.

$$\mathbf{P}^i \cdot \mathbf{x} = \sum \beta_j \mathbf{P}^i \cdot \mathbf{P}_j = \beta_i \tag{23.45}$$

$$\beta_i = \mathbf{P}^i \cdot \mathbf{x} \tag{23.46}$$

Or,

$$\mathrm{Proj}_{P_k}(\mathbf{x}) = \sum_{j=1}^{k} \mathbf{P}_j \mathbf{P}^j \cdot \mathbf{x} \tag{23.47}$$

In matrix form, with inner product $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^* A \mathbf{v}$, and writing $P = \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_l \end{bmatrix}$, and $Q = \begin{bmatrix} \mathbf{P}^1 & \mathbf{P}^2 & \cdots & \mathbf{P}^l \end{bmatrix}$, this is:

$$\mathrm{Proj}_{P_k}(\mathbf{x}) = \left(P \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q^* A \right) \mathbf{x} \tag{23.48}$$

Observe that the reciprocal frame vectors can be expressed as the rows of the matrix

$$Q^* A = \frac{1}{P^* A P} P^* A$$

Assuming the matrix of dot product is invertible, the reciprocal frame vectors are the columns of:

$$Q = P \frac{1}{P^* A^* P} \tag{23.49}$$

I had expect that the matrix of most dot product forms would also have
$A = A^*$ (ie: Hermitian symmetric).

That is certainly true for all the vector dot products I am interested in utilizing. ie: the standard euclidean dot product, Minkowski space time metrics, and complex field vector space inner products (all of those are not only real symmetric, but are also all diagonal). For completeness, exploring this form for a more generalized form of inner product was also explored in E.

23.6.1 *Calculation efficiency*

It would be interesting to compare the computational complexity for a set of reciprocal frame vectors calculated with the GA method (where overhat indicates omission) :

$$P^i = (-1)^{i-1} P_1 \wedge \cdots \hat{P}_i \cdots \wedge P_k \frac{1}{P_1 \wedge P_2 \wedge \cdots \wedge P_k} \tag{23.50}$$

The wedge in the denominator can be done just once for all the frame vectors. Is there a way to use that for the numerator too (dividing out the wedge product with the vector in question)?

Calculation of the $\frac{1}{P^{\mathrm{T}} P}$ term could be avoided by using SVD.

Writing $P = U \Sigma V^{\mathrm{T}}$, the reciprocal frame vectors will be $Q = U \Sigma \frac{1}{\Sigma^{\mathrm{T}} \Sigma} V^{\mathrm{T}}$.

Would that be any more efficient, or perhaps more importantly, for larger degree vectors is that a more numerically stable calculation?

23.7 FOLLOWUP

Review: Have questionable GA algebra reduction earlier for grade > 1 (following eq. (23.24)).

Q: Can a directed frame vector projection be defined in terms of an “oblique” dot product.

Q: What applications would a non-diagonal bilinear form have?

Editorial: I have defined the inner product in matrix form with:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* A \mathbf{v} \tag{23.51}$$

This is slightly irregular since it is the conjugate of the normal complex inner product, so in retrospect I would have been better to express things as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathrm{T}} A \bar{\mathbf{v}} \tag{23.52}$$

Editorial: I have used the term oblique projection. In retrospect I think I have really been describing what is closer to an affine (non-metric) projection so that would probably have been better to use.

24 PROJECTION AND MOORE-PENROSE VECTOR INVERSE

24.1 PROJECTION AND MOORE-PENROSE VECTOR INVERSE

One can observe that the Moore Penrose left vector inverse $\mathbf{v}^+$ shows up in the projection matrix for a projection onto a line with a direction vector $\mathbf{v}$:

$$\text{Proj}_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} \overbrace{\boxed{\frac{1}{\mathbf{v}^\text{T}\mathbf{v}}\mathbf{v}^\text{T}}}^{\mathbf{v}^+} \mathbf{x} \tag{24.1}$$

I do not know of any other "application" of this Moore-Penrose vector inverse in traditional matrix algebra. As stated it is an interesting mathematical curiosity that yes one can define a vector inverse, however what would you do with it?

In geometric algebra we also have a vector inverse, but it plays a much more fundamental role, and does not have the restriction of only acting from the left and producing a scalar result. As an example consider the projection, and rejection decomposition of a vector:

$$\begin{aligned}
\mathbf{x} &= \mathbf{v}\frac{1}{\mathbf{v}}\mathbf{x} \\
&= \mathbf{v}\left(\frac{1}{\mathbf{v}} \cdot \mathbf{x}\right) + \mathbf{v}\left(\frac{1}{\mathbf{v}} \wedge \mathbf{x}\right) \\
&= \mathbf{v}\left(\frac{\mathbf{v}}{\mathbf{v}^2} \cdot \mathbf{x}\right) + \mathbf{v}\left(\frac{\mathbf{v}}{\mathbf{v}^2} \wedge \mathbf{x}\right)
\end{aligned} \tag{24.2}$$

In the above, $\frac{\mathbf{v}}{\mathbf{v}^2}\cdot = \frac{\mathbf{v}^\text{T}}{\mathbf{v}^\text{T}\mathbf{v}} = \mathbf{v}^+$. We can therefore describe the Moore Penrose vector left inverse as the matrix of the GA linear transformation $\frac{1}{\mathbf{v}}\cdot$.

Unlike the GA vector inverse, whos associativity allowed for the projection/rejection derivation above, this Moore-Penrose vector inverse has only left action, so in the above, you can not further write:

$$\mathbf{v}\mathbf{v}^+ = 1 \tag{24.3}$$

(ie: $\mathbf{v}\mathbf{v}^+$ is a projection matrix not scalar or matrix unity).

24.1.1 *matrix of wedge project transformation?*

Q: What is the matrix of the linear transformation $\frac{1}{\mathbf{v}}\wedge$?

In rigid body dynamics we see the matrix of the linear transformation $T_{\mathbf{v}}(\mathbf{x}) = (\mathbf{v}\times)(\mathbf{x})$. This is the completely antisymmetric matrix as follows:

$$\mathbf{v} \times \mathbf{x} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{24.4}$$

In order to specify the matrix of a vector-vector wedge product linear transformation we must introduce bivector coordinate vectors. For the matrix of the cross product linear transformation the standard vector basis was the obvious choice.

Let us pick the following orthonormal basis:

$$\sigma = \{\sigma_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j\}_{i<j} \tag{24.5}$$

and construct the matrix of the wedge project $T_{\mathbf{v}} : \mathbb{R}^N \rightarrow \bigwedge^2$

$$T_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} \wedge \mathbf{x} = \sum_{\mu=ij,i<j} \begin{vmatrix} v_i & v_j \\ x_i & x_j \end{vmatrix} \sigma_\mu \tag{24.6}$$

$$\implies T_{\mathbf{v}}(\mathbf{e}_k) \cdot {\sigma_{ij}}^\dagger = \sum_{k\in ij,i<j} \begin{vmatrix} v_i & v_j \\ x_i & x_j \end{vmatrix} = v_i\delta_{kj} - v_j\delta_{ki} \tag{24.7}$$

Since k cannot be simultaneously equal to both i, and j, this is:

$$T_{\mathbf{v}}(\mathbf{e}_k) \cdot {\sigma_{ij}}^\dagger = \left\{ \begin{array}{rl} v_i & k = j \\ -v_j & k = i \\ 0 & k \neq i, j \end{array} \right\} \tag{24.8}$$

Unlike the left Moore-Penrose vector inverse that we find as the matrix of the linear transformation $v \cdot (\cdot)$, except for $\mathbb{R}^3$ where we have the cross product, I do not recognize this as the matrix of any common linear transformation.

25 ANGLE BETWEEN GEOMETRIC ELEMENTS

Have the calculation for the angle between bivectors done elsewhere

$$\cos\theta = -\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \tag{25.1}$$

For $\theta \in [0, \pi]$.

The vector/vector result is well known and also works fine in $\mathbb{R}^N$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \tag{25.2}$$

25.1 CALCULATION FOR A LINE AND A PLANE

Given a line with unit direction vector $\mathbf{u}$, and plane with unit direction bivector $\mathbf{A}$, the component of that vector in the plane is:

$$-\mathbf{u}\cdot\mathbf{A}\mathbf{A}. \tag{25.3}$$

So the direction cosine is available immediately

$$\cos\theta = \mathbf{u}\cdot\frac{-\mathbf{u}\cdot\mathbf{A}\mathbf{A}}{|\mathbf{u}\cdot\mathbf{A}\mathbf{A}|} \tag{25.4}$$

However, this can be reduced significantly. Start with the denominator

$$\begin{aligned}|\mathbf{u}\cdot\mathbf{A}\mathbf{A}|^2 &= (\mathbf{u}\cdot\mathbf{A}\mathbf{A})(\mathbf{A}\mathbf{A}\cdot\mathbf{u}) \\ &= (\mathbf{u}\cdot\mathbf{A})^2.\end{aligned} \tag{25.5}$$

And in the numerator we have:

$$
\begin{aligned}
\mathbf{u} \cdot (\mathbf{u} \cdot \mathbf{A}\mathbf{A}) &= \frac{1}{2}(\mathbf{u}(\mathbf{u} \cdot \mathbf{A}\mathbf{A}) + (\mathbf{u} \cdot \mathbf{A}\mathbf{A})\mathbf{u}) \\
&= \frac{1}{2}((\mathbf{u}\mathbf{u} \cdot \mathbf{A})\mathbf{A} + (\mathbf{u} \cdot \mathbf{A})\mathbf{A}\mathbf{u}) \\
&= \frac{1}{2}((\mathbf{A} \cdot \mathbf{u}\mathbf{u})\mathbf{A} - (\mathbf{A} \cdot \mathbf{u})\mathbf{A}\mathbf{u}) \\
&= (\mathbf{A} \cdot \mathbf{u})\frac{1}{2}(\mathbf{u}\mathbf{A} - \mathbf{A}\mathbf{u}) \\
&= -(\mathbf{A} \cdot \mathbf{u})^2.
\end{aligned}
\tag{25.6}
$$

Putting things back together

$$
\cos\theta = \frac{(\mathbf{A} \cdot \mathbf{u})^2}{|\mathbf{u} \cdot \mathbf{A}|} = |\mathbf{u} \cdot \mathbf{A}|
$$

The strictly positive value here is consistent with the fact that theta as calculated is in the $[0, \pi/2]$ range.

Restated for consistency with equations eq. (25.2) and eq. (25.1) in terms of not necessarily unit vector and bivectors $\mathbf{u}$ and $\mathbf{A}$, we have

$$
\cos\theta = \frac{|\mathbf{u} \cdot \mathbf{A}|}{|\mathbf{u}||\mathbf{A}|}
\tag{25.7}
$$

26 ORTHOGONAL DECOMPOSITION TAKE II

26.1 LEMMA. ORTHOGONAL DECOMPOSITION

To do so we first need to be able to express a vector $\mathbf{x}$ in terms of components parallel and perpendicular to the blade $\mathbf{A} \in \wedge^k$.

$$
\begin{aligned}
\mathbf{x} &= \mathbf{x}\mathbf{A}\frac{1}{\mathbf{A}} \\
&= (\mathbf{x}\cdot\mathbf{A} + \mathbf{x}\wedge\mathbf{A})\frac{1}{\mathbf{A}} \\
&= (\mathbf{x}\cdot\mathbf{A})\cdot\frac{1}{\mathbf{A}} + \sum_{i=3,5,\cdots,2k-1}\left\langle(\mathbf{x}\cdot\mathbf{A})\frac{1}{\mathbf{A}}\right\rangle_i \\
&\quad + (\mathbf{x}\wedge\mathbf{A})\cdot\frac{1}{\mathbf{A}} + \sum_{i=3,5,\cdots,2k-1}\left\langle(\mathbf{x}\wedge\mathbf{A})\frac{1}{\mathbf{A}}\right\rangle_i + \overbrace{\boxed{(\mathbf{x}\wedge\mathbf{A})\wedge\frac{1}{\mathbf{A}}}}^{=0}
\end{aligned}
\tag{26.1}
$$

Since the LHS and RHS must both be vectors all the non-grade one terms are either zero or cancel out. This can be observed directly since:

$$
\begin{aligned}
\left\langle\mathbf{x}\cdot\mathbf{A}\frac{1}{\mathbf{A}}\right\rangle_i &= \left\langle\frac{\mathbf{x}\mathbf{A} - (-1)^k\mathbf{A}\mathbf{x}}{2}\frac{1}{\mathbf{A}}\right\rangle_i \\
&= -\frac{(-1)^k}{2}\left\langle\mathbf{A}\mathbf{x}\frac{1}{\mathbf{A}}\right\rangle_i
\end{aligned}
\tag{26.2}
$$

and

$$
\begin{aligned}
\left\langle\mathbf{x}\wedge\mathbf{A}\frac{1}{\mathbf{A}}\right\rangle_i &= \left\langle\frac{\mathbf{x}\mathbf{A} + (-1)^k\mathbf{A}\mathbf{x}}{2}\frac{1}{\mathbf{A}}\right\rangle_i \\
&= +\frac{(-1)^k}{2}\left\langle\mathbf{A}\mathbf{x}\frac{1}{\mathbf{A}}\right\rangle_i
\end{aligned}
\tag{26.3}
$$

Thus all of the grade $3, \cdots, 2k-1$ terms cancel each other out. Some terms like $(\mathbf{x}\cdot\mathbf{A})\wedge\frac{1}{\mathbf{A}}$ are also independently zero.

(This is a result I have got in other places, but I thought it is worth writing down since I thought the direct cancellation is elegant).

27

MATRIX OF GRADE K MULTIVECTOR LINEAR TRANSFORMATIONS

27.1 MOTIVATION

The following shows explicitly the calculation required to form the matrix of a linear transformation between two grade k multivector subspaces (is there a name for a multivector of fixed grade that is not neccessarily a blade?). This is nothing fancy or original, but just helpful to have written out showing naturally how one generates this matrix from a consideration of the two sets of basis vectors. After so much exposure to linear transformations only in matrix form it is good to write this out in a way so that it is clear exactly how the coordinate matrices come in to the picture when and if they are introduced.

27.2 CONTENT

Given T, a linear transformation between two grade k multivector subspaces, let $\sigma = \{\sigma_i\}_{i=1}^m$ be a basis for a grade k multivector subspace. For $T(x) \in span\{\beta_i\}$ (ie: image of T contained in this span). Let $\beta = \{\beta_i\}_{i=1}^n$ be a basis for this (possibly different) grade k multivector subspace.

Additionally introduce a set of respective reciprocal frames $\{\sigma^i\}$, and $\{\beta^i\}$. Define the reciprocal frame with respect to the dot product for the space. For a linearly independent, but not necessary orthogonal (or orthonormal), set of vectors $\{u_i\}$ this set has as its defining property:

$$u_i \cdot u^j = \delta_{ij} \tag{27.1}$$

I have chosen to use this covariant, contravariant coordinate notation since that works well for both vectors (not necessarily orthogonal or orthonormal), as well as higher grade vectors. When the basis is orthonormal these reciprocal frame grade k multivectors can be computed with just reversion. For example, suppose that $\{\beta_i\}$ is an orthonormal bivector basis for the image of T, then the reciprocal frame bivectors are just $\beta^i = {\beta_i}^\dagger$.

With this we can decompose the linear transformation into components generated by each of the σ_i grade k multivectors:

$$T(x) = T(\sum x \cdot \sigma_j \sigma^j) = \sum x_j T(\sigma^j) \tag{27.2}$$

This we can write as a matrix equation:

$$T(x) = \begin{bmatrix} T(\sigma^1) & T(\sigma^1) & \cdots & T(\sigma^n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{27.3}$$

Now, decompose the $T(\sigma^j)$ in terms of the basis β:

$$T(\sigma^j) = \sum T(\sigma^j) \cdot \beta^i \beta_i \tag{27.4}$$

This we can also write as a matrix equation

$$\begin{aligned} T(\sigma^j) &= \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix} \begin{bmatrix} T(\sigma^j) \cdot \beta^1 \\ T(\sigma^j) \cdot \beta^2 \\ \vdots \\ T(\sigma^j) \cdot \beta^m \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix} \begin{bmatrix} \beta^1 \cdot T(\sigma^j) \\ \beta^2 \cdot T(\sigma^j) \\ \vdots \\ \beta^m \cdot T(\sigma^j) \end{bmatrix} \end{aligned} \tag{27.5}$$

These two sets of matrix equations, can be combined into a single equation:

$$T(x) = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \beta^1 \cdot T(\sigma^1) & \beta^1 \cdot T(\sigma^2) & \cdots & \beta^1 \cdot T(\sigma^n) \\ \beta^2 \cdot T(\sigma^1) & \beta^2 \cdot T(\sigma^2) & \cdots & \beta^2 \cdot T(\sigma^n) \\ \vdots & \cdots & \ddots & \vdots \\ \beta^m \cdot T(\sigma^1) & \beta^m \cdot T(\sigma^2) & \cdots & \beta^m \cdot T(\sigma^n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{27.6}$$

Here the matrix $(x_1, x_2, \cdots, x_n)$ is a coordinate vector with respect to basis σ, but the vector $(\beta_1, \beta_2, \cdots, \beta_n)$ is matrix of the basis vectors $\beta_i \in \beta$. This makes sense since the end result has not been defined in terms of a coordinate vector space, but the space of T itself.

This can all be written more compactly as

$$T(x) = \begin{bmatrix} \beta_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \beta^i \cdot T(\sigma^j) \end{bmatrix} \begin{bmatrix} x_i \end{bmatrix} \tag{27.7}$$

We can also recover the original result from this by direct expansion and then regrouping:

$$
\begin{aligned}
\begin{bmatrix}\beta_i\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\sum_j \beta^i \cdot T(\sigma^j) x_j\end{bmatrix} &= \begin{bmatrix}\sum_{kj} \beta_k \beta^k \cdot T(\sigma^j) x_j\end{bmatrix} \\
&= \sum_{kj} \beta_k \beta^k \cdot T(\sigma^j) x_j \\
&= \sum_k \beta_k \beta^k \cdot T(\sum_j \sigma^j x_j) \\
&= \sum_k \beta_k \beta^k \cdot T(x) \\
&= T(x)
\end{aligned}
\tag{27.8}
$$

Observe that this demonstrates that we can write the coordinate vector $[T]_\beta$ as the two left most matrices above

$$
\begin{aligned}
[T(x)]_\beta &= \begin{bmatrix}\beta^i \cdot T(x)\end{bmatrix}_i \\
&= \begin{bmatrix}\sum_j \beta^i \cdot T(\sigma^j) x_j\end{bmatrix}_i \\
&= \begin{bmatrix}\beta^i \cdot T(\sigma^j)\end{bmatrix} \begin{bmatrix}x_i\end{bmatrix}
\end{aligned}
\tag{27.9}
$$

Looking at the above I found it interesting that eq. (27.6) which embeds the coordinate vector of $T(x)$ has the structure of a bilinear form, so in a sense one can view the matrix of a linear transformation:

$$
[T]_\sigma^\beta = \begin{bmatrix}\beta^i \cdot T(\sigma^j)\end{bmatrix}
\tag{27.10}
$$

as a bilinear form that can act as a mapping from the generating basis to the image basis.

28 VECTOR FORM OF JULIA FRACTAL

28.1 MOTIVATION

As outlined in [11], 2-D and N-D Julia fractals can be computed using the geometric product, instead of complex numbers. Explore a couple of details related to that here.

28.2 GUTS

Fractal patterns like the Mandelbrot and Julia sets are typically using iterative computations in the complex plane. For the Julia set, our iteration has the form

$$Z \rightarrow Z^p + C \tag{28.1}$$

where p is an integer constant, and Z, and C are complex numbers. For $p = 2$ I believe we obtain the Mandelbrot set. Given the isomorphism between complex numbers and vectors using the geometric product, we can use write

$$\begin{aligned} Z &= \mathbf{x}\hat{\mathbf{n}} \\ C &= \mathbf{c}\hat{\mathbf{n}}, \end{aligned} \tag{28.2}$$

and re-express the Julia iterator as

$$\mathbf{x} \rightarrow (\mathbf{x}\hat{\mathbf{n}})^p \hat{\mathbf{n}} + \mathbf{c} \tag{28.3}$$

It is not obvious that the RHS of this equation is a vector and not a multivector, especially when the vector $\mathbf{x}$ lies in $\mathbb{R}^3$ or higher dimensional space. To get a feel for this, let us start by write this out in components for $\hat{\mathbf{n}} = \mathbf{e}_1$ and $p = 2$. We obtain for the product term

$$\begin{aligned} (\mathbf{x}\hat{\mathbf{n}})^p \hat{\mathbf{n}} &= \mathbf{x}\hat{\mathbf{n}}\mathbf{x}\hat{\mathbf{n}}\hat{\mathbf{n}} \\ &= \mathbf{x}\hat{\mathbf{n}}\mathbf{x} \\ &= (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2)\mathbf{e}_1(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) \\ &= (x_1 + x_2 \mathbf{e}_2 \mathbf{e}_1)(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) \\ &= (x_1^2 - x_2^2)\mathbf{e}_1 + 2 x_1 x_2 \mathbf{e}_2 \end{aligned} \tag{28.4}$$

Looking at the same square in coordinate representation for the $\mathbb{R}^n$ case (using summation notation unless otherwise specified), we have

$$
\begin{aligned}
\mathbf{x}\hat{\mathbf{n}}\mathbf{x} &= x_k \mathbf{e}_k \mathbf{e}_1 x_m \mathbf{e}_m \\
&= \left(x_1 + \sum_{k>1} x_k \mathbf{e}_k \mathbf{e}_1 \right) x_m \mathbf{e}_m \\
&= x_1 x_m \mathbf{e}_m + \sum_{k>1} x_k x_m \mathbf{e}_k \mathbf{e}_1 \mathbf{e}_m \\
&= x_1 x_m \mathbf{e}_m + \sum_{k>1} x_k x_1 \mathbf{e}_k + \sum_{k>1,m>1} x_k x_m \mathbf{e}_k \mathbf{e}_1 \mathbf{e}_m \\
&= \left(x_1^2 - \sum_{k>1} x_k^2 \right) \mathbf{e}_1 + 2 \sum_{k>1} x_1 x_k \mathbf{e}_k + \sum_{1<k<m,1<m<k} x_k x_m \mathbf{e}_k \mathbf{e}_1 \mathbf{e}_m
\end{aligned}
\tag{28.5}
$$

This last term is zero since $\mathbf{e}_k \mathbf{e}_1 \mathbf{e}_m = -\mathbf{e}_m \mathbf{e}_1 \mathbf{e}_k$, and we are left with

$$
\mathbf{x}\hat{\mathbf{n}}\mathbf{x} = \left(x_1^2 - \sum_{k>1} x_k^2 \right) \mathbf{e}_1 + 2 \sum_{k>1} x_1 x_k \mathbf{e}_k,
\tag{28.6}
$$

a vector, even for non-planar vectors. How about for an arbitrary orientation of the unit vector in $\mathbb{R}^n$? For that we get

$$
\begin{aligned}
\mathbf{x}\hat{\mathbf{n}}\mathbf{x} &= (\mathbf{x} \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} + \mathbf{x} \wedge \hat{\mathbf{n}}\hat{\mathbf{n}})\hat{\mathbf{n}}\mathbf{x} \\
&= (\mathbf{x} \cdot \hat{\mathbf{n}} + \mathbf{x} \wedge \hat{\mathbf{n}})(\mathbf{x} \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} + \mathbf{x} \wedge \hat{\mathbf{n}}\hat{\mathbf{n}}) \\
&= ((\mathbf{x} \cdot \hat{\mathbf{n}})^2 + (\mathbf{x} \wedge \hat{\mathbf{n}})^2)\hat{\mathbf{n}} + 2(\mathbf{x} \cdot \hat{\mathbf{n}})(\mathbf{x} \wedge \hat{\mathbf{n}})\hat{\mathbf{n}}
\end{aligned}
\tag{28.7}
$$

We can read eq. (28.6) off of this result by inspection for the $\hat{\mathbf{n}} = \mathbf{e}_1$ case.

It is now straightforward to show that the product $(\mathbf{x}\hat{\mathbf{n}})^p \hat{\mathbf{n}}$ is a vector for integer $p \geq 2$. We have covered the $p = 2$ case, justifying an assumption that this product has the following form

$$
(\mathbf{x}\hat{\mathbf{n}})^{p-1} \hat{\mathbf{n}} = a\hat{\mathbf{n}} + b(\mathbf{x} \wedge \hat{\mathbf{n}})\hat{\mathbf{n}},
\tag{28.8}
$$

for scalars a and b. The induction test becomes

$$
\begin{aligned}
(\mathbf{x}\hat{\mathbf{n}})^p \hat{\mathbf{n}} &= (\mathbf{x}\hat{\mathbf{n}})^{p-1} (\mathbf{x}\hat{\mathbf{n}})\hat{\mathbf{n}} \\
&= (\mathbf{x}\hat{\mathbf{n}})^{p-1} \mathbf{x} \\
&= (a + b(\mathbf{x} \wedge \hat{\mathbf{n}}))((\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x} \wedge \hat{\mathbf{n}})\hat{\mathbf{n}}) \\
&= (a(\mathbf{x} \cdot \hat{\mathbf{n}})^2 - b(\mathbf{x} \wedge \hat{\mathbf{n}})^2)\hat{\mathbf{n}} + (a + b(\mathbf{x} \cdot \hat{\mathbf{n}}))(\mathbf{x} \wedge \hat{\mathbf{n}})\hat{\mathbf{n}}.
\end{aligned}
\tag{28.9}
$$

Again we have a vector split nicely into projective and rejective components, so for any integer power of p our iterator eq. (28.3) employing the geometric product is a mapping from vectors to vectors.

There is a striking image in the text of such a Julia set for such a 3D iterator, and an exercise left for the adventurous reader to attempt to code that based on the 2D $p = 2$ sample code they provide.

Part III

ROTATION

29

ROTOR NOTES

29.1 ROTATIONS STRICTLY IN A PLANE

For a plane rotation, a rotation does not have to be expressed in terms of left and right half angle rotations, as is the case with complex numbers. Starting with this "natural" one sided rotation we will see why the half angle double sided Rotor formula works.

29.1.1 *Identifying a plane with a bivector. Justification*

Given a bivector $\mathbf{B}$, we can say this defines the orientation of a plane (through the origin) since for any vector in the plane we have $\mathbf{B} \wedge \mathbf{x} = 0$, or any vector strictly normal to the plane $\mathbf{B} \cdot \mathbf{x} = 0$.

Note that this naturally compares to the equation of a line (through the origin) expressed in terms of a direction vector $\mathbf{b}$, where $\mathbf{b} \wedge \mathbf{x} = 0$ if $\mathbf{x}$ lies on the line, and $\mathbf{b} \cdot \mathbf{x} = 0$ if $\mathbf{x}$ is normal to the line.

Given this it is not unreasonable to identify the plane with its bivector. This will be done below, and it should be clear that loose language such as "the plane $\mathbf{B}$", should really be interpreted as "the plane with direction bivector $\mathbf{B}$", where the direction bivector has the wedge and dot product properties noted above.

29.1.2 *Components of a vector in and out of a plane*

To calculate the components of a vector in and out of a plane, we can form the product

$$\mathbf{x} = \mathbf{x}\mathbf{B}\frac{1}{\mathbf{B}} = \mathbf{x} \cdot \mathbf{B}\frac{1}{\mathbf{B}} + \mathbf{x} \wedge \mathbf{B}\frac{1}{\mathbf{B}} \tag{29.1}$$

This is an orthogonal decomposition of the vector $\mathbf{x}$ where the first part is the projective term onto the plane $\mathbf{B}$, and the second is the rejective term, the component not in the plane. Let us verify this.

Write $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$, where $\mathbf{x}_\parallel$, and $\mathbf{x}_\perp$ are the components of $\mathbf{x}$ parallel and perpendicular to the plane. Also write $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2$, where $\mathbf{b}_i$ are non-colinear vectors in the plane $\mathbf{B}$.

If $\mathbf{x} = \mathbf{x}_\parallel$, a vector entirely in the plane $\mathbf{B}$, then one can write

$$\mathbf{x} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 \tag{29.2}$$

and the wedge product term is zero

$$\begin{aligned}\mathbf{x} \wedge \mathbf{B} &= \left(a_1 \mathbf{b}_1 + a_2 \mathbf{b}_2\right) \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \\ &= a_1 (\mathbf{b}_1 \wedge \mathbf{b}_1) \wedge \mathbf{b}_2 - a_2 (\mathbf{b}_2 \wedge \mathbf{b}_2) \wedge \mathbf{b}_1 \\ &= 0\end{aligned} \tag{29.3}$$

Thus the component parallel to the plane is composed strictly of the dot product term

$$\mathbf{x}_\parallel = \mathbf{x} \cdot \mathbf{B} \frac{1}{\mathbf{B}} \tag{29.4}$$

Or for a general vector not necessarily in the plane the component of that vector in the plane, its projection onto the plane is,

$$\mathrm{Proj}_{\mathbf{B}}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{B} \frac{1}{\mathbf{B}} = \frac{1}{|\mathbf{B}|^2} (\mathbf{B} \cdot \mathbf{x}) \mathbf{B} = (\hat{\mathbf{B}} \cdot \mathbf{x}) \hat{\mathbf{B}} \tag{29.5}$$

Now, for a vector that lies completely perpendicular to the plane $\mathbf{x} = \mathbf{x}_\perp$, the dot product term with the plane is zero. To verify this observe

$$\begin{aligned}\mathbf{x}_\perp \cdot \mathbf{B} &= \mathbf{x}_\perp \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2) \\ &= (\mathbf{x}_\perp \cdot \mathbf{b}_1) \mathbf{b}_2 - (\mathbf{x}_\perp \cdot \mathbf{b}_2) \mathbf{b}_1\end{aligned} \tag{29.6}$$

Each of these dot products are zero since $\mathbf{x}$ has no components that lie in the plane (those components if they existed could be expressed as linear combinations of $\mathbf{b}_i$).

Thus only the component perpendicular to the plane is composed strictly of the wedge product term

$$\mathbf{x}_\perp = \mathbf{x} \wedge \mathbf{B} \frac{1}{\mathbf{B}} \tag{29.7}$$

And again for a general vector the component that lies out of the plane as, the rejection of the plane from the vector is

$$\mathrm{Rej}_{\mathbf{B}}(\mathbf{x}) = \mathbf{x} \wedge \mathbf{B} \frac{1}{\mathbf{B}} = -\frac{1}{|\mathbf{B}|^2} \mathbf{x} \wedge \mathbf{B} \mathbf{B} = -\mathbf{x} \wedge \hat{\mathbf{B}} \hat{\mathbf{B}} \tag{29.8}$$

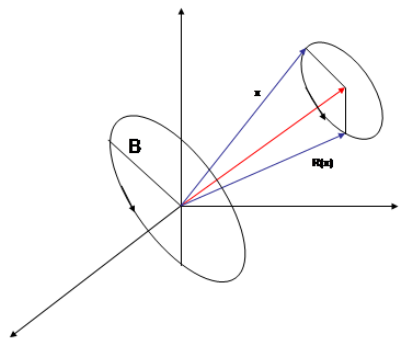

Figure 29.1: Rotation of Vector

29.2 ROTATION AROUND NORMAL TO ARBITRARILY ORIENTED PLANE THROUGH ORIGIN

Having established the preliminaries, we can now express a rotation around the normal to a plane (with the plane and that normal through the origin).

Such a rotation is illustrated in fig. 29.1 preserves all components of the vector that are perpendicular to the plane, and operates only on the components parallel to the plane.

Expressed in terms of exponentials and the projective and rejective decompositions above, this is

$$\begin{aligned} R_\theta(\mathbf{x}) &= \mathbf{x} \wedge \mathbf{B}\frac{1}{\mathbf{B}} + \left(\mathbf{x} \cdot \mathbf{B}\frac{1}{\mathbf{B}}\right) e^{\hat{\mathbf{B}}\theta} \\ &= \mathbf{x} \wedge \mathbf{B}\frac{1}{\mathbf{B}} + e^{-\hat{\mathbf{B}}\theta}\left(\mathbf{x} \cdot \mathbf{B}\frac{1}{\mathbf{B}}\right) \end{aligned} \tag{29.9}$$

Where we have made explicit note that a plane rotation does not commute with a vector in a plane (its reverse is required).

To demonstrate this write $i = \mathbf{e}_2\mathbf{e}_1$, a unit bivector in some plane with unit vectors $\mathbf{e}_i$ also in the plane. If a vector lies in that plane we can write the rotation

$$\begin{aligned}
\mathbf{x}e^{i\theta} &= \left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2\right)\left(\cos\theta + i\sin\theta\right) \\
&= \cos\theta\left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2\right) + \left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2\right)\left(\mathbf{e}_2\mathbf{e}_1\sin\theta\right) \\
&= \cos\theta\left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2\right) + \sin\theta\left(-a_1\mathbf{e}_2 + a_2\mathbf{e}_1\right) \\
&= \cos\theta\left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2\right) - \mathbf{e}_2\mathbf{e}_1\sin\theta\left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2\right) \\
&= e^{-i\theta}\mathbf{x}
\end{aligned} \tag{29.10}$$

Similarly for a vector that lies outside of the plane we can write

$$\begin{aligned}
\mathbf{x}e^{i\theta} &= (\sum_{j\neq 1,2} a_j\mathbf{e}_j)(\cos\theta + \mathbf{e}_2\mathbf{e}_1\sin\theta) \\
&= (\cos\theta + \mathbf{e}_2\mathbf{e}_1\sin\theta)(\sum_{j\neq 1,2} a_j\mathbf{e}_j) \\
&= e^{i\theta}\mathbf{x}
\end{aligned} \tag{29.11}$$

The multivector for a rotation in a plane perpendicular to a vector commutes with that vector. The properties of the exponential allow us to factor a rotation

$$R(\theta) = R(\alpha\theta)R((1-\alpha)\theta) \tag{29.12}$$

where $\alpha <= 1$, and in particular we can set $\alpha = 1/2$, and write

$$\begin{aligned}
R_\theta(\mathbf{x}) &= \mathbf{x}\wedge\mathbf{B}\frac{1}{\mathbf{B}} + \left(\mathbf{x}\cdot\mathbf{B}\frac{1}{\mathbf{B}}\right)e^{\hat{\mathbf{B}}\theta} \\
&= \left(\mathbf{x}\wedge\mathbf{B}\frac{1}{\mathbf{B}}\right)e^{-\hat{\mathbf{B}}\theta/2}e^{\hat{\mathbf{B}}\theta/2} + \left(\mathbf{x}\cdot\mathbf{B}\frac{1}{\mathbf{B}}\right)e^{\hat{\mathbf{B}}\theta/2}e^{\hat{\mathbf{B}}\theta/2} \\
&= e^{-\hat{\mathbf{B}}\theta/2}\left(\mathbf{x}\wedge\mathbf{B}\frac{1}{\mathbf{B}}\right)e^{\hat{\mathbf{B}}\theta/2} + e^{-\hat{\mathbf{B}}\theta/2}\left(\mathbf{x}\cdot\mathbf{B}\frac{1}{\mathbf{B}}\right)e^{\hat{\mathbf{B}}\theta/2} \\
&= e^{-\hat{\mathbf{B}}\theta/2}\left(\mathbf{x}\wedge\mathbf{B} + \mathbf{x}\cdot\mathbf{B}\right)\frac{1}{\mathbf{B}}e^{\hat{\mathbf{B}}\theta/2} \\
&= e^{-\hat{\mathbf{B}}\theta/2}\left(\mathbf{x}\mathbf{B}\frac{1}{\mathbf{B}}\right)e^{\hat{\mathbf{B}}\theta/2}
\end{aligned} \tag{29.13}$$

This takes us full circle from dot and wedge products back to $\mathbf{x}$, and allows us to express the rotated vector as:

$$R_\theta(\mathbf{x}) = e^{-\hat{\mathbf{B}}\theta/2}\mathbf{x}e^{\hat{\mathbf{B}}\theta/2} \tag{29.14}$$

Only when the vector lies in the plane ($\mathbf{x} = \mathbf{x}_\parallel$, or $\mathbf{x} \wedge \mathbf{B} = 0$) can be written using the familiar left or right "full angle" rotation exponential that we are used to from complex arithmetic:

$$R_\theta(\mathbf{x}) = e^{-\hat{\mathbf{B}}\theta}\mathbf{x} = \mathbf{x}e^{\hat{\mathbf{B}}\theta} \tag{29.15}$$

29.3 ROTOR EQUATION IN TERMS OF NORMAL TO PLANE

The rotor equation above is valid for any number of dimensions. For $\mathbb{R}^3$ we can alternatively parametrize the plane in terms of a unit normal $\mathbf{n}$:

$$\mathbf{B} = ki\mathbf{n} \tag{29.16}$$

Here i is the $\mathbb{R}^3$ pseudoscalar $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$.
Thus we can write

$$\hat{\mathbf{B}} = i\mathbf{n} \tag{29.17}$$

and expressing eq. (29.14) in terms of the unit normal becomes trivial

$$R_\theta(\mathbf{x}) = e^{-i\mathbf{n}\theta/2}\mathbf{x}e^{i\mathbf{n}\theta/2} \tag{29.18}$$

Expressing this in terms of components and the unit normal is a bit harder

$$\begin{aligned} R_\theta(\mathbf{x}) &= \mathbf{x} \wedge \mathbf{B}\frac{1}{\mathbf{B}} + \left(\mathbf{x} \cdot \mathbf{B}\frac{1}{\mathbf{B}}\right)e^{\hat{\mathbf{B}}\theta} \\ &= \mathbf{x} \wedge (i\mathbf{n})\frac{1}{i\mathbf{n}} + \left(\mathbf{x} \cdot (i\mathbf{n})\frac{1}{i\mathbf{n}}\right)e^{i\mathbf{n}\theta} \end{aligned} \tag{29.19}$$

Now,

$$\begin{aligned} \mathbf{x} \wedge (i\mathbf{n}) &= \frac{1}{2}(\mathbf{x}i\mathbf{n} + i\mathbf{n}\mathbf{x}) \\ &= \frac{i}{2}(\mathbf{x}\mathbf{n} + \mathbf{n}\mathbf{x}) \\ &= (\mathbf{x} \cdot \mathbf{n})i \end{aligned} \tag{29.20}$$

And

$$\begin{aligned} \frac{1}{i\mathbf{n}} &= \frac{1}{i\mathbf{n}}\frac{1}{\mathbf{n}i}\mathbf{n}i \\ &= -i\mathbf{n} \end{aligned} \tag{29.21}$$

So the rejective term becomes

$$\begin{aligned} \mathbf{x} \wedge \mathbf{B}\frac{1}{\mathbf{B}} &= \mathbf{x} \wedge (i\mathbf{n})\frac{1}{i\mathbf{n}} \\ &= \mathbf{x} \wedge (i\mathbf{n})\frac{1}{i\mathbf{n}} \\ &= (\mathbf{x} \cdot \mathbf{n})i(-i)\mathbf{n} \\ &= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} \\ &= \mathrm{Proj}_{\mathbf{n}}(\mathbf{x}) \end{aligned} \tag{29.22}$$

Now, for the dot product with the plane term, we have

$$\begin{aligned} \mathbf{x} \cdot \mathbf{B} &= \mathbf{x} \cdot (i\mathbf{n}) \\ &= \frac{1}{2}(\mathbf{x}i\mathbf{n} - i\mathbf{n}\mathbf{x}) \\ &= (\mathbf{x} \wedge \mathbf{n})i \end{aligned} \tag{29.23}$$

Putting it all together we have

$$R_\theta(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + (\mathbf{x} \wedge \mathbf{n})\mathbf{n}e^{i\mathbf{n}\theta} \tag{29.24}$$

In terms of explicit sine and cosine terms this is (observe that $(i\mathbf{n})^2 = -1$),

$$R_\theta(\mathbf{x}) = \left(\mathbf{x} \cdot \mathbf{n}\right)\mathbf{n} + \left(\mathbf{x} \wedge \mathbf{n}\right)\mathbf{n}\left(\cos\theta + i\mathbf{n}\sin\theta\right) \tag{29.25}$$

$$R_\theta(\mathbf{x}) = \left(\mathbf{x} \cdot \mathbf{n}\right)\mathbf{n} + \left(\mathbf{x} \wedge \mathbf{n}\right)\mathbf{n}\cos\theta + (\mathbf{x} \wedge \mathbf{n})i\sin\theta \tag{29.26}$$

This triplet of mutually orthogonal direction vectors, $\mathbf{n}$, $(\mathbf{x} \wedge \mathbf{n})\mathbf{n}$, and $(\mathbf{x} \wedge \mathbf{n})i$ are illustrated in fig. 29.2. The component of the vector in the direction of the normal $\mathrm{Proj}_{\mathbf{n}}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{n}\mathbf{n}$ is unaltered by the rotation. The rotation is applied to the remaining component of $\mathbf{x}$, $\mathrm{Rej}_{\mathbf{n}}(\mathbf{x}) = (\mathbf{x} \wedge \mathbf{n})\mathbf{n}$, and we rotate in the direction $(\mathbf{x} \wedge \mathbf{n})i$

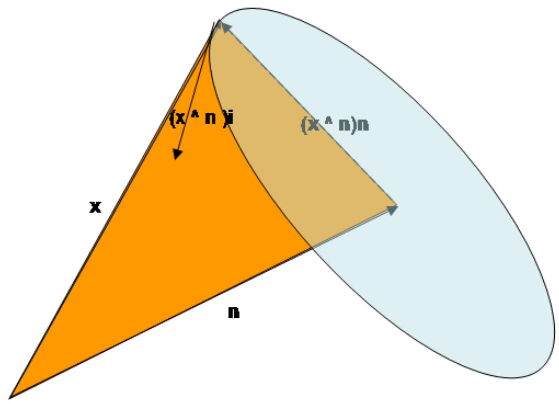

Figure 29.2: Direction vectors associated with rotation

29.3.1 *Vector rotation in terms of dot and cross products only*

Expression of this rotation formula eq. (29.26) in terms of "vector" relations is also possible, by removing the wedge products and the pseudoscalar references.

First the rejective term

$$
\begin{aligned}
(\mathbf{x} \wedge \mathbf{n})\mathbf{n} &= ((\mathbf{x} \times \mathbf{n})i)\mathbf{n} \\
&= ((\mathbf{x} \times \mathbf{n})i) \cdot \mathbf{n} \\
&= \frac{1}{2}(((\mathbf{x} \times \mathbf{n})i)\mathbf{n} - \mathbf{n}((\mathbf{x} \times \mathbf{n})i)) \\
&= \frac{i}{2}((\mathbf{x} \times \mathbf{n})\mathbf{n} - \mathbf{n}(\mathbf{x} \times \mathbf{n})) \\
&= i((\mathbf{x} \times \mathbf{n}) \wedge \mathbf{n}) \\
&= i^2((\mathbf{x} \times \mathbf{n}) \times \mathbf{n}) \\
&= \mathbf{n} \times (\mathbf{x} \times \mathbf{n})
\end{aligned}
\tag{29.27}
$$

The next term expressed in terms of the cross product is

$$
\begin{aligned}
(\mathbf{x} \wedge \mathbf{n})i &= (\mathbf{x} \times \mathbf{n})i^2 \\
&= \mathbf{n} \times \mathbf{x}
\end{aligned}
\tag{29.28}
$$

And putting it all together we have

$$R_\theta(\mathbf{x}) = \left(\mathbf{x} \cdot \mathbf{n}\right) \mathbf{n} + \left(\mathbf{n} \times \mathbf{x}\right) \times \mathbf{n} \cos\theta + \mathbf{n} \times \mathbf{x} \sin\theta \tag{29.29}$$

Compare eq. (29.29) to eq. (29.26) and eq. (29.24), and then back to eq. (29.14).

29.4 GIVING A MEANING TO THE SIGN OF THE BIVECTOR

For a rotation between two vectors in the plane containing those vectors, we can write the rotation in terms of the exponential as either a left or right rotation operator:

$$\mathbf{b} = \mathbf{a} e^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta} \mathbf{a} \tag{29.30}$$

$$\mathbf{b} = e^{\mathbf{j}\theta} \mathbf{a} = \mathbf{a} e^{-\mathbf{j}\theta/2} \tag{29.31}$$

Here both $\mathbf{i}$ and $\mathbf{j} = -\mathbf{i}$ are unit bivectors with the property $\mathbf{i}^2 = \mathbf{j}^2 = -1$. Thus in order to write a rotation in exponential form a meaning must be assigned to the sign of the unit bivector that describes the plane and the orientation of the rotation.

Consider for example the case of a rotation by $\pi/2$. For this is the exponential is:

$$e^{\mathbf{i}\pi/2} = \cos(\pi/2) + \mathbf{i}\sin(\pi/2) = \mathbf{i} \tag{29.32}$$

Thus for perpendicular unit vectors $\mathbf{u}$ and $\mathbf{v}$, if we wish $\mathbf{i}$ to act as a $\pi/2$ rotation left acting operator on $\mathbf{u}$ towards $\mathbf{v}$ its value must be:

$$\mathbf{i} = \mathbf{u} \wedge \mathbf{v} \tag{29.33}$$

$$\mathbf{u}\mathbf{i} = \mathbf{u}\mathbf{u} \wedge \mathbf{v} = \mathbf{u}\mathbf{u}\mathbf{v} = \mathbf{v} \tag{29.34}$$

For that same rotation if the bivector is employed as a right acting operator, the reverse is required:

$$\mathbf{j} = \mathbf{v} \wedge \mathbf{u} \tag{29.35}$$

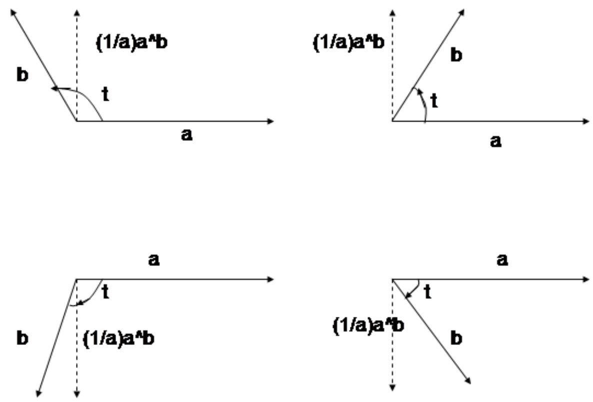

Figure 29.3: Orientation of unit imaginary

$$\mathbf{j}\mathbf{u} = \mathbf{v} \wedge \mathbf{u} = \mathbf{v}\mathbf{u}\mathbf{u} = \mathbf{v} \tag{29.36}$$

In general, for any two vectors, one can find an angle θ in the range $0 \le \theta \le \pi$ between those vectors. If one lets that angle define the orientation of the rotation between the vectors, and implicitly define a sort of "imaginary axis" for that plane, that imaginary axis will have direction

$$\frac{1}{\mathbf{a}}\mathbf{a} \wedge \mathbf{b} = \mathbf{b} \wedge \mathbf{a}\frac{1}{\mathbf{a}}. \tag{29.37}$$

This is illustrated in fig. 29.3.
Thus the bivector

$$\mathbf{i} = \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|} \tag{29.38}$$

When acting as an operator to the left ($\mathbf{a}\mathbf{i}$) with a vector in the plane can be interpreted as acting as a rotation by $\pi/2$ towards $\mathbf{b}$.

Similarly the bivector

$$\mathbf{j} = \mathbf{i}^\dagger = -\mathbf{i} = \frac{\mathbf{b} \wedge \mathbf{a}}{|\mathbf{b} \wedge \mathbf{a}|} \tag{29.39}$$

also applied to a vector in the plane produces the same rotation when acting as an operator to the right. Thus, in general we can write a rotation by theta in the plane containing non-colinear vectors $\mathbf{a}$ and $\mathbf{b}$ in the direction of minimal angle from $\mathbf{a}$ towards $\mathbf{b}$ in one of the three forms:

$$R_{\theta:\mathbf{a}\to\mathbf{b}}(\mathbf{a}) = \mathbf{a}e^{\frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}\wedge\mathbf{b}|}\theta} = e^{\frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{b}\wedge\mathbf{a}|}\theta}\mathbf{a} \tag{29.40}$$

Or,

$$R_{\theta:\mathbf{a}\to\mathbf{b}}(\mathbf{x}) = e^{\frac{\mathbf{b}\wedge\mathbf{a}}{|\mathbf{b}\wedge\mathbf{a}|}\theta/2}\mathbf{x}e^{\frac{\mathbf{a}\wedge\mathbf{b}}{|\mathbf{a}\wedge\mathbf{b}|}\theta/2} \tag{29.41}$$

This last (writing $\mathbf{x}$ instead of $\mathbf{a}$ since it also applies to vectors that lie outside of the $\mathbf{a} \wedge \mathbf{b}$ plane), is our rotor formula eq. (29.14), reexpressed in a way that removes the sign ambiguity of the bivector $\mathbf{i}$ in that equation.

29.5 ROTATION BETWEEN TWO UNIT VECTORS

As illustrated in fig. 29.4, when the angle between two vectors is less than π the fact that the sum of two arbitrarily oriented unit vectors bisects those vectors provides a convenient way to compute the half angle rotation exponential.

Thus we can write

$$\frac{\mathbf{a} + \mathbf{b}}{|\mathbf{a} + \mathbf{b}|} = \mathbf{a}e^{\mathbf{i}\theta/2} = e^{\mathbf{j}\theta/2}\mathbf{a}$$

Where $\mathbf{i} = \mathbf{j}^\dagger$ are unit bivectors of appropriate sign. Multiplication through by $\mathbf{a}$ gives

$$e^{\mathbf{i}\theta/2} = \frac{1 + \mathbf{ab}}{|\mathbf{a} + \mathbf{b}|}$$

Or,

$$e^{\mathbf{j}\theta/2} = \frac{1 + \mathbf{ba}}{|\mathbf{a} + \mathbf{b}|}$$

Thus we can write the total rotation from $\mathbf{a}$ to $\mathbf{b}$ as

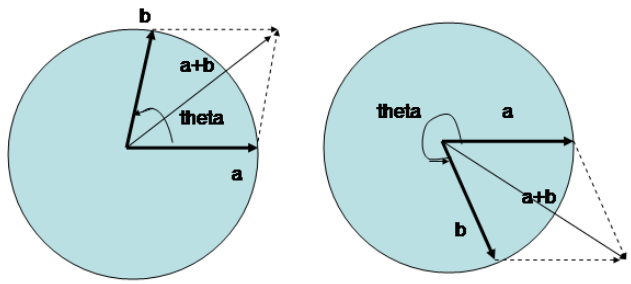

Figure 29.4: Sum of unit vectors bisects angle between

$$\mathbf{b} = e^{-\mathbf{i}\theta/2}\mathbf{a}e^{\mathbf{i}\theta/2} = e^{\mathbf{j}\theta/2}\mathbf{a}e^{-\mathbf{j}\theta/2} = \left(\frac{1+\mathbf{ba}}{|\mathbf{a}+\mathbf{b}|}\right)\mathbf{a}\left(\frac{1+\mathbf{ab}}{|\mathbf{a}+\mathbf{b}|}\right)$$

For the case where the rotation is through an angle θ where $\pi < \theta < 2\pi$, again employing a left acting exponential operator we have

$$\begin{aligned}\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|} &= \mathbf{b}e^{\mathbf{i}(2\pi-\theta)/2} \\ &= \mathbf{b}e^{\mathbf{i}\pi}e^{-\mathbf{i}\theta/2} \\ &= -\mathbf{b}e^{-\mathbf{i}\theta/2}\end{aligned} \tag{29.42}$$

Or,

$$e^{-\mathbf{i}\theta/2} = -\frac{\mathbf{ba}+1}{|\mathbf{a}+\mathbf{b}|} \tag{29.43}$$

Thus

$$\mathbf{b} = e^{-\mathbf{i}\theta/2}\mathbf{a}e^{\mathbf{i}\theta/2} = \left(-\frac{1+\mathbf{ba}}{|\mathbf{a}+\mathbf{b}|}\right)\mathbf{a}\left(-\frac{1+\mathbf{ab}}{|\mathbf{a}+\mathbf{b}|}\right) \tag{29.44}$$

Note that the two negatives cancel, giving the same result as in the $\theta < \pi$ case. Thus eq. (29.44) is valid for all vectors $\mathbf{a} \neq -\mathbf{b}$ (this can be verified by direct multiplication.)

These half angle exponentials are called rotors, writing the rotor as

$$R = \frac{1 + \mathbf{a}\mathbf{b}}{|\mathbf{a} + \mathbf{b}|} \tag{29.45}$$

and the rotation in terms of rotors is:

$$\mathbf{b} = R^{\dagger}\mathbf{a}R \tag{29.46}$$

The angle associated with this rotor R is the minimal angle between the two vectors ($0 < \theta < \pi$), and is directed from $\mathbf{a}$ to $\mathbf{b}$. Inverting the rotor will not change the net effect of the rotation, but has the geometric meaning that the rotation from $\mathbf{a}$ to $\mathbf{b}$ rotates in the opposite direction through the larger angle ($\pi < \theta < 2\pi$) between the vectors.

29.6 EIGENVALUES, VECTORS AND COORDINATE VECTOR AND MATRIX OF THE ROTATION LINEAR TRANSFORMATION

Given the plane containing two orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$, we can form a unit bivector for the plane

$$\mathbf{B} = \mathbf{u}\mathbf{v} \tag{29.47}$$

A normal to this plane is $\mathbf{n} = \mathbf{v}\mathbf{u}I$.

The rotation operator for a rotation around $\mathbf{n}$ in that plane (directed from $\mathbf{u}$ towards $\mathbf{v}$) is

$$R_\theta(\mathbf{x}) = e^{\mathbf{v}\mathbf{u}\theta/2}\mathbf{x}e^{\mathbf{u}\mathbf{v}\theta/2} \tag{29.48}$$

To form the matrix of this linear transformation assume an orthonormal basis $\sigma = \{\mathbf{e}_i\}$.

In terms of these basis vectors we can write

$$R_\theta(\mathbf{e}_j) = e^{-\mathbf{v}\mathbf{u}\theta/2}\mathbf{e}_j e^{\mathbf{u}\mathbf{v}\theta/2} = \sum_i \left(e^{-\mathbf{v}\mathbf{u}\theta/2}\mathbf{e}_j e^{\mathbf{u}\mathbf{v}\theta/2}\right)\cdot\mathbf{e}_i\mathbf{e}_i \tag{29.49}$$

Thus the coordinate vector for this basis is

$$\left[R_\theta(\mathbf{e}_j)\right]_\sigma = \begin{bmatrix} \left(e^{-\mathbf{v}\mathbf{u}\theta/2}\mathbf{e}_j e^{\mathbf{u}\mathbf{v}\theta/2}\right)\cdot\mathbf{e}_1 \\ \vdots \\ \left(e^{-\mathbf{v}\mathbf{u}\theta/2}\mathbf{e}_j e^{\mathbf{u}\mathbf{v}\theta/2}\right)\cdot\mathbf{e}_n \end{bmatrix} \tag{29.50}$$

We can use this to form the matrix for the linear operator that takes coordinate vectors from the basis σ to σ:

$$\left[R_\theta(\mathbf{x})\right]_\sigma = \left[R_\theta\right]_\sigma^\sigma \left[\mathbf{x}\right]_\sigma \tag{29.51}$$

Where

$$\left[R_\theta\right]_\sigma^\sigma = \left[\left[R_\theta(\mathbf{e}_1)\right]_\sigma \cdots \left[R_\theta(\mathbf{e}_n)\right]_\sigma\right] = \left[\left(e^{-\mathbf{vu}\theta/2}\mathbf{e}_j e^{\mathbf{uv}\theta/2}\right) \cdot \mathbf{e}_i\right]_{ij} \tag{29.52}$$

If one uses the plane and its normal to form an alternate orthonormal basis $\alpha = \{\mathbf{u}, \mathbf{v}, \mathbf{n}\}$. The transformation matrix for coordinate vectors in this basis is

$$\left[R_\theta\right]_\alpha^\alpha = \begin{bmatrix} \left(\mathbf{u}e^{\mathbf{uv}\theta}\right) \cdot \mathbf{u} & \left(\mathbf{v}e^{\mathbf{uv}\theta}\right) \cdot \mathbf{u} & 0 \\ \left(\mathbf{u}e^{\mathbf{uv}\theta}\right) \cdot \mathbf{v} & \left(\mathbf{v}e^{\mathbf{uv}\theta}\right) \cdot \mathbf{v} & 0 \\ 0 & 0 & \mathbf{n} \cdot \mathbf{n} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29.53}$$

This matrix has eigenvalues $e^{i\theta}, e^{-i\theta}, 1$, with (coordinate) eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{29.54}$$

Its interesting to observe that without introducing coordinate vectors an eigensolution is possible directly from the linear transformation itself.

The rotation linear operator has right and left eigenvalues $e^{\mathbf{uv}\theta}$, $e^{\mathbf{vu}\theta}$ (respectively), where the eigenvectors for these are any vectors in the plane. There is also a scalar eigenvalue 1 (both left and right eigenvalue), for the eigenvector $\mathbf{n}$:

$$\begin{aligned} R_\theta(\mathbf{u}) &= e^{\mathbf{vu}\theta}\mathbf{x} = \mathbf{x}e^{\mathbf{uv}\theta} \\ R_\theta(\mathbf{u}) &= e^{\mathbf{vu}\theta}\mathbf{x} = \mathbf{x}e^{\mathbf{uv}\theta} \\ R_\theta(\mathbf{n}) &= \mathbf{n}(1) \end{aligned} \tag{29.55}$$

Observe that the eigenvalues here are not all scalars, which is likely related to the fact that the coordinate matrix was not diagonalizable with real vectors.

the matrix of the linear transformation. Given this, one can write:

$$\begin{aligned}\begin{bmatrix} R_\theta(\mathbf{u}) & R_\theta(\mathbf{v}) & R_\theta(\mathbf{n}) \end{bmatrix} &= \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{n} \end{bmatrix} \begin{bmatrix} e^{\mathbf{uv}\theta} & 0 & 0 \\ 0 & e^{\mathbf{uv}\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\mathbf{vu}\theta} & 0 & 0 \\ 0 & e^{\mathbf{vu}\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{n} \end{bmatrix}\end{aligned} \tag{29.56}$$

But neither of these can be used to diagonalize the matrix of the transformation. To do that we require dot products that span the matrix product to form the coordinate vector columns.

Observe that interestingly enough the left and right eigenvalues of the operator in the plane are of complex exponential form ($e^{\pm\mathbf{n}I\theta}$) just as the eigenvalues for coordinate vectors restricted to the plane are complex exponentials ($e^{\pm i\theta}$).

29.7 MATRIX FOR ROTATION LINEAR TRANSFORMATION

Let us expand the terms in eq. (29.52) to calculate explicitly the rotation matrix for an arbitrary rotation. Also, as before, write $\mathbf{n} = \mathbf{vu}I$, and parametrize the Rotor as follows:

$$R = e^{\mathbf{n}I\theta/2} = \cos\theta/2 + \mathbf{n}I\sin\theta/2 = \alpha + I\boldsymbol{\beta} \tag{29.57}$$

Thus the ij terms in the matrix are:

$$\begin{aligned}\mathbf{e}_i \cdot \left(e^{-\mathbf{n}I\theta/2}\mathbf{e}_j e^{\mathbf{n}I\theta/2}\right) &= \langle \mathbf{e}_i(\alpha - I\boldsymbol{\beta})\mathbf{e}_j(\alpha + I\boldsymbol{\beta})\rangle \\ &= \langle \mathbf{e}_i(\mathbf{e}_j\alpha - I\boldsymbol{\beta}\mathbf{e}_j)(\alpha + I\boldsymbol{\beta})\rangle \\ &= \langle \mathbf{e}_i\left(\mathbf{e}_j\alpha^2 - I\alpha(\boldsymbol{\beta}\mathbf{e}_j - \mathbf{e}_j\boldsymbol{\beta}) + \boldsymbol{\beta}\mathbf{e}_j\boldsymbol{\beta}\right)\rangle \\ &= \delta_{ij}\alpha^2 + \langle \mathbf{e}_i\left(-2I\alpha(\boldsymbol{\beta}\wedge\mathbf{e}_j) + \boldsymbol{\beta}\mathbf{e}_j\boldsymbol{\beta}\right)\rangle \\ &= \delta_{ij}\alpha^2 + 2\alpha\mathbf{e}_i\cdot(\boldsymbol{\beta}\times\mathbf{e}_j) + \langle \mathbf{e}_i\boldsymbol{\beta}\mathbf{e}_j\boldsymbol{\beta}\rangle\end{aligned} \tag{29.58}$$

Lets expand the last term separately:

$$\begin{aligned}\langle \mathbf{e}_i\boldsymbol{\beta}\mathbf{e}_j\boldsymbol{\beta}\rangle &= \langle(\mathbf{e}_i\cdot\boldsymbol{\beta} + \mathbf{e}_i\wedge\boldsymbol{\beta})(\mathbf{e}_j\cdot\boldsymbol{\beta} + \mathbf{e}_j\wedge\boldsymbol{\beta})\rangle \\ &= (\mathbf{e}_i\cdot\boldsymbol{\beta})(\mathbf{e}_j\cdot\boldsymbol{\beta}) + \langle(\mathbf{e}_i\wedge\boldsymbol{\beta})(\mathbf{e}_j\wedge\boldsymbol{\beta})\rangle\end{aligned} \tag{29.59}$$

And once more considering first the $i = j$ case (writing $s \neq i \neq t$).

$$\begin{aligned}
\langle(\mathbf{e}_i \wedge \boldsymbol{\beta})^2\rangle &= \left(\sum_{k\neq i} \mathbf{e}_{ik}\beta_k\right)^2 \\
&= (\mathbf{e}_{is}\beta_s + \mathbf{e}_{it}\beta_t)(\mathbf{e}_{is}\beta_s + \mathbf{e}_{it}\beta_t) \\
&= -\beta_s^2 - \beta_t^2 - \mathbf{e}_{st}\beta_s\beta_t + \mathbf{e}_{ts}\beta_t\beta_s \\
&= -\beta_s^2 - \beta_t^2 \\
&= -\boldsymbol{\beta}^2 + \beta_i^2
\end{aligned} \tag{29.60}$$

For the $i \neq j$ term, writing $i \neq j \neq k$

$$\begin{aligned}
\langle(\mathbf{e}_i \wedge \boldsymbol{\beta})(\mathbf{e}_j \wedge \boldsymbol{\beta})\rangle &= \langle\sum_{s\neq i} \mathbf{e}_{is}\beta_s \sum_{t\neq i} \mathbf{e}_{it}\beta_t\rangle \\
&= \langle(\mathbf{e}_{ij}\beta_j + \mathbf{e}_{ik}\beta_k)(\mathbf{e}_{ji}\beta_i + \mathbf{e}_{jk}\beta_k)\rangle \\
&= \beta_i\beta_j + \langle\mathbf{e}_{ji}\beta_k^2 + \mathbf{e}_{ik}\beta_j\beta_k + \mathbf{e}_{kj}\beta_k\beta_i\rangle \\
&= \beta_i\beta_j
\end{aligned} \tag{29.61}$$

Thus

$$\langle(\mathbf{e}_i \wedge \boldsymbol{\beta})(\mathbf{e}_j \wedge \boldsymbol{\beta})\rangle = \delta_{ij}(-\boldsymbol{\beta}^2 + \beta_i^2) + (1 - \delta_{ij})\beta_i\beta_j = \beta_i\beta_j - \delta_{ij}\boldsymbol{\beta}^2 \tag{29.62}$$

And putting it all back together

$$\mathbf{e}_i \cdot \left(e^{-\mathbf{n}I\theta/2}\mathbf{e}_j e^{\mathbf{n}I\theta/2}\right) = \delta_{ij}(\alpha^2 - \boldsymbol{\beta}^2) + 2\alpha\mathbf{e}_i \cdot (\boldsymbol{\beta} \times \mathbf{e}_j) + 2\beta_i\beta_j \tag{29.63}$$

The α and β terms can be expanded in terms of θ. we see that The δ_{ij} coefficient is

$$\alpha^2 - \boldsymbol{\beta}^2 = 2\cos^2\theta - 1 = \cos\theta. \tag{29.64}$$

The triple product $\mathbf{e}_i \cdot (\boldsymbol{\beta} \times \mathbf{e}_j)$ is zero along the diagonal where $i = j$ since an $\mathbf{e}_j = \mathbf{e}_i$ cross has no $\mathbf{e}_i$ component, so for $k \neq i \neq j$, the triple product term is

$$\begin{aligned}
2\alpha\mathbf{e}_i \cdot (\boldsymbol{\beta} \times \mathbf{e}_j) &= 2\alpha\beta_k\mathbf{e}_i \cdot (\mathbf{e}_k \times \mathbf{e}_j) \\
&= 2\alpha\beta_k \operatorname{sgn}\pi_{ikj} \\
&= 2n_k \cos(\theta/2)\sin(\theta/2) \operatorname{sgn}\pi_{ikj} \\
&= n_k \sin\theta \operatorname{sgn}\pi_{ikj}
\end{aligned} \tag{29.65}$$

The last term is:

$$2\beta_i\beta_j = 2n_in_j\sin^2(\theta/2) = n_in_j(1-\cos\theta) \tag{29.66}$$

Thus we can alternatively write eq. (29.63)

$$\mathbf{e}_i \cdot \left(e^{-\mathbf{n}I\theta/2}\mathbf{e}_je^{\mathbf{n}I\theta/2}\right) = \delta_{ij}\cos\theta + n_k\sin\theta\epsilon_{ikj} + n_in_j(1-\cos\theta) \tag{29.67}$$

This is enough to easily and explicitly write out the complete rotation matrix for a rotation about unit vector $\mathbf{n} = (n_1, n_2, n_3)$: (with basis $\sigma = \{\mathbf{e}_i\}$):

$$[R_\theta]_\sigma^\sigma = \begin{bmatrix} \cos\theta(1-n_1^2)+n_1^2 & n_1n_2(1-\cos\theta)-n_3\sin\theta & n_1n_3(1-\cos\theta)+n_2\sin\theta \\ n_1n_2(1-\cos\theta)+n_3\sin\theta & \cos\theta(1-n_2^2)+n_2^2 & n_2n_3(1-\cos\theta)-n_1\sin\theta \\ n_1n_3(1-\cos\theta)-n_2\sin\theta & n_2n_3(1-\cos\theta)+n_1\sin\theta & \cos\theta(1-n_3^2)+n_3^2 \end{bmatrix} \tag{29.68}$$

Note also that the n_i terms are the direction cosines of the unit normal for the rotation, so all the terms above are really strictly sums of sine and cosine products, so we have the rotation matrix completely described in terms of four angles. Also observe how much additional complexity we have to express a rotation in terms of the matrix. This representation also does not work for plane rotations, just vectors (whereas that is not the case for the rotor form).

It is actually somewhat simpler looking to leave things in terms of the α, and β parameters. We can rewrite eq. (29.63) as:

$$\mathbf{e}_i \cdot \left(e^{-\mathbf{n}I\theta/2}\mathbf{e}_je^{\mathbf{n}I\theta/2}\right) = \delta_{ij}(2\alpha^2-1) + 2\alpha\beta_k\epsilon_{ikj} + 2\beta_i\beta_j \tag{29.69}$$

and the rotation matrix:

$$[R_\theta]_\sigma^\sigma = 2\begin{bmatrix} \alpha^2-\frac{1}{2}+\beta_1^2 & \beta_1\beta_2-\beta_3\alpha & \beta_1\beta_3+\beta_2\alpha \\ \beta_1\beta_2+\beta_3\alpha & \alpha^2-\frac{1}{2}+\beta_2^2 & \beta_2\beta_3-\beta_1\alpha \\ \beta_1\beta_3-\beta_2\alpha & \beta_2\beta_3+\beta_1\alpha & \alpha^2-\frac{1}{2}+\beta_3^2 \end{bmatrix} \tag{29.70}$$

Not really that much simpler, but a bit. The trade off is that the similarity to the standard $2x2$ rotation matrix is not obvious.

30

EULER ANGLE NOTES

30.1 REMOVING THE ROTORS FROM THE EXPONENTIALS

In [10] section 2.7.5 the Euler angle formula is developed for $\{z, x', z''\}$ axis rotations by $\{\phi, \theta, \psi\}$ respectively.

Other than a few details the derivation is pretty straightforward. Equation 2.153 would be clearer with a series expansion hint like

$$
\begin{aligned}
\exp(R\alpha i R^\dagger) &= \sum_k \frac{1}{k!}(R\alpha i R^\dagger)^k \\
&= \sum_k \frac{1}{k!} R(\alpha i)^k R^\dagger \\
&= R \exp(\alpha i) R^\dagger
\end{aligned}
\tag{30.1}
$$

where i is a bivector, and R is a rotor ($RR^\dagger = 1$).

The first rotation is straightforward, by an angle ϕ around the z axis

$$
R_\phi(x) = \exp(-Ie_3\phi/2) x \exp(Ie_3\phi/2) = R_\phi x R_\phi^\dagger
\tag{30.2}
$$

The next rotation is around the transformed x axis, for which the rotational plane is

$$
IR_\phi e_1 R_\phi^\dagger = R_\phi I e_1 R_\phi^\dagger
\tag{30.3}
$$

So the rotor for this plane by angle θ is

$$
\begin{aligned}
R_\theta &= \exp(R_\phi I e_1 R_\phi^\dagger) \\
&= R_\phi \exp(Ie_1\theta/2) R_\phi^\dagger,
\end{aligned}
\tag{30.4}
$$

resulting in the composite rotor

$$
\begin{aligned}
R_{\theta\phi} &= R_\theta R_\phi \\
&= R_\phi \exp(Ie_1\theta/2) R_\phi^\dagger R_\phi \\
&= R_\phi \exp(Ie_1\theta/2) \\
&= \exp(-Ie_3\phi/2) \exp(-Ie_1\theta/2)
\end{aligned}
\tag{30.5}
$$

The composition of the simple rotation around the e_3 axis, followed by the rotation around the e_1' axis ends up as a product of rotors around the original e_1 and e_3 axis, but curiously enough in inverted order.

30.2 EXPANDING THE ROTOR PRODUCT

Completing the calculation outlined above follows in the same fashion. The end result is that the composite Euler rotations has the following rotor form

$$\begin{aligned} R(x) &= RxR^\dagger \\ R &= \exp(-e_{12}\phi/2)\exp(-e_{23}\theta/2)\exp(-e_{12}\psi/2) \end{aligned} \tag{30.6}$$

Then there are notes saying this is easier to visualize and work with than the equivalent matrix formula. Let us see what the equivalent matrix formula is. First calculate the rotor action on $\mathbf{e}_1$

$$\begin{aligned} R\mathbf{e}_1R^\dagger &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}e^{-e_{12}\psi/2}\mathbf{e}_1e^{e_{12}\psi/2}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}\mathbf{e}_1e^{e_{12}\psi}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}(\mathbf{e}_1C_\psi + \mathbf{e}_2S_\psi)e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}(\mathbf{e}_1C_\psi + \mathbf{e}_2S_\psi e^{e_{23}\theta})e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}(\mathbf{e}_1C_\psi + \mathbf{e}_2S_\psi C_\theta + \mathbf{e}_3S_\psi S_\theta)e^{e_{12}\phi/2} \\ &= (\mathbf{e}_1C_\psi + \mathbf{e}_2S_\psi C_\theta)e^{e_{12}\phi} + \mathbf{e}_3S_\psi S_\theta \\ &= \mathbf{e}_1(C_\psi C_\phi - S_\psi C_\theta S_\phi) + \mathbf{e}_2(C_\psi S_\phi + S_\psi C_\theta C_\phi) + \mathbf{e}_3S_\psi S_\theta \end{aligned} \tag{30.7}$$

Now $\mathbf{e}_2$:

$$\begin{aligned} R\mathbf{e}_2R^\dagger &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}e^{-e_{12}\psi/2}\mathbf{e}_2e^{e_{12}\psi/2}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}(\mathbf{e}_2C_\psi - \mathbf{e}_1S_\psi)e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\ &= (\mathbf{e}_2C_\psi C_\theta - \mathbf{e}_1S_\psi)e^{e_{12}\phi} + \mathbf{e}_3C_\psi S_\theta \\ &= \mathbf{e}_1(-C_\psi C_\theta S_\phi - S_\psi C_\phi) + \mathbf{e}_2(-S_\psi S_\phi + C_\psi C_\theta C_\phi) + \mathbf{e}_3C_\psi S_\theta \end{aligned} \tag{30.8}$$

And finally $\mathbf{e}_3$

$$\begin{aligned} R\mathbf{e}_3R^\dagger &= e^{-e_{12}\phi/2}e^{-e_{23}\theta/2}e^{-e_{12}\psi/2}\mathbf{e}_3e^{e_{12}\psi/2}e^{e_{23}\theta/2}e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}\mathbf{e}_3e^{e_{23}\theta}e^{e_{12}\phi/2} \\ &= e^{-e_{12}\phi/2}(\mathbf{e}_3C_\theta - \mathbf{e}_2S_\theta)e^{e_{12}\phi/2} \\ &= -\mathbf{e}_2S_\theta e^{e_{12}\phi} + \mathbf{e}_3C_\theta \\ &= \mathbf{e}_1S_\theta S_\phi - \mathbf{e}_2S_\theta C_\phi + \mathbf{e}_3C_\theta \end{aligned} \tag{30.9}$$

This can now be assembled into matrix form

$$\begin{bmatrix} R(x)\cdot\mathbf{e}_1 \\ R(x)\cdot\mathbf{e}_2 \\ R(x)\cdot\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} (R\mathbf{e}_1R^\dagger)\cdot\mathbf{e}_1 & (R\mathbf{e}_2R^\dagger)\cdot\mathbf{e}_1 & (R\mathbf{e}_3R^\dagger)\cdot\mathbf{e}_1 \\ (R\mathbf{e}_1R^\dagger)\cdot\mathbf{e}_2 & (R\mathbf{e}_2R^\dagger)\cdot\mathbf{e}_2 & (R\mathbf{e}_3R^\dagger)\cdot\mathbf{e}_2 \\ (R\mathbf{e}_1R^\dagger)\cdot\mathbf{e}_3 & (R\mathbf{e}_2R^\dagger)\cdot\mathbf{e}_3 & (R\mathbf{e}_3R^\dagger)\cdot\mathbf{e}_3 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \mathbf{R}\mathbf{x} \tag{30.10}$$

Therefore we have the composite matrix form for the Euler angle rotations

$$\mathbf{R} = \begin{bmatrix} C_\psi C_\phi - S_\psi C_\theta S_\phi & -S_\psi C_\phi - C_\psi C_\theta S_\phi & S_\theta S_\phi \\ C_\psi S_\phi + S_\psi C_\theta C_\phi & -S_\psi S_\phi + C_\psi C_\theta C_\phi & -S_\theta C_\phi \\ S_\psi S_\theta & C_\psi S_\theta & C_\theta \end{bmatrix} \tag{30.11}$$

Lots of opportunity to make sign errors here. Let us check with matrix multiplication, which should give the same result

30.3 WITH COMPOSITION OF ROTATION MATRICES (DONE WRONG, BUT WITH DISCUSSION AND REQUIRED CORRECTION)

$$\begin{aligned} R(x) &= \mathbf{R}\mathbf{x} \\ &= \mathbf{R}_{\psi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}\mathbf{R}_{\phi\mathbf{e}_3}\mathbf{x} \end{aligned} \tag{30.12}$$

Now, that first rotation is

$$\begin{aligned} R_\phi(x) &= e^{-e_{12}\phi/2}(\mathbf{e}_i x^i)e^{e_{12}\phi/2} \\ &= (\mathbf{e}_1 x^1 + \mathbf{e}_2 x^2)e^{e_{12}\phi} + \mathbf{e}_3 x^3 \\ &= x^1(\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi) + x^2(\mathbf{e}_2\cos\phi - \mathbf{e}_1\sin\phi) + x^3\mathbf{e}_3 \\ &= \mathbf{e}_1(x^1\cos\phi - x^2\sin\phi) + \mathbf{e}_2(x^1\sin\phi + x^2\cos\phi) + \mathbf{e}_3 x^3 \end{aligned} \tag{30.13}$$

Which has the expected matrix form

$$\mathbf{R}_{\phi\mathbf{e}_3}\mathbf{x} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{30.14}$$

Using $C_x = \cos(x)$, and $S_x = \sin(x)$ for brevity, the composite rotation is

$$\begin{aligned}
\mathbf{R}_{\psi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}\mathbf{R}_{\phi\mathbf{e}_3} &= \begin{bmatrix} C_\psi & -S_\psi & 0 \\ S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_\psi & -S_\psi C_\theta & S_\psi S_\theta \\ S_\psi & C_\psi C_\theta & -C_\psi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_\psi C_\phi - S_\psi C_\theta S_\phi & -C_\psi S_\phi - S_\psi C_\theta C_\phi & S_\psi S_\theta \\ S_\psi C_\phi + C_\psi C_\theta S_\phi & -S_\psi S_\phi + C_\psi C_\theta C_\phi & -C_\psi S_\theta \\ S_\theta S_\phi & S_\theta C_\phi & C_\theta \end{bmatrix}
\end{aligned} \tag{30.15}$$

This is different from the rotor generated result above, although with a ϕ, and ψ interchange things appear to match?

$$\begin{bmatrix} C_\phi C_\psi - S_\phi C_\theta S_\psi & -S_\phi C_\psi - C_\phi C_\theta S_\psi & S_\theta S_\psi \\ C_\phi S_\psi + S_\phi C_\theta C_\psi & -S_\phi S_\psi + C_\phi C_\theta C_\psi & -S_\theta C_\psi \\ S_\phi S_\theta & C_\phi S_\theta & C_\theta \end{bmatrix} \tag{30.16}$$

Where is the mistake? I suspect it is in the matrix formulation, where the plain old rotations for the axis were multiplied. Because the rotations need to be along the transformed axis I bet there is a reversion of matrix products as there was an reversion of rotors? How would one show if this is the case?

What is needed is more careful treatment of the rotation about the transformed axis. Considering the first, for a rotation around the $\mathbf{e}'_1 = (C_\phi, S_\phi, 0)$ axis. From 29 we have the rotation matrix for a θ rotation about an arbitrary normal $\mathbf{n} = (n_1, n_2, n_3)$

$$
\begin{aligned}
R_\theta &= \begin{bmatrix} C_\theta(1-n_1^2)+n_1^2 & n_1n_2(1-C_\theta)-n_3S_\theta & n_1n_3(1-C_\theta)+n_2S_\theta \\ n_1n_2(1-C_\theta)+n_3S_\theta & C_\theta(1-n_2^2)+n_2^2 & n_2n_3(1-C_\theta)-n_1S_\theta \\ n_1n_3(1-C_\theta)-n_2S_\theta & n_2n_3(1-C_\theta)+n_1S_\theta & C_\theta(1-n_3^2)+n_3^2 \end{bmatrix} \\
&= \begin{bmatrix} C_\theta(1-{C_\phi}^2)+{C_\phi}^2 & C_\phi S_\phi(1-C_\theta) & S_\phi S_\theta \\ C_\phi S_\phi(1-C_\theta) & C_\theta(1-{S_\phi}^2)+{S_\phi}^2 & -C_\phi S_\theta \\ -S_\phi S_\theta & C_\phi S_\theta & C_\theta \end{bmatrix} \\
&= \begin{bmatrix} C_\theta({S_\phi}^2)+{C_\phi}^2 & C_\phi S_\phi(1-C_\theta) & S_\phi S_\theta \\ C_\phi S_\phi(1-C_\theta) & C_\theta({C_\phi}^2)+{S_\phi}^2 & -C_\phi S_\theta \\ -S_\phi S_\theta & C_\phi S_\theta & C_\theta \end{bmatrix}
\end{aligned}
\tag{30.17}
$$

Now the composite rotation for the sequence of ϕ, and θ rotations about the z, and x' axis is

$$
\begin{aligned}
\mathbf{R}_{\phi\mathbf{e}_3,\theta\mathbf{e}'_1} &= \begin{bmatrix} C_\theta({S_\phi}^2)+{C_\phi}^2 & C_\phi S_\phi(1-C_\theta) & S_\phi S_\theta \\ C_\phi S_\phi(1-C_\theta) & C_\theta({C_\phi}^2)+{S_\phi}^2 & -C_\phi S_\theta \\ -S_\phi S_\theta & C_\phi S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_\phi & -C_\theta S_\phi & S_\phi S_\theta \\ S_\phi & C_\theta C_\phi & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \\
&= \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \\
&= \mathbf{R}_{\phi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}
\end{aligned}
\tag{30.18}
$$

Wow, sure enough the composite rotation matrix is the result of the inverted order product of the two elementary rotation matrices. The algebra here is fairly messy, so it would not be terribly fun to go one step further using just matrices that the final triple rotation is not the product of eq. (30.12), but instead requires the matrix product

$$
\begin{aligned}
R(x) &= \mathbf{R}\mathbf{x} \\
&= \mathbf{R}_{\phi\mathbf{e}_3}\mathbf{R}_{\theta\mathbf{e}_1}\mathbf{R}_{\psi\mathbf{e}_3}\mathbf{x}
\end{aligned}
\tag{30.19}
$$

Assuming that this is true, the swapping of angles to match the rotor expression is fully accounted for, and it is understood how to correctly do the same calculation in matrix form.

30.4 RELATION TO CAYLEY-KLEIN PARAMETERS

Exercise 2.9 from [10] is to relate the rotation matrix expressed in terms of Cayley-Klein parameters back to the rotor formulation. That matrix has the look of something that involves the half angles. Use of software to expressing the rotation in terms of the half angle signs and cosines, plus some manually factoring (which could be carried further), produces the following mess

$$
\begin{aligned}
R_{11} &= S_\phi^2 S_\psi^2 - C_\psi^2 S_\phi^2 - C_\phi^2 S_\psi^2 + C_\phi^2 C_\psi^2 - 4 C_\phi S_\phi C_\psi S_\psi (C_\theta^2 - S_\theta^2) \\
R_{21} &= 2 C_\psi S_\psi (C_\phi^2 - S_\phi^2)(C_\theta^2 - S_\theta^2) + 2 C_\phi S_\phi (C_\psi^2 - S_\psi^2) \\
R_{31} &= 4 C_\psi C_\theta S_\psi S_\theta \\
R_{12} &= -2 C_\phi S_\phi (C_\psi^2 - S_\psi^2)(C_\theta^2 - S_\theta^2) - 2 C_\psi S_\psi (C_\phi^2 - S_\phi^2) \\
R_{22} &= (C_\theta^2 - S_\theta^2)(-S_\phi^2 + C_\phi^2)(C_\psi^2 - S_\psi^2) - 4 C_\phi S_\phi C_\psi S_\psi \\
R_{32} &= 2 C_\theta S_\theta (C_\psi^2 - S_\psi^2) \\
R_{13} &= 4 C_\phi C_\theta S_\phi S_\theta \\
R_{23} &= -2 C_\theta S_\theta (C_\phi^2 - S_\phi^2) \\
R_{33} &= +C_\theta^2 S_\phi^2 - S_\psi^2 S_\theta^2 - C_\psi^2 S_\theta^2 + C_\phi^2 C_\theta^2
\end{aligned}
\tag{30.20}
$$

It kind of looks like the terms $C_\theta S_\phi$ may be related to these parameters. error prone. A Google search for Cayley Klein also verifies that those parameters are expressed in terms of half angle relations, but even with the hint, I was not successful getting something tidy out of all this.

An alternate approach is to just expand the rotor, so terms may be grouped before that rotor and its reverse is applied to the object to be rotated. Again in terms of half angle signs and cosines this is

$$
\begin{aligned}
R &= \exp(-e_{12}\phi/2)\exp(-e_{23}\theta/2)\exp(-e_{12}\psi/2) \\
&= (C_\phi C_\psi - S_\phi S_\psi) C_\theta - (C_\psi S_\phi + C_\phi S_\psi) C_\theta \mathbf{e}_{12} \\
&+ (C_\phi S_\psi - C_\psi S_\phi) S_\theta \mathbf{e}_{31} - (S_\phi S_\psi + C_\phi C_\psi) S_\theta \mathbf{e}_{23}
\end{aligned}
\tag{30.21}
$$

Grouping terms produces

$$
R = \cos\left(\frac{\theta}{2}\right)\exp\left(\frac{-\mathbf{e}_{12}}{2}(\psi+\phi)\right) + \sin\left(\frac{\theta}{2}\right)\exp\left(\frac{\mathbf{e}_{12}}{2}(\psi-\phi)\right)\mathbf{e}_{32}
\tag{30.22}
$$

Okay... now I see how you naturally get four parameters out of this. Also see why it was hard to get there from the fully expanded rotation product ... it would first be required to group all the ϕ and ψ terms just right in terms of sums and differences.

With

$$\begin{aligned}
R &= \alpha + \delta \mathbf{e}_{12} + \beta \mathbf{e}_{23} + \gamma \mathbf{e}_{31} \\
\alpha &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{1}{2}\left(\psi + \phi\right)\right) \\
\delta &= -\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{1}{2}\left(\psi + \phi\right)\right) \\
\beta &= -\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{1}{2}\left(\psi - \phi\right)\right) \\
\gamma &= \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{1}{2}\left(\psi - \phi\right)\right)
\end{aligned} \tag{30.23}$$

By inspection, these have the required property $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$, and multiplying out the rotors yields the rotation matrix

$$\mathbf{U} = \begin{bmatrix} -\gamma^2 + \beta^2 - \delta^2 + \alpha^2 & +2\beta\gamma + 2\alpha\delta & +2\delta\beta - 2\alpha\gamma \\ +2\beta\gamma - 2\alpha\delta & +\gamma^2 - \beta^2 - \delta^2 + \alpha^2 & +2\delta\gamma + 2\alpha\beta \\ +2\delta\beta + 2\alpha\gamma & +2\delta\gamma - 2\alpha\beta & -\gamma^2 - \beta^2 + \delta^2 + \alpha^2 \end{bmatrix} \tag{30.24}$$

Now that particular choice of sign and permutation of the α, β, γ, and δ is not at all obvious, and is also arbitrary. Of the 120 different sign and permutation variations that can be tried, this one results in the particular desired matrix $\mathbf{U}$ from the problem. The process of performing the multiplications was well suited to a symbolic GA calculator and one was written with this and other problems in mind.

[16] also treats these Cayley-Klein parametrizations, but does so considerably differently. He has complex parametrizations and quaternion matrix representations, and it will probably be worthwhile to reconcile all of these.

30.5 OMITTED DETAILS

30.5.1 *Cayley Klein details*

Equation (30.22) was obtained with the following manipulations

$$
\begin{aligned}
R &= (C_\phi C_\psi - S_\phi S_\psi)C_\theta - (C_\psi S_\phi + C_\phi S_\psi)C_\theta \mathbf{e}_{12} \\
&+ (C_\phi S_\psi - C_\psi S_\phi)S_\theta \mathbf{e}_{31} - (S_\phi S_\psi + C_\phi C_\psi)S_\theta \mathbf{e}_{23} \\
&= \cos\left(\frac{1}{2}(\phi+\psi)\right)C_\theta - \sin\left(\frac{1}{2}(\phi+\psi)\right)C_\theta \mathbf{e}_{12} \\
&- \sin\left(\frac{1}{2}(\phi-\psi)\right)S_\theta \mathbf{e}_{31} - \cos\left(\frac{1}{2}(\phi-\psi)\right)S_\theta \mathbf{e}_{23} \\
&= \cos\left(\frac{1}{2}(\psi+\phi)\right)C_\theta - \sin\left(\frac{1}{2}(\psi+\phi)\right)C_\theta \mathbf{e}_{12} \\
&+ \sin\left(\frac{1}{2}(\psi-\phi)\right)S_\theta \mathbf{e}_{31} - \cos\left(\frac{1}{2}(\psi-\phi)\right)S_\theta \mathbf{e}_{23} \\
&= C_\theta \left(\cos\left(\frac{1}{2}(\psi+\phi)\right) - \sin\left(\frac{1}{2}(\psi+\phi)\right)\mathbf{e}_{12}\right) \\
&+ S_\theta \left(\sin\left(\frac{1}{2}(\psi-\phi)\right)\mathbf{e}_{31} - \cos\left(\frac{1}{2}(\psi-\phi)\right)\mathbf{e}_{23}\right) \\
&= C_\theta \exp\left(\frac{-\mathbf{e}_{12}}{2}\,(\psi+\phi)\right) \\
&+ S_\theta \mathbf{e}_3 \left(\sin\left(\frac{1}{2}(\psi-\phi)\right)\mathbf{e}_1 + \cos\left(\frac{1}{2}(\psi-\phi)\right)\mathbf{e}_2\right) \\
&= C_\theta \exp\left(\frac{-\mathbf{e}_{12}}{2}\,(\psi+\phi)\right) + S_\theta \exp\left(\frac{\mathbf{e}_{12}}{2}\,(\psi-\phi)\right)\mathbf{e}_{32}
\end{aligned}
\tag{30.25}
$$

In that last step an arbitrary but convenient decision to write the complex number i as $\mathbf{e}_{12}$ was employed.

31 SPHERICAL POLAR COORDINATES

31.1 MOTIVATION

Reading the math intro of [47], I found the statement that the gradient in spherical polar form is:

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{31.1}$$

There was no picture or description showing the conventions for measurement of the angles or directions. To clarify things and leave a margin note I decided to derive the coordinates and unit vector transformation relationships, gradient, divergence and curl in spherical polar coordinates.

Although details for this particular result can be found in many texts, including the excellent review article [14], the exercise of personally working out the details was thought to be a worthwhile learning exercise. Additionally, some related ideas about rotating frame systems seem worth exploring, and that will be done here.

31.2 NOTES

31.2.1 *Conventions*

Figure 31.1 illustrates the conventions used in these notes. By inspection, the coordinates can be read off the diagram.

$$\begin{aligned}
u &= r\cos\phi \\
x &= u\cos\theta = r\cos\phi\cos\theta \\
y &= u\sin\theta = r\cos\phi\sin\theta \\
z &= r\sin\phi
\end{aligned} \tag{31.2}$$

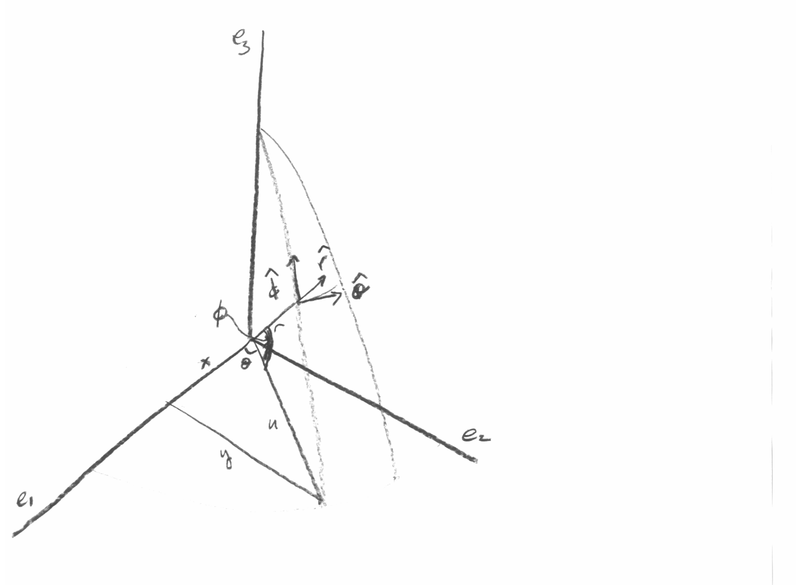

Figure 31.1: Angles and lengths in spherical polar coordinates

31.2.2 *The unit vectors*

To calculate the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ in the spherical polar frame we need to apply two sets of rotations. The first is a rotation in the x, y plane, and the second in the x', z plane.

For the intermediate frame after just the x, y plane rotation we have

$$\begin{aligned} R_\theta &= \exp(-\mathbf{e}_{12}\theta/2) \\ \mathbf{e}'_i &= R_\theta \mathbf{e}_i R_\theta^\dagger \end{aligned} \tag{31.3}$$

Now for the rotational plane for the ϕ rotation is

$$\begin{aligned} \mathbf{e}'_1 \wedge \mathbf{e}_3 &= (R_\theta \mathbf{e}_1 R_\theta^\dagger) \wedge \mathbf{e}_3 \\ &= \frac{1}{2}(R_\theta \mathbf{e}_1 R_\theta^\dagger \mathbf{e}_3 - \mathbf{e}_3 R_\theta \mathbf{e}_1 R_\theta^\dagger) \end{aligned} \tag{31.4}$$

The rotor (or quaternion) R_θ has scalar and $\mathbf{e}_{12}$ components, so it commutes with $\mathbf{e}_3$ leaving

$$\begin{aligned} \mathbf{e}'_1 \wedge \mathbf{e}_3 &= R_\theta \frac{1}{2}(\mathbf{e}_1 \mathbf{e}_3 - \mathbf{e}_3 \mathbf{e}_1) R_\theta^\dagger \\ &= R_\theta \mathbf{e}_1 \wedge \mathbf{e}_3 R_\theta^\dagger \end{aligned} \tag{31.5}$$

Therefore the rotor for the second stage rotation is

$$
\begin{aligned}
R_\phi &= \exp(-R_\theta \mathbf{e}_1 \wedge \mathbf{e}_3 R_\theta^\dagger \phi/2) \\
&= \sum \frac{1}{k!} \left(-R_\theta \mathbf{e}_1 \wedge \mathbf{e}_3 R_\theta^\dagger \phi/2\right)^k \\
&= R_\theta \sum \frac{1}{k!} (-\mathbf{e}_1 \wedge \mathbf{e}_3 \phi/2)^k R_\theta^\dagger \\
&= R_\theta \exp(-\mathbf{e}_{13}\phi/2) R_\theta^\dagger
\end{aligned}
\tag{31.6}
$$

Composing both sets of rotations one has

$$
\begin{aligned}
R(\mathbf{x}) &= R_\theta \exp(-\mathbf{e}_{13}\phi/2) R_\theta^\dagger R_\theta \mathbf{x} R_\theta^\dagger R_\theta \exp(\mathbf{e}_{13}\phi/2) R_\theta^\dagger \\
&= \exp(-\mathbf{e}_{12}\theta/2) \exp(-\mathbf{e}_{13}\phi/2) \mathbf{x} \exp(\mathbf{e}_{13}\phi/2) \exp(\mathbf{e}_{12}\theta/2)
\end{aligned}
\tag{31.7}
$$

Or, more compactly

$$
\begin{aligned}
R(\mathbf{x}) &= R\mathbf{x}R^\dagger \\
R &= R_\theta R_\phi \\
R_\phi &= \exp(-\mathbf{e}_{13}\phi/2) \\
R_\theta &= \exp(-\mathbf{e}_{12}\theta/2)
\end{aligned}
\tag{31.8}
$$

Application of these to the $\{\mathbf{e}_i\}$ basis produces the $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ basis. First application of R_ϕ yields the basis vectors for the intermediate rotation.

$$
\begin{aligned}
R_\phi \mathbf{e}_1 {R_\phi}^\dagger &= \mathbf{e}_1(\cos\phi + \mathbf{e}_{13}\sin\phi) &&= \mathbf{e}_1 \cos\phi + \mathbf{e}_3 \sin\phi \\
R_\phi \mathbf{e}_2 {R_\phi}^\dagger &= \mathbf{e}_2 R_\phi {R_\phi}^\dagger &&= \mathbf{e}_2 \\
R_\phi \mathbf{e}_3 {R_\phi}^\dagger &= \mathbf{e}_3(\cos\phi + \mathbf{e}_{13}\sin\phi) &&= \mathbf{e}_3 \cos\phi - \mathbf{e}_1 \sin\phi
\end{aligned}
\tag{31.9}
$$

Applying the second rotation to $R_\phi(\mathbf{e}_i)$ we have

$$
\begin{aligned}
\hat{\mathbf{r}} &= R_\theta(\mathbf{e}_1 \cos\phi + \mathbf{e}_3 \sin\phi){R_\theta}^\dagger \\
&= \mathbf{e}_1 \cos\phi(\cos\theta + \mathbf{e}_{12}\sin\theta) + \mathbf{e}_3 \sin\phi \\
&= \mathbf{e}_1 \cos\phi\cos\theta + \mathbf{e}_2 \cos\phi\sin\theta + \mathbf{e}_3 \sin\phi \\
\hat{\boldsymbol{\theta}} &= R_\theta(\mathbf{e}_2){R_\theta}^\dagger \\
&= \mathbf{e}_2(\cos\theta + \mathbf{e}_{12}\sin\theta) \\
&= -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta \\
\hat{\boldsymbol{\phi}} &= R_\theta(\mathbf{e}_3 \cos\phi - \mathbf{e}_1 \sin\phi){R_\theta}^\dagger \\
&= \mathbf{e}_3 \cos\phi - \mathbf{e}_1 \sin\phi(\cos\theta + \mathbf{e}_{12}\sin\theta) \\
&= -\mathbf{e}_1 \sin\phi\cos\theta - \mathbf{e}_2 \sin\phi\sin\theta + \mathbf{e}_3 \cos\phi
\end{aligned}
\tag{31.10}
$$

In summary these are

$$
\begin{aligned}
\hat{\mathbf{r}} &= \mathbf{e}_1 \cos\phi\cos\theta + \mathbf{e}_2 \cos\phi\sin\theta + \mathbf{e}_3 \sin\phi \\
\hat{\boldsymbol{\theta}} &= -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta \\
\hat{\boldsymbol{\phi}} &= -\mathbf{e}_1 \sin\phi\cos\theta - \mathbf{e}_2 \sin\phi\sin\theta + \mathbf{e}_3 \cos\phi
\end{aligned}
\tag{31.11}
$$

31.2.3 *An alternate pictorial derivation of the unit vectors*

Somewhat more directly, $\hat{\mathbf{r}}$ can be calculated from the coordinate expression of eq. (31.2)

$$
\hat{\mathbf{r}} = \frac{1}{r}(x, y, z),
\tag{31.12}
$$

which was found by inspection of the diagram.

For $\hat{\boldsymbol{\theta}}$, again from the figure, observe that it lies in an latitudinal plane (ie: x, y plane), and is perpendicular to the outwards radial vector in that plane. That is

$$
\hat{\boldsymbol{\theta}} = (\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2)\mathbf{e}_1\mathbf{e}_2
\tag{31.13}
$$

Lastly, $\hat{\boldsymbol{\phi}}$ can be calculated from the dual of $\hat{\mathbf{r}} \wedge \hat{\boldsymbol{\theta}}$

$$
\hat{\boldsymbol{\phi}} = -\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\hat{\mathbf{r}} \wedge \hat{\boldsymbol{\theta}})
\tag{31.14}
$$

Completing the algebra for the expressions above we have

$$
\begin{aligned}
\hat{\mathbf{r}} &= \cos\phi\cos\theta\mathbf{e}_1 + \cos\phi\sin\theta\mathbf{e}_2 + \sin\phi\mathbf{e}_3 \\
\hat{\boldsymbol{\theta}} &= \cos\theta\mathbf{e}_2 - \sin\theta\mathbf{e}_1 \\
\hat{\mathbf{r}}\wedge\hat{\boldsymbol{\theta}} &= \sin\phi\sin\theta\mathbf{e}_1\mathbf{e}_3 + \sin\phi\cos\theta\mathbf{e}_3\mathbf{e}_2 + \cos\phi\mathbf{e}_1\mathbf{e}_2 \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\cos\theta\mathbf{e}_1 - \sin\phi\sin\theta\mathbf{e}_2 + \cos\phi\mathbf{e}_3
\end{aligned}
\tag{31.15}
$$

Sure enough this produces the same result as with the rotor logic.

The rotor approach was purely algebraically and does not have the same reliance on pictures. That may have an additional advantage since one can then study any frame transformations of the general form $\{\mathbf{e}'_i\} = \{R\mathbf{e}_i R^\dagger\}$, and produce results that apply to not only spherical polar coordinate systems but others such as the cylindrical polar.

31.2.4 *Tensor transformation*

Considering a linear transformation providing a mapping from one basis to another of the following form

$$
f_i = \mathcal{L}(e_i) = L e_i L^{-1}
\tag{31.16}
$$

The coordinate representation, or Fourier decomposition, of the vectors in each of these frames is

$$
x = x^i e_i = y^j f_j.
\tag{31.17}
$$

Utilizing a reciprocal frame (ie: not yet requiring an orthonormal frame here), such that $e^i \cdot e_j = {\delta^i}_j$, then dot product provide the coordinate transformations

$$
\begin{aligned}
x^k e_k \cdot e^k &= y^j f_j \cdot e^k \\
y^j f_j \cdot f^i &= x^k e_k \cdot f^i \\
&\implies \\
x^i &= y^j f_j \cdot e^i \\
y^i &= x^j e_j \cdot f^i
\end{aligned}
\tag{31.18}
$$

The transformed reciprocal frame vectors can be expressed directly in terms of the initial reciprocal frame $f^i = \mathcal{L}(e^i)$. Taking dot products confirms this

$$\begin{aligned}
(Le_i L^{-1}) \cdot (Le^j L^{-1}) &= \left\langle Le_i L^{-1} Le^j L^{-1} \right\rangle \\
&= \left\langle Le_i e^j L^{-1} \right\rangle \\
&= e_i \cdot e^j \left\langle LL^{-1} \right\rangle \\
&= e_i \cdot e^j
\end{aligned} \tag{31.19}$$

This implies that the forward and inverse coordinate transformations may be summarized as

$$\begin{aligned}
y^i &= x^j e_j \cdot \mathcal{L}(e^i) \\
x^i &= y^j \mathcal{L}(e_j) \cdot e^i
\end{aligned} \tag{31.20}$$

Or in matrix form

$$\begin{aligned}
\Lambda^i{}_j &= \mathcal{L}(e^i) \cdot e_j \\
\{\Lambda^{-1}\}^i{}_j &= \mathcal{L}(e_j) \cdot e^i \\
y^i &= \Lambda^i{}_j x^j \\
x^i &= \{\Lambda^{-1}\}^i{}_j y^j
\end{aligned} \tag{31.21}$$

The use of inverse notation is justified by the following

$$\begin{aligned}
x^i &= \{\Lambda^{-1}\}^i{}_k y^k \\
&= \{\Lambda^{-1}\}^i{}_k \Lambda^k{}_j x^j \\
\Longrightarrow& \\
\{\Lambda^{-1}\}^i{}_k \Lambda^k{}_j &= \delta^i_j
\end{aligned} \tag{31.22}$$

For the special case where the basis is orthonormal ($e_i \cdot e^j = \delta_i{}^j$), then it can be observed here that the inverse must also be the transpose since the forward and reverse transformation tensors then differ only be a swap of indices.

On notation. Some references such as [34] use $\Lambda^i{}_j$ for both the forward and inverse transformations, with specific conventions about which index is varied to distinguish the two matrices. I have found that confusing and have instead used the explicit inverse notation of [41].

31.2.5 *Gradient after change of coordinates*

With the transformation matrices enumerated above we are now equipped to take the gradient expressed in initial frame

$$\nabla = \sum e^i \frac{\partial}{\partial x^i}, \tag{31.23}$$

and express it in the transformed frame. The chain rule is required for the derivatives in terms of the transformed coordinates

$$\begin{aligned}
\frac{\partial}{\partial x^i} &= \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \\
&= {\Lambda^j}_i \frac{\partial}{\partial y^j} \\
&= \mathcal{L}(e^j) \cdot e_i \frac{\partial}{\partial y^j} \\
&= f^j \cdot e_i \frac{\partial}{\partial y^j}
\end{aligned} \tag{31.24}$$

Therefore the gradient is

$$\begin{aligned}
\nabla &= \sum e^i (f^j \cdot e_i) \frac{\partial}{\partial y^j} \\
&= \sum f^j \frac{\partial}{\partial y^j}
\end{aligned} \tag{31.25}$$

This gets us most of the way towards the desired result for the spherical polar gradient since all that remains is a calculation of the $\partial/\partial y^j$ values for each of the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ directions.

It is also interesting to observe (as in [7]) that the gradient can also be written as

$$\nabla = \frac{1}{f_j} \frac{\partial}{\partial y^j} \tag{31.26}$$

Observe the similarity to the Fourier component decomposition of the vector itself $x = f_i y^i$. Thus, roughly speaking, the differential operator parts of the gradient can be seen to be directional derivatives along the directions of each of the frame vectors.

This is sufficient to read the elements of distance in each of the directions off the figure

$$\begin{aligned}
\delta \mathbf{x} \cdot \hat{\mathbf{r}} &= \delta r \\
\delta \mathbf{x} \cdot \hat{\boldsymbol{\theta}} &= r \cos\phi \delta\theta \\
\delta \mathbf{x} \cdot \hat{\boldsymbol{\phi}} &= r \delta\theta
\end{aligned} \tag{31.27}$$

Therefore the gradient is just

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r\cos\phi}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial\phi} \tag{31.28}$$

Although this last bit has been derived graphically, and not analytically, it does clarify the original question of exactly angle and unit vector conventions were intended in the text (polar angle measured from the North pole, not equator, and θ, and ϕ reversed).

This was the long way to that particular result, but this has been an exploratory treatment of frame rotation concepts that I personally felt the need to clarity for myself.

There are still some additional details that I will explore before concluding (including an analytic treatment of the above).

31.3 TRANSFORMATION OF FRAME VECTORS VS. COORDINATES

To avoid confusion it is worth noting how the frame vectors vs. the components themselves differ under rotational transformation.

31.3.1 *Example. Two dimensional plane rotation*

Consideration of the example of a pair of orthonormal unit vectors for the plane illustrates this

$$\begin{aligned}
\mathbf{e}_1' &= \mathbf{e}_1 \exp(\mathbf{e}_{12}\theta) = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta \\
\mathbf{e}_2' &= \mathbf{e}_2 \exp(\mathbf{e}_{12}\theta) = \mathbf{e}_2 \cos\theta - \mathbf{e}_1 \sin\theta
\end{aligned} \tag{31.29}$$

Forming a matrix for the transformation of these unit vectors we have

$$\begin{bmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} \tag{31.30}$$

Now compare this to the transformation of a vector in its entirety

$$\begin{aligned}
y^1\mathbf{e}_1' + y^2\mathbf{e}_2' &= (x^1\mathbf{e}_1 + x^2\mathbf{e}_2)\exp(\mathbf{e}_{12}\theta) \\
&= x^1(\mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta) + x^2(\mathbf{e}_2\cos\theta - \mathbf{e}_1\sin\theta)
\end{aligned} \tag{31.31}$$

If one uses the standard basis to specify both the rotated point and the original, then taking dot products with $\mathbf{e}_i$ yields the equivalent matrix representation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{31.32}$$

Note how this inverts (transposes) the transformation matrix here compared to the matrix for the transformation of the frame vectors.

31.3.2 *Inverse relations for spherical polar transformations*

The relations of eq. (31.11) can be summarized in matrix form

$$\begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{bmatrix} = \begin{bmatrix}\cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi\\ -\sin\theta & \cos\theta & 0\\ -\sin\phi\cos\theta & -\sin\phi\sin\theta & \cos\phi\end{bmatrix}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix} \tag{31.33}$$

Or, more compactly

$$\begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{bmatrix} = \mathbf{U}\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix} \tag{31.34}$$

This composite rotation can be inverted with a transpose operation, which becomes clear with the factorization

$$\mathbf{U} = \begin{bmatrix}\cos\phi & 0 & \sin\phi\\ 0 & 1 & 0\\ -\sin\phi & 0 & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \tag{31.35}$$

Thus

$$\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix} = \begin{bmatrix}\cos\phi\cos\theta & -\sin\theta & -\sin\phi\cos\theta\\ \cos\phi\sin\theta & \cos\theta & -\sin\phi\sin\theta\\ \sin\phi & 0 & \cos\phi\end{bmatrix}\begin{bmatrix}\hat{\mathbf{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\phi}}\end{bmatrix} \tag{31.36}$$

31.3.3 *Transformation of coordinate vector under spherical polar rotation*

In eq. (31.32) the matrix for the rotation of a coordinate vector for the plane rotation was observed to be the transpose of the matrix that transformed the frame vectors themselves. This is also the case in this spherical polar case, as can be seen by forming a general vector and applying equation eq. (31.33) to the standard basis vectors.

$$\begin{aligned}
x^1\mathbf{e}_1 &\rightarrow x^1(\cos\phi\cos\theta\mathbf{e}_1 + \cos\phi\sin\theta\mathbf{e}_2 + \sin\phi\mathbf{e}_3)\\
x^2\mathbf{e}_2 &\rightarrow x^2(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2)\\
x^3\mathbf{e}_3 &\rightarrow x^3(-\sin\phi\cos\theta\mathbf{e}_1 - \sin\phi\sin\theta\mathbf{e}_2 + \cos\phi\mathbf{e}_3)
\end{aligned} \tag{31.37}$$

Summing this and regrouping (ie: a transpose operation) one has:

$$\begin{aligned}
x^i \mathbf{e}_i &\rightarrow y^i \mathbf{e}_i \\
&\mathbf{e}_1(x^1 \cos\phi \cos\theta - x^2 \sin\theta - x^3 \sin\phi \cos\theta) \\
&+ \mathbf{e}_2(x^1 \cos\phi \sin\theta + x^2 \cos\theta - x^3 \sin\phi \sin\theta) \\
&+ \mathbf{e}_3(x^1 \sin\phi + x^3 \cos\phi)
\end{aligned} \tag{31.38}$$

taking dot products with $\mathbf{e}_i$ produces the matrix form

$$\begin{aligned}
\begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} &= \begin{bmatrix} \cos\phi\cos\theta & -\sin\theta & -\sin\phi\cos\theta \\ \cos\phi\sin\theta & \cos\theta & -\sin\phi\sin\theta \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}
\end{aligned} \tag{31.39}$$

As observed in 30 the matrix for this transformation of the coordinate vector under the composite x, y rotation followed by an x', z rotation ends up expressed as the product of the elementary rotations, but applied in reverse order!

32

ROTOR INTERPOLATION CALCULATION

The aim is to compute the interpolating rotor r that takes an object from one position to another in n steps. Here the initial and final positions are given by two rotors R_1, and R_2 like so

$$
\begin{aligned}
X_1 &= R_1 X {R_1}^\dagger \\
X_2 &= R_2 X {R_2}^\dagger = r^n R_1 X {R_1}^\dagger {r^n}^\dagger
\end{aligned}
\tag{32.1}
$$

So, writing

$$
a = r^n = R_2 \frac{1}{R_1} = \frac{R_2 {R_1}^\dagger}{R_1 {R_1}^\dagger} = \cos\theta + I \sin\theta
\tag{32.2}
$$

So,

$$
\begin{aligned}
\frac{\langle a \rangle_2}{\langle a \rangle} &= \frac{\langle a \rangle_2}{|\langle a \rangle_2|} \frac{|\langle a \rangle_2|}{\langle a \rangle} \\
&= I \tan\theta
\end{aligned}
\tag{32.3}
$$

Therefore the interpolating rotor is:

$$
\begin{aligned}
I &= \frac{\langle a \rangle_2}{|\langle a \rangle_2|} \\
\theta &= \operatorname{atan2}\left(|\langle a \rangle_2|, \langle a \rangle\right) \\
r &= \cos(\theta/n) + I \sin(\theta/n)
\end{aligned}
\tag{32.4}
$$

In [11], equation 10.15, they have got something like this for a fractional angle, but then say that they do not use that in software, instead using r directly, with a comment about designing more sophisticated algorithms (bivector splines). That spline comment in particular sounds interesting. Sounds like the details on that are to be found in the journals mentioned in Further Reading section of chapter 10.

33

EXPONENTIAL OF A BLADE

33.1 MOTIVATION

Exponentials of bivectors and complex numbers are useful as generators of rotations, and exponentials of square matrices can be used in linear differential equation solution.

How about exponentials of vectors?

Because any power of a vector can be calculated it should be perfectly well defined to use the exponential infinite series with k-vector parameters. An exponential function of this form will be expanded explicitly and compared to the real number result. The first derivative will also be calculated to examine its form.

In addition for completeness, the bivector and quaternion exponential forms will be examined.

33.2 VECTOR EXPONENTIAL

The infinite series representation of the exponential defines a function for any x that can be repeatedly multiplied with it self.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{33.1}$$

Depending on the type of the parameter x this may or may not have properties consistent with the real number exponential function. For a vector $\mathbf{x} = \hat{\mathbf{x}}|\mathbf{x}|$, after splitting the sum into even and odd terms this infinite series takes the following form:

$$e^{\pm\mathbf{x}} = \sum_{k=0}^{\infty} \frac{\mathbf{x}^{2k}}{(2k)!} \pm \sum_{k=0}^{\infty} \frac{|\mathbf{x}|^{2k}|\mathbf{x}|\hat{\mathbf{x}}}{(2k+1)!}$$

$$\implies e^{\pm\mathbf{x}} = \cosh|\mathbf{x}| \pm \hat{\mathbf{x}}\sinh|\mathbf{x}| \tag{33.2}$$

One can also employ symmetric and antisymmetric sums to write the hyperbolic functions in terms of the vector exponentials:

$$\cosh|\mathbf{x}| = \frac{e^{\mathbf{x}} + e^{-\mathbf{x}}}{2} \tag{33.3}$$

$$\sinh|\mathbf{x}| = \frac{e^{\mathbf{x}} - e^{-\mathbf{x}}}{2\hat{\mathbf{x}}} \tag{33.4}$$

33.2.1 *Vector Exponential derivative*

One of the defining properties of the exponential is that its derivative is related to itself

$$\frac{de^x}{du} = \frac{dx}{du} e^x = e^x \frac{dx}{du} \tag{33.5}$$

For a vector parameter $\mathbf{x}$ one should not generally expect that. Let us expand this to see the form of this derivative:

$$\begin{aligned}\frac{de^{\mathbf{x}}}{du} &= \frac{d}{du}(\cosh|\mathbf{x}| + \hat{\mathbf{x}}\sinh|\mathbf{x}|) \\ &= (\sinh|\mathbf{x}| + \hat{\mathbf{x}}\cosh|\mathbf{x}|)\frac{d|\mathbf{x}|}{du} + \frac{d\hat{\mathbf{x}}}{du}\sinh|\mathbf{x}|\end{aligned} \tag{33.6}$$

Can calculate $\frac{d|\mathbf{x}|}{du}$ with the usual trick:

$$\frac{d|\mathbf{x}|^2}{du} = 2|\mathbf{x}|\frac{d|\mathbf{x}|}{du} = \frac{d\mathbf{x}}{du}\mathbf{x} + \mathbf{x}\frac{d\mathbf{x}}{du} = 2\frac{d\mathbf{x}}{du} \cdot \mathbf{x} \tag{33.7}$$

$$\implies \frac{d|\mathbf{x}|}{du} = \frac{d\mathbf{x}}{du} \cdot \hat{\mathbf{x}} \tag{33.8}$$

Calculation of $\frac{d\hat{\mathbf{x}}}{du}$ uses this result:

$$\begin{aligned}
\frac{d\hat{\mathbf{x}}}{du} &= \frac{d}{du}\frac{\mathbf{x}}{|\mathbf{x}|} \\
&= \frac{d\mathbf{x}}{du}\frac{1}{|\mathbf{x}|} - \frac{\mathbf{x}}{|\mathbf{x}|^2}\frac{d|\mathbf{x}|}{du} \\
&= \frac{d\mathbf{x}}{du}\frac{1}{|\mathbf{x}|} - \frac{\mathbf{x}}{|\mathbf{x}|^2}\frac{d\mathbf{x}}{du}\cdot\hat{\mathbf{x}} \\
&= \frac{1}{|\mathbf{x}|}\left(\frac{d\mathbf{x}}{du} - \hat{\mathbf{x}}\left(\frac{d\mathbf{x}}{du}\cdot\hat{\mathbf{x}}\right)\right) \\
&= \frac{\hat{\mathbf{x}}}{|\mathbf{x}|}\left(\hat{\mathbf{x}}\wedge\frac{d\mathbf{x}}{du}\right) \\
&= \frac{1}{|\mathbf{x}|}\mathrm{Rej}_{\hat{\mathbf{x}}}\left(\frac{d\mathbf{x}}{du}\right)
\end{aligned} \tag{33.9}$$

Putting these together one write the derivative in a few ways:

$$\begin{aligned}
\frac{de^{\mathbf{x}}}{du} &= \left(\frac{d\mathbf{x}}{du}\cdot\hat{\mathbf{x}}\right)\hat{\mathbf{x}}(\hat{\mathbf{x}}\sinh|\mathbf{x}| + \cosh|\mathbf{x}|) + \frac{\hat{\mathbf{x}}}{|\mathbf{x}|}\left(\hat{\mathbf{x}}\wedge\frac{d\mathbf{x}}{du}\right)\sinh|\mathbf{x}| \\
&= \hat{\mathbf{x}}\left(\frac{d\mathbf{x}}{du}\cdot\hat{\mathbf{x}}\right)e^{\mathbf{x}} + \frac{\hat{\mathbf{x}}}{|\mathbf{x}|}\left(\hat{\mathbf{x}}\wedge\frac{d\mathbf{x}}{du}\right)\sinh|\mathbf{x}| \\
&= \mathrm{Proj}_{\hat{\mathbf{x}}}\left(\frac{d\mathbf{x}}{du}\right)e^{\mathbf{x}} + \frac{1}{|\mathbf{x}|}\mathrm{Rej}_{\hat{\mathbf{x}}}\left(\frac{d\mathbf{x}}{du}\right)\sinh|\mathbf{x}|
\end{aligned} \tag{33.10}$$

This is considerably different from the real number case. Only when the vector $\mathbf{x}$ and all its variation $\frac{d\mathbf{x}}{du}$ are colinear does $\frac{d\mathbf{x}}{du} = \mathrm{Proj}_{\hat{\mathbf{x}}}\left(\frac{d\mathbf{x}}{du}\right)$ for the real number like result:

$$\frac{de^{\mathbf{x}}}{du} = \frac{d\mathbf{x}}{du}e^{\mathbf{x}} = e^{\mathbf{x}}\frac{d\mathbf{x}}{du} \tag{33.11}$$

Note that the sinh term can be explicitly removed

$$\frac{de^{\mathbf{x}}}{du} = \left(\hat{\mathbf{x}}\left(\frac{d\mathbf{x}}{du}\cdot\hat{\mathbf{x}}\right) - \frac{1}{2|\mathbf{x}|}\left(\hat{\mathbf{x}}\wedge\frac{d\mathbf{x}}{du}\right)\right)e^{\mathbf{x}} - \frac{1}{2|\mathbf{x}|}\left(\hat{\mathbf{x}}\wedge\frac{d\mathbf{x}}{du}\right)e^{-\mathbf{x}} \tag{33.12}$$

, but without a $\mathrm{Rej}_{\hat{\mathbf{x}}}\left(\frac{d\mathbf{x}}{du}\right) = 0$ constraint, there will always be a term that is not proportional to $e^{\mathbf{x}}$.

33.3 BIVECTOR EXPONENTIAL

The bivector exponential can be expanded utilizing its complex number equivalence:

$$
\begin{aligned}
e^{\mathbf{B}} &= e^{\hat{\mathbf{B}}|\mathbf{B}|} \\
&= \cos|\mathbf{B}| + \hat{\mathbf{B}}\sin|\mathbf{B}|
\end{aligned}
\tag{33.13}
$$

So, taking the derivative we have

$$
\begin{aligned}
(e^{\mathbf{B}})' &= \left(-\sin|\mathbf{B}| + \hat{\mathbf{B}}\cos|\mathbf{B}|\right)|\mathbf{B}|' + \hat{\mathbf{B}}'\sin|\mathbf{B}| \\
&= \hat{\mathbf{B}}\left(\hat{\mathbf{B}}\sin|\mathbf{B}| + \cos|\mathbf{B}|\right)|\mathbf{B}|' + \hat{\mathbf{B}}'\sin|\mathbf{B}| \\
&= \hat{\mathbf{B}}e^{\mathbf{B}}|\mathbf{B}|' + \hat{\mathbf{B}}'\sin|\mathbf{B}| \\
&= e^{\mathbf{B}}\hat{\mathbf{B}}|\mathbf{B}|' + \hat{\mathbf{B}}'\sin|\mathbf{B}|
\end{aligned}
\tag{33.14}
$$

33.3.1 *bivector magnitude derivative*

As with the vector case we have got a couple helper derivatives required. Here is the first:

$$
\begin{aligned}
&(|\mathbf{B}|^2)' = 2|\mathbf{B}||\mathbf{B}|' = -(\mathbf{B}\mathbf{B}' + \mathbf{B}'\mathbf{B}) \\
&\implies \\
&|\mathbf{B}|' = -\frac{\hat{\mathbf{B}}\mathbf{B}' + \mathbf{B}'\hat{\mathbf{B}}}{2}
\end{aligned}
\tag{33.15}
$$

Unlike the vector case this last expression is not a bivector dot product $= -\hat{\mathbf{B}} \cdot \mathbf{B}'$ since there could be a $\langle\rangle_4$ term that this symmetric sum would also include. That wedge term would be zero for example if $\mathbf{B} = \mathbf{x} \wedge \mathbf{k}$ for a constant vector $\mathbf{k}$.

33.3.2 *Unit bivector derivative*

Now calculate $\hat{\mathbf{B}}'$:

$$
\begin{aligned}
\hat{\mathbf{B}}' &= \frac{\mathbf{B}'}{|\mathbf{B}|} - \frac{\mathbf{B}}{|\mathbf{B}|^2}|\mathbf{B}|' \\
&= \frac{1}{|\mathbf{B}|}\left(\mathbf{B}' + \hat{\mathbf{B}}\frac{\hat{\mathbf{B}}\mathbf{B}' + \mathbf{B}'\hat{\mathbf{B}}}{2}\right) \\
&= \frac{1}{2|\mathbf{B}|}\left(\mathbf{B}' + \hat{\mathbf{B}}\mathbf{B}'\hat{\mathbf{B}}\right) \\
&= \frac{\hat{\mathbf{B}}}{|\mathbf{B}|}\frac{-\hat{\mathbf{B}}\mathbf{B}' + \mathbf{B}'\hat{\mathbf{B}}}{2}
\end{aligned}
\tag{33.16}
$$

Thus, the derivative is a scaled bivector rejection:

$$
\hat{\mathbf{B}}' = \frac{1}{\mathbf{B}}\left\langle\hat{\mathbf{B}}\mathbf{B}'\right\rangle_2
\tag{33.17}
$$

Although this appears different from a unit vector derivative, a slight adjustment highlights the similarities:

$$
\begin{aligned}
\hat{\mathbf{r}}' &= \frac{\hat{\mathbf{r}}}{|\mathbf{r}|}\hat{\mathbf{r}} \wedge \mathbf{r}' \\
&= \frac{1}{\mathbf{r}}\langle\hat{\mathbf{r}}\mathbf{r}'\rangle_2
\end{aligned}
\tag{33.18}
$$

Note however the sign inversion that is built into the bivector inversion.

33.3.3 *combining results*

Putting the individual results back together we have:

$$
(e^{\mathbf{B}})' = \frac{1}{\hat{\mathbf{B}}}\frac{\hat{\mathbf{B}}\mathbf{B}' + \mathbf{B}'\hat{\mathbf{B}}}{2}e^{\mathbf{B}} + \frac{1}{\mathbf{B}}\left\langle\hat{\mathbf{B}}\mathbf{B}'\right\rangle_2 \sin|\mathbf{B}|
\tag{33.19}
$$

In general with bivectors we can have two sorts of perpendicularity. The first is perpendicular but intersecting (generated by the grade 2 term of the product), and perpendicular with no common line (generated by the grade 4 term). In $\mathbb{R}^3$ we have only the first sort.

With a restriction that the derivative only changes the bivector enough to introduce the first term, this exponential derivative is reduced to:

$$\begin{aligned}(e^{\mathbf{B}})' &= \frac{1}{\hat{\mathbf{B}}}\hat{\mathbf{B}} \cdot \mathbf{B}' e^{\mathbf{B}} + \frac{1}{\mathbf{B}}\left\langle \hat{\mathbf{B}}\mathbf{B}' \right\rangle_2 \sin|\mathbf{B}| \\ &= \mathrm{Proj}_{\hat{\mathbf{B}}}(\mathbf{B}')e^{\mathbf{B}} + \frac{1}{|\mathbf{B}|}\mathrm{Rej}_{\hat{\mathbf{B}}}(\mathbf{B}') \sin|\mathbf{B}|\end{aligned} \tag{33.20}$$

Only if the bivector variation is in the same plane as the bivector itself can the $\langle\rangle_2$ term be dropped in which case, since the derivative will equal its projection one has:

$$(e^{\mathbf{B}})' = \mathbf{B}' e^{\mathbf{B}} = e^{\mathbf{B}}\mathbf{B}' \tag{33.21}$$

33.4 QUATERNION EXPONENTIAL DERIVATIVE

Using the phrase somewhat loosely a quaternion, or complex number is a multivector of the form

$$\alpha + \mathbf{B} \tag{33.22}$$

Where α is a scalar, and $\mathbf{B}$ is a bivector.

Using the results above, the derivative of a quaternion exponential (ie: a rotation operator) will be

$$\begin{aligned}(e^{\alpha+\mathbf{B}})' &= (e^{\alpha})' e^{\mathbf{B}} + e^{\alpha}(e^{\mathbf{B}})' \\ &= \alpha' e^{\alpha+\mathbf{B}} + e^{\alpha}\frac{1}{\hat{\mathbf{B}}}\frac{\hat{\mathbf{B}}\mathbf{B}' + \mathbf{B}'\hat{\mathbf{B}}}{2} e^{\mathbf{B}} + \frac{1}{\mathbf{B}}\left\langle \hat{\mathbf{B}}\mathbf{B}' \right\rangle_2 e^{\alpha} \sin|\mathbf{B}|\end{aligned} \tag{33.23}$$

For the total derivative:

$$(e^{\alpha+\mathbf{B}})' = \left(\alpha' + \frac{1}{\hat{\mathbf{B}}}\frac{\hat{\mathbf{B}}\mathbf{B}' + \mathbf{B}'\hat{\mathbf{B}}}{2}\right) e^{\alpha+\mathbf{B}} + \frac{1}{\mathbf{B}}\left\langle \hat{\mathbf{B}}\mathbf{B}' \right\rangle_2 e^{\alpha} \sin|\mathbf{B}| \tag{33.24}$$

As with the bivector case, the two restrictions $\left\langle \hat{\mathbf{B}}\mathbf{B}' \right\rangle_2 = 0$, and $\left\langle \hat{\mathbf{B}}\mathbf{B}' \right\rangle_4 = 0$ are required to get a real number like exponential derivative:

$$(e^{\alpha+\mathbf{B}})' = (\alpha + \mathbf{B})' e^{\alpha+\mathbf{B}} = e^{\alpha+\mathbf{B}} (\alpha + \mathbf{B})' \tag{33.25}$$

Note that both of these are true for the important class of multivectors, the complex number.

33.4.1 *bivector with only one degree of freedom*

For a bivector that includes a constant vector such as $\mathbf{B} = \mathbf{x} \wedge \mathbf{k}$ there will be no $\langle\rangle_4$ term.

$$\left\langle \hat{\mathbf{B}}\mathbf{B}' \right\rangle_4 \propto \langle \mathbf{x} \wedge \mathbf{k}\mathbf{x}' \wedge \mathbf{k} \rangle_4 = \mathbf{x} \wedge \mathbf{k} \wedge \mathbf{x}' \wedge \mathbf{k} = 0 \tag{33.26}$$

Suppose $\alpha + \mathbf{B} = \mathbf{x}\mathbf{k} = \mathbf{x} \cdot \mathbf{k} + \mathbf{x} \wedge \mathbf{k}$. In this case this quaternion exponential derivative reduces to

$$\begin{aligned}(e^{\mathbf{x}\mathbf{k}})' &= \left(\mathbf{x}' \cdot \mathbf{k} + \frac{1}{\mathbf{x} \wedge \mathbf{k}}(\mathbf{x} \wedge \mathbf{k}) \cdot (\mathbf{x}' \wedge \mathbf{k})\right) e^{\mathbf{x}\mathbf{k}} \\ &+ \frac{1}{\mathbf{x} \wedge \mathbf{k}} \left\langle \frac{\mathbf{x} \wedge \mathbf{k}}{|\mathbf{x} \wedge \mathbf{k}|} \mathbf{x}' \wedge \mathbf{k} \right\rangle_2 e^{\mathbf{x} \cdot \mathbf{k}} \sin|\mathbf{x} \wedge \mathbf{k}|\end{aligned} \tag{33.27}$$

It is only with the addition restriction that all the bivector variation lies in the plane $\mathbf{i} = \frac{\mathbf{x} \wedge \mathbf{k}}{|\mathbf{x} \wedge \mathbf{k}|}$. ie:

$$\langle \mathbf{x} \wedge \mathbf{k}\mathbf{x}' \wedge \mathbf{k} \rangle_2 = 0 \tag{33.28}$$

does one have:

$$\begin{aligned}(e^{\mathbf{x}\mathbf{k}})' &= \left(\mathbf{x}' \cdot \mathbf{k} + \mathrm{Proj}_{\mathbf{i}}(\mathbf{x}' \wedge \mathbf{k})\right) e^{\mathbf{x}\mathbf{k}} \\ &= \left(\mathbf{x}' \cdot \mathbf{k} + \mathbf{x}' \wedge \mathbf{k}\right) e^{\mathbf{x}\mathbf{k}}\end{aligned} \tag{33.29}$$

Thus with these two restrictions to the variation of the bivector term we have:

$$(e^{\mathbf{x}\mathbf{k}})' = \mathbf{x}'\mathbf{k}e^{\mathbf{x}\mathbf{k}} = e^{\mathbf{x}\mathbf{k}}\mathbf{x}'\mathbf{k} \tag{33.30}$$

34

GENERATOR OF ROTATIONS IN ARBITRARY DIMENSIONS

34.1 MOTIVATION

Eli in his recent blog post on angular momentum operators used an exponential operator to generate rotations

$$R_{\Delta\theta} = e^{\Delta\theta \hat{\mathbf{n}} \cdot (\mathbf{x} \times \boldsymbol{\nabla})} \tag{34.1}$$

This is something I hhad not seen before, but is comparable to the vector shift operator expressed in terms of directional derivatives $\mathbf{x} \cdot \boldsymbol{\nabla}$

$$f(\mathbf{x} + \mathbf{a}) = e^{\mathbf{a} \cdot \boldsymbol{\nabla}} f(\mathbf{x}) \tag{34.2}$$

The translation operator of eq. (34.2) translates easily to higher dimensions. Of particular interest is the Minkowski metric 4D spacetime case, where we can use the four gradient $\nabla = \gamma^\mu \partial_\mu$, and a vector spacetime translation of $x = x^\mu \gamma_\mu \rightarrow (x^\mu + a^\mu)\gamma_\mu$ to translate "trivially" translate this

$$f(x + a) = e^{a \cdot \nabla} f(x) \tag{34.3}$$

Since we do not have a cross product of two vectors in a 4D space, re-expressing eq. (34.1) in a form that is not tied to three dimensions is desirable. A duality transformation with $\hat{\mathbf{n}} = i\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ accomplishes this, where i is a unit bivector for the plane perpendicular to $\hat{\mathbf{n}}$ (i.e. product of two perpendicular unit vectors in the plane). That duality transformation, expressing the rotation direction using an oriented plane instead of the normal to the plane gives us

$$\begin{aligned}
\hat{\mathbf{n}} \cdot (\mathbf{x} \times \boldsymbol{\nabla}) &= \langle \hat{\mathbf{n}} (\mathbf{x} \times \boldsymbol{\nabla}) \rangle \\
&= \langle (i\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(-\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{x} \wedge \boldsymbol{\nabla}) \rangle \\
&= \langle i(\mathbf{x} \wedge \boldsymbol{\nabla}) \rangle
\end{aligned} \tag{34.4}$$

This is just $i \cdot (\mathbf{x} \wedge \boldsymbol{\nabla})$, so the generator of the rotation in 3D is

$$R_{\Delta\theta} = e^{\Delta\theta i \cdot (\mathbf{x} \wedge \boldsymbol{\nabla})} \tag{34.5}$$

It is reasonable to guess then that we could substitute the spacetime gradient and allow i to be any 4D unit spacetime bivector, where a spacelike product pair will generate rotations and a spacetime bivector will generate boosts. That is really just a notational shift, and we would write

$$R_{\Delta\theta} = e^{\Delta\theta i \cdot (x \wedge \nabla)} \tag{34.6}$$

This is very likely correct, but building up to this guess in a logical sequence from a known point will be the aim of this particular exploration.

34.2 SETUP AND CONVENTIONS

Rather than expressing the rotation in terms of coordinates, here the rotation will be formulated in terms of dual sided multivector operators (using Geometric Algebra) on vectors. Then employing the chain rule an examination of the differential change of a multivariable scalar valued function on the underlying rotation will be made.

Following conventions of [10] vectors will be undecorated rather than boldface since we are deriving results applicable to four vector (and higher) spaces, and not requiring an Euclidean metric.

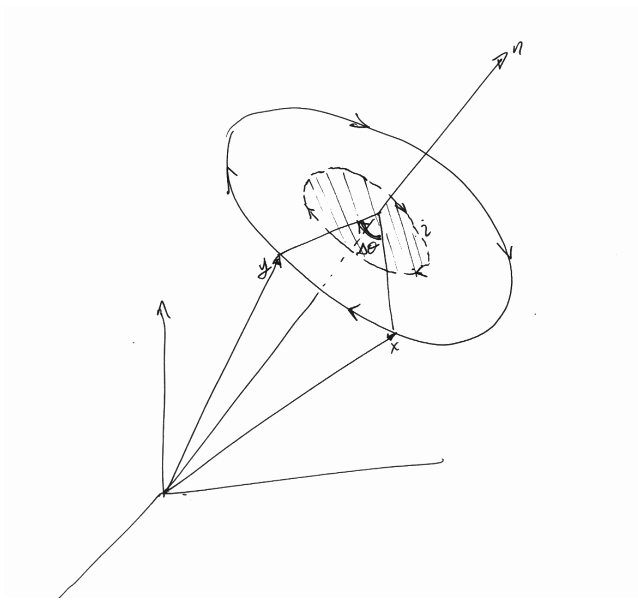

Figure 34.1: Rotating vector in the plane with bivector i

The fig. 34.1 has a pair of vectors related by rotation, where the vector $x(\theta)$ is rotated to $y(\theta) = x(\theta + \Delta\theta)$. We choose here to express this rotation using a quaternion-ic operator $R = \alpha + ab$, where α is a scalar and a, and b are vectors.

$$y = \tilde{R}xR \tag{34.7}$$

Required of R is an invertability property, but without loss of generality we can impose a strictly unitary property $\tilde{R}R = 1$. Here $\tilde{R}$ denotes the multivector reverse of a Geometric product

$$(ab)\tilde{} = \tilde{b}\tilde{a} \tag{34.8}$$

Where for individual vectors the reverse is itself $\tilde{a} = a$. A singly parametrized rotation or boost can be conveniently expressed using the half angle exponential form

$$R = e^{i\theta/2} \tag{34.9}$$

where $i = \hat{u}\hat{v}$ is a unit bivector, a product of two perpendicular unit vectors ($\hat{u}\hat{v} = -\hat{v}\hat{u}$). For rotations $\hat{u}$, and $\hat{v}$ are both spatial vectors, implying $i^2 = -1$. For boosts i is the product of a unit timelike vector and unit spatial vector, and with a Minkowski metric condition $\hat{u}^2\hat{v}^2 = -1$, we have a positive square $i^2 = 1$ for our spacetime rotation plane i.

A general Lorentz transformation, containing a composition of rotations and boosts can be formed by application of successive transformations

$$\mathcal{L}(x) = (\tilde{U}(\tilde{T} \cdots (\tilde{S}xS)T) \cdots U) = \tilde{U}\tilde{T} \cdots \tilde{S}xST \cdots U \tag{34.10}$$

The composition still has the unitary property $(ST \cdots U)\tilde{}ST \cdots U = 1$, so when the specifics of the parametrization are not required we will allow the rotation operator $R = ST \cdots U$ to be a general composition of individual rotations and boosts.

We will have brief use of coordinates and employ a reciprocal basis pair $\{\gamma^\mu\}$ and $\{\gamma_\nu\}$ where $\gamma^\mu \cdot \gamma_\nu = \delta^\mu{}_\nu$. A vector, employing summation convention, is then denoted

$$x = x^\mu\gamma_\mu = x_\mu\gamma^\mu \tag{34.11}$$

Where

$$\begin{aligned} x_\mu &= x \cdot \gamma_\mu \\ x_\mu &= x \cdot \gamma_\mu \end{aligned} \tag{34.12}$$

Shorthand for partials

$$
\begin{aligned}
\partial_\mu &\equiv \frac{\partial}{\partial x^\mu} \\
\partial^\mu &\equiv \frac{\partial}{\partial x_\mu}
\end{aligned}
\tag{34.13}
$$

will allow the gradient to be expressed as

$$
\nabla \equiv \gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu \tag{34.14}
$$

The perhaps unintuitive mix of upper and lower indices is required to make the indices in the direction derivative come out right when expressed as a dot product

$$
\lim_{\tau \to 0} \frac{f(x + a\tau) - f(x)}{\tau} = a^\mu \partial_\mu f(x) = a \cdot \nabla f(x) \tag{34.15}
$$

34.3 ROTOR EXAMPLES

While not attempting to discuss the exponential rotor formulation in any depth, at least illustrating by example for a spatial rotation and Lorentz boost seems called for.

Application of either of these is most easily performed with a split of the vector into components parallel and perpendicular to the "plane" of rotation i. For example suppose we decompose a vector $x = p + n$ where n is perpendicular to the rotation plane i (i.e. $ni = in$), and p is the components in the plane ($pi = -ip$). A consequence is that n commutes with R and p induces a conjugate effect in the rotor

$$
\begin{aligned}
\tilde{R} x R &= e^{-i\theta/2} (p + n) e^{i\theta/2} \\
&= p e^{i\theta/2} e^{i\theta/2} + n e^{-i\theta/2} e^{i\theta/2}
\end{aligned}
\tag{34.16}
$$

This is then just

$$
\tilde{R} x R = p e^{i\theta} + n \tag{34.17}
$$

To expand any further the metric details are required. The half angle rotors of eq. (34.9) can be expanded in series, where the metric properties of the bivector dictate the behavior. In the spatial bivector case, where $i^2 = -1$ we have

$$
R = e^{i\theta/2} = \cos(\theta/2) + i \sin(\theta/2) \tag{34.18}
$$

whereas when $i^2 = 1$, the series expansion yields a hyperbolic pair

$$R = e^{i\theta/2} = \cosh(\theta/2) + i \sinh(\theta/2) \tag{34.19}$$

To make things more specific, and relate to the familiar, consider a rotation in the Euclidean x, y plane where we pick $i = \mathbf{e}_1\mathbf{e}_2$, and rotate $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Applying eq. (34.17), and eq. (34.18) we have

$$\tilde{R}\mathbf{x}R = (x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\theta + \mathbf{e}_1\mathbf{e}_2 \sin\theta) + z\mathbf{e}_3 \tag{34.20}$$

We have $\mathbf{e}_1{}^2 = \mathbf{e}_2{}^2 = 1$ and $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2$, so with some rearrangement

$$\tilde{R}\mathbf{x}R = \mathbf{e}_1(x\cos\theta - y\sin\theta) + \mathbf{e}_2(x\sin\theta + y\cos\theta) + \mathbf{e}_3 z \tag{34.21}$$

This is the familiar x, y plane rotation up to a possible sign preference. Observe that we have the flexibility to adjust the sign of the rotation by altering either θ or i (we could use $i = \mathbf{e}_2\mathbf{e}_1$ for example). Because of this Hestenes [19] chooses to make the angle bivector valued, so instead of $i\theta$ writes

$$R = e^B \tag{34.22}$$

where B is bivector valued, and thus contains the sign or direction of the rotation or boost as well as the orientation.

For completeness lets also expand a rotor application for an x-axis boost in the spacetime plane $i = \gamma_1\gamma_0$. Following [10], we use the $(+, -, -, -)$ metric convention $1 = \gamma_0{}^2 = -\gamma_1{}^2 = -\gamma_2{}^2 = -\gamma_3{}^2$. Switching variable conventions to match the norm lets use α for the rapidity angle, with x-axis boost rotor

$$R = e^{\gamma_1\gamma_0\alpha/2} \tag{34.23}$$

for the rapidity angle α. The rotor application then gives

$$\begin{aligned}
\mathcal{L}(x) &= \tilde{R}(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3)R \\
&= \tilde{R}(x^0\gamma_0 + x^1\gamma_1)R + x^2\gamma_2 + x^3\gamma_3 \\
&= (x^0\gamma_0 + x^1\gamma_1)(\cosh(\theta) + \gamma_1\gamma_0 \sinh(\theta/2)) + x^2\gamma_0 + x^3\gamma_3
\end{aligned} \tag{34.24}$$

A final bit of rearrangement yields the familiar

$$\begin{aligned} \mathcal{L}(x) &= \gamma_0(x^0 \cosh(\theta) - x^1 \sinh(\theta/2)) \\ &\qquad + \gamma_1(-x^0 \sinh(\theta/2) + x^1 \cosh(\theta)) + x^2\gamma_0 + x^3\gamma_3 \end{aligned} \tag{34.25}$$

Again observe the flexibility to adjust the sign as desired by either the bivector orientation or the sign of the scalar rapidity angle.

34.4 THE ROTATION OPERATOR

Moving on to the guts. From eq. (34.7) we can express x in terms of y using the inverse transformation

$$x = Ry\tilde{R} \tag{34.26}$$

Assuming R is parametrized by θ, and that both x and y are not directly dependent on θ, we have

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{dR}{d\theta}y\tilde{R} + Ry\frac{d\tilde{R}}{d\theta} \\ &= \left(\frac{dR}{d\theta}\tilde{R}\right)(Ry\tilde{R}) + (Ry\tilde{R})\left(R\frac{d\tilde{R}}{d\theta}\right) \\ &= \left(\frac{dR}{d\theta}\tilde{R}\right)x + x\left(R\frac{d\tilde{R}}{d\theta}\right) \end{aligned} \tag{34.27}$$

Since we also have $R\tilde{R} = 1$, this product has zero derivative

$$0 = \frac{d(R\tilde{R})}{d\theta} = \frac{dR}{d\theta}\tilde{R} + R\frac{d\tilde{R}}{d\theta} \tag{34.28}$$

Labeling one of these, say

$$\Omega \equiv \frac{dR}{d\theta}\tilde{R} \tag{34.29}$$

The multivector Ω must in fact be a bivector. As the product of a grade $0, 2$ multivector with another $0, 2$ multivector, the product may have grades $0, 2, 4$. Since reversing Ω negates it, this product can only have grade 2 components. In particular, employing the exponential representation of R from eq. (34.9) for a simply parametrized rotation (or boost), we have

$$\Omega = \frac{i}{2} e^{i\theta/2} e^{-i\theta/2} = \frac{i}{2} \tag{34.30}$$

With this definition we have a

complete description of the incremental (first order) rotational along the curve from x to y induced by R via the commutator of this bivector Ω with the initial position vector x.

$$\frac{dx}{d\theta} = [\Omega, x] = \frac{1}{2}(ix - xi) \tag{34.31}$$

This commutator is in fact the generalize bivector-vector dot product $[\Omega, x] = i \cdot x$, and is vector valued.

Now consider a scalar valued function $f = f(x(\theta))$. Employing the chain rule, for the *theta* derivative of f we have a contribution from each coordinate x^μ. That is

$$\begin{aligned} \frac{df}{d\theta} &= \sum_\mu \frac{dx^\mu}{d\theta} \frac{\partial f}{\partial x^\mu} \\ &= \frac{dx^\mu}{d\theta} \partial_\mu f \\ &= \left(\frac{dx^\mu}{d\theta} \gamma_\mu \right) \cdot \left(\gamma^\nu \partial_\nu \right) f \end{aligned} \tag{34.32}$$

But this is just

$$\frac{df}{d\theta} = \frac{dx}{d\theta} \cdot \nabla f \tag{34.33}$$

Or in operator form

$$\frac{d}{d\theta} = (i \cdot x) \cdot \nabla \tag{34.34}$$

The complete Taylor expansion of $f(\theta) = f(x(\theta))$ is therefore

$$\begin{aligned} f(x(\theta + \Delta\theta)) &= \sum_{k=0}^\infty \frac{1}{k!} \left(\Delta\theta \frac{d}{d\theta} \right)^k f(x(\theta)) \\ &= \sum_{k=0}^\infty \frac{1}{k!} \left(\Delta\theta (i \cdot x) \cdot \nabla \right)^k f(x(\theta)) \end{aligned} \tag{34.35}$$

Expressing this sum formally as an exponential we have

$$f(x(\theta + \Delta\theta)) = e^{\Delta\theta(i\cdot x)\cdot\nabla} f(x(\theta)) \tag{34.36}$$

In this form, the product $(i \cdot x) \cdot \nabla$ does not look much like the cross or wedge product representations of the angular momentum operator that was initially guessed at. Referring to fig. 34.2 let us make a couple observations about this particular form before translating back to the wedge formulation.

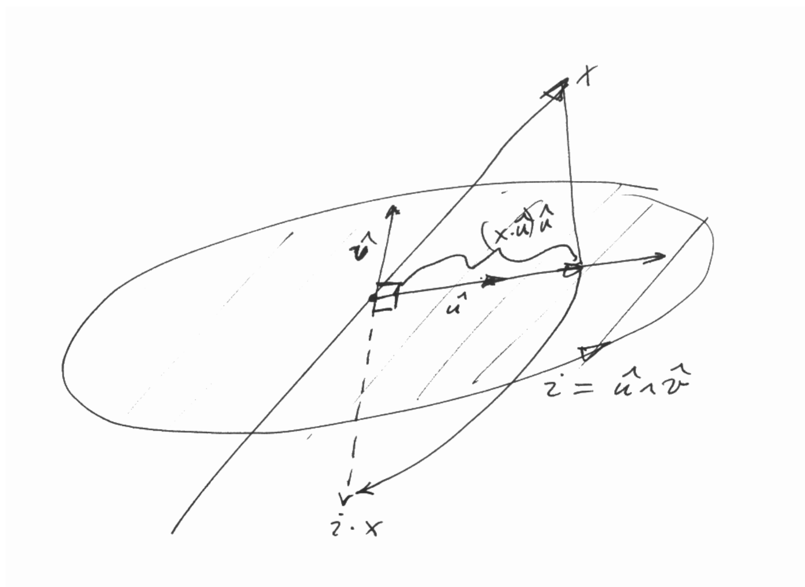

Figure 34.2: Bivector dot product with vector

It is worth pointing out that any bivector has no unique vector factorization. For example any of the following are equivalent

$$\begin{aligned} i &= \hat{\mathbf{u}} \wedge \hat{\mathbf{v}} \\ &= (2\hat{\mathbf{u}}) \wedge (\hat{\mathbf{v}}/2 + \alpha\hat{\mathbf{u}}) \\ &= \frac{1}{\alpha b - \beta a}(\alpha\hat{\mathbf{u}} + \beta\hat{\mathbf{v}}) \wedge (a\hat{\mathbf{u}} + b\hat{\mathbf{v}}) \end{aligned} \tag{34.37}$$

For this reason if we factor a bivector into two vectors within the plane we are free to pick one of these in any direction we please and can pick the other in one of the perpendiculars within the plane. In the figure exactly this was done, factoring the bivector into two perpendicular vectors

$i = \hat{\mathbf{u}}\hat{\mathbf{v}}$, where $\hat{\mathbf{u}}$ was picked to be in the direction of the projection of the vector $\mathbf{x}$ onto the plane spanned by $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\}$. Suppose that projection of $\mathbf{x}$ onto the plane is $\alpha\hat{\mathbf{u}}$. We then have for the bivector vector dot product

$$\begin{aligned} i \cdot \mathbf{x} &= (\hat{\mathbf{u}}\hat{\mathbf{v}}) \cdot (\alpha\hat{\mathbf{u}}) \\ &= \alpha\hat{\mathbf{u}}\hat{\mathbf{v}}\hat{\mathbf{u}} \\ &= -\alpha \underbrace{\boxed{\hat{\mathbf{u}}\hat{\mathbf{u}}}}_{=1} \hat{\mathbf{v}} \end{aligned} \tag{34.38}$$

So we have for the dot product $i \cdot \mathbf{x} = -\alpha\hat{\mathbf{v}}$, a rotation in the plane of the projection of the vector $\mathbf{x}$ onto the plane by 90 degrees. The direction of the rotation is metric dependent, and a spatially positive metric was used in this example. Observe that the action of a bivector product on a vector, provided that vector is in the plane spanned by the factors of the bivector is very much like the complex imaginary action. In both cases we have a 90 degree rotation. This complex number correspondence is not entirely equivalent though, since we also have $i \cdot \mathbf{x} = -\mathbf{x} \cdot i$, a negation on reversal of the product ordering, whereas we do not have to worry about commuting the imaginary of complex arithmetic.

This shows how the bivector dot product naturally encodes a rotation. We could leave things this way, but we also want to see how to put this in a more "standard" form. This is possible by rewriting the scalar product using a scalar grade selection operator. Also employing the cyclic reordering identity $\langle abc \rangle = \langle bca \rangle$, we have

$$\begin{aligned} (i \cdot x) \cdot \nabla &= \frac{1}{2}\langle (ix - xi)\nabla \rangle \\ &= \frac{1}{2}\langle ix\nabla - \nabla xi \rangle \end{aligned} \tag{34.39}$$

A pause is required to note that this reordering needs to be interpreted with x fixed with respect to the gradient so that the gradient is acting only to the extreme right. Then we have

$$(i \cdot x) \cdot \nabla = \frac{1}{2}\langle i(x \cdot \nabla) - (x \cdot \nabla)i \rangle + \frac{1}{2}\langle i(x \wedge \nabla) + (x \cdot \nabla)i \rangle \tag{34.40}$$

The rightmost action of the gradient allows the gradient dot and wedge products to be reordered (with interchange of sign for the wedge). The product in the first scalar selector has only bivector terms, so we are left with

$$(i \cdot x) \cdot \nabla = i \cdot (x \wedge \nabla) \tag{34.41}$$

and the rotation operator takes the postulated form

$$f(x(\theta + \Delta\theta)) = e^{\Delta\theta i\cdot(x\wedge\nabla)} f(x(\theta)) \tag{34.42}$$

While the cross product formulation of this is fine for 3D, this works in a plane when desired, as well as higher dimensional spaces as well as optionally non-Euclidean spaces like the Minkowski space required for electrodynamics and relativity.

34.5 COORDINATE EXPANSION

We have seen the structure of the scalar angular momentum operator of eq. (34.41) in the context of components of the cross product angular momentum operator in 3D spaces. For a more general space what do we have?

Let $i = \gamma_\beta\gamma_\alpha$, then we have

$$\begin{aligned} i\cdot(x\wedge\nabla) &= (\gamma_\beta\wedge\gamma_\alpha)\cdot(\gamma^\mu\wedge\gamma^\nu)x_\mu\partial_\nu \\ &= ({\delta_\beta}^\nu{\delta_\alpha}^\mu - {\delta_\beta}^\mu{\delta_\alpha}^\nu)x_\mu\partial_\nu \end{aligned} \tag{34.43}$$

which is

$$(\gamma_\beta\wedge\gamma_\alpha)\cdot(x\wedge\nabla) = x_\alpha\partial_\beta - x_\beta\partial_\alpha \tag{34.44}$$

In particular, in the four vector Minkowski space, when the pair α,β includes both space and time indices we loose (or gain) negation in this operator sum. For example with $i = \gamma_1\gamma_0$, we have

$$(\gamma_1\wedge\gamma_0)\cdot(x\wedge\nabla) = x^0\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^0} \tag{34.45}$$

We can also generalize the coordinate expansion of eq. (34.44) to a more general plane of rotation. Suppose that u and v are two perpendicular unit vectors in the plane of rotation. For this rotational plane we have $i = uv = u\wedge v$, and our expansion is

$$\begin{aligned} i\cdot(x\wedge\nabla) &= (\gamma_\beta\wedge\gamma_\alpha)\cdot(\gamma^\mu\wedge\gamma^\nu)u^\beta v^\alpha x_\mu\partial_\nu \\ &= ({\delta_\beta}^\nu{\delta_\alpha}^\mu - {\delta_\beta}^\mu{\delta_\alpha}^\nu)u^\beta v^\alpha x_\mu\partial_\nu \end{aligned} \tag{34.46}$$

So we have

$$i\cdot(x\wedge\nabla) = (u^\nu v^\mu - u^\mu v^\nu)x_\mu\partial_\nu \tag{34.47}$$

This scalar antisymmetric mixed index object is apparently called a vierbien (not a tensor) and written

$$\epsilon^{\nu\mu} = (u^\nu v^\mu - u^\mu v^\nu) \tag{34.48}$$

It would be slightly prettier to raise the index on x^μ (and correspondingly lower the μs in ϵ). We then have a completely non Geometric Algebra representation of the angular momentum operator for higher dimensions (and two dimensions) as well as for the Minkowski (and other if desired) metrics.

$$i \cdot (x \wedge \nabla) = \epsilon^\nu{}_\mu x^\mu \partial_\nu \tag{34.49}$$

34.6 MATRIX TREATMENT

It should be more accessible to do the same sort of treatment with matrices than the Geometric Algebra approach. It did not occur to me to try it that way initially, and it is worthwhile to do a comparative derivation. Setup should be similar

$$\begin{aligned}\mathbf{y} &= R\mathbf{x} \\ \mathbf{x} &= R^\mathrm{T}\mathbf{y}\end{aligned} \tag{34.50}$$

Taking derivatives we then have

$$\begin{aligned}\frac{d\mathbf{x}}{d\theta} &= \frac{dR^\mathrm{T}}{d\theta}\mathbf{y} \\ &= \frac{dR^\mathrm{T}}{d\theta}RR^\mathrm{T}\mathbf{y} \\ &= \left(\frac{dR^\mathrm{T}}{d\theta}R\right)\mathbf{x}\end{aligned} \tag{34.51}$$

Introducing an $\Omega = (dR^\mathrm{T}/d\theta)R$ very much like before we can write this

$$\frac{d\mathbf{x}}{d\theta} = \Omega\mathbf{x} \tag{34.52}$$

For Euclidean spaces (where $R^{-1} = R^\mathrm{T}$ as assumed above), we have $R^\mathrm{T}R = 1$, and thus

$$\Omega = \frac{dR^\mathrm{T}}{d\theta}R = -R^\mathrm{T}\frac{dR}{d\theta} \tag{34.53}$$

Transposition shows that this matrix Ω is completely antisymmetric since we have

$$\Omega^{\mathrm{T}} = -\Omega \tag{34.54}$$

Now, is there a convenient formulation for a general plane rotation in matrix form, perhaps like the Geometric exponential form? Probably can be done, but considering an x,y plane rotation should give the rough idea.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{34.55}$$

After a bit of algebra we have

$$\Omega = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{34.56}$$

In general we must have

$$\Omega = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ b & a & 0 \end{bmatrix} \tag{34.57}$$

For some a, b, c. This procedure is not intrinsically three dimension, but in the specific 3D case, we can express this antisymetrization using the cross product. Writing $\hat{\mathbf{n}} = (a, b, c)$ for the vector with these components, we have in the 3D case only

$$\Omega\mathbf{x} = \hat{\mathbf{n}} \times \mathbf{x} \tag{34.58}$$

The first order rotation of a function $f(\mathbf{x}(\theta))$ now follows from the chain rule as before

$$\begin{aligned} \frac{df}{d\theta} &= \frac{dx^m}{d\theta}\frac{\partial f}{\partial x^m} \\ &= \frac{d\mathbf{x}}{d\theta} \cdot \boldsymbol{\nabla} f \\ &= (\mathbf{n} \times \mathbf{x}) \cdot \boldsymbol{\nabla} f \end{aligned} \tag{34.59}$$

We have then for the first order rotation derivative operator in 3D

$$\frac{d}{d\theta} = \mathbf{n} \cdot (\mathbf{x} \times \boldsymbol{\nabla}) \tag{34.60}$$

For higher (or 2D) spaces one cannot use the cross product so a more general expression of the result eq. (34.60) would be

$$\frac{d}{d\theta} = (\Omega \mathbf{x}) \cdot \boldsymbol{\nabla} \tag{34.61}$$

Now, in this outline was a fair amount of cheating. We know that $\hat{\mathbf{n}}$ is the unit normal to the rotational plane, but that has not been shown here. Instead it was a constructed quantity just pulled out of thin air knowing it would be required. If one were interested in pursuing a treatment of the rotation generator operator strictly using matrix algebra, that would have to be considered. More troublesome and non-obvious is how this would translate to other metric spaces, where we do not necessarily have the transpose relationships to exploit.

35 SPHERICAL POLAR UNIT VECTORS IN EXPONENTIAL FORM

35.1 MOTIVATION

In 110 I blundered on a particularly concise exponential non-coordinate form for the unit vectors in a spherical polar coordinate system. For future reference outside of a quantum mechanical context here is a separate and more concise iteration of these results.

35.2 THE ROTATION AND NOTATION

The spherical polar rotor is a composition of rotations, expressed as half angle exponentials. Following the normal physics conventions we first apply a z, x plane rotation by angle theta, then an x, y plane rotation by angle ϕ. This produces the rotor

$$R = e^{\mathbf{e}_{31}\theta/2} e^{\mathbf{e}_{12}\phi/2} \tag{35.1}$$

Our triplet of Cartesian unit vectors is therefore rotated as

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \tilde{R} \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} R \tag{35.2}$$

In the quantum mechanical context it was convenient to denote the x, y plane unit bivector with the imaginary symbol

$$i = \mathbf{e}_1 \mathbf{e}_2 \tag{35.3}$$

reserving for the spatial pseudoscalar the capital

$$I = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 = \hat{\mathbf{r}} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\phi}} = i \mathbf{e}_3 \tag{35.4}$$

Note the characteristic differences between these two "imaginaries". The planar quantity $i = \mathbf{e}_1 \mathbf{e}_2$ commutes with $\mathbf{e}_3$, but anticommutes with either $\mathbf{e}_1$ or $\mathbf{e}_2$. On the other hand the spatial pseudoscalar I commutes with any vector, bivector or trivector in the algebra.

35.3 APPLICATION OF THE ROTOR. THE SPHERICAL POLAR UNIT VECTORS

Having fixed notation, lets apply the rotation to each of the unit vectors in sequence, starting with the calculation for $\hat{\boldsymbol{\phi}}$. This is

$$\begin{aligned}\hat{\boldsymbol{\phi}} &= e^{-i\phi/2}e^{-\mathbf{e}_{31}\theta/2}(\mathbf{e}_2)e^{\mathbf{e}_{31}\theta/2}e^{i\phi/2} \\ &= \mathbf{e}_2 e^{i\phi}\end{aligned} \tag{35.5}$$

Here, since $\mathbf{e}_2$ commutes with the rotor bivector $\mathbf{e}_3\mathbf{e}_1$ the innermost exponentials cancel, leaving just the $i\phi$ rotation. For $\hat{\mathbf{r}}$ it is a bit messier, and we have

$$\begin{aligned}\hat{\mathbf{r}} &= e^{-i\phi/2}e^{-\mathbf{e}_{31}\theta/2}(\mathbf{e}_3)e^{\mathbf{e}_{31}\theta/2}e^{i\phi/2} \\ &= e^{-i\phi/2}\mathbf{e}_3 e^{\mathbf{e}_{31}\theta}e^{i\phi/2} \\ &= e^{-i\phi/2}(\mathbf{e}_3\cos\theta + \mathbf{e}_1\sin\theta)e^{i\phi/2} \\ &= \mathbf{e}_3\cos\theta + \mathbf{e}_1\sin\theta e^{i\phi} \\ &= \mathbf{e}_3\cos\theta + \mathbf{e}_1\mathbf{e}_2\sin\theta\mathbf{e}_2 e^{i\phi} \\ &= \mathbf{e}_3\cos\theta + i\sin\theta\hat{\boldsymbol{\phi}} \\ &= \mathbf{e}_3(\cos\theta + \mathbf{e}_3 i\sin\theta\hat{\boldsymbol{\phi}}) \\ &= \mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta}\end{aligned} \tag{35.6}$$

Finally for $\hat{\boldsymbol{\theta}}$, we have a similar messy expansion

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= e^{-i\phi/2}e^{-\mathbf{e}_{31}\theta/2}(\mathbf{e}_1)e^{\mathbf{e}_{31}\theta/2}e^{i\phi/2} \\ &= e^{-i\phi/2}\mathbf{e}_1 e^{\mathbf{e}_{31}\theta}e^{i\phi/2} \\ &= e^{-i\phi/2}(\mathbf{e}_1\cos\theta - \mathbf{e}_3\sin\theta)e^{i\phi/2} \\ &= \mathbf{e}_1\cos\theta e^{i\phi} - \mathbf{e}_3\sin\theta \\ &= i\cos\theta\mathbf{e}_2 e^{i\phi} - \mathbf{e}_3\sin\theta \\ &= i\hat{\boldsymbol{\phi}}\cos\theta - \mathbf{e}_3\sin\theta \\ &= i\hat{\boldsymbol{\phi}}(\cos\theta + \hat{\boldsymbol{\phi}}i\mathbf{e}_3\sin\theta) \\ &= i\hat{\boldsymbol{\phi}}e^{I\hat{\boldsymbol{\phi}}\theta}\end{aligned} \tag{35.7}$$

Summarizing the three of these relations we have for the rotated unit vectors

$$\begin{aligned}\hat{\mathbf{r}} &= \mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta} \\ \hat{\boldsymbol{\theta}} &= i\hat{\boldsymbol{\phi}}e^{I\hat{\boldsymbol{\phi}}\theta} \\ \hat{\boldsymbol{\phi}} &= \mathbf{e}_2 e^{i\phi}\end{aligned} \tag{35.8}$$

and in particular for the radial position vector from the origin, rotating from the polar axis, we have

$$\mathbf{x} = r\hat{\mathbf{r}} = r\mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta} \tag{35.9}$$

Compare this to the coordinate representation

$$\mathbf{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{35.10}$$

it is not initially obvious that these θ and ϕ rotations admit such a tidy factorization. In retrospect, this does not seem so surprising, since we can form a quaternion product that acts via multiplication to map a vector to a rotated position. In fact those quaternions, acting from the right on the initial vectors are

$$\begin{aligned}
\mathbf{e}_3 &\rightarrow \hat{\mathbf{r}} = \mathbf{e}_3\left(e^{I\hat{\boldsymbol{\phi}}\theta}\right) \\
\mathbf{e}_1 &\rightarrow \hat{\boldsymbol{\theta}} = \mathbf{e}_1\left(\mathbf{e}_2\hat{\boldsymbol{\phi}}e^{I\hat{\boldsymbol{\phi}}\theta}\right) \\
\mathbf{e}_2 &\rightarrow \hat{\boldsymbol{\phi}} = \mathbf{e}_2\left(e^{i\phi}\right)
\end{aligned} \tag{35.11}$$

FIXME: it should be possible to reduce the quaternion that rotates $\mathbf{e}_1 \rightarrow \hat{\boldsymbol{\theta}}$ to a single exponential. What is it?

35.4 A CONSISTENCY CHECK

We expect that the dot product between a north pole oriented vector $\mathbf{z} = Z\mathbf{e}_3$ and the spherically polar rotated vector $\mathbf{x} = r\mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta}$ is just

$$\mathbf{x}\cdot\mathbf{z} = Zr\cos\theta \tag{35.12}$$

Lets verify this

$$\begin{aligned}
\mathbf{x}\cdot\mathbf{z} &= \left\langle Z\mathbf{e}_3\mathbf{e}_3 r e^{I\hat{\boldsymbol{\phi}}\theta}\right\rangle \\
&= Zr\left\langle\cos\theta + I\hat{\boldsymbol{\phi}}\sin\theta\right\rangle \\
&= Zr\cos\theta
\end{aligned} \tag{35.13}$$

$\square$

35.5 AREA AND VOLUME ELEMENTS

Let us use these results to compute the spherical polar volume element. Pictorially this can be read off simply from a diagram. If one is less trusting of pictorial means (or want a method more generally applicable), we can also do this particular calculation algebraically, expanding the determinant of partials

$$\begin{vmatrix} \frac{\partial \mathbf{x}}{\partial r} & \frac{\partial \mathbf{x}}{\partial \theta} & \frac{\partial \mathbf{x}}{\partial \phi} \end{vmatrix} dr d\theta d\phi = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} r^2 dr d\theta d\phi \tag{35.14}$$

One can chug through the trig reduction for this determinant with not too much trouble, but it is not particularly fun.

Now compare to the same calculation proceeding directly with the exponential form. We do still need to compute the partials

$$\frac{\partial \mathbf{x}}{\partial r} = \hat{\mathbf{r}} \tag{35.15}$$

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \theta} &= r\mathbf{e}_3 \frac{\partial}{\partial \theta} e^{I\hat{\boldsymbol{\phi}}\theta} \\ &= r\hat{\mathbf{r}} I\hat{\boldsymbol{\phi}} \\ &= r\hat{\mathbf{r}}(\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}})\hat{\boldsymbol{\phi}} \\ &= r\hat{\boldsymbol{\theta}} \end{aligned} \tag{35.16}$$

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \phi} &= r\mathbf{e}_3 \frac{\partial}{\partial \phi}(\cos\theta + I\hat{\boldsymbol{\phi}}\sin\theta) \\ &= -r\mathbf{e}_3 I i\hat{\boldsymbol{\phi}}\sin\theta \\ &= r\hat{\boldsymbol{\phi}}\sin\theta \end{aligned} \tag{35.17}$$

So the area element, the oriented area of the parallelogram between the two vectors $d\theta\partial\mathbf{x}/\partial\theta$, and $d\phi\partial\mathbf{x}/\partial\phi$ on the spherical surface at radius r is

$$d\mathbf{S} = \left(d\theta\frac{\partial \mathbf{x}}{\partial \theta}\right) \wedge \left(d\phi\frac{\partial \mathbf{x}}{\partial \phi}\right) = r^2\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\sin\theta d\theta d\phi \tag{35.18}$$

and the volume element in trivector form is just the product

$$d\mathbf{V} = \left(dr\frac{\partial \mathbf{x}}{\partial r}\right) \wedge d\mathbf{S} = r^2 \sin\theta I dr d\theta d\phi \tag{35.19}$$

35.6 LINE ELEMENT

The line element for the particle moving on a spherical surface can be calculated by calculating the derivative of the spherical polar unit vector $\hat{\mathbf{r}}$

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{\partial \hat{\mathbf{r}}}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial \hat{\mathbf{r}}}{\partial \theta}\frac{d\theta}{dt} \tag{35.20}$$

than taking the magnitude of this vector. We can start either in coordinate form

$$\hat{\mathbf{r}} = \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta \cos\phi + \mathbf{e}_2 \sin\theta \sin\phi \tag{35.21}$$

or, instead do it the fun way, first grouping this into a complex exponential form. This factorization was done above, but starting over allows this to be done a bit more effectively for this particular problem. As above, writing $i = \mathbf{e}_1\mathbf{e}_2$, the first factorization is

$$\hat{\mathbf{r}} = \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{i\phi} \tag{35.22}$$

The unit vector $\boldsymbol{\rho} = \mathbf{e}_1 e^{i\phi}$ lies in the x, y plane perpendicular to $\mathbf{e}_3$, so we can form the unit bivector $\mathbf{e}_3\boldsymbol{\rho}$ and further factor the unit vector terms into a $\cos + i \sin$ form

$$\begin{aligned}\hat{\mathbf{r}} &= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{i\phi} \\ &= \mathbf{e}_3(\cos\theta + \mathbf{e}_3\boldsymbol{\rho} \sin\theta)\end{aligned} \tag{35.23}$$

This allows the spherical polar unit vector to be expressed in complex exponential form (really a vector-quaternion product)

$$\hat{\mathbf{r}} = \mathbf{e}_3 e^{\mathbf{e}_3\boldsymbol{\rho}\theta} = e^{-\mathbf{e}_3\boldsymbol{\rho}\theta}\mathbf{e}_3 \tag{35.24}$$

Now, calculating the unit vector velocity, we get

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \mathbf{e}_3\mathbf{e}_3\boldsymbol{\rho} e^{\mathbf{e}_3\boldsymbol{\rho}\theta}\dot{\theta} + \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 \sin\theta e^{i\phi}\dot{\phi} \\ &= \boldsymbol{\rho} e^{\mathbf{e}_3\boldsymbol{\rho}\theta}\left(\dot{\theta} + e^{-\mathbf{e}_3\boldsymbol{\rho}\theta}\boldsymbol{\rho}\sin\theta e^{-i\phi}\mathbf{e}_2\dot{\phi}\right) \\ &= \left(\dot{\theta} + \mathbf{e}_2 \sin\theta e^{i\phi}\dot{\phi}\boldsymbol{\rho} e^{\mathbf{e}_3\boldsymbol{\rho}\theta}\right) e^{-\mathbf{e}_3\boldsymbol{\rho}\theta}\boldsymbol{\rho}\end{aligned} \tag{35.25}$$

The last two lines above factor the $\boldsymbol{\rho}$ vector and the $e^{\mathbf{e}_3\rho\theta}$ quaternion to the left and to the right in preparation for squaring this to calculate the magnitude.

$$
\begin{aligned}
\left(\frac{d\hat{\mathbf{r}}}{dt}\right)^2 &= \left\langle\left(\frac{d\hat{\mathbf{r}}}{dt}\right)^2\right\rangle \\
&= \left\langle\left(\dot{\theta} + \mathbf{e}_2 \sin\theta e^{i\phi}\dot{\phi}\boldsymbol{\rho}e^{\mathbf{e}_3\rho\theta}\right)\left(\dot{\theta} + e^{-\mathbf{e}_3\rho\theta}\boldsymbol{\rho}\sin\theta e^{-i\phi}\mathbf{e}_2\dot{\phi}\right)\right\rangle \\
&= \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 + \sin\theta\dot{\phi}\dot{\theta}\left\langle \mathbf{e}_2 e^{i\phi}\boldsymbol{\rho}e^{\mathbf{e}_3\rho\theta} + e^{-\mathbf{e}_3\rho\theta}\boldsymbol{\rho}e^{-i\phi}\mathbf{e}_2\right\rangle
\end{aligned}
\tag{35.26}
$$

This last term ($\in \mathrm{span}\{\rho\mathbf{e}_1, \rho\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3\}$) has only grade two components, so the scalar part is zero. We are left with the line element

$$
\left(\frac{d(r\hat{\mathbf{r}})}{dt}\right)^2 = r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)
\tag{35.27}
$$

In retrospect, at least once one sees the answer, it seems obvious. Keeping θ constant the length increment moving in the plane is $ds = \sin\theta d\phi$, and keeping ϕ constant, we have $ds = d\theta$. Since these are perpendicular directions we can add the lengths using the Pythagorean theorem.

35.6.1 *Line element using an angle and unit bivector parameterization*

Parameterizing using scalar angles is not the only approach that we can take to calculate the line element on the unit sphere. Proceding directly with a alternate polar representation, utilizing a unit bivector j, and scalar angle θ is

$$
\mathbf{x} = r\mathbf{e}_3 e^{j\theta}
\tag{35.28}
$$

For this product to be a vector j must contain $\mathbf{e}_3$ as a factor ($j = \mathbf{e}_3 \wedge m$ for some vector m.) Setting $r = 1$ for now, the deriviate of $\mathbf{x}$ is

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{e}_3\frac{d}{dt}\left(\cos\theta + j\sin\theta\right) \\
&= \mathbf{e}_3\dot{\theta}\left(-\sin\theta + j\cos\theta\right) + \mathbf{e}_3\frac{dj}{dt}\sin\theta \\
&= \mathbf{e}_3\dot{\theta}j\left(j\sin\theta + \cos\theta\right) + \mathbf{e}_3\frac{dj}{dt}\sin\theta
\end{aligned}
\tag{35.29}
$$

This is

$$
\dot{\mathbf{x}} = \mathbf{e}_3\left(\frac{d\theta}{dt}je^{j\theta} + \frac{dj}{dt}\sin\theta\right)
\tag{35.30}
$$

Alternately, we can take derivatives of $\mathbf{x} = re^{-j\theta}\mathbf{e}_3$, for
Or

$$\dot{\mathbf{x}} = -\left(\frac{d\theta}{dt} je^{-j\theta} + \frac{dj}{dt}\sin\theta\right)\mathbf{e}_3 \tag{35.31}$$

Together with eq. (35.30), the line element for position change on the unit sphere is then

$$\begin{aligned}
\dot{\mathbf{x}}^2 &= \left\langle -\left(\frac{d\theta}{dt} je^{-j\theta} + \frac{dj}{dt}\sin\theta\right)\mathbf{e}_3\mathbf{e}_3\left(\frac{d\theta}{dt} je^{j\theta} + \frac{dj}{dt}\sin\theta\right)\right\rangle \\
&= \left\langle -\left(\frac{d\theta}{dt} je^{-j\theta} + \frac{dj}{dt}\sin\theta\right)\left(\frac{d\theta}{dt} je^{j\theta} + \frac{dj}{dt}\sin\theta\right)\right\rangle \\
&= \left(\frac{d\theta}{dt}\right)^2 - \left(\frac{dj}{dt}\right)^2 \sin^2\theta - \frac{d\theta}{dt}\sin\theta\left\langle \frac{dj}{dt} je^{j\theta} + je^{-j\theta}\frac{dj}{dt}\right\rangle
\end{aligned} \tag{35.32}$$

Starting with cyclic reordering of the last term, we get zero

$$\begin{aligned}
\left\langle \frac{dj}{dt} je^{j\theta} + je^{-j\theta}\frac{dj}{dt}\right\rangle &= \left\langle \frac{dj}{dt} j\left(e^{j\theta} + e^{-j\theta}\right)\right\rangle \\
&= \left\langle \frac{dj}{dt} j 2 j \sin\theta\right\rangle \\
&= -2\sin\theta \frac{d}{dt}\underbrace{\boxed{\langle j\rangle}}_{=0}
\end{aligned} \tag{35.33}$$

The line element (for constant r) is therefore

$$\dot{\mathbf{x}}^2 = r^2\left(\dot{\theta}^2 - \left(\frac{dj}{dt}\right)^2 \sin^2\theta\right) \tag{35.34}$$

This is essentially the same result as we got starting with an explicit r, θ, ϕ. Repeating for comparision that was

$$\dot{\mathbf{x}}^2 = r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{35.35}$$

The bivector that we have used this time encodes the orientation of the plane of rotation from the polar axis down to the position on the sphere corresponds to the angle ϕ in the scalar parameterization. The negation in sign is expected due to the negative bivector square.

Also comparing to previous results it is notable that we can explicitly express this bivector in terms of the scalar angle if desired as

$$\begin{aligned}\boldsymbol{\rho} &= \mathbf{e}_1 e^{\mathbf{e}_1\mathbf{e}_2\phi} = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi \\ j &= \mathbf{e}_3 \wedge \boldsymbol{\rho} = \mathbf{e}_3\boldsymbol{\rho}\end{aligned} \tag{35.36}$$

The inverse mapping, expressing the scalar angle using the bivector representation is also possible, but not unique. The principle angle for that inverse mapping is

$$\phi = -\mathbf{e}_1\mathbf{e}_2 \ln(\mathbf{e}_1\mathbf{e}_3 j) \tag{35.37}$$

35.6.2 *Allowing the magnitude to vary*

Writing a vector in polar form

$$\mathbf{x} = r\hat{\mathbf{r}} \tag{35.38}$$

and also allowing r to vary, we have

$$\begin{aligned}\left(\frac{d\mathbf{x}}{dt}\right)^2 &= \left(\frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt}\right)^2 \\ &= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\hat{\mathbf{r}}}{dt}\right)^2 + 2r\frac{dr}{dt}\hat{\mathbf{r}} \cdot \frac{d\hat{\mathbf{r}}}{dt}\end{aligned} \tag{35.39}$$

The squared unit vector derivative was previously calculated to be

$$\left(\frac{d\hat{\mathbf{r}}}{dt}\right)^2 = \dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 \tag{35.40}$$

Picturing the geometry is enough to know that $\dot{\hat{\mathbf{r}}} \cdot \hat{\mathbf{r}} = 0$ since $\dot{\hat{\mathbf{r}}}$ is always tangential to the sphere. It should also be possible to algebraically show this, but without going through the effort we at least now know the general line element

$$\dot{\mathbf{x}}^2 = \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{35.41}$$

36 INFINITESIMAL ROTATIONS

36.1 MOTIVATION

In a classical mechanics lecture (which I audited) Prof. Poppitz made the claim that an infinitesimal rotation in direction $\hat{\mathbf{n}}$ of magnitude $\delta\phi$ has the form

$$\mathbf{x} \rightarrow \mathbf{x} + \delta\boldsymbol{\phi} \times \mathbf{x}, \tag{36.1}$$

where

$$\delta\boldsymbol{\phi} = \hat{\mathbf{n}}\delta\phi. \tag{36.2}$$

I believe he expressed things in terms of the differential displacement

$$\delta\mathbf{x} = \delta\boldsymbol{\phi} \times \mathbf{x} \tag{36.3}$$

This was verified for the special case $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ and $\mathbf{x} = x\hat{\mathbf{x}}$. Let us derive this in the general case too.

36.2 WITH GEOMETRIC ALGEBRA

Let us temporarily dispense with the normal notation and introduce two perpendicular unit vectors $\hat{\mathbf{u}}$, and $\hat{\mathbf{v}}$ in the plane of the rotation. Relate these to the unit normal with

$$\hat{\mathbf{n}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}. \tag{36.4}$$

A rotation through an angle ϕ (infinitesimal or otherwise) is then

$$\mathbf{x} \rightarrow e^{-\hat{\mathbf{u}}\hat{\mathbf{v}}\phi/2}\mathbf{x}e^{\hat{\mathbf{u}}\hat{\mathbf{v}}\phi/2}. \tag{36.5}$$

Suppose that we decompose $\mathbf{x}$ into components in the plane and in the direction of the normal $\hat{\mathbf{n}}$. We have

$$\mathbf{x} = x_u \hat{\mathbf{u}} + x_v \hat{\mathbf{v}} + x_n \hat{\mathbf{n}}. \tag{36.6}$$

The exponentials commute with the $\hat{\mathbf{n}}$ vector, and anticommute otherwise, leaving us with

$$\begin{aligned} \mathbf{x} &\rightarrow x_n \hat{\mathbf{n}} + (x_u \hat{\mathbf{u}} + x_v \hat{\mathbf{v}}) e^{\hat{\mathbf{u}}\hat{\mathbf{v}}\phi} \\ &= x_n \hat{\mathbf{n}} + (x_u \hat{\mathbf{u}} + x_v \hat{\mathbf{v}})(\cos\phi + \hat{\mathbf{u}}\hat{\mathbf{v}} \sin\phi) \\ &= x_n \hat{\mathbf{n}} + \hat{\mathbf{u}}(x_u \cos\phi - x_v \sin\phi) + \hat{\mathbf{v}}(x_v \cos\phi + x_u \sin\phi). \end{aligned} \tag{36.7}$$

In the last line we use $\hat{\mathbf{u}}^2 = 1$ and $\hat{\mathbf{u}}\hat{\mathbf{v}} = -\hat{\mathbf{v}}\hat{\mathbf{u}}$. Making the angle infinitesimal $\phi \rightarrow \delta\phi$ we have

$$\begin{aligned} \mathbf{x} &\rightarrow x_n \hat{\mathbf{n}} + \hat{\mathbf{u}}(x_u - x_v \delta\phi) + \hat{\mathbf{v}}(x_v + x_u \delta\phi) \\ &= \mathbf{x} + \delta\phi(x_u \hat{\mathbf{v}} - x_v \hat{\mathbf{u}}) \end{aligned} \tag{36.8}$$

We have only to confirm that this matches the assumed cross product representation

$$\begin{aligned} \hat{\mathbf{n}} \times \mathbf{x} &= \begin{vmatrix} \hat{\mathbf{u}} & \hat{\mathbf{v}} & \hat{\mathbf{n}} \\ 0 & 0 & 1 \\ x_u & x_v & x_n \end{vmatrix} \\ &= -\hat{\mathbf{u}} x_v + \hat{\mathbf{v}} x_u \end{aligned} \tag{36.9}$$

Taking the two last computations we find

$$\delta\mathbf{x} = \delta\phi \hat{\mathbf{n}} \times \mathbf{x} = \delta\boldsymbol{\phi} \times \mathbf{x}, \tag{36.10}$$

as desired.

36.3 WITHOUT GEOMETRIC ALGEBRA

We have also done the setup above to verify this result without GA. Here we wish to apply the rotation to the coordinate vector of $\mathbf{x}$ in the $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}}\}$ basis which gives us

$$
\begin{aligned}
\begin{bmatrix} x_u \\ x_v \\ x_n \end{bmatrix} &\rightarrow \begin{bmatrix} \cos\delta\phi & -\sin\delta\phi & 0 \\ \sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u \\ x_v \\ x_n \end{bmatrix} \\
&\approx \begin{bmatrix} 1 & -\delta\phi & 0 \\ \delta\phi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u \\ x_v \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} x_u \\ x_v \\ x_n \end{bmatrix} + \begin{bmatrix} 0 & -\delta\phi & 0 \\ \delta\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_u \\ x_v \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} x_u \\ x_v \\ x_n \end{bmatrix} + \delta\phi \begin{bmatrix} -x_v \\ x_u \\ 0 \end{bmatrix}
\end{aligned}
\tag{36.11}
$$

But as we have shown, this last coordinate vector is just $\hat{\mathbf{n}} \times \mathbf{x}$, and we get our desired result using plain old fashioned matrix algebra as well.

Really the only difference between this and what was done in class is that there is no assumption here that $\mathbf{x} = x\hat{\mathbf{x}}$.

Part IV

CALCULUS

37

DEVELOPING SOME INTUITION FOR MULTIVARIABLE AND MULTIVECTOR TAYLOR SERIES

The book [10] uses Geometric Calculus heavily in its Lagrangian treatment. In particular it is used in some incomprehensible seeming ways in the stress energy tensor treatment.

In the treatment of transformation of the dependent variables (not the field variables themselves) of field Lagrangians, there is one bit that appears to be the first order linear term from a multivariable Taylor series expansion. Play with multivariable Taylor series here a bit to develop some intuition with it.

37.1 SINGLE VARIABLE CASE, AND GENERALIZATION OF IT

For the single variable case, Taylor series takes the form

$$f(x) = \sum \frac{x^k}{k!} \frac{d^k f(x)}{dx^k}\bigg|_{x=0} \tag{37.1}$$

or

$$f(x_0 + \epsilon) = \sum \frac{\epsilon^k}{k!} \frac{d^k f(x)}{dx^k}\bigg|_{x=x_0} \tag{37.2}$$

As pointed out in [5], this can (as they demonstrated for polynomials) be put into exponential operator form

$$f(x_0 + \epsilon) = e^{\epsilon d/dx} f(x)\big|_{x=x_0} \tag{37.3}$$

Without proof, the multivector generalization of this is

$$f(x_0 + \epsilon) = e^{\epsilon\cdot\nabla} f(x)\big|_{x=x_0} \tag{37.4}$$

Or in full,

$$f(x_0 + \epsilon) = \sum \frac{1}{k!} (\epsilon \cdot \nabla)^k f(x)\big|_{x=x_0} \tag{37.5}$$

Let us work with this, and develop some comfort with what it means, then revisit the proof.

37.2 DIRECTIONAL DERIVATIVES

First a definition of directional derivative is required.

In standard two variable vector calculus the directional derivative is defined in one of the following ways

$$\begin{aligned}\boldsymbol{\nabla}_{\mathbf{u}} f(x, y) &= \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h} \\ \mathbf{u} &= (a, b)\end{aligned} \tag{37.6}$$

Or in a more general vector form as

$$\boldsymbol{\nabla}_{\mathbf{u}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \tag{37.7}$$

Or in terms of the gradient as

$$\boldsymbol{\nabla}_{\mathbf{u}} f(\mathbf{x}) = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \boldsymbol{\nabla} f \tag{37.8}$$

Each of these was for a vector parametrized scalar function, although the wikipedia article does mention a vector valued form that is identical to that use by [10]. Specifically, that is

$$\begin{aligned}(\epsilon \cdot \nabla) f(x) &= \lim_{h \to 0} \frac{f(x + h\epsilon) - f(x)}{h} \\ &= \left.\frac{\partial f(x + h\epsilon)}{\partial h}\right|_{h=0}\end{aligned} \tag{37.9}$$

Observe that this definition as a limit avoids the requirement to define the gradient upfront. That definition is not necessarily obvious especially for multivector valued functions.

37.3 WORK SOME EXAMPLES

37.3.1 *First order linear vector polynomial*

Let

$$f(x) = a + x \tag{37.10}$$

For this simplest of vector valued vector parametrized functions we have

$$\begin{aligned}\frac{\partial f(x+h\epsilon)}{\partial h} &= \frac{\partial}{\partial h}(a+x+h\epsilon) \\ &= \epsilon \\ &= (\epsilon \cdot \nabla) f\end{aligned} \tag{37.11}$$

with no requirement to evaluate at $h = 0$ to complete the directional derivative computation. The Taylor series expansion about 0 is thus

$$\begin{aligned}f(\epsilon) &= (\epsilon \cdot \nabla)^0 f\Big|_{x=0} + (\epsilon \cdot \nabla)^1 f\Big|_{x=0} \\ &= a + \epsilon\end{aligned} \tag{37.12}$$

Nothing else could be expected.

37.3.2 *Second order vector parametrized multivector polynomial*

Now, step up the complexity slightly, and introduce a multivector valued second degree polynomial, say,

$$f(x) = \alpha + a + xy + wx + cx^2 + dxe + xgx \tag{37.13}$$

Here α is a scalar, and all the other variables are vectors, so we have grades ≤ 3.
For the first order partial we have

$$\begin{aligned}&\frac{\partial f(x+h\epsilon)}{\partial h} \\ &= \frac{\partial}{\partial h}(\alpha + a + (x+h\epsilon)y + w(x+h\epsilon) + c(x+h\epsilon)^2 + d(x+h\epsilon)e + (x+h\epsilon)g(x+h\epsilon)) \\ &= \epsilon y + w\epsilon + c\epsilon(x+h\epsilon) + c(x+h\epsilon)\epsilon + c\epsilon + d\epsilon e + \epsilon g(x+h\epsilon) + (x+h\epsilon)g\epsilon\end{aligned} \tag{37.14}$$

Evaluation at $h = 0$ we have

$$(\epsilon \cdot \nabla) f = \epsilon y + w\epsilon + c\epsilon x + cx\epsilon + c\epsilon + d\epsilon e + \epsilon g x + x g \epsilon \tag{37.15}$$

By inspection we have

$$(\epsilon \cdot \nabla)^2 f = +2c\epsilon^2 + 2\epsilon g \epsilon \tag{37.16}$$

Combining things forming the Taylor series expansion about the origin we should recover our function

$$
\begin{aligned}
f(\epsilon) &= \frac{1}{0!}\left(\epsilon\cdot\nabla\right)^0 f\Big|_{x=0} + \frac{1}{1!}\left(\epsilon\cdot\nabla\right)^1 f\Big|_{x=0} + \frac{1}{2}\left(\epsilon\cdot\nabla\right)^2 f\Big|_{x=0} \\
&= \frac{1}{1}(\alpha + a) + \frac{1}{1}(\epsilon y + w\epsilon + c\epsilon + d\epsilon e) + \frac{1}{2}(2c\epsilon^2 + 2\epsilon g\epsilon) \\
&= \alpha + a + \epsilon y + w\epsilon + c\epsilon + d\epsilon e + c\epsilon^2 + \epsilon g\epsilon
\end{aligned}
\tag{37.17}
$$

This should match eq. (37.13), with an $x = \epsilon$ substitution, and does. With the vector factors in these functions commutativity assumptions could not be made. These calculations help provide a small verification that this form of Taylor series does in fact work out fine with such non-commutative variables.

Observe as well that there was really no requirement in this example that x or any of the other factors to be vectors. If they were all bivectors or trivectors or some mix the calculations would have had the same results.

37.4 PROOF OF THE MULTIVECTOR TAYLOR EXPANSION

A peek back into [19] shows that eq. (37.5) was in fact proved, but it was done in a very sneaky and clever way. Rather than try to prove treat the multivector parameters explicitly, the following scalar parametrized hybrid function was created

$$
G(\tau) = F(\mathbf{x}_0 + \tau\mathbf{a}) \tag{37.18}
$$

The scalar parametrized function $G(\tau)$ can be Taylor expanded about the origin, and then evaluated at 1 resulting in eq. (37.5) in terms of powers of $(\mathbf{a}\cdot\nabla)$. I will not reproduce or try to enhance that proof for myself here since it is actually quite clear in the text. Obviously the trick is non-intuitive enough that when thinking about how to prove this myself it did not occur to me.

37.5 EXPLICIT EXPANSION FOR A SCALAR FUNCTION

Now, despite the $a\cdot\nabla$ notation being unfamiliar seeming, the end result is not. Explicit expansion of this for a vector to scalar mapping will show this. In fact this will also account for the Hessian matrix, as in

$$
y = f(\mathbf{x} + \Delta\mathbf{x}) \approx f(\mathbf{x}) + J(\mathbf{x})\Delta\mathbf{x} \tag{37.19}
$$

providing not only the background on where this comes from, but also the so often omitted third order and higher generalizations (most often referred to as $\cdots$). Poking around a bit I see that the wikipedia Taylor Series does explicitly define the higher order case, but if I had seen that before the connection to the Hessian was not obvious.

37.5.1 *Two variable case*

Rather than start with the general case, the expansion of the first few powers of $(\mathbf{a} \cdot \boldsymbol{\nabla}) f$ for the two variable case is enough to show the pattern. How to further generalize this scalar function case will be clear from inspection.

Starting with the first order term, writing $\mathbf{a} = (a, b)$ we have

$$
\begin{aligned}
(\mathbf{a} \cdot \boldsymbol{\nabla}) f(x, y) &= \frac{\partial}{\partial \tau} f(x + a\tau, y + b\tau)\bigg|_{\tau=0} \\
&= \left(\frac{\partial}{\partial x + a\tau} f(x + a\tau, y + b\tau) \frac{\partial (x + a\tau)}{\partial \tau} \right)\bigg|_{\tau=0} \\
&\quad + \left(\frac{\partial}{\partial y + b\tau} f(x + a\tau, y + b\tau) \frac{\partial (y + b\tau)}{\partial \tau} \right)\bigg|_{\tau=0} \\
&= a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} \\
&= \mathbf{a} \cdot (\boldsymbol{\nabla} f)
\end{aligned}
\tag{37.20}
$$

For the second derivative operation we have

$$
\begin{aligned}
(\mathbf{a} \cdot \boldsymbol{\nabla})^2 f(x, y) &= (\mathbf{a} \cdot \boldsymbol{\nabla}) \big((\mathbf{a} \cdot \boldsymbol{\nabla}) f(x, y) \big) \\
&= (\mathbf{a} \cdot \boldsymbol{\nabla}) \left(a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} \right) \\
&= \frac{\partial}{\partial \tau} \left(a \frac{\partial f}{\partial x} (x + a\tau, y + b\tau) + b \frac{\partial f}{\partial y} (x + a\tau, y + b\tau) \right)\bigg|_{\tau=0}
\end{aligned}
\tag{37.21}
$$

Especially if one makes a temporary substitution of the partials for some other named variables, it is clear this follows as before, and one gets

$$
(\mathbf{a} \cdot \boldsymbol{\nabla})^2 f(x, y) = a^2 \frac{\partial^2 f}{\partial x^2} + ba \frac{\partial^2 f}{\partial y \partial x} + ab \frac{\partial^2 f}{\partial x \partial y} + b^2 \frac{\partial^2 f}{\partial y^2}
\tag{37.22}
$$

Similarly the third order derivative operator gives us

$$\begin{aligned}
(\mathbf{a}\cdot\boldsymbol{\nabla})^3 f(x,y) &= aaa\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial x}f + aba\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial x}f \\
&+ aab\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial x}f + abb\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}f \\
&\quad + baa\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}f + bba\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial x}f \\
&\quad + bab\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial x}f + bbb\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial y}f \\
&= a^3\frac{\partial^3 f}{\partial x^3} + 3a^2 b\frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial y} + 3ab^2\frac{\partial}{\partial x}\frac{\partial^2 f}{\partial y^2} + b^3\frac{\partial^3 f}{\partial y^3}
\end{aligned} \tag{37.23}$$

We no longer have the notational nicety of being able to use the gradient notation as was done for the first derivative term. For the first and second order derivative operations, one has the option of using the gradient and Hessian matrix notations

$$\begin{aligned}
(\mathbf{a}\cdot\boldsymbol{\nabla}) f(x,y) &= \mathbf{a}^{\mathrm{T}}\begin{bmatrix} f_x \\ f_y \end{bmatrix} \\
(\mathbf{a}\cdot\boldsymbol{\nabla})^2 f(x,y) &= \mathbf{a}^{\mathrm{T}}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\mathbf{a}
\end{aligned} \tag{37.24}$$

But this will not be helpful past the second derivative.

Additionally, if we continue to restrict oneself to the two variable case, it is clear that we have

$$(\mathbf{a}\cdot\boldsymbol{\nabla})^n f(x,y) = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\left(\frac{\partial}{\partial x}\right)^{n-k}\left(\frac{\partial}{\partial y}\right)^{k} f(x,y) \tag{37.25}$$

But it is also clear that if we switch to more than two variables, a binomial series expansion of derivative powers in this fashion will no longer work. For example for three (or more) variables, writing for example $\mathbf{a} = (a_1, a_2, a_3)$, we have

$$\begin{aligned}
(\mathbf{a}\cdot\boldsymbol{\nabla}) f(\mathbf{x}) &= \sum_i \left(a_i\frac{\partial}{\partial x_i}\right) f(\mathbf{x}) \\
(\mathbf{a}\cdot\boldsymbol{\nabla})^2 f(\mathbf{x}) &= \sum_{ij} \left(a_i\frac{\partial}{\partial x_i}\right)\left(a_j\frac{\partial}{\partial x_j}\right) f(\mathbf{x}) \\
(\mathbf{a}\cdot\boldsymbol{\nabla})^3 f(\mathbf{x}) &= \sum_{ijk} \left(a_i\frac{\partial}{\partial x_i}\right)\left(a_j\frac{\partial}{\partial x_j}\right)\left(a_k\frac{\partial}{\partial x_k}\right) f(\mathbf{x})
\end{aligned} \tag{37.26}$$

If the partials are all collected into a single indexed object, one really has a tensor. For the first and second orders we can represent this tensor in matrix form (as the gradient and Hessian respectively)

37.6 GRADIENT WITH NON-EUCLIDEAN BASIS

The directional derivative has been calculated above for a scalar function. There is nothing intrinsic to that argument that requires an orthonormal basis.

Suppose we have a basis $\{\gamma_\mu\}$, and a reciprocal frame $\{\gamma^\mu\}$. Let

$$\begin{aligned} x &= x^\mu \gamma_\mu = x_\mu \gamma^\mu \\ a &= a^\mu \gamma_\mu = a_\mu \gamma^\mu \end{aligned} \tag{37.27}$$

The first order directional derivative is then

$$(a \cdot \nabla) f(x) = \left. \frac{\partial f}{\partial \tau}(x + \tau a) \right|_{\tau=0} \tag{37.28}$$

This is

$$(a \cdot \nabla) f(x) = \sum_\mu a^\mu \frac{\partial f}{\partial x^\mu}(x) \tag{37.29}$$

Now, we are used to ∇ as a standalone object, and want that operator defined such that we can also write eq. (37.29) as

$$a \cdot (\nabla f(x)) = \left(a^\mu \gamma_\mu\right) \cdot (\nabla f(x)) \tag{37.30}$$

Comparing these we see that our partials in eq. (37.29) do the job provided that we form the vector operator

$$\nabla = \sum_\mu \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{37.31}$$

The text [10] defines ∇ in this fashion, but has no logical motivation of this idea. One sees quickly enough that this definition works, and is the required form, but building up to the construction in a way that builds on previously established ideas is still desirable. We see here that this reciprocal frame definition of the gradient follows inevitably from the definition of the directional derivative. Additionally this is a definition with how the directional derivative is defined in a standard Euclidean space with an orthonormal basis.

37.7 WORK OUT GRADIENT FOR A FEW SPECIFIC MULTIVECTOR SPACES

The directional derivative result expressed in eq. (37.29) holds for arbitrarily parametrized multivector spaces, and the image space can also be a generalized one. However, the corresponding result eq. (37.31) for the gradient itself is good only when the parameters are vectors. These vector parameters may be non-orthonormal, and the function this is applied to does not have to be a scalar function.

If we switch to functions parametrized by multivector spaces the vector dot gradient notation also becomes misleading. The natural generalization of the Taylor expansion for such a function, instead of eq. (37.4), or eq. (37.5) should instead be

$$f(x_0 + \epsilon) = e^{\langle \epsilon \nabla \rangle} f(x) \Big|_{x=x_0} \tag{37.32}$$

Or in full,

$$f(x_0 + \epsilon) = \sum \frac{1}{k!} \langle \epsilon \nabla \rangle^k f(x) \Big|_{x=x_0} \tag{37.33}$$

One could alternately express this in a notationally less different form using the scalar product operator instead of grade selection, if one writes

$$\epsilon * \nabla \equiv \langle \epsilon \nabla \rangle \tag{37.34}$$

However, regardless of the notation used, the fundamental definition is still going to be the same (and the same as in the vector case), which operationally is

$$\epsilon * \nabla f(x) = \langle \epsilon \nabla \rangle f(x) = \frac{\partial f(x + h\epsilon)}{\partial h} \Big|_{h=0} \tag{37.35}$$

37.7.1 *Complex numbers*

The simplest grade mixed multivector space is that of the complex numbers. Let us write out the directional derivative and gradient in this space explicitly. Writing

$$\begin{aligned} z_0 &= u + iv \\ z &= x + iy \end{aligned} \tag{37.36}$$

So we have

$$
\begin{aligned}
\langle z_0 \nabla \rangle f(z) &= u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} \\
&= u \frac{\partial f}{\partial x} + i v \frac{1}{i} \frac{\partial f}{\partial y} \\
&= \left\langle z_0 \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) \right\rangle f(z)
\end{aligned}
\tag{37.37}
$$

and we can therefore identify the gradient operator as

$$
\nabla_{0,2} = \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y}
\tag{37.38}
$$

Observe the similarity here between the vector gradient for a 2D Euclidean space, where we can form complex numbers by (left) factoring out a unit vector, as in

$$
\begin{aligned}
\mathbf{x} &= e_1 x + e_2 y \\
&= e_1 (x + e_1 e_2 y) \\
&= e_1 (x + i y) \\
&= e_1 z
\end{aligned}
\tag{37.39}
$$

It appears that we can form this complex gradient, by (right) factoring out of the same unit vector from the vector gradient

$$
\begin{aligned}
e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial x} + e_2 e_1 \frac{\partial}{\partial y} \right) e_1 \\
&= \left(\frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right) e_1 \\
&= \nabla_{0,2} e_1
\end{aligned}
\tag{37.40}
$$

So, if we write $\boldsymbol{\nabla}$ as the $\mathbb{R}^2$ vector gradient, with $\mathbf{x} = e_1 x + e_2 y = e_1 z$ as above, we have

$$
\begin{aligned}
\boldsymbol{\nabla} \mathbf{x} &= \nabla_{0,2} e_1 e_1 z \\
&= \nabla_{0,2} z
\end{aligned}
\tag{37.41}
$$

This is a rather curious equivalence between 2D vectors and complex numbers.

37.7.1.1 *Comparison of contour integral and directional derivative Taylor series*

Having a complex gradient is not familiar from standard complex variable theory. Then again, neither is a non-contour integral formulation of complex Taylor series. The two of these ought to be equivalent, which seems to imply there is a contour integral representation of the gradient in a complex number space too (one of the Hestenes paper's mentioned this but I did not understand the notation).

Let us do an initial comparison of the two. We need a reminder of the contour integral form of the complex derivative. For a function $f(z)$ and its derivatives regular in a neighborhood of a point z_0, we can evaluate

$$
\begin{aligned}
\oint \frac{f(z)dz}{(z-z_0)^k} &= -\frac{1}{k-1}\oint f(z)dz\left(\frac{1}{(z-z_0)^{k-1}}\right)' \\
&= \frac{1}{k-1}\oint f'(z)dz\left(\frac{1}{(z-z_0)^{k-1}}\right) \\
&= \frac{1}{(k-1)(k-2)}\oint f^2(z)dz\left(\frac{1}{(z-z_0)^{k-2}}\right) \\
&= \frac{1}{(k-1)(k-2)\cdots(k-n)}\oint f^n(z)dz\left(\frac{1}{(z-z_0)^{k-n}}\right) \\
&= \frac{1}{(k-1)(k-2)\cdots(1)}\oint \frac{f^{k-1}(z)dz}{z-z_0} \\
&= \frac{2\pi i}{(k-1)!}f^{k-1}(z_0)
\end{aligned}
\tag{37.42}
$$

So we have

$$
\left.\frac{d^k}{dz^k}f(z)\right|_{z_0} = \frac{k!}{2\pi i}\oint \frac{f(z)dz}{(z-z_0)^{k+1}} \tag{37.43}
$$

Given this we now have a few alternate forms of complex Taylor series

$$
\begin{aligned}
f(z_0+\epsilon) &= \sum \frac{1}{k!}\left.\langle\epsilon\nabla\rangle^k f(z)\right|_{z=z_0} \\
&= \sum \frac{1}{k!}\epsilon^k \left.\frac{d^k}{dz^k}f(z)\right|_{z_0} \\
&= \frac{1}{2\pi i}\sum \epsilon^k \oint \frac{f(z)dz}{(z-z_0)^{k+1}}
\end{aligned}
\tag{37.44}
$$

Observe that the the $0,2$ subscript for the gradient has been dropped above (ie: this is the complex gradient, not the vector form).

37.7.1.2 *Complex gradient compared to the derivative*

A gradient operator has been identified by factoring it out of the directional derivative. Let us compare this to a plain old complex derivative.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{37.45}$$

In particular, evaluating this limit for $z = z_0 + h$, approaching z_0 along the x-axis, we have

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \frac{\partial f}{\partial x}(z_0) \end{aligned} \tag{37.46}$$

Evaluating this limit for $z = z_0 + ih$, approaching z_0 along the y-axis, we have

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + ih) - f(z_0)}{ih} \\ &= -i\frac{\partial f}{\partial y}(z_0) \end{aligned} \tag{37.47}$$

We have the Cauchy equations by equating these, and if the derivative exists (ie: independent of path) we require at least

$$\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0) \tag{37.48}$$

Or

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0) \\ &= \tilde{\nabla} f(z_0) \end{aligned} \tag{37.49}$$

Premultiplying by ∇ produces the harmonic equation

$$\nabla\tilde{\nabla} f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f \tag{37.50}$$

37.7.1.3 *First order expansion around a point*

The above, while interesting or curious, does not provide a way to express the differential operator directly in terms of the gradient.

We can write

$$\begin{aligned} \langle \epsilon \nabla \rangle f(z)|_{z_0} &= \frac{\epsilon}{2\pi i} \oint \frac{f(z) dz}{(z - z_0)^2} \\ &= \epsilon f'(z_0) \end{aligned} \tag{37.51}$$

One can probably integrate this in some circumstances (perhaps when f(z) is regular along the straight path from z_0 to $z = z_0 + \epsilon$). If so, then we have

$$\epsilon \int_{s=z_0}^{z} f'(s) ds = \int_{s=z_0}^{z} \langle \epsilon \nabla \rangle f(z)|_{z=s} \, ds \tag{37.52}$$

Or

$$f(z) = f(z_0) + \int_{s=z_0}^{z} \frac{1}{\epsilon} \langle \epsilon \nabla \rangle f(z) \Big|_{z=s} ds \tag{37.53}$$

Is there any validity to doing this? The idea here is to play with some circumstances where we could see where the multivector gradient may show up. Much more play is required, some of which for discovery and the rest to do things more rigorously.

37.7.2 *4D scalar plus bivector space*

Suppose we form a scalar, bivector space by factoring out the unit time vector in a Dirac vector representation

$$\begin{aligned} x &= x^\mu \gamma_\mu \\ &= \left(x^0 + x^k \gamma_k \gamma_0 \right) \gamma_0 \\ &= \left(x^0 + x^k \sigma_k \right) \gamma_0 \\ &= q \gamma_0 \end{aligned} \tag{37.54}$$

This q has the structure of a quaternion-like object (scalar, plus bivector), but the bivectors all have positive square. Our directional derivative, for multivector direction $Q = Q^0 + Q^k \sigma_k$ is

$$\langle Q \nabla \rangle f(q) = Q^0 \frac{\partial f}{\partial x^0} + \sum_k Q^k \frac{\partial f}{\partial x^k} \tag{37.55}$$

So, we can write

$$\nabla = \frac{\partial}{\partial x^0} + \sum_k \sigma_k \frac{\partial}{\partial x^k} \tag{37.56}$$

We can do something similar for an Euclidean four vector space

$$\begin{aligned} x &= x^\mu e_\mu \\ &= \left(x^0 + x^k e_k e_0\right) e_0 \\ &= \left(x^0 + x^k i_k\right) e_0 \\ &= q e_0 \end{aligned} \tag{37.57}$$

Here each of the bivectors i_k have a negative square, much more quaternion-like (and could easily be defined in an isomorphic fashion). This time we have

$$\nabla = \frac{\partial}{\partial x^0} + \sum_k \frac{1}{i_k} \frac{\partial}{\partial x^k} \tag{37.58}$$

38

EXTERIOR DERIVATIVE AND CHAIN RULE COMPONENTS OF THE GRADIENT

38.1 GRADIENT FORMULATION IN TERMS OF RECIPROCAL FRAMES

We have seen how to calculate reciprocal frames as a method to find components of a vector with respect to an arbitrary basis (does not have to be orthogonal).

This can be applied to any vector:

$$\mathbf{x} = \sum \mathbf{a}_i(\mathbf{a}^i \cdot \mathbf{x}) = \sum \mathbf{a}^i(\mathbf{a}_i \cdot \mathbf{x}) \tag{38.1}$$

so why not the gradient operator too.

$$\nabla = \sum \mathbf{a}_i(\mathbf{a}^i \cdot \nabla) = \sum \mathbf{a}^i(\mathbf{a}_i \cdot \nabla) \tag{38.2}$$

The dot product part:

$$(\mathbf{a} \cdot \nabla) f(\mathbf{u}) = \lim_{\tau \to 0} \frac{f(\mathbf{u} + \mathbf{a}\tau) - f(\mathbf{u})}{\tau} \tag{38.3}$$

we know how to calculate explicitly (from NFCM) and is the direction derivative.

So this gives us an explicit factorization of the gradient into components in some arbitrary set of directions, all weighted appropriately.

38.2 ALLOWING THE BASIS TO VARY ACCORDING TO A PARAMETRIZATION

Now, if one allows the vector basis $\mathbf{a}_i$ to vary along a curve, it is interesting to observe the consequences of this to the gradient expressed as a component along the curve and perpendicular components.

Suppose that one has a parametrization $\phi(u_1, u_2, \cdots, u_{n-1}) \in \mathbb{R}^n$, defining a generalized surface of degree one less than the space.

Provided these surface direction vectors are linearly independent and non zero, we can writing $\boldsymbol{\phi}_{u_i} = \frac{\partial \boldsymbol{\phi}}{\partial u_i}$, and form a basis for the space by extension with a reciprocal frame vector:

$$\{\boldsymbol{\phi}_{u_1}, \boldsymbol{\phi}_{u_2}, \cdots, \boldsymbol{\phi}_{u_{n-1}}, (\boldsymbol{\phi}_{u_1} \wedge \boldsymbol{\phi}_{u_2} \cdots \wedge \boldsymbol{\phi}_{u_{n-1}})\frac{1}{\mathbf{I}_n}\} \tag{38.4}$$

38.2.1 *Try it with the simplest case*

Let us calculate ∇ in this basis. Intuition says this will produce something like the exterior derivative from differential forms for the component that is normal to the surface.

To make things easy, consider the absolutely simplest case, a curve in $\mathbb{R}^2$, with parametrization $\mathbf{r} = \boldsymbol{\phi}(t)$. The basis associated with this curve at some point is

$$\{\mathbf{a}_1, \mathbf{a}_2\} = \{\boldsymbol{\phi}_t, \mathbf{I}\boldsymbol{\phi}_t\} \tag{38.5}$$

with a reciprocal basis of:

$$\{\mathbf{a}^1, \mathbf{a}^2\} = \{\frac{1}{\boldsymbol{\phi}_t}, -\frac{1}{\boldsymbol{\phi}_t}\mathbf{I}\} \tag{38.6}$$

In terms of this components, the gradient along the curve at the specified point is:

$$\begin{aligned}
\nabla f &= \left(\mathbf{a}^1\mathbf{a}_1 \cdot \nabla + \mathbf{a}^2\mathbf{a}_2 \cdot \nabla\right) f \\
&= \left(\frac{1}{\boldsymbol{\phi}_t}\boldsymbol{\phi}_t \cdot \nabla + \left(\mathbf{I}\frac{1}{\boldsymbol{\phi}_t}\right)(\mathbf{I}\boldsymbol{\phi}_t) \cdot \nabla\right) f \\
&= \frac{1}{\boldsymbol{\phi}_t}\left(\boldsymbol{\phi}_t \cdot \nabla - \mathbf{I}\left(\mathbf{I} \cdot \boldsymbol{\phi}_t\right) \cdot \nabla\right) f \\
&= \frac{1}{\boldsymbol{\phi}_t}\left(\boldsymbol{\phi}_t \cdot \nabla - \mathbf{I}\left(\mathbf{I} \cdot \left(\boldsymbol{\phi}_t \wedge \nabla\right)\right)\right) f \\
&= \frac{1}{\boldsymbol{\phi}_t}\left(\boldsymbol{\phi}_t \cdot \nabla - \mathbf{I}\left(\mathbf{I}\left(\boldsymbol{\phi}_t \wedge \nabla\right)\right)\right) f \\
&= \frac{1}{\boldsymbol{\phi}_t}\left(\boldsymbol{\phi}_t \cdot \nabla + \boldsymbol{\phi}_t \wedge \nabla\right) f
\end{aligned} \tag{38.7}$$

Lo and behold, we come full circle through a mass of identities back to the geometric product. As with many things in math, knowing the answer we can be clever and start from the answer going backwards. This would have allowed the standard factorization of the gradient vector into orthogonal components in the usual fashion:

$$\nabla = \frac{1}{\boldsymbol{\phi}_t}\left(\boldsymbol{\phi}_t\nabla\right) = \frac{1}{\boldsymbol{\phi}_t}\left(\boldsymbol{\phi}_t \cdot \nabla + \boldsymbol{\phi}_t \wedge \nabla\right) \tag{38.8}$$

Let us continue writing $\boldsymbol{\phi}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2$. Then

$$\boldsymbol{\phi}' = x'\mathbf{e}_1 + y'\mathbf{e}_2 \tag{38.9}$$

$$\boldsymbol{\phi}' \cdot \nabla = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} \tag{38.10}$$

$$\boldsymbol{\phi}' \wedge \nabla = \mathbf{I}\left(\frac{dx}{dt}\frac{\partial}{\partial y} - \frac{dy}{dt}\frac{\partial}{\partial x}\right) \tag{38.11}$$

Combining these and inserting back into eq. (38.8) we have

$$\begin{aligned}
\nabla &= \frac{\boldsymbol{\phi}'}{(\boldsymbol{\phi}')^2}\left(\boldsymbol{\phi}' \cdot \nabla + \boldsymbol{\phi}' \wedge \nabla\right) \\
&= \frac{1}{(x')^2 + (y')^2}(x', y')\left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}\right) + (-y', x')\left(\frac{dx}{dt}\frac{\partial}{\partial y} - \frac{dy}{dt}\frac{\partial}{\partial x}\right)
\end{aligned} \tag{38.12}$$

Now, here it is worth pointing out that the choice of the parametrization can break some of the assumptions made. In particular the curve can be completely continuous, but the parametrization could allow it to be zero for some interval since $(x(t), y(t))$ can be picked to be constant for a "time" before continuing.

This problem is eliminated by picking an arc length parametrization. Provided the curve is not degenerate (ie: a point), then we have at least one of $dx/ds \neq 0$, or $dy/ds \neq 0$. Additionally, by parametrization using arc length we have $(dx/ds)^2 + (dy/ds)^2 = (ds/ds)^2 = 1$. This eliminates the denominator leaving the following decomposition of the $\mathbb{R}^2$ gradient

$$\nabla = \overset{\text{Unit tangent vector}}{\boxed{(x', y')}}\underset{\text{Chain Rule in operator form.}}{\boxed{\left(\frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y}\right)}} + \overset{\text{Unit normal vector}}{\boxed{(-y', x')}}\underset{\text{Exterior derivative operator.}}{\boxed{\left(\frac{dx}{ds}\frac{\partial}{\partial y} - \frac{dy}{ds}\frac{\partial}{\partial x}\right)}} \tag{38.13}$$

Thus, loosely speaking we have the chain rule as the scalar component of the unit tangent vector along a parametrized curve, and we have the exterior derivative as the component of the gradient that lies colinear to a unit normal to the curve (believe this is the unit normal that points inwards to the curvature if there is any).

38.2.2 *Extension to higher dimensional curves*

In $\mathbb{R}^3$ or above we can perform the same calculation. The result is similar and straightforward to derive:

$$\nabla = \underbrace{\boxed{\left(\frac{dx_1}{ds}, \frac{dx_2}{ds}, \cdots, \frac{dx_n}{ds}\right)}}_{\text{Unit tangent.}} \sum_{i=1}^{n} \frac{dx_i}{ds}\frac{\partial}{\partial x_i} + \sum_{1\le i<j\le n} \underbrace{\boxed{\left(\mathbf{e}_j \frac{dx_i}{ds} - \mathbf{e}_i \frac{dx_j}{ds}\right)}}_{\text{Normal to unit tangent.}} \left(\frac{dx_i}{ds}\frac{\partial}{\partial x_j} - \frac{dx_j}{ds}\frac{\partial}{\partial x_i}\right) \tag{38.14}$$

Here we have a set of normals to the unit tangent. For $\mathbb{R}^3$, we have $ij = \{12, 13, 23\}$. One of these unit normals must be linearly dependent on the other two (or zero). The exterior scalar factors here loose some of their resemblance to the exterior derivative here. Perhaps a parametrized (hyper-)surface is required to get the familiar form for $\mathbb{R}^3$ or above.

39 SPHERICAL AND HYPERSPHERICAL PARAMETRIZATION

39.1 MOTIVATION

In 118 a 4D Fourier transform solution of Maxwell's equation yielded a Green's function of the form

$$G(x) = \iiiint \frac{e^{ik_\mu x^\mu}}{k_\nu k^\nu} dk_1 dk_2 dk_3 dk_4 \tag{39.1}$$

To attempt to "evaluate" this integral, as done in 114 to produce the retarded time potentials, a hypervolume equivalent to spherical polar coordinate parametrization is probably desirable.

Before attempting to tackle the problem of interest, the basic question of how to do volume and weighted volume integrals over a hemispherical volumes must be considered. Doing this for both Euclidean and Minkowski metrics will have to be covered.

39.2 EUCLIDEAN N-VOLUME

39.2.1 *Parametrization*

The wikipedia article on n-volumes gives a parametrization, which I will write out explicitly for the first few dimensions

- 1-sphere (circle)

$$\begin{aligned} x^1 &= r\cos\phi_1 \\ x^2 &= r\sin\phi_1 \end{aligned} \tag{39.2}$$

- 2-sphere (sphere)

$$\begin{aligned} x^1 &= r\cos\phi_1 \\ x^2 &= r\sin\phi_1\cos\phi_2 \\ x^3 &= r\sin\phi_1\sin\phi_2 \end{aligned} \tag{39.3}$$

- 3-sphere (hypersphere)

$$\begin{aligned} x^1 &= r\cos\phi_1 \\ x^2 &= r\sin\phi_1\cos\phi_2 \\ x^3 &= r\sin\phi_1\sin\phi_2\cos\phi_3 \\ x^4 &= r\sin\phi_1\sin\phi_2\sin\phi_3 \end{aligned} \tag{39.4}$$

By inspection one can see that we have the desired $r^2 = \sum_i (x^i)^2$ relation. Each of these can be vectorized to produce a parametrized vector that can trace out all the possible points on the volume

$$\mathbf{r} = \sigma_k x^k \tag{39.5}$$

39.2.2 *Volume elements*

We can form a parallelogram area (or parallelepiped volume, ...) element for any parametrized surface by taking wedge products, as in fig. 39.1. This can also be done for this spherical parametrization too.

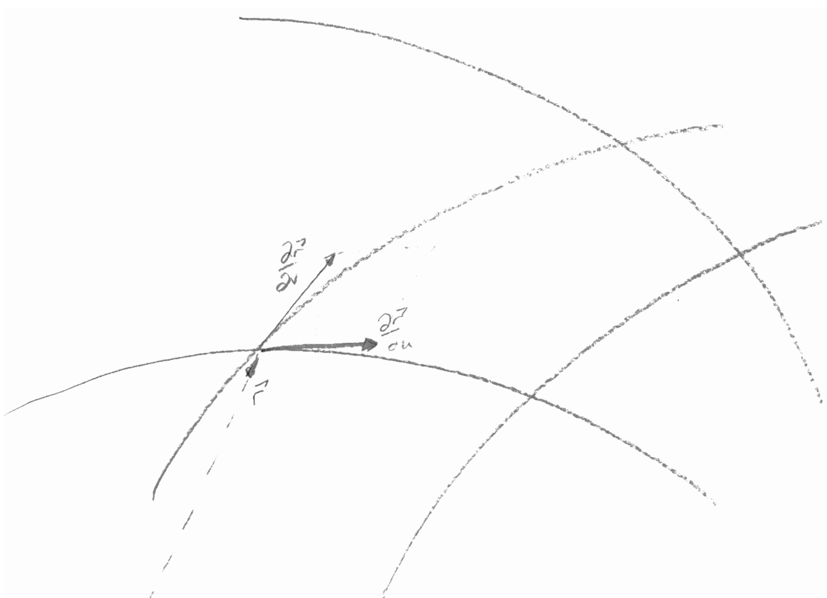

Figure 39.1: Tangent vector along curves of parametrized vector

For example for the circle we have

$$
\begin{aligned}
dV_{\mathbb{R}^2} &= \frac{\partial \mathbf{r}}{\partial r} \wedge \frac{\partial \mathbf{r}}{\partial \phi_1} dr d\phi_1 \\
&= \left(\frac{\partial}{\partial r} r(\cos\phi_1, \sin\phi_1) \right) \wedge \left(\frac{\partial}{\partial \phi_1} r(\cos\phi_1, \sin\phi_1) \right) dr d\phi_1 \\
&= (\cos\phi_1, \sin\phi_1) \wedge (-\sin\phi_1, \cos\phi_1) r dr d\phi_1 \\
&= (\cos^2\phi_1 \sigma_1\sigma_2 - \sin^2\phi_1 \sigma_2\sigma_1) r dr d\phi_1 \\
&= r dr d\phi_1 \sigma_1\sigma_2
\end{aligned}
\tag{39.6}
$$

And for the sphere

$$
\begin{aligned}
dV_{\mathbb{R}^3} &= \frac{\partial \mathbf{r}}{\partial r} \wedge \frac{\partial \mathbf{r}}{\partial \phi_1} \wedge \frac{\partial \mathbf{r}}{\partial \phi_2} dr d\phi_1 d\phi_2 \\
&= (\cos\phi_1, \sin\phi_1\cos\phi_2, \sin\phi_1\sin\phi_2) \\
&\wedge (-\sin\phi_1, \cos\phi_1\cos\phi_2, \cos\phi_1\sin\phi_2) \\
&\wedge (0, -\sin\phi_1\sin\phi_2, \sin\phi_1\cos\phi_2) r^2 dr d\phi_1 d\phi_2 \\
&= \begin{vmatrix} \cos\phi_1 & \sin\phi_1\cos\phi_2 & \sin\phi_1\sin\phi_2 \\ -\sin\phi_1 & \cos\phi_1\cos\phi_2 & \cos\phi_1\sin\phi_2 \\ 0 & -\sin\phi_1\sin\phi_2 & \sin\phi_1\cos\phi_2 \end{vmatrix} r^2 dr d\phi_1 d\phi_2 \sigma_1\sigma_2\sigma_3 \\
&= r^2 dr \sin\phi_1 d\phi_1 d\phi_2 \sigma_1\sigma_2\sigma_3
\end{aligned}
\tag{39.7}
$$

And finally for the hypersphere

$$
\begin{aligned}
dV_{\mathbb{R}^4} &= \begin{vmatrix} \cos\phi_1 & \sin\phi_1\cos\phi_2 & \sin\phi_1\sin\phi_2\cos\phi_3 & \sin\phi_1\sin\phi_2\sin\phi_3 \\ -\sin\phi_1 & \cos\phi_1\cos\phi_2 & \cos\phi_1\sin\phi_2\cos\phi_3 & \cos\phi_1\sin\phi_2\sin\phi_3 \\ 0 & -\sin\phi_1\sin\phi_2 & \sin\phi_1\cos\phi_2\cos\phi_3 & \sin\phi_1\cos\phi_2\sin\phi_3 \\ 0 & 0 & -\sin\phi_1\sin\phi_2\sin\phi_3 & \sin\phi_1\sin\phi_2\cos\phi_3 \end{vmatrix} \\
&\quad r^3 dr d\phi_1 d\phi_2 d\phi_3 \sigma_1\sigma_2\sigma_3\sigma_4 \\
&= r^3 dr \sin^2\phi_1 d\phi_1 \sin\phi_2 d\phi_2 d\phi_3 \sigma_1\sigma_2\sigma_3\sigma_4
\end{aligned}
\tag{39.8}
$$

Each of these is consistent with the result in the wiki page.

39.2.3 *Some volume computations*

Let us apply the above results to compute the corresponding n-volume's.

- 1-sphere (circle)

$$
\begin{aligned}
V_{\mathbb{R}^2} &= 4 \int_0^R r dr \int_0^{\pi/2} d\phi_1 \\
&= \pi R^2
\end{aligned}
\tag{39.9}
$$

- 2-sphere (sphere)

$$
\begin{aligned}
V_{\mathbb{R}^3} &= 8 \int_0^R r^2 dr \int_0^{\pi/2} \sin\phi_1 d\phi_1 \int_0^{\pi/2} d\phi_2 \\
&= 8 \frac{1}{3} R^3 \left(-\cos\phi_1 |_0^{\pi/2} \right) \frac{\pi}{2} \\
&= \frac{4\pi R^3}{3}
\end{aligned}
\tag{39.10}
$$

Okay, so far so good.

- 3-sphere (hypersphere)

$$
\begin{aligned}
V_{\mathbb{R}^3} &= 16 \int_0^R r^3 dr \int_0^{\pi/2} \sin^2\phi_1 d\phi_1 \int_0^{\pi/2} \sin\phi_2 d\phi_2 \int_0^{\pi/2} d\phi_3 \\
&= 2\pi R^4 \int_0^{\pi/2} \sin^2\phi_1 d\phi_1 \\
&= \pi R^4 \left(\phi_1 - \cos\phi_1 \sin\phi_1 |_0^{\pi/2} \right) \\
&= \frac{\pi^2 R^4}{2}
\end{aligned}
\tag{39.11}
$$

This is also consistent with the formula supplied in the wiki article.

39.2.4 *Range determination*

What I have done here though it integrate over only one of the quadrants, and multiply by 2^n. This avoided the more tricky issue of what exact range of angles is required for a complete and non-overlapping cover of the surface.

The wiki article says that the range is $[0, 2\pi]$ for the last angle and $[0, \pi]$ for the others. Reevaluating the integrals above shows that this does work, but that is a bit of a cheat, and it is not obvious to me past $\mathbb{R}^3$ that this should be the case.

How can this be rationalized?

- circle For the case of the circle what are the end points in each of the quadrants? These are (with $r = 1$)

$$
\begin{aligned}
(\cos\phi_1, \sin\phi_1)_{\phi_1=0} &= (1, 0) = \sigma_1 \\
(\cos\phi_1, \sin\phi_1)_{\phi_1=\pi/2} &= (0, 1) = \sigma_2 \\
(\cos\phi_1, \sin\phi_1)_{\phi_1=\pi} &= (-1, 0) = -\sigma_1 \\
(\cos\phi_1, \sin\phi_1)_{\phi_1=3\pi/2} &= (0, -1) = -\sigma_2
\end{aligned}
\tag{39.12}
$$

  As expected, each of the $\pi/2$ increments traces out the points in successive quadrants.

- sphere

  Again with $r = 1$, some representative points on the circle are

ϕ_1	ϕ_2	$(\cos\phi_1, \sin\phi_1\cos\phi_2, \sin\phi_1\sin\phi_2)$	$\mathbf{r}$
0	0	$(1, 0, 0)$	σ_1
0	$\pi/2$	$(1, 0, 0)$	σ_1
0	π	$(1, 0, 0)$	σ_1
0	$3\pi/2$	$(1, 0, 0)$	σ_1
$\pi/2$	0	$(0, 1, 0)$	σ_2
$\pi/2$	$\pi/2$	$(0, 0, 1)$	σ_3
$\pi/2$	π	$(0, -1, 0)$	$-\sigma_2$
$\pi/2$	$3\pi/2$	$(0, 0, -1)$	$-\sigma_3$
π	0	$(-1, 0, 0)$	$-\sigma_1$
π	$\pi/2$	$(-1, 0, 0)$	$-\sigma_1$
π	π	$(-1, 0, 0)$	$-\sigma_1$
π	$3\pi/2$	$(-1, 0, 0)$	$-\sigma_1$

  The most informative of these is for $\phi_1 = \pi/2$, where we had $\mathbf{r} = (0, \cos\phi_2, \sin\phi_2)$, and our points trace out a path along the unit circle of the y, z plane. At $\phi_1 = 0$ our point $\mathbf{r} = \sigma_1$ did not move, and at $\phi_1 = \pi$ we are at the other end of the sphere, also fixed. A reasonable guess is that at each ϕ_1 we trace out a different circle in the y, z plane.

  We can write, with $\sigma_{23} = \sigma_1 \wedge \sigma_2 = \sigma_1\sigma_2$,

$$
\begin{aligned}
\mathbf{r} &= \cos\phi_1\sigma_1 + \sin\phi_1(\cos\phi_2\sigma_2 + \sin\phi_2\sigma_3) \\
&= \cos\phi_1\sigma_1 + \sin\phi_1\sigma_2(\cos\phi_2 + \sin\phi_2\sigma_2\sigma_3)
\end{aligned}
\tag{39.13}
$$

Or, in exponential form

$$\mathbf{r} = \cos\phi_1\sigma_1 + \sin\phi_1\sigma_2 \exp(\sigma_{23}\phi_2) \tag{39.14}$$

Put this way the effects of the parametrization is clear. For each fixed ϕ_1, the exponential traces out a circle in the y, z plane, starting at the point $\mathbf{r} = \cos\phi_1\sigma_1 + \sin\phi_1\sigma_2$. ϕ_1 traces out a semi-circle in the x, y plane.

FIXME: picture.

This would have been easy enough to understand if starting from a picture and constructing the parametrization. Seeing what the geometry is from the algebra requires a bit more (or different) work. Having done it, are we now prepared to understand the geometry of the hypersphere parametrization.

- hypersphere.

  The vector form in the spherical case was convenient for extracting geometric properties. Can we do that here too?

$$\begin{aligned}
\mathbf{r} &= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \cos\phi_2 + \sigma_3 \sin\phi_1 \sin\phi_2 \cos\phi_3 + \sigma_4 \sin\phi_1 \sin\phi_2 \sin\phi_3 \\
&= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \cos\phi_2 + \sigma_3 \sin\phi_1 \sin\phi_2 (\cos\phi_3 + \sigma_{34} \sin\phi_3) \\
&= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \cos\phi_2 + \sigma_3 \sin\phi_1 \sin\phi_2 \exp(\sigma_{34}\phi_3) \\
&= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 (\cos\phi_2 + \sigma_{23} \sin\phi_2 \exp(\sigma_{34}\phi_3))
\end{aligned} \tag{39.15}$$

Observe that if $\phi_3 = 0$ we have

$$\mathbf{r} = \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 \exp(\sigma_{23}\phi_2) \tag{39.16}$$

Which is exactly the parametrization of a half sphere ($\phi_2 \in [0, \pi]$). Contrast this to the semi-circle that ϕ_1 traced out in the spherical case.

In the spherical case, the points $\phi_1 = \pi/2$ were nicely representative. For the hypersphere those points are

$$\mathbf{r} = \sigma_2 \cos\phi_2 + \sigma_3 \sin\phi_2 \exp(\sigma_{34}\phi_3) \tag{39.17}$$

We saw above that this is the parametrization of a sphere.

Also like the spherical case, we have $\mathbf{r} = \pm\sigma_1$ at $\phi_1 = 0$, and $\phi_1 = \pi$ respectively.

The geometrical conclusion is that for each $\phi_1 \in [0, \pi/2]$ range the points $\mathbf{r}$ trace out increasingly larger spheres, and after that decreasing sized spheres until we get to a point again at $\phi_1 = \pi$.

39.3 MINKOWSKI METRIC SPHERE

39.3.1 *2D hyperbola*

Our 1-sphere equation was all the points on the curve

$$x^2 + y^2 = r^2 \tag{39.18}$$

The hyperbolic equivalent to this is

$$x^2 - y^2 = r^2 \tag{39.19}$$

Although this is not a closed curve like the circle. To put this in a more natural physical context, lets write (with $c = 1$)

$$\mathbf{r} = \gamma_0 t + \gamma_1 x \tag{39.20}$$

So the equation of the 1-hyperboloid becomes

$$\mathbf{r}^2 = t^2 - x^2 = r^2 \tag{39.21}$$

We can parametrize this with complex angles $i\phi$

$$\mathbf{r} = r(\gamma_0 \cosh\phi + \gamma_1 \sinh\phi) \tag{39.22}$$

This gives us

$$\mathbf{r}^2 = r^2(\cosh^2\phi - \sinh^2\phi) = r^2 \tag{39.23}$$

as desired. Like the circle, writing $\gamma_{01} = \gamma_0 \wedge \gamma_1$, an exponential form also works nicely

$$\mathbf{r} = r\gamma_0 \exp(\gamma_{01}\phi) \tag{39.24}$$

Here the square is

$$
\begin{aligned}
\mathbf{r}^2 &= r^2 \gamma_0 \exp(\gamma_{01}\phi)\gamma_0 \exp(\gamma_{01}\phi) \\
&= r^2 \exp(-\gamma_{01}\phi)(\gamma_0)^2 \exp(\gamma_{01}\phi) \\
&= r^2 \exp(-\gamma_{01}\phi) \exp(\gamma_{01}\phi) \\
&= r^2
\end{aligned}
\tag{39.25}
$$

Again as desired.

39.3.2 *3D hyperbola*

Unlike the circle, a pure hyperbolic parametrization does not work to construct a Minkowski square signature. Consider for example

$$
\mathbf{r} = \cosh\phi\gamma_0 + \gamma_1 \sinh\phi \cosh\psi + \gamma_2 \sinh\phi \sinh\psi \tag{39.26}
$$

Squaring this we have

$$
\mathbf{r}^2 = \cosh^2\phi - \sinh^2\phi(\cosh^2\psi + \sinh^2\psi) \tag{39.27}
$$

We would get the desired result if we chop off the h in all the ψ hyperbolic functions. This shows that an appropriate parametrization is instead

$$
\mathbf{r} = \cosh\phi\gamma_0 + \gamma_1 \sinh\phi \cos\psi + \gamma_2 \sinh\phi \sin\psi \tag{39.28}
$$

This now squares to 1. To see how to extend this to higher dimensions (of which we only need one more) we can factor out a γ_0

$$
\mathbf{r} = \gamma_0(\cosh\phi - \sinh\phi(\overbrace{\boxed{\sigma_1 \cos\psi + \sigma_2 \sin\psi}}^{\text{spatial vector parametrization of circle}})) \tag{39.29}
$$

Now to extend this to three dimensions we have just to substituted the spherical parametrization from eq. (39.14)

$$
\begin{aligned}
\mathbf{r} &= r\gamma_0(\cosh\phi_0 - \sinh\phi_0(\cos\phi_1\sigma_1 + \sin\phi_1\sigma_2 \exp(\sigma_{23}\phi_2))) \\
&= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0(\cos\phi_1\gamma_1 + \sin\phi_1\gamma_2 \exp(\gamma_{32}\phi_2)))
\end{aligned}
\tag{39.30}
$$

39.3.3 *Summarizing the hyperbolic vector parametrization*

Our parametrization in two, three, and four dimensions, respectively, are

$$
\begin{aligned}
\mathbf{r}_2 &= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0\gamma_1) \\
\mathbf{r}_3 &= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0\gamma_1 \exp(\gamma_{21}\phi_1)) \\
\mathbf{r}_4 &= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0(\cos\phi_1\gamma_1 + \sin\phi_1\gamma_2 \exp(\gamma_{32}\phi_2)))
\end{aligned}
\tag{39.31}
$$

39.3.4 *Volume elements*

What are our volume elements using this parametrization can be calculated as above.

39.3.4.1 *For one spatial dimension we have*

$$
\begin{aligned}
dV_2\gamma_0\gamma_1 &= \begin{vmatrix} \cosh\phi_0 & \sinh\phi_0 \\ \sinh\phi_0 & \cosh\phi_0 \end{vmatrix} rdrd\phi_0 \\
&= rdrd\phi_0
\end{aligned}
\tag{39.32}
$$

39.3.4.2 *For two spatial dimensions we have*

$$
\mathbf{r}_3 = r(\gamma_0 \cosh\phi_0 + \gamma_1 \sinh\phi_0 \cos\phi_1 + \gamma_2 \sinh\phi_0 \sin\phi_2)
\tag{39.33}
$$

The derivatives are

$$
\begin{aligned}
\frac{\partial \mathbf{r}_3}{\partial r} &= \gamma_0 \cosh\phi_0 + \gamma_1 \sinh\phi_0 \cos\phi_1 + \gamma_2 \sinh\phi_0 \sin\phi_1 \\
\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_0} &= \gamma_0 \sinh\phi_0 + \gamma_1 \cosh\phi_0 \cos\phi_1 + \gamma_2 \cosh\phi_0 \sin\phi_1 \\
\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_1} &= -\gamma_1 \sinh\phi_0 \sin\phi_1 + \gamma_2 \sinh\phi_0 \cos\phi_1
\end{aligned}
\tag{39.34}
$$

Or

$$
\begin{aligned}
\frac{\partial \mathbf{r}_3}{\partial r} &= \gamma_0 \cosh\phi_0 + \gamma_1 \sinh\phi_0 \exp(\gamma_{21}\phi_1) \\
\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_0} &= \gamma_0 \sinh\phi_0 + \cosh\phi_0 \exp(-\gamma_{21}\phi_1)\gamma_1 \\
\frac{1}{r}\frac{\partial \mathbf{r}_3}{\partial \phi_1} &= \sinh\phi_0\gamma_2 \exp(\gamma_{21}\phi_1)
\end{aligned}
\tag{39.35}
$$

Multiplying this out, discarding non-grade three terms we have

$$
\begin{aligned}
&(\gamma_{10} \sinh^2 \phi_0 \exp(\gamma_{21}\phi_1) + \gamma_{01} \cosh^2 \phi_0 \exp(\gamma_{21}\phi_1)) \sinh \phi_0 \gamma_2 \exp(\gamma_{21}\phi_1) \\
&= \gamma_{01} \exp(\gamma_{21}\phi_1) \sinh \phi_0 \exp(-\gamma_{21}\phi_1)\gamma_2 \\
&= \gamma_{01} \sinh \phi_0 \gamma_2
\end{aligned}
\tag{39.36}
$$

This gives us

$$
dV_3 = r^2 \sinh \phi_0 dr d\phi_0 d\phi_1 \tag{39.37}
$$

39.3.4.3 *For three spatial dimensions we have*

$$
\mathbf{r}_4 = r(\gamma_0 \cosh \phi_0 + \sinh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2))) \tag{39.38}
$$

So our derivatives are

$$
\begin{aligned}
\frac{\partial \mathbf{r}_4}{\partial r} &= \gamma_0 \cosh \phi_0 + \sinh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2)) \\
\frac{1}{r}\frac{\partial \mathbf{r}_4}{\partial \phi_0} &= \gamma_0 \sinh \phi_0 + \cosh \phi_0 (\cos \phi_1 \gamma_1 + \sin \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2)) \\
\frac{1}{r}\frac{\partial \mathbf{r}_4}{\partial \phi_1} &= \sinh \phi_0 (-\sin \phi_1 \gamma_1 + \cos \phi_1 \gamma_2 \exp(\gamma_{32}\phi_2)) \\
\frac{1}{r}\frac{\partial \mathbf{r}_4}{\partial \phi_2} &= \sinh \phi_0 \sin \phi_1 \gamma_3 \exp(\gamma_{32}\phi_2)
\end{aligned}
\tag{39.39}
$$

In shorthand, writing C and S for the trig and hyperbolic functions as appropriate, we have

$$
\begin{aligned}
\gamma_0 C_0 + S_0 C_1 \gamma_1 + S_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) \\
\gamma_0 S_0 + C_0 C_1 \gamma_1 + C_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) \\
-S_0 S_1 \gamma_1 + S_0 C_1 \gamma_2 \exp(\gamma_{32}\phi_2) \\
S_0 S_1 \gamma_3 \exp(\gamma_{32}\phi_2)
\end{aligned}
\tag{39.40}
$$

Multiplying these out and dropping terms that will not contribute grade four bits is needed to calculate the volume element. The full product for the first two derivatives is

$$
\begin{aligned}
&\gamma_0 C_0 \gamma_0 S_0 + \gamma_0 C_0 C_0 C_1 \gamma_1 + \gamma_0 C_0 C_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) \\
&+ S_0 C_1 \gamma_1 \gamma_0 S_0 + S_0 C_1 \gamma_1 C_0 C_1 \gamma_1 + S_0 C_1 \gamma_1 C_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) \\
&+ S_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) \gamma_0 S_0 + S_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) C_0 C_1 \gamma_1 + S_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) C_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2)
\end{aligned}
$$

$$(39.41)$$

We can discard the scalar terms:

$$\gamma_0 C_0 \gamma_0 S_0 + S_0 C_1 \gamma_1 C_0 C_1 \gamma_1 + S_0 S_1 \gamma_2 \exp(\gamma_{32}\phi_2) C_0 S_1 \exp(-\gamma_{32}\phi_2)\gamma_2 \tag{39.42}$$

Two of these terms cancel out

$$S_0 C_0 C_1 S_1 \gamma_{12} \exp(\gamma_{32}\phi_2) + S_0 C_0 C_1 S_1 \gamma_{21} \exp(\gamma_{32}\phi_2) \tag{39.43}$$

and we are left with two bivector contributors to the eventual four-pseudoscalar

$$\gamma_{01} C_1 + \gamma_{02} S_1 \exp(\gamma_{32}\phi_2) \tag{39.44}$$

Multiplying out the last two derivatives we have

$$-S_0^2 S_1^2 \gamma_{13} \exp(\gamma_{32}\phi_2) + S_0^2 C_1 S_1 \gamma_{23} \tag{39.45}$$

Almost there. A final multiplication of these sets of products gives

$$\begin{aligned}
&- \gamma_{01} C_1 S_0^2 S_1^2 \gamma_{13} \exp(\gamma_{32}\phi_2) - \gamma_{02} S_1 \exp(\gamma_{32}\phi_2) S_0^2 S_1^2 \gamma_{13} \exp(\gamma_{32}\phi_2) \\
&+ \gamma_{01} C_1 S_0^2 C_1 S_1 \gamma_{23} + \gamma_{02} S_1 \exp(\gamma_{32}\phi_2) S_0^2 C_1 S_1 \gamma_{23} \\
&= \gamma_{03} C_1 S_0^2 S_1^2 \exp(\gamma_{32}\phi_2) - \gamma_{02} \exp(\gamma_{32}\phi_2) S_0^2 S_1^3 \exp(-\gamma_{32}\phi_2)\gamma_{13} \\
&\qquad + \gamma_{0123} C_1^2 S_0^2 S_1 - \gamma_{03} \exp(\gamma_{32}\phi_2) S_0^2 C_1 S_1^2 \\
&= \gamma_{03} C_1 S_0^2 S_1^2 \exp(\gamma_{32}\phi_2) + \gamma_{0123} S_0^2 S_1 (S_1^2 + C_1^2) - \gamma_{03} \exp(\gamma_{32}\phi_2) S_0^2 C_1 S_1^2 \\
&= \gamma_{03} C_1 S_1^2 (S_0^2 - S_0^2) \exp(\gamma_{32}\phi_2) + \gamma_{0123} S_0^2 S_1
\end{aligned} \tag{39.46}$$

Therefore our final result is

$$dV_4 = \sinh^2\phi_0 \sin\phi_1 r^3 dr d\phi_0 d\phi_1 d\phi_2 \tag{39.47}$$

39.4 SUMMARY

39.4.1 *Vector parametrization*

N-Spherical parametrization

$$\begin{aligned}
\mathbf{r}_2 &= \sigma_1 \exp(\sigma_{12}\phi_1) \\
\mathbf{r}_3 &= \cos\phi_1 \sigma_1 + \sin\phi_1 \sigma_2 \exp(\sigma_{23}\phi_2) \\
\mathbf{r}_4 &= \sigma_1 \cos\phi_1 + \sigma_2 \sin\phi_1 (\cos\phi_2 + \sigma_{23} \sin\phi_2 \exp(\sigma_{34}\phi_3))
\end{aligned} \tag{39.48}$$

N-Hypersphere parametrization

$$\begin{aligned}
\mathbf{r}_2 &= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0\gamma_1) \\
\mathbf{r}_3 &= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0\gamma_1 \exp(\gamma_{21}\phi_1)) \\
\mathbf{r}_4 &= r(\gamma_0 \cosh\phi_0 + \sinh\phi_0(\cos\phi_1\gamma_1 + \sin\phi_1\gamma_2 \exp(\gamma_{32}\phi_2)))
\end{aligned} \tag{39.49}$$

39.4.2 *Volume elements*

To summarize the mess of algebra we have shown that our hyperbolic volume elements are given by

$$\begin{aligned}
dV_2 &= (rdr)\, d\phi_0 \\
dV_3 &= \left(r^2 dr\right)(\sinh\phi_0 d\phi_0)\, d\phi_1 \\
dV_4 &= \left(r^3 dr\right)\left(\sinh^2\phi_0 d\phi_0\right)(\sin\phi_1 d\phi_1)\, d\phi_2
\end{aligned} \tag{39.50}$$

Compare this to the volume elements for the n-spheres

$$\begin{aligned}
dV_2 &= (rdr)\, d\phi_1 \\
dV_3 &= \left(r^2 dr\right)(\sin\phi_1 d\phi_1)\, d\phi_2 \\
dV_4 &= \left(r^3 dr\right)\left(\sin^2\phi_1 d\phi_1\right)(\sin\phi_2 d\phi_2)\, d\phi_3
\end{aligned} \tag{39.51}$$

Besides labeling variations the only difference in the form is a switch from trig to hyperbolic functions for the first angle (which has an implied range difference as well).

40 VECTOR DIFFERENTIAL IDENTITIES

40.1 SOME IDENTITIES

[12] electrodynamics chapter II lists a number of differential vector identities.

1. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) = \boldsymbol{\nabla}^2 T$ = a scalar field
2. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = 0$
3. $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{h})$ = a vector field
4. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{h}) = 0$
5. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{h}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{h}) - \boldsymbol{\nabla}^2 \mathbf{h}$
6. $(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{h}$ = a vector field

Let us see how all these translate to GA form.

40.1.1 *Divergence of a gradient*

This one has the same form, but expanding it can be evaluated by grade selection

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) &= \langle \boldsymbol{\nabla}\boldsymbol{\nabla} T \rangle \\
&= (\boldsymbol{\nabla}^2) T
\end{aligned}
\tag{40.1}
$$

A less sneaky expansion would be by coordinates

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) &= \sum_{k,j} (\sigma_k \partial_k) \cdot (\sigma_j \partial_j T) \\
&= \left\langle \sum_{k,j} (\sigma_k \partial_k)(\sigma_j \partial_j T) \right\rangle \\
&= \left\langle \left(\sum_{k,j} \sigma_k \partial_k \sigma_j \partial_j \right) T \right\rangle \\
&= \left\langle \boldsymbol{\nabla}^2 T \right\rangle \\
&= \boldsymbol{\nabla}^2 T
\end{aligned}
\tag{40.2}
$$

40.1.2 *Curl of a gradient is zero*

The duality analogue of this is

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = -i(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} T)) \tag{40.3}$$

Let us verify that this bivector curl is zero. This can also be done by grade selection

$$\begin{aligned}
\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} T) &= \langle \boldsymbol{\nabla}(\boldsymbol{\nabla} T) \rangle_2 \\
&= \langle (\boldsymbol{\nabla}\boldsymbol{\nabla}) T \rangle_2 \\
&= (\boldsymbol{\nabla} \wedge \boldsymbol{\nabla}) T \\
&= 0
\end{aligned} \tag{40.4}$$

Again, this is sneaky and side steps the continuity requirement for mixed partial equality. Again by coordinates is better

$$\begin{aligned}
\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} T) &= \left\langle \sum_{k,j} \sigma_k \partial_k (\sigma_j \partial_j T) \right\rangle_2 \\
&= \left\langle \sum_{k<j} \sigma_k \sigma_j (\partial_k \partial_j - \partial_j \partial_k) T \right\rangle_2 \\
&= \sum_{k<j} \sigma_k \wedge \sigma_j (\partial_k \partial_j - \partial_j \partial_k) T
\end{aligned} \tag{40.5}$$

So provided the mixed partials are zero the curl of a gradient is zero.

40.1.3 *Gradient of a divergence*

Nothing more to say about this one.

40.1.4 *Divergence of curl*

This one looks like it will have a dual form using bivectors.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{h}) &= \boldsymbol{\nabla} \cdot (-i(\boldsymbol{\nabla} \wedge \mathbf{h})) \\
&= \langle \boldsymbol{\nabla}(-i(\boldsymbol{\nabla} \wedge \mathbf{h})) \rangle \\
&= \langle -i\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{h}) \rangle \\
&= -(i\boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{h})
\end{aligned} \tag{40.6}$$

Is this any better than the cross product relationship?

I do not really think so. They both say the same thing, and only possible value to this duality form is if more than three dimensions are required (in which case the sign of the pseudoscalar i has to be dealt with more carefully). Geometrically one has the dual of the gradient (a plane normal to the vector itself) dotted with the plane formed by the gradient and the vector operated on. The corresponding statement for the cross product form is that we have a dot product of a vector with a vector normal to it, so also intuitively expect a zero. In either case, because we are talking about operators here just saying this is zero because of geometrical arguments is not necessarily convincing. Let us evaluate this explicitly in coordinates to verify

$$\begin{aligned}(i\boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) &= \langle i\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{h})\rangle \\ &= \left\langle i \sum_{k,j,l} \sigma_k \partial_k \left((\sigma_j \wedge \sigma_l)\partial_j h^l\right)\right\rangle \\ &= -i \sum_l \sigma_l \wedge \left(\sum_{k<j} (\sigma_k \wedge \sigma_j)(\partial_k \partial_j - \partial_j \partial_k) h^l\right)\end{aligned} \tag{40.7}$$

This inner quantity is zero, again by equality of mixed partials. While the dual form of this identity was not really any better than the cross product form, there is nothing in this zero equality proof that was tied to the dimension of the vectors involved, so we do have a more general form than can be expressed by the cross product, which could be of value in Minkowski space later.

40.1.5 *Curl of a curl*

This will also have a dual form. That is

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{h}) &= -i(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \times \mathbf{h})) \\ &= -i(\boldsymbol{\nabla} \wedge (-i(\boldsymbol{\nabla} \wedge \mathbf{h}))) \\ &= -i\langle \boldsymbol{\nabla}(-i(\boldsymbol{\nabla} \wedge \mathbf{h}))\rangle_2 \\ &= i\langle i\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{h})\rangle_2 \\ &= i^2 \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) \\ &= -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h})\end{aligned} \tag{40.8}$$

Now, let us expand this quantity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) \tag{40.9}$$

If the gradient could be treated as a plain old vector we could just do

$$\mathbf{a} \cdot (\mathbf{a} \wedge \mathbf{h}) = \mathbf{a}^2\mathbf{h} - \mathbf{a}(\mathbf{a} \cdot \mathbf{h}) \tag{40.10}$$

With the gradient substituted this is exactly the desired identity (with the expected sign difference)

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{h}) = \boldsymbol{\nabla}^2\mathbf{h} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{h}) \tag{40.11}$$

A coordinate expansion to truly verify that this is valid is logically still required, but having done the others above, it is clear how this would work out.

40.1.6 *Laplacian of a vector*

This one is not interesting seeming.

40.2 TWO MORE THEOREMS

Two theorems without proof are mentioned in the text.

40.2.1 *Zero curl implies gradient solution*

Theorem was

$$\begin{aligned}
&\text{If} && \boldsymbol{\nabla} \times \mathbf{A} &&= 0 \\
&\text{there is a} && \psi && \\
&\text{such that} && \mathbf{A} &&= \boldsymbol{\nabla}\psi
\end{aligned}$$

This appears to be half of an if and only if theorem. The unstated part is if one has a gradient then the curl is zero

$$\begin{aligned}
&\mathbf{A} = \boldsymbol{\nabla}\psi \\
&\quad \implies \\
&\quad \boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0
\end{aligned} \tag{40.12}$$

This last was proven above, and follows from the assumed mixed partial equality. Now, the real problem here is to find ψ given $\mathbf{A}$. First note that we can remove the three dimensionality of the theorem by duality writing $\boldsymbol{\nabla} \times \mathbf{A} = -i(\boldsymbol{\nabla} \wedge \mathbf{A})$. In one sense changing the theorem to use the wedge instead of cross makes the problem harder since the wedge product is defined not just for $\mathbb{R}^3$. However, this also allows starting with the simpler $\mathbb{R}^2$ case, so let us do that one first.

Write

$$\mathbf{A} = \sigma^1 A_1 + \sigma^2 A_2 = \sigma^1(\partial_1\psi) + \sigma^2(\partial_2\psi) \tag{40.13}$$

The gradient is

$$\boldsymbol{\nabla} = \sigma^1\partial_1 + \sigma^2\partial_2 \tag{40.14}$$

Our curl is then

$$(\sigma^1\partial_1 + \sigma^2\partial_2) \wedge (\sigma^1 A_1 + \sigma^2 A_2) = (\sigma^1 \wedge \sigma^2)(\partial_1 A_2 - \partial_2 A_1) \tag{40.15}$$

So we have

$$\partial_1 A_2 = \partial_2 A_1 \tag{40.16}$$

Now from eq. (40.13) this means we must have

$$\partial_1\partial_2\psi = \partial_2\partial_1\psi \tag{40.17}$$

This is just a statement of mixed partial equality, and does not look particularly useful for solving for ψ. It really shows that the is redundancy in the problem, and instead of substituting for both of A_1 and A_2 in eq. (40.16), we can use one or the other.

Doing so we have two equations, either of which we can solve for

$$\begin{aligned}
\partial_2\partial_1\psi &= \partial_1 A_2 \\
\partial_1\partial_2\psi &= \partial_2 A_1
\end{aligned} \tag{40.18}$$

Integrating once gives

$$\begin{aligned}
\partial_1\psi &= \int \partial_1 A_2 dy + B(x) \\
\partial_2\psi &= \int \partial_2 A_1 dx + C(y)
\end{aligned} \tag{40.19}$$

And a second time produces solutions for ψ in terms of the vector coordinates

$$\begin{aligned}
\psi &= \iint \frac{\partial A_2}{\partial x} dydx + \int B(x)dx + D(y) \\
\psi &= \iint \frac{\partial A_1}{\partial y} dxdy + \int C(y)dy + E(x)
\end{aligned} \tag{40.20}$$

Is there a natural way to merge these so that ψ can be expressed more symmetrically in terms of both coordinates? Looking at eq. (40.20) I am led to guess that its possible to combine these into a single equation expressing ψ in terms of both A_1 and A_2. One way to do so is perhaps just to average the two as in

$$\psi = \alpha \iint \frac{\partial A_2}{\partial x} dydx + (1-\alpha) \iint \frac{\partial A_1}{\partial y} dxdy + \int C(y)dy + E(x) + \int B(x)dx + D(y) \tag{40.21}$$

But that seems pretty arbitrary. Perhaps that is the point?

FIXME: work some examples.

FIXME: look at more than the $\mathbb{R}^2$ case.

40.2.2 *Zero divergence implies curl solution*

Theorem was

$$\begin{aligned}
&\text{If} && \nabla \cdot \mathbf{D} &&= 0 \\
&\text{there is a} && \mathbf{C} && \\
&\text{such that} && \mathbf{D} &&= \nabla \times \mathbf{C}
\end{aligned}$$

As above, if $\mathbf{D} = \nabla \times \mathbf{C}$, then we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\nabla \times \mathbf{C}) \tag{40.22}$$

and this has already been shown to be zero. So the problem becomes find $\mathbf{C}$ given $\mathbf{D}$.

Also, as before an equivalent generalized (or de-generalized) problem can be expressed. That is

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \langle \nabla \mathbf{D} \rangle \\
&= \langle \nabla (\nabla \times \mathbf{C}) \rangle \\
&= \langle \nabla - i(\nabla \wedge \mathbf{C}) \rangle \\
&= -\langle i\nabla \cdot (\nabla \wedge \mathbf{C}) \rangle - \langle i\nabla \wedge (\nabla \wedge \mathbf{C}) \rangle \\
&= -\langle i\nabla \cdot (\nabla \wedge \mathbf{C}) \rangle
\end{aligned} \tag{40.23}$$

So if $\boldsymbol{\nabla} \cdot \mathbf{D}$ it is also true that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{C}) = 0$
Thus the (de)generalized theorem to prove is

$$
\begin{aligned}
&\text{If} && \boldsymbol{\nabla} \cdot D && = 0 \\
&\text{there is a} && C && \\
&\text{such that} && D && = \boldsymbol{\nabla} \wedge C
\end{aligned}
$$

In the $\mathbb{R}^3$ case, to prove the original theorem we want a bivector $D = -i\mathbf{D}$, and seek a vector C such that $D = \boldsymbol{\nabla} \wedge C$ ($\mathbf{D} = -i(\boldsymbol{\nabla} \wedge C)$).

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot D &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge C) \\
&= (\sigma^k \partial_k) \cdot (\sigma^i \wedge \sigma^j \partial_i C_j) \\
&= \sigma^k \cdot (\sigma^i \wedge \sigma^j) \partial_k \partial_i C_j \\
&= (\sigma^j \delta^{ki} - \sigma^i \delta^{kj}) \partial_k \partial_i C_j \\
&= \sigma^j \partial_i \partial_i C_j - \sigma^i \partial_j \partial_i C_j \\
&= \sigma^j \partial_i (\partial_i C_j - \partial_j C_i)
\end{aligned}
\tag{40.24}
$$

If this is to equal zero we must have the following constraint on C

$$
\partial_{ii} C_j = \partial_{ij} C_i \tag{40.25}
$$

If the following equality was also true

$$
\partial_i C_j = \partial_j C_i \tag{40.26}
$$

Then this would also work, but would also mean D equals zero so that is not an interesting solution. So, we must go back to eq. (40.25) and solve for C_k in terms of D.

Suppose we have D explicitly in terms of coordinates

$$
\begin{aligned}
D &= D_{ij} \sigma^i \wedge \sigma^j \\
&= \sum_{i<j} (D_{ij} - D_{ji}) \sigma^i \wedge \sigma^j
\end{aligned}
\tag{40.27}
$$

compare this to $\boldsymbol{\nabla} \wedge C$

$$
\begin{aligned}
C &= (\partial_i C_j) \sigma^i \wedge \sigma^j \\
&= \sum_{i<j} (\partial_i C_j - \partial_j C_i) \sigma^i \wedge \sigma^j
\end{aligned}
\tag{40.28}
$$

With the identity

$$\partial_i C_j = D_i j \tag{40.29}$$

Equation (40.25) becomes

$$\begin{aligned} &\partial_{ij} C_i = \partial_i D_{ij} \\ &\implies \\ &\partial_j C_i = D_{ij} + \alpha_{ij}(x^{k \neq i}) \end{aligned} \tag{40.30}$$

Where $\alpha_{ij}(x^{k \neq i})$ is some function of all the $x^k \neq x^i$.
Integrating once more we have

$$C_i = \int \left(D_{ij} + \alpha_{ij}(x^{k \neq i}) \right) dx^j + \beta_{ij}(x^{k \neq j}) \tag{40.31}$$

41

WAS PROFESSOR DMITREVSKY RIGHT ABOUT INSISTING THE LAPLACIAN IS NOT GENERALLY $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$

41.1 DEDICATION

To all tyrannical old Professors driven to cruelty by an unending barrage of increasingly ill prepared students.

41.2 MOTIVATION

The text [5] has an excellent general derivation of a number of forms of the gradient, divergence, curl and Laplacian.

This is actually done, not starting with the usual Cartesian forms, but more general definitions.

$$
\begin{aligned}
(\text{grad } \phi)_i &= \lim_{ds_i \rightarrow 0} \frac{\phi(q_i + dq_i) - \phi(q_i)}{ds_i} \\
\text{div } \mathbf{V} &= \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \int_\sigma \mathbf{V} \cdot d\boldsymbol{\sigma} \\
(\text{curl } \mathbf{V}) \cdot \mathbf{n} &= \lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} \oint_\lambda \mathbf{V} \cdot d\boldsymbol{\lambda} \\
\text{Laplacian } \phi &= \text{div}(\text{grad } \phi).
\end{aligned}
\tag{41.1}
$$

These are then shown to imply the usual Cartesian definitions, plus provide the means to calculate the general relationships in whatever coordinate system you like. All in all one can not beat this approach, and I am not going to try to replicate it, because I can not improve it in any way by doing so.

Given that, what do I have to say on this topic? Well, way way back in first year electricity and magnetism, my dictator of a prof, the intimidating but diminutive Dmitrevsky, yelled at us repeatedly that one cannot just dot the gradient to form the Laplacian. As far as he was concerned one can only say

$$
\text{Laplacian } \phi = \text{div}(\text{grad } \phi),
\tag{41.2}
$$

and never never never, the busted way

$$\text{Laplacian } \phi = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\phi. \tag{41.3}$$

Because "this only works in Cartesian coordinates". He probably backed up this assertion with a heartwarming and encouraging statement like "back in the days when University of Toronto was a real school you would have learned this in kindergarten".

This detail is actually something that has bugged me ever since, because my assumption was that, provided one was careful, why would a change to an alternate coordinate system matter? The gradient is still the gradient, so it seems to me that this ought to be a general way to calculate things.

Here we explore the validity of the dictatorial comments of Prof Dmitrevsky. The key to reconciling intuition and his statement turns out to lie with the fact that one has to let the gradient operate on the unit vectors in the non Cartesian representation as well as the partials, something that was not clear as a first year student. Provided that this is done, the plain old dot product procedure yields the expected results.

This exploration will utilize a two dimensional space as a starting point, transforming from Cartesian to polar form representation. I will also utilize a geometric algebra representation of the polar unit vectors.

41.3 THE GRADIENT IN POLAR FORM

Lets start off with a calculation of the gradient in polar form starting with the Cartesian form. Writing $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $\partial_r = \partial/\partial r$, and $\partial_\theta = \partial/\partial\theta$, we want to map

$$\boldsymbol{\nabla} = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \partial_1 \\ \partial_2 \end{bmatrix}, \tag{41.4}$$

into the same form using $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, ∂_r, and ∂_θ. With $i = \mathbf{e}_1\mathbf{e}_2$ we have

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = e^{i\theta} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \end{bmatrix}. \tag{41.5}$$

Next we need to do a chain rule expansion of the partial operators to change variables. In matrix form that is

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix}. \tag{41.6}$$

To calculate these partials we drop back to coordinates

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \frac{y}{x} &= \tan\theta \\ \frac{x}{y} &= \cot\theta. \end{aligned} \tag{41.7}$$

From this we calculate

$$\begin{aligned} \frac{\partial r}{\partial x} &= \cos\theta \\ \frac{\partial r}{\partial y} &= \sin\theta \\ \frac{1}{r\cos\theta} &= \frac{\partial\theta}{\partial y}\frac{1}{\cos^2\theta} \\ \frac{1}{r\sin\theta} &= -\frac{\partial\theta}{\partial x}\frac{1}{\sin^2\theta}, \end{aligned} \tag{41.8}$$

for

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\theta} \end{bmatrix}. \tag{41.9}$$

We can now write down the gradient in polar form, prior to final simplification

$$\boldsymbol{\nabla} = e^{i\theta} \begin{bmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\theta} \end{bmatrix}. \tag{41.10}$$

Observe that we can factor a unit vector

$$\begin{bmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} \end{bmatrix} = \hat{\mathbf{r}} \begin{bmatrix} 1 & i \end{bmatrix} = \begin{bmatrix} i & 1 \end{bmatrix} \hat{\boldsymbol{\theta}} \tag{41.11}$$

so the $1, 1$ element of the matrix product in the interior is

$$\begin{bmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \hat{\mathbf{r}} e^{i\theta} = e^{-i\theta} \hat{\mathbf{r}}. \tag{41.12}$$

Similarly, the $1, 2$ element of the matrix product in the interior is

$$\begin{bmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} \end{bmatrix} \begin{bmatrix} -\sin\theta/r \\ \cos\theta/r \end{bmatrix} = \frac{1}{r} e^{-i\theta} \hat{\boldsymbol{\theta}}. \tag{41.13}$$

The exponentials cancel nicely, leaving after a final multiplication with the polar form for the gradient

$$\boldsymbol{\nabla} = \hat{\mathbf{r}} \partial_r + \hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta \tag{41.14}$$

That was a fun way to get the result, although we could have just looked it up. We want to use this now to calculate the Laplacian.

41.4 POLAR FORM LAPLACIAN FOR THE PLANE

We are now ready to look at the Laplacian. First let us do it the first year electricity and magnetism course way. We look up the formula for polar form divergence, the one we were supposed to have memorized in kindergarten, and find it to be

$$\text{div}\, \mathbf{A} = \partial_r A_r + \frac{1}{r} A_r + \frac{1}{r} \partial_\theta A_\theta \tag{41.15}$$

We can now apply this to the gradient vector in polar form which has components $\boldsymbol{\nabla}_r = \partial_r$, and $\boldsymbol{\nabla}_\theta = (1/r)\partial_\theta$, and get

$$\text{div grad} = \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r} \partial_{\theta\theta} \tag{41.16}$$

This is the expected result, and what we should get by performing $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ in polar form. Now, let us do it the wrong way, dotting our gradient with itself.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} &= \left(\partial_r, \frac{1}{r} \partial_\theta \right) \cdot \left(\partial_r, \frac{1}{r} \partial_\theta \right) \\
&= \partial_{rr} + \frac{1}{r} \partial_\theta \left(\frac{1}{r} \partial_\theta \right) \\
&= \partial_{rr} + \frac{1}{r^2} \partial_{\theta\theta}
\end{aligned} \tag{41.17}$$

This is wrong! So is Dmitrevsky right that this procedure is flawed, or do you spot the mistake? I have also cruelly written this out in a way that obscures the error and highlights the source of the confusion.

The problem is that our unit vectors are functions, and they must also be included in the application of our partials. Using the coordinate polar form without explicitly putting in the unit vectors is how we go wrong. Here is the right way

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} &= \left(\hat{\mathbf{r}}\partial_r + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\right) \cdot \left(\hat{\mathbf{r}}\partial_r + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\right) \\
&= \hat{\mathbf{r}} \cdot \partial_r \left(\hat{\mathbf{r}}\partial_r\right) + \hat{\mathbf{r}} \cdot \partial_r \left(\hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\right) + \hat{\boldsymbol{\theta}} \cdot \frac{1}{r}\partial_\theta \left(\hat{\mathbf{r}}\partial_r\right) + \hat{\boldsymbol{\theta}} \cdot \frac{1}{r}\partial_\theta \left(\hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta\right)
\end{aligned} \tag{41.18}$$

Now we need the derivatives of our unit vectors. The ∂_r derivatives are zero since these have no radial dependence, but we do have θ partials

$$\begin{aligned}
\partial_\theta \hat{\mathbf{r}} &= \partial_\theta \left(\mathbf{e}_1 e^{i\theta}\right) \\
&= \mathbf{e}_1 \mathbf{e}_1 \mathbf{e}_2 e^{i\theta} \\
&= \mathbf{e}_2 e^{i\theta} \\
&= \hat{\boldsymbol{\theta}},
\end{aligned} \tag{41.19}$$

and

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\theta}} &= \partial_\theta \left(\mathbf{e}_2 e^{i\theta}\right) \\
&= \mathbf{e}_2 \mathbf{e}_1 \mathbf{e}_2 e^{i\theta} \\
&= -\mathbf{e}_1 e^{i\theta} \\
&= -\hat{\mathbf{r}}.
\end{aligned} \tag{41.20}$$

(One should be able to get the same results if these unit vectors were written out in full as $\hat{\mathbf{r}} = \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta$, and $\hat{\boldsymbol{\theta}} = \mathbf{e}_2 \cos\theta - \mathbf{e}_1 \sin\theta$, instead of using the obscure geometric algebra quaterionic rotation exponential operators.)

Having calculated these partials we now have

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}) = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} \tag{41.21}$$

Exactly what it should be, and what we got with the coordinate form of the divergence operator when applying the "Laplacian equals the divergence of the gradient" rule blindly. We see that the expectation that $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ is the Laplacian in more than the Cartesian coordinate system is not invalid, but that care is required to apply the chain rule to all functions. We also see that expressing a vector in coordinate form when the basis vectors are position dependent is also a path to danger.

Is this anything that our electricity and magnetism prof did not know? Unlikely. Is this something that our prof felt that could not be explained to a mob of first year students? Probably.

42

DERIVATION OF THE SPHERICAL POLAR LAPLACIAN

42.1 MOTIVATION

In 41 was a Geometric Algebra derivation of the 2D polar Laplacian by squaring the gradient. In 35 was a factorization of the spherical polar unit vectors in a tidy compact form. Here both these ideas are utilized to derive the spherical polar form for the Laplacian, an operation that is strictly algebraic (squaring the gradient) provided we operate on the unit vectors correctly.

42.2 OUR ROTATION MULTIVECTOR

Our starting point is a pair of rotations. We rotate first in the x, y plane by ϕ

$$\begin{aligned}
\mathbf{x} \rightarrow \mathbf{x}' &= \tilde{R}_\phi \mathbf{x} R_\phi \\
i &\equiv \mathbf{e}_1 \mathbf{e}_2 \\
R_\phi &= e^{i\phi/2}
\end{aligned} \tag{42.1a}$$

Then apply a rotation in the $\mathbf{e}_3 \wedge (\tilde{R}_\phi \mathbf{e}_1 R_\phi) = \tilde{R}_\phi \mathbf{e}_3 \mathbf{e}_1 R_\phi$ plane

$$\begin{aligned}
\mathbf{x}' \rightarrow \mathbf{x}'' &= \tilde{R}_\theta \mathbf{x}' R_\theta \\
R_\theta &= e^{\tilde{R}_\phi \mathbf{e}_3 \mathbf{e}_1 R_\phi \theta/2} = \tilde{R}_\phi e^{\mathbf{e}_3 \mathbf{e}_1 \theta/2} R_\phi
\end{aligned} \tag{42.2a}$$

The composition of rotations now gives us

$$\begin{aligned}
\mathbf{x} \rightarrow \mathbf{x}'' &= \tilde{R}_\theta \tilde{R}_\phi \mathbf{x} R_\phi R_\theta = \tilde{R} \mathbf{x} R \\
R &= R_\phi R_\theta = e^{\mathbf{e}_3 \mathbf{e}_1 \theta/2} e^{\mathbf{e}_1 \mathbf{e}_2 \phi/2}.
\end{aligned} \tag{42.3}$$

42.3 EXPRESSIONS FOR THE UNIT VECTORS

The unit vectors in the rotated frame can now be calculated. With $I = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ we can calculate

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \tilde{R} \mathbf{e}_2 R \\
\hat{\mathbf{r}} &= \tilde{R} \mathbf{e}_3 R \\
\hat{\boldsymbol{\theta}} &= \tilde{R} \mathbf{e}_1 R
\end{aligned} \tag{42.4a}$$

Performing these we get

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= e^{-\mathbf{e}_1\mathbf{e}_2\phi/2} e^{-\mathbf{e}_3\mathbf{e}_1\theta/2} \mathbf{e}_2 e^{\mathbf{e}_3\mathbf{e}_1\theta/2} e^{\mathbf{e}_1\mathbf{e}_2\phi/2} \\
&= \mathbf{e}_2 e^{i\phi},
\end{aligned} \tag{42.5}$$

and

$$\begin{aligned}
\hat{\mathbf{r}} &= e^{-\mathbf{e}_1\mathbf{e}_2\phi/2} e^{-\mathbf{e}_3\mathbf{e}_1\theta/2} \mathbf{e}_3 e^{\mathbf{e}_3\mathbf{e}_1\theta/2} e^{\mathbf{e}_1\mathbf{e}_2\phi/2} \\
&= e^{-\mathbf{e}_1\mathbf{e}_2\phi/2} (\mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta) e^{\mathbf{e}_1\mathbf{e}_2\phi/2} \\
&= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{\mathbf{e}_1\mathbf{e}_2\phi} \\
&= \mathbf{e}_3 (\cos\theta + \mathbf{e}_3\mathbf{e}_1 \sin\theta e^{\mathbf{e}_1\mathbf{e}_2\phi}) \\
&= \mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta},
\end{aligned} \tag{42.6}$$

and

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= e^{-\mathbf{e}_1\mathbf{e}_2\phi/2} e^{-\mathbf{e}_3\mathbf{e}_1\theta/2} \mathbf{e}_1 e^{\mathbf{e}_3\mathbf{e}_1\theta/2} e^{\mathbf{e}_1\mathbf{e}_2\phi/2} \\
&= e^{-\mathbf{e}_1\mathbf{e}_2\phi/2} (\mathbf{e}_1 \cos\theta - \mathbf{e}_3 \sin\theta) e^{\mathbf{e}_1\mathbf{e}_2\phi/2} \\
&= \mathbf{e}_1 \cos\theta e^{\mathbf{e}_1\mathbf{e}_2\phi/2} - \mathbf{e}_3 \sin\theta \\
&= i\hat{\boldsymbol{\phi}} \cos\theta - \mathbf{e}_3 \sin\theta \\
&= i\hat{\boldsymbol{\phi}} (\cos\theta + \hat{\boldsymbol{\phi}} i \mathbf{e}_3 \sin\theta) \\
&= i\hat{\boldsymbol{\phi}} e^{I\hat{\boldsymbol{\phi}}\theta}.
\end{aligned} \tag{42.7}$$

Summarizing these are

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \mathbf{e}_2 e^{i\phi} \\
\hat{\mathbf{r}} &= \mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta} \\
\hat{\boldsymbol{\theta}} &= i\hat{\boldsymbol{\phi}} e^{I\hat{\boldsymbol{\phi}}\theta}.
\end{aligned} \tag{42.8a}$$

42.4 DERIVATIVES OF THE UNIT VECTORS

We will need the partials. Most of these can be computed from eq. (42.8) by inspection, and are

$$\begin{aligned}
\partial_r \hat{\boldsymbol{\phi}} &= 0 \\
\partial_r \hat{\mathbf{r}} &= 0 \\
\partial_r \hat{\boldsymbol{\theta}} &= 0 \\
\partial_\theta \hat{\boldsymbol{\phi}} &= 0 \\
\partial_\theta \hat{\mathbf{r}} &= \hat{\mathbf{r}} I \hat{\boldsymbol{\phi}} \\
\partial_\theta \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{\theta}} I \hat{\boldsymbol{\phi}} \\
\partial_\phi \hat{\boldsymbol{\phi}} &= \hat{\boldsymbol{\phi}} i \\
\partial_\phi \hat{\mathbf{r}} &= \hat{\boldsymbol{\phi}} \sin\theta \\
\partial_\phi \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{\phi}} \cos\theta
\end{aligned} \tag{42.9a}$$

42.5 EXPANDING THE LAPLACIAN

We note that the line element is $ds = dr + rd\theta + r\sin\theta d\phi$, so our gradient in spherical coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi. \tag{42.10}$$

We can now evaluate the Laplacian

$$\boldsymbol{\nabla}^2 = \left(\hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi\right) \cdot \left(\hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi\right). \tag{42.11}$$

Evaluating these one set at a time we have

$$\hat{\mathbf{r}}\partial_r \cdot \left(\hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi\right) = \partial_{rr}, \tag{42.12}$$

and

$$\begin{aligned}
\frac{1}{r}\hat{\boldsymbol{\theta}}\partial_\theta \cdot \left(\hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi\right) &= \frac{1}{r}\left\langle \hat{\boldsymbol{\theta}}\left(\hat{\mathbf{r}} I \hat{\boldsymbol{\phi}}\partial_r + \hat{\mathbf{r}}\partial_{\theta r} + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_{\theta\theta} + \frac{1}{r}\hat{\boldsymbol{\theta}} I \hat{\boldsymbol{\phi}}\partial_\theta + \hat{\boldsymbol{\phi}}\partial_\theta \frac{1}{r\sin\theta}\partial_\phi\right)\right\rangle \\
&= \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta},
\end{aligned}$$

(42.13)

and

$$
\begin{aligned}
&\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi \cdot \left(\hat{\mathbf{r}}\partial_r + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_\theta + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\partial_\phi\right) \\
&\qquad = \frac{1}{r\sin\theta}\left\langle \hat{\boldsymbol{\phi}}\left(\hat{\boldsymbol{\phi}}\sin\theta\partial_r + \hat{\mathbf{r}}\partial_{\phi r} + \hat{\boldsymbol{\phi}}\cos\theta\frac{1}{r}\partial_\theta + \frac{\hat{\boldsymbol{\theta}}}{r}\partial_{\phi\theta} + \hat{\boldsymbol{\phi}}i\frac{1}{r\sin\theta}\partial_\phi + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_{\phi\phi}\right)\right\rangle \\
&\qquad = \frac{1}{r}\partial_r + \frac{\cot\theta}{r^2}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi}
\end{aligned}
\tag{42.14}
$$

Summing these we have

$$
\boldsymbol{\nabla}^2 = \partial_{rr} + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \frac{\cot\theta}{r^2}\partial_\theta + \frac{1}{r^2\sin^2\theta}\partial_{\phi\phi} \tag{42.15}
$$

This is often written with a chain rule trick to consolidate the r and θ partials

$$
\boldsymbol{\nabla}^2\Psi = \frac{1}{r}\partial_{rr}(r\Psi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Psi\right) + \frac{1}{r^2\sin^2\theta}\partial_{\psi\psi}\Psi \tag{42.16}
$$

It is simple to verify that this is identical to eq. (42.15).

43

TANGENT PLANES AND NORMALS IN THREE AND FOUR DIMENSIONS

43.1 MOTIVATION

I was reviewing the method of Lagrange in my old first year calculus book [38] and found that I needed a review of some of the geometry ideas associated with the gradient (that it is normal to the surface). The approach in the text used 3D level surfaces $f(x, y, z) = c$, which is general but not the most intuitive.

If we define a surface in the simpler explicit form $z = f(x, y)$, then how would you show this normal property? Here we explore this in 3D and 4D, using geometric and wedge products to express the tangent planes and tangent volumes respectively.

In the 4D approach, with a vector x defined by coordinates x^μ and basis $\{\gamma_\mu\}$ so that

$$x = \gamma_\mu x^\mu, \tag{43.1}$$

the reciprocal basis γ^μ is defined implicitly by the dot product relations

$$\gamma^\mu \cdot \gamma_\nu = \delta^\mu{}_\nu. \tag{43.2}$$

Assuming such a basis makes the result general enough that the 4D (or a trivial generalization to N dimensions) holds for both Euclidean spaces as well as mixed metric (i.e. Minkowski) spaces, and avoids having to detail the specific metric in question.

43.2 3D SURFACE

We start by considering fig. 43.1. We wish to determine the bivector for the tangent plane in the neighborhood of the point $\mathbf{p}$

$$\mathbf{p} = (x, y, f(x, y)), \tag{43.3}$$

then using a duality transformation (multiplication by the pseudoscalar for the space) determine the normal vector to that plane at this point. Holding either of the two free parameters constant, we find the tangent vectors on that surface to be

$$\begin{aligned}\mathbf{p}_1 &= \left(dx, 0, \frac{\partial f}{\partial x} dx\right) \\ &\propto \left(1, 0, \frac{\partial f}{\partial x}\right)\end{aligned} \tag{43.4a}$$

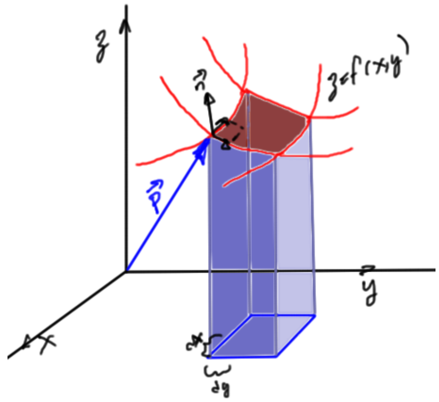

Figure 43.1: A portion of a surface in 3D

$$
\begin{aligned}
\mathbf{p}_2 &= \left(0, dy, \frac{\partial f}{\partial y} dy\right) \\
&\propto \left(0, 1, \frac{\partial f}{\partial y}\right)
\end{aligned}
\tag{43.4b}
$$

The tangent plane is then

$$
\begin{aligned}
\mathbf{p}_1 \wedge \mathbf{p}_2 &= \left(1, 0, \frac{\partial f}{\partial x}\right) \wedge \left(0, 1, \frac{\partial f}{\partial y}\right) \\
&= \left(\mathbf{e}_1 + \mathbf{e}_3 \frac{\partial f}{\partial x}\right) \wedge \left(\mathbf{e}_2 + \mathbf{e}_3 \frac{\partial f}{\partial y}\right) \\
&= \mathbf{e}_1 \mathbf{e}_2 + \mathbf{e}_1 \mathbf{e}_3 \frac{\partial f}{\partial y} + \mathbf{e}_3 \mathbf{e}_2 \frac{\partial f}{\partial x}.
\end{aligned}
\tag{43.5}
$$

We can factor out the pseudoscalar 3D volume element $I = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$, assuming a Euclidean space for which $\mathbf{e}_k^2 = 1$. That is

$$
\mathbf{p}_1 \wedge \mathbf{p}_2 = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \left(\mathbf{e}_3 - \mathbf{e}_2 \frac{\partial f}{\partial y} - \mathbf{e}_1 \frac{\partial f}{\partial x}\right)
\tag{43.6}
$$

Multiplying through by the volume element I we find that the normal to the surface at this point is

$$
\begin{aligned}
\mathbf{n} &\propto -I(\mathbf{p}_1 \wedge \mathbf{p}_2) \\
&= \mathbf{e}_3 - \mathbf{e}_1 \frac{\partial f}{\partial x} - \mathbf{e}_2 \frac{\partial f}{\partial y}.
\end{aligned}
\tag{43.7}
$$

Observe that we can write this as

$$\boxed{\mathbf{n} = \boldsymbol{\nabla}(z - f(x, y)).} \tag{43.8}$$

Let's see how this works in 4D, so that we know how to handle the Minkowski spaces we find in special relativity.

43.3 4D SURFACE

Now, let's move up to one additional direction, with

$$x^3 = f(x^0, x^1, x^2). \tag{43.9}$$

the differential of this is

$$dx^3 = \sum_{k=0}^{2} \frac{\partial f}{\partial x^k} dx^k = \sum_{k=0}^{2} \partial_k f dx^k. \tag{43.10}$$

We are going to look at the 3-surface in the neighborhood of the point

$$p = \left(x^0, x^1, x^2, x^3\right), \tag{43.11}$$

so that the tangent vectors in the neighborhood of this point are in the span of

$$dp = \left(x^0, x^1, x^2, \sum_{k=0}^{2} \partial_k dx^k\right). \tag{43.12}$$

In particular, in each of the directions we have

$$p_0 \propto (1, 0, 0, d_0 f) \tag{43.13a}$$

$$p_1 \propto (0, 1, 0, d_1 f) \tag{43.13b}$$

$$p_2 \propto (0, 0, 1, d_2 f) \tag{43.13c}$$

Our tangent volume in this neighborhood is

$$\begin{aligned} p_0 \wedge p_1 \wedge p_2 &= (\gamma_0 + \gamma_3 \partial_0 f) \wedge (\gamma_1 + \gamma_3 \partial_1 f) \wedge (\gamma_2 + \gamma_3 \partial_2 f) \\ &= (\gamma_0 \gamma_1 + \gamma_0 \gamma_3 \partial_1 f + \gamma_3 \gamma_1 \partial_0 f) \wedge (\gamma_2 + \gamma_3 \partial_2 f) \\ &= \gamma_{012} - \gamma_{023} \partial_1 f + \gamma_{123} \partial_0 f + \gamma_{013} \partial_2 f. \end{aligned} \tag{43.14}$$

Here the shorthand $\gamma_{ijk} = \gamma_i \gamma_j \gamma_k$ has been used. Can we factor out a 4D pseudoscalar from this and end up with a coherent result? We have

$$\gamma_{0123} \gamma^3 = \gamma_{012} \tag{43.15a}$$

$$\gamma_{0123} \gamma^1 = \gamma_{023} \tag{43.15b}$$

$$\gamma_{0123} \gamma^0 = -\gamma_{123} \tag{43.15c}$$

$$\gamma_{0123} \gamma^2 = -\gamma_{013}. \tag{43.15d}$$

This gives us

$$d^3 p = p_0 \wedge p_1 \wedge p_2 = \gamma_{0123} \left(\gamma^3 - \gamma^1 \partial_1 f - \gamma^0 \partial_0 f - \gamma^2 \partial_2 f \right). \tag{43.16}$$

With the usual 4d gradient definition (sum implied)

$$\nabla = \gamma^\mu \partial_\mu, \tag{43.17}$$

we have

$$\begin{aligned} \nabla x^3 &= \gamma^\mu \partial_\mu x^3 \\ &= \gamma^\mu {\delta_\mu}^3 \\ &= \gamma^3, \end{aligned} \tag{43.18}$$

so we can write

$$d^3 p = \gamma_{0123} \nabla \left(x^3 - f(x^0, x^1, x^2) \right), \tag{43.19}$$

so, finally, the "normal" to this surface volume element at this point is

$$\boxed{n = \nabla \left(x^3 - f(x^0, x^1, x^2) \right).} \tag{43.20}$$

This is just like the 3D Euclidean result, with the exception that we need to look at the dual of a 3-volume “surface” instead of our normal 2D surface.

It may seem curious that we had to specify an Euclidean metric for the 3D case, but did not here. That doesn’t mean this is a metric free result. Instead, the metric choice is built into the definition of the gradient eq. (43.17) and its associated reciprocal basis. For example with a $1, 3$ metric where $\gamma_0^2 = 1, \gamma_k^2 = -1$, we have $\gamma^0 = \gamma_0$ and $\gamma^k = -\gamma_k$.

44

STOKES THEOREM

The subject of differential forms is one way to obtain an understanding of how to apply Stokes theorem to higher dimensional spaces, non-Euclidean metrics, and curvilinear coordinate systems. The formalism of differential forms requires reexpressing physical quantities as “forms”. A notable example, as given in [13], introduces a two-form for the Faraday field

$$\alpha = \left(E_1 dx^1 + E_2 dx^2 + E_3 dx^3\right) \wedge (cdt) + \left(H_1 dx^2 \wedge dx^3 H_2 dx^3 \wedge dx^1 H_3 dx^1 \wedge dx^2\right). \tag{44.1}$$

With a metric for which $(cdt, cdt) = -1$, and a cooresponding notion of duality (*), Maxwell’s equations for freespace become

$$\begin{aligned} d\alpha &= 0 \\ d{*}\alpha &= 0. \end{aligned} \tag{44.2}$$

These are equivalent to the usual pair of tensor equations

$$\begin{aligned} \partial_i F^{ij} &= 0 \\ \epsilon^{ijkl} \partial_j F_{kl} &= 0 \end{aligned}, \tag{44.3}$$

or the Geometric Algebra equation

$$\nabla F = 0. \tag{44.4}$$

An aspect of differential forms that I found unintuitive, is that all physical quantities have to be expressed as forms, even when we have no pressing desire to integrate them. It also seemed to me that it ought to be possible to express volume and area elements in parameterized spaces directly as wedge products. For example, given a two parameter surface of all the points that can be traced out on $\mathbf{x}(u, v)$, we can express the (oriented) area of a patch of that surface directly as

$$dA = \left(du \frac{\partial \mathbf{x}}{\partial u}\right) \wedge \left(dv \frac{\partial \mathbf{x}}{\partial v}\right). \tag{44.5}$$

With such a capability is the abstract notion of a form really required? Can we stick with the vector notation that we are comfortable with, perhaps just generalizing slightly? How would something like Stokes theorem, a basic tool needed to tackle so many problems in electromagnetism, be expressed so that it acted on vectors directly?

An answer to this question was found in Denker’s straight wire treatment [8], which states that the geometric algebra formulation of Stokes theorem has the form

$$\int_S \nabla \wedge F = \int_{\partial S} F. \tag{44.6}$$

This looks simple enough, but there are some important details left out. In particular the grades do not match, so there must be some sort of implied projection or dot product operations too. We also need to understand how to express the hypervolume and hypersurfaces when evaluating these integrals, especially when we want to use curvilinear coordinates.

If one restricts attention to the special case where the dimension of the integration volume also equaled the dimension of the vector space, so that the grade of the curl matches the grade of the space (i.e. integration of a two form $\int d^2\mathbf{x} \cdot (\boldsymbol{\nabla} \wedge \mathbf{f})$ in a two dimensional space), then some of those important details are not too hard to work out.

To treat a more general case, such as the same two form $\int d^2\mathbf{x} \cdot (\boldsymbol{\nabla} \wedge \mathbf{f})$ in a space of dimension greater than two, we need to introduce the notion of tangent space. That concept can also be found in differential forms, but can also be expressed directly in vector algebra. Suppose, for example, that $\mathbf{x}(u, v, w)$ parameterizes a subspace, the the tangent space at the point of evaluation is the space that is spanned by $\{\partial\mathbf{x}/\partial u, \partial\mathbf{x}/\partial v, \partial\mathbf{x}/\partial w\}$. Stokes theorem is expressed not in terms of the gradient $\boldsymbol{\nabla}$, but the projection of the gradient onto the tangent space, which will be denoted by $\boldsymbol{\partial}$ and called the vector derivative. The concept of tangent space and and vector derivative are covered thoroughly in [31], which also introduces Stokes theorem as a special case of a more fundamental theorem for integration of geometric algebraic objects.

The objective of this chapter is to detail the Geometric algebra form of Stokes theorem, and not the fundamental theorem of geometric calculus. We wish to cover the generalization of Stokes theorem to higher dimensional spaces and non-Euclidean metrics (i.e. especially those used for special relativity and electromagnetism), and understanding how to properly deal with curvilinear coordinates. This generalization has the form

Theorem 44.1: Stokes' Theorem

For blades $F \in \bigwedge^s$, and m volume element $d^k\mathbf{x}$, $s < k$,

$$\int_V d^k\mathbf{x} \cdot (\boldsymbol{\partial} \wedge F) = \int_{\partial V} d^{k-1}\mathbf{x} \cdot F.$$

Here the volume integral is over a m dimensional surface (manifold), $\boldsymbol{\partial}$ is the projection of the gradient onto the tangent space of the manifold, and ∂V indicates integration over the boundary of V.

It takes some work to give this more concrete meaning. I will attempt to do so in a gradual fashion, and provide a number of examples that illustrate some of the relevant details.

44.1 CURVILINEAR COORDINATES

A finite vector space, not necessarily Euclidean, with basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots\}$ will be assumed to be the generator of the geometric algebra. A dual or reciprocal basis $\left\{\mathbf{e}^1, \mathbf{e}^2, \cdots\right\}$ for this basis can be calculated, defined by the property

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i{}^j. \tag{44.7}$$

This is an Euclidean space when $\mathbf{e}_i = \mathbf{e}^i, \forall i$.

For our purposes a manifold can be loosely defined as a parameterized surface. For example, a 2D manifold can be considered a surface in an n dimensional vector space, parameterized by two variables

$$\mathbf{x} = \mathbf{x}(a, b) = \mathbf{x}(u^1, u^2). \tag{44.8}$$

Note that the indices here do not represent exponentiation. We can construct a basis for the manifold as

$$\mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u^i}. \tag{44.9}$$

On the manifold we can calculate a reciprocal basis $\left\{\mathbf{x}^i\right\}$, defined by requiring, at each point on the surface

$$\mathbf{x}^i \cdot \mathbf{x}_j = \delta^i{}_j. \tag{44.10}$$

Associated implicitly with this basis is a curvilinear coordinate representation defined by the projection operation

$$\mathbf{x} = x^i \mathbf{x}_i, \tag{44.11}$$

(sums over mixed indices are implied). These coordinates can be calculated by taking dot products with the reciprocal frame vectors

$$\begin{aligned} \mathbf{x} \cdot \mathbf{x}^i &= x^j \mathbf{x}_j \cdot \mathbf{x}^i \\ &= x^j \delta_j{}^i \\ &= x^i. \end{aligned} \tag{44.12}$$

In this document all coordinates are with respect to a specific curvilinear basis , and not with respect to the standard basis $\{\mathbf{e}_i\}$ or its dual basis unless otherwise noted.

Similar to the usual notation for derivatives with respect to the standard basis coordinates we form a lower index partial derivative operator

$$\frac{\partial}{\partial u^i} \equiv \partial_i, \tag{44.13}$$

so that when the complete vector space is spanned by $\{\mathbf{x}_i\}$ the gradient has the curvilinear representation

$$\boldsymbol{\nabla} = \mathbf{x}^i \frac{\partial}{\partial u^i}. \tag{44.14}$$

When the basis $\{\mathbf{x}_i\}$ does not span the space, the projection of the gradient onto the tangent space at the point of evaluation is

$$\boldsymbol{\partial} = \mathbf{x}^i \partial_i = \sum_i \mathbf{x}_i \frac{\partial}{\partial u^i}. \tag{44.15}$$

This is called the vector derivative.

44.2 GREEN'S THEOREM

Given a two parameter (u, v) surface parameterization, the curvilinear coordinate representation of a vector $\mathbf{f}$ has the form

$$\mathbf{f} = f_u \mathbf{x}^u + f_v \mathbf{x}^v + f_\perp \mathbf{x}^\perp. \tag{44.16}$$

We assume that the vector space is of dimension two or greater but otherwise unrestricted, and need not have an Euclidean basis. Here $f_\perp \mathbf{x}^\perp$ denotes the rejection of $\mathbf{f}$ from the tangent space at the point of evaluation. Green's theorem relates the integral around a closed curve to an "area" integral on that surface

Theorem 44.2: Green's Theorem

$$\oint \mathbf{f} \cdot d\mathbf{l} = \iint \left(-\frac{\partial f_u}{\partial v} + \frac{\partial f_v}{\partial u} \right) du dv$$

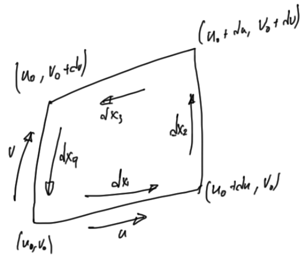

Figure 44.1: Infinitesimal loop integral

Following the arguments used in [39] for Stokes theorem in three dimensions, we first evaluate the loop integral along the differential element of the surface at the point $\mathbf{x}(u_0, v_0)$ evaluated over the range (du, dv), as shown in the infinitesimal loop of fig. 44.1.

Over the infinitesimal area, the loop integral decomposes into

$$\oint \mathbf{f} \cdot d\mathbf{l} = \int \mathbf{f} \cdot d\mathbf{x}_1 + \int \mathbf{f} \cdot d\mathbf{x}_2 + \int \mathbf{f} \cdot d\mathbf{x}_3 + \int \mathbf{f} \cdot d\mathbf{x}_4, \tag{44.17}$$

where the differentials along the curve are

$$\begin{aligned} d\mathbf{x}_1 &= \left.\frac{\partial \mathbf{x}}{\partial u}\right|_{v=v_0} du \\ d\mathbf{x}_2 &= \left.\frac{\partial \mathbf{x}}{\partial v}\right|_{u=u_0+du} dv \\ d\mathbf{x}_3 &= -\left.\frac{\partial \mathbf{x}}{\partial u}\right|_{v=v_0+dv} du \\ d\mathbf{x}_4 &= -\left.\frac{\partial \mathbf{x}}{\partial v}\right|_{u=u_0} dv. \end{aligned} \tag{44.18}$$

It is assumed that the parameterization change (du, dv) is small enough that this loop integral can be considered planar (regardless of the dimension of the vector space). Making use of the fact that $\mathbf{x}^\perp \cdot \mathbf{x}_\alpha = 0$ for $\alpha \in \{u, v\}$, the loop integral is

$$\begin{aligned} \oint \mathbf{f} \cdot d\mathbf{l} &= \int \left(f_u \mathbf{x}^u + f_v \mathbf{x}^v + f_\perp \mathbf{x}^\perp \right) \cdot \left(\mathbf{x}_u(u, v_0) du - \mathbf{x}_u(u, v_0 + dv) du + \mathbf{x}_v(u_0 + du, v) dv - \mathbf{x}_v(u_0, v) dv \right) \\ &= \int f_u(u, v_0) du - f_u(u, v_0 + dv) du + f_v(u_0 + du, v) dv - f_v(u_0, v) dv \end{aligned} \tag{44.19}$$

With the distances being infinitesimal, these differences can be rewritten as partial differentials

$$\oint \mathbf{f} \cdot d\mathbf{l} = \iint \left(-\frac{\partial f_u}{\partial v} + \frac{\partial f_v}{\partial u} \right) dudv. \tag{44.20}$$

We can now sum over a larger area as in fig. 44.2

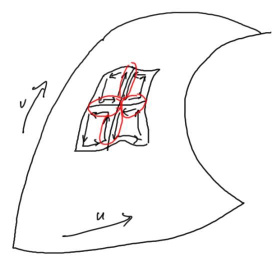

Figure 44.2: Sum of infinitesimal loops

All the opposing oriented loop elements cancel, so the integral around the complete boundary of the surface $\mathbf{x}(u, v)$ is given by the u, v area integral of the partials difference.

We will see that Green's theorem is a special case of the Stokes theorem. This observation will also provide a geometric interpretation of the right hand side area integral of theorem 44.2, and allow for a coordinate free representation.

Special case: An important special case of Green's theorem is for a Euclidean two dimensional space where the vector function is

$$\mathbf{f} = P\mathbf{e}_1 + Q\mathbf{e}_2. \tag{44.21}$$

Here Green's theorem takes the form

$$\boxed{\oint Pdx + Qdy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.} \tag{44.22}$$

44.3 STOKES THEOREM, TWO VOLUME VECTOR FIELD

Having examined the right hand side of theorem 44.1 for the very simplest geometric object $\mathbf{f}$, let's look at the right hand side, the area integral in more detail. We restrict our attention for now to vectors $\mathbf{f}$ still defined by eq. (44.16).

First we need to assign a meaning to $d^2\mathbf{x}$. By this, we mean the wedge products of the two differential elements. With

$$
\begin{aligned}
d\mathbf{x}_i &= du^i \frac{\partial \mathbf{x}}{\partial u^i} \\
&= du^i \mathbf{x}_i,
\end{aligned}
\tag{44.23}
$$

that area element is

$$
d^2\mathbf{x} = d\mathbf{x}_1 \wedge d\mathbf{x}_2 = du^1 du^2 \mathbf{x}_1 \wedge \mathbf{x}_2. \tag{44.24}
$$

This is the oriented area element that lies in the tangent plane at the point of evaluation, and has the magnitude of the area of that segment of the surface, as depicted in fig. 44.3.

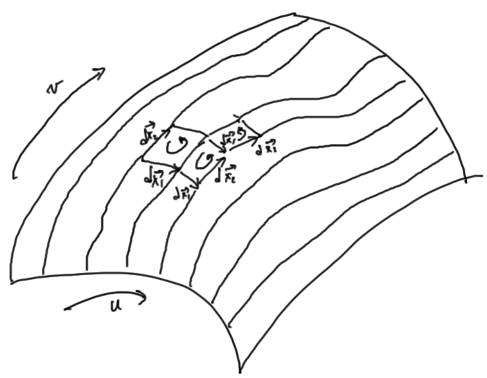

Figure 44.3: Oriented area element tiling of a surface

Observe that we have no requirement to introduce a normal to the surface to describe the direction of the plane. The wedge product provides the information about the orientation of the place in the space, even when the vector space that our vector lies in has dimension greater than three.

Proceeding with the expansion of the dot product of the area element with the curl, using **??**, **??**, and **??**, and a scalar selection operation, we have

$$\begin{aligned}
d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= \left\langle d^2\mathbf{x} \left(\boldsymbol{\partial} \wedge \mathbf{f} \right) \right\rangle \\
&= \left\langle d^2\mathbf{x} \frac{1}{2} \left(\overrightarrow{\boldsymbol{\partial}} \mathbf{f} - \mathbf{f} \overleftarrow{\boldsymbol{\partial}} \right) \right\rangle \\
&= \frac{1}{2} \left\langle d^2\mathbf{x} \left(\mathbf{x}^i \left(\partial_i \mathbf{f} \right) - \left(\partial_i \mathbf{f} \right) \mathbf{x}^i \right) \right\rangle \\
&= \frac{1}{2} \left\langle \left(\partial_i \mathbf{f} \right) d^2\mathbf{x}\mathbf{x}^i - \left(\partial_i \mathbf{f} \right) \mathbf{x}^i d^2\mathbf{x} \right\rangle \\
&= \left\langle \left(\partial_i \mathbf{f} \right) \left(d^2\mathbf{x} \cdot \mathbf{x}^i \right) \right\rangle \\
&= \partial_i \mathbf{f} \cdot \left(d^2\mathbf{x} \cdot \mathbf{x}^i \right).
\end{aligned} \tag{44.25}$$

Let's proceed to expand the inner dot product

$$\begin{aligned}
d^2\mathbf{x} \cdot \mathbf{x}^i &= du^1 du^2 \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \right) \cdot \mathbf{x}^i \\
&= du^1 du^2 \left(\mathbf{x}_2 \cdot \mathbf{x}^i \mathbf{x}_1 - \mathbf{x}_1 \cdot \mathbf{x}^i \mathbf{x}_2 \right) \\
&= du^1 du^2 \left({\delta_2}^i \mathbf{x}_1 - {\delta_1}^i \mathbf{x}_2 \right).
\end{aligned} \tag{44.26}$$

The complete curl term is thus

$$d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = du^1 du^2 \left(\frac{\partial \mathbf{f}}{\partial u^2} \cdot \mathbf{x}_1 - \frac{\partial \mathbf{f}}{\partial u^1} \cdot \mathbf{x}_2 \right) \tag{44.27}$$

This almost has the form of eq. (44.20), although that is not immediately obvious. Working backwards, using the shorthand $u = u^1, v = u^2$, we can show that this coordinate representation can be eliminated

$$\begin{aligned}
-dudv \left(\frac{\partial f_v}{\partial u} - \frac{\partial f_u}{\partial v} \right) &= dudv \left(\frac{\partial}{\partial v} \left(\mathbf{f} \cdot \mathbf{x}_u \right) - \frac{\partial}{\partial u} \left(\mathbf{f} \cdot \mathbf{x}_v \right) \right) \\
&= dudv \left(\frac{\partial \mathbf{f}}{\partial v} \cdot \mathbf{x}_u - \frac{\partial \mathbf{f}}{\partial u} \cdot \mathbf{x}_v + \mathbf{f} \cdot \left(\frac{\partial \mathbf{x}_u}{\partial v} - \frac{\partial \mathbf{x}_v}{\partial u} \right) \right) \\
&= dudv \left(\frac{\partial \mathbf{f}}{\partial v} \cdot \mathbf{x}_u - \frac{\partial \mathbf{f}}{\partial u} \cdot \mathbf{x}_v + \mathbf{f} \cdot \left(\frac{\partial^2 \mathbf{x}}{\partial v \partial u} - \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \right) \right) \\
&= dudv \left(\frac{\partial \mathbf{f}}{\partial v} \cdot \mathbf{x}_u - \frac{\partial \mathbf{f}}{\partial u} \cdot \mathbf{x}_v \right) \\
&= d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}).
\end{aligned} \tag{44.28}$$

This relates the two parameter surface integral of the curl to the loop integral over its boundary

$$\boxed{\int d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = \oint \mathbf{f} \cdot d\mathbf{l}.} \tag{44.29}$$

This is the very simplest special case of Stokes theorem. When written in the general form of Stokes theorem 44.1

$$\int_A d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = \int_{\partial A} d^1\mathbf{x} \cdot \mathbf{f} = \int_{\partial A} (d\mathbf{x}_1 - d\mathbf{x}_2) \cdot \mathbf{f}, \tag{44.30}$$

we must remember (the ∂A is to remind us of this) that it is implied that both the vector $\mathbf{f}$ and the differential elements are evaluated on the boundaries of the integration ranges respectively. A more exact statement is

$$\boxed{\int_{\partial A} d^1\mathbf{x} \cdot \mathbf{f} = \int \mathbf{f} \cdot d\mathbf{x}_1|_{\Delta u^2} - \mathbf{f} \cdot d\mathbf{x}_2|_{\Delta u^1} = \int f_1|_{\Delta u^2} du^1 - f_2|_{\Delta u^1} du^2.} \tag{44.31}$$

Expanded out in full this is

$$\int \mathbf{f} \cdot d\mathbf{x}_1|_{u^2(1)} - \mathbf{f} \cdot d\mathbf{x}_1|_{u^2(0)} + \mathbf{f} \cdot d\mathbf{x}_2|_{u^1(0)} - \mathbf{f} \cdot d\mathbf{x}_2|_{u^1(1)}, \tag{44.32}$$

which can be cross checked against fig. 44.4 to demonstrate that this specifies a clockwise orientation. For the surface with oriented area $d\mathbf{x}_1 \wedge d\mathbf{x}_2$, the clockwise loop is designated with line elements (1)-(4), we see that the contributions around this loop (in boxes) match eq. (44.32).

Example 44.1: Green's theorem, a 2D Cartesian parameterization for a Euclidean space

For a Cartesian 2D Euclidean parameterization of a vector field and the integration space, Stokes theorem should be equivalent to Green's theorem eq. (44.22). Let's expand both sides of eq. (44.29) independently to verify equality. The parameterization is

$$\mathbf{x}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2. \tag{44.33}$$

Here the dual basis is the basis, and the projection onto the tangent space is just the gradient

$$\boldsymbol{\partial} = \boldsymbol{\nabla} = \mathbf{e}_1 \frac{\partial}{\partial x} + \mathbf{e}_2 \frac{\partial}{\partial y}. \tag{44.34}$$

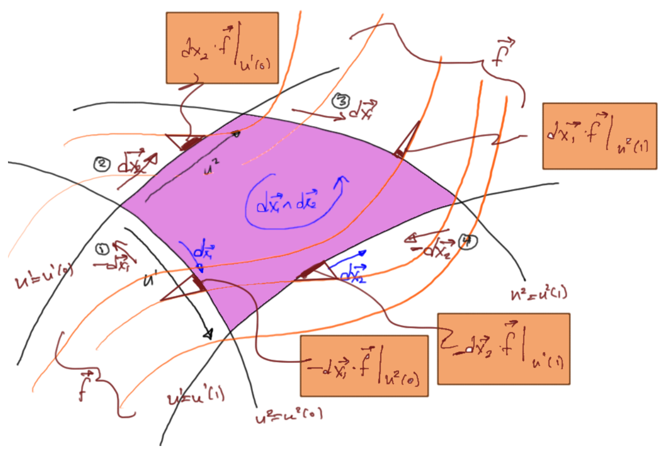

Figure 44.4: Clockwise loop

The volume element is an area weighted pseudoscalar for the space

$$d^2\mathbf{x} = dxdy\frac{\partial \mathbf{x}}{\partial x} \wedge \frac{\partial \mathbf{x}}{\partial y} = dxdy\mathbf{e}_1\mathbf{e}_2, \tag{44.35}$$

and the curl of a vector $\mathbf{f} = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ is

$$\begin{aligned}\boldsymbol{\partial} \wedge \mathbf{f} &= \left(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\right) \wedge (f_1\mathbf{e}_1 + f_2\mathbf{e}_2) \\ &= \mathbf{e}_1\mathbf{e}_2\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).\end{aligned} \tag{44.36}$$

So, the LHS of Stokes theorem takes the coordinate form

$$\int d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = \iint dxdy \overbrace{\boxed{\langle \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 \rangle}}^{=-1} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right). \tag{44.37}$$

For the RHS, following fig. 44.5, we have

$$\begin{aligned}\oint \mathbf{f} \cdot d\mathbf{x} &= f_2(x_0, y)dy + f_1(x, y_1)dx - f_2(x_1, y)dy - f_1(x, y_0)dx \\ &= \int dx\left(f_1(x, y_1) - f_1(x, y_0)\right) - \int dy\left(f_2(x_1, y) - f_2(x_0, y)\right).\end{aligned} \tag{44.38}$$

As expected, we can also obtain this by integrating eq. (44.37).

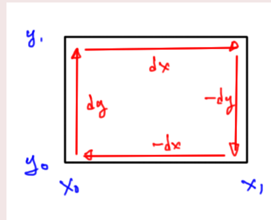

Figure 44.5: Euclidean 2D loop

Example 44.2: Cylindrical parameterization

Let's now consider a cylindrical parameterization of a 4D space with Euclidean metric $++++$ or Minkowski metric $+++-$. For such a space let's do a brute force expansion of both sides of Stokes theorem to gain some confidence that all is well.

With $\kappa = \mathbf{e}_3 \mathbf{e}_4$, such a space is conveniently parameterized as illustrated in fig. 44.6 as

$$\mathbf{x}(\rho, \theta, h) = x\mathbf{e}_1 + y\mathbf{e}_2 + \rho \mathbf{e}_3 e^{\kappa\theta}. \tag{44.39}$$

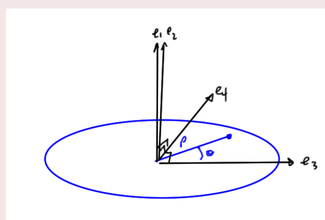

Figure 44.6: Cylindrical polar parameterization

Note that the Euclidean case where $(\mathbf{e}_4)^2 = 1$ rejection of the non-axial components of $\mathbf{x}$ expands to

$$\Big((\mathbf{x} \wedge \mathbf{e}_1 \wedge \mathbf{e}_2) \cdot \mathbf{e}^2 \Big) \cdot \mathbf{e}^1 = \rho \left(\mathbf{e}_3 \cos\theta + \mathbf{e}_4 \sin\theta \right), \tag{44.40}$$

whereas for the Minkowski case where $(\mathbf{e}_4)^2 = -1$ we have a hyperbolic expansion

$$\Big((\mathbf{x} \wedge \mathbf{e}_1 \wedge \mathbf{e}_2) \cdot \mathbf{e}^2 \Big) \cdot \mathbf{e}^1 = \rho \left(\mathbf{e}_3 \cosh\theta + \mathbf{e}_4 \sinh\theta \right). \tag{44.41}$$

Within such a space consider the surface along $x = c, y = d$, for which the vectors are parameterized by

$$\mathbf{x}(\rho, \theta) = c\mathbf{e}_1 + d\mathbf{e}_2 + \rho\mathbf{e}_3 e^{\kappa\theta}. \tag{44.42}$$

The tangent space unit vectors are

$$\begin{aligned} \mathbf{x}_\rho &= \frac{\partial \mathbf{x}}{\partial \rho} \\ &= \mathbf{e}_3 e^{\kappa\theta}, \end{aligned} \tag{44.43}$$

and

$$\begin{aligned} \mathbf{x}_\theta &= \frac{\partial \mathbf{x}}{\partial \theta} \\ &= \rho\mathbf{e}_3\mathbf{e}_3\mathbf{e}_4 e^{\kappa\theta} \\ &= \rho\mathbf{e}_4 e^{\kappa\theta}. \end{aligned} \tag{44.44}$$

Observe that both of these vectors have their origin at the point of evaluation, and aren't relative to the absolute origin used to parameterize the complete space.

We wish to compute the volume element for the tangent plane. Noting that $\mathbf{e}_3$ and $\mathbf{e}_4$ both anticommute with κ we have for $\mathbf{a} \in \text{span}\{\mathbf{e}_3, \mathbf{e}_4\}$

$$\mathbf{a}e^{\kappa\theta} = e^{-\kappa\theta}\mathbf{a}, \tag{44.45}$$

so

$$\begin{aligned} \mathbf{x}_\theta \wedge \mathbf{x}_\rho &= \left\langle \mathbf{e}_3 e^{\kappa\theta} \rho \mathbf{e}_4 e^{\kappa\theta} \right\rangle_2 \\ &= \rho \left\langle \mathbf{e}_3 e^{\kappa\theta} e^{-\kappa\theta} \mathbf{e}_4 \right\rangle_2 \\ &= \rho\mathbf{e}_3\mathbf{e}_4. \end{aligned} \tag{44.46}$$

The tangent space volume element is thus

$$d^2\mathbf{x} = \rho d\rho d\theta \mathbf{e}_3\mathbf{e}_4. \tag{44.47}$$

With the tangent plane vectors both perpendicular we don't need the general theorem B.6 to compute the reciprocal basis, but can do so by inspection

$$\mathbf{x}^\rho = e^{-\kappa\theta}\mathbf{e}^3, \tag{44.48}$$

and

$$\mathbf{x}^\theta = e^{-\kappa\theta}\mathbf{e}^4 \frac{1}{\rho}. \tag{44.49}$$

Observe that the latter depends on the metric signature.

The vector derivative, the projection of the gradient on the tangent space, is

$$\begin{aligned}
\boldsymbol{\partial} &= \mathbf{x}^\rho \frac{\partial}{\partial \rho} + \mathbf{x}^\theta \frac{\partial}{\partial \theta} \\
&= e^{-\kappa\theta} \left(\mathbf{e}^3 \partial_\rho + \frac{\mathbf{e}^4}{\rho} \partial_\theta \right).
\end{aligned} \tag{44.50}$$

From this we see that acting with the vector derivative on a scalar radial only dependent function $f(\rho)$ is a vector function that has a radial direction, whereas the action of the vector derivative on an azimuthal only dependent function $g(\theta)$ is a vector function that has only an azimuthal direction. The interpretation of the geometric product action of the vector derivative on a vector function is not as simple since the product will be a multivector.

Expanding the curl in coordinates is messier, but yields in the end when tackled with sufficient care

$$\begin{aligned}
\boldsymbol{\partial} \wedge \mathbf{f} &= \left\langle e^{-\kappa\theta} \left(e^3 \partial_\rho + \frac{e^4}{\rho} \partial_\theta \right) \left(\cancel{e_1 x} + \cancel{e_2 y} + e_3 e^{\kappa\theta} f_\rho + \frac{e^4}{\rho} e^{\kappa\theta} f_\theta \right) \right\rangle_2 \\
&= \cancel{\left\langle e^{-\kappa\theta} e^3 \partial_\rho \left(e_3 e^{\kappa\theta} f_\rho \right) \right\rangle_2} + \left\langle \cancel{e^{-\kappa\theta}} e^3 \partial_\rho \left(\frac{e^4}{\rho} \cancel{e^{\kappa\theta}} f_\theta \right) \right\rangle_2 \\
&\quad + \left\langle e^{-\kappa\theta} \frac{e^4}{\rho} \partial_\theta \left(e_3 e^{\kappa\theta} f_\rho \right) \right\rangle_2 + \left\langle e^{-\kappa\theta} \frac{e^4}{\rho} \partial_\theta \left(\frac{e^4}{\rho} e^{\kappa\theta} f_\theta \right) \right\rangle_2 \\
&= \mathbf{e}^3 \mathbf{e}^4 \left(-\frac{f_\theta}{\rho^2} + \frac{1}{\rho} \partial_\rho f_\theta - \frac{1}{\rho} \partial_\theta f_\rho \right) + \frac{1}{\rho^2} \left\langle e^{-\kappa\theta} \left(\mathbf{e}^4 \right)^2 \left(\mathbf{e}_3 \mathbf{e}_4 f_\theta + \cancel{\partial_\theta f_\theta} \right) e^{\kappa\theta} \right\rangle_2 \\
&= \mathbf{e}^3 \mathbf{e}^4 \left(-\frac{f_\theta}{\rho^2} + \frac{1}{\rho} \partial_\rho f_\theta - \frac{1}{\rho} \partial_\theta f_\rho \right) + \frac{1}{\rho^2} \left\langle \cancel{e^{-\kappa\theta}} \mathbf{e}_3 \mathbf{e}^4 f_\theta \cancel{e^{\kappa\theta}} \right\rangle_2 \\
&= \frac{\mathbf{e}^3 \mathbf{e}^4}{\rho} \left(\partial_\rho f_\theta - \partial_\theta f_\rho \right).
\end{aligned} \tag{44.51}$$

After all this reduction, we can now state in coordinates the LHS of Stokes theorem explicitly

$$
\begin{aligned}
\int d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= \int \rho d\rho d\theta \left\langle \mathbf{e}_3 \mathbf{e}_4 \mathbf{e}^3 \mathbf{e}^4 \right\rangle \frac{1}{\rho} \left(\partial_\rho f_\theta - \partial_\theta f_\rho \right) \\
&= \int d\rho d\theta \left(\partial_\theta f_\rho - \partial_\rho f_\theta \right) \\
&= \int d\rho\, f_\rho \big|_{\Delta\theta} - \int d\theta\, f_\theta \big|_{\Delta\rho}.
\end{aligned}
\tag{44.52}
$$

Now compare this to the direct evaluation of the loop integral portion of Stokes theorem. Expressing this using eq. (44.31), we have the same result

$$
\int d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = \int f_\rho \big|_{\Delta\theta} d\rho - f_\theta \big|_{\Delta\rho} d\theta
\tag{44.53}
$$

This example highlights some of the power of Stokes theorem, since the reduction of the volume element differential form was seen to be quite a chore (and easy to make mistakes doing.)

Example 44.3: Composition of boost and rotation

Working in a $\bigwedge^{1,3}$ space with basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ where $(\gamma_0)^2 = 1$ and $(\gamma_k)^2 = -1, k \in \{1, 2, 3\}$, an active composition of boost and rotation has the form

$$
\begin{aligned}
\mathbf{x}' &= e^{i\alpha/2} \mathbf{x}_0 e^{-i\alpha/2} \\
\mathbf{x}'' &= e^{-j\theta/2} \mathbf{x}' e^{j\theta/2},
\end{aligned}
\tag{44.54}
$$

where i is a bivector of a timelike unit vector and perpendicular spacelike unit vector, and j is a bivector of two perpendicular spacelike unit vectors. For example, $i = \gamma_0\gamma_1$ and $j = \gamma_1\gamma_2$. For such i, j the respective Lorentz transformation matrices are

$$
\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}' =
\begin{bmatrix}
\cosh\alpha & -\sinh\alpha & 0 & 0 \\
-\sinh\alpha & \cosh\alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix},
\tag{44.55}
$$

and

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}'. \tag{44.56}$$

Let's calculate the tangent space vectors for this parameterization, assuming that the particle is at an initial spacetime position of $\mathbf{x}_0$. That is

$$\mathbf{x} = e^{-j\theta/2} e^{i\alpha/2} \mathbf{x}_0 e^{-i\alpha/2} e^{j\theta/2}. \tag{44.57}$$

To calculate the tangent space vectors for this subspace we note that

$$\frac{\partial \mathbf{x}'}{\partial \alpha} = \frac{i}{2}\mathbf{x}_0 - \mathbf{x}_0 \frac{i}{2} = i \cdot \mathbf{x}_0, \tag{44.58}$$

and

$$\frac{\partial \mathbf{x}''}{\partial \theta} = -\frac{j}{2}\mathbf{x}' + \mathbf{x}' \frac{j}{2} = \mathbf{x}' \cdot j. \tag{44.59}$$

The tangent space vectors are therefore

$$\begin{aligned} \mathbf{x}_\alpha &= e^{-j\theta/2} \left(i \cdot \mathbf{x}_0 \right) e^{j\theta/2} \\ \mathbf{x}_\theta &= \left(e^{i\alpha/2} \mathbf{x}_0 e^{-i\alpha/2} \right) \cdot j. \end{aligned} \tag{44.60}$$

Continuing a specific example where $i = \gamma_0\gamma_1$, $j = \gamma_1\gamma_2$ let's also pick $\mathbf{x}_0 = \gamma_0$, the spacetime position of a particle at the origin of a frame at that frame's $ct = 1$. The tangent space vectors for the subspace parameterized by this transformation and this initial position is then reduced to

$$\begin{aligned} \mathbf{x}_\alpha &= -\gamma_1 e^{j\theta} \\ &= \gamma_1 \sin\theta + \gamma_2 \cos\theta, \end{aligned} \tag{44.61}$$

and

$$\begin{aligned}
\mathbf{x}_\theta &= \left(\gamma_0 e^{-i\alpha}\right) \cdot j \\
&= (\gamma_0 (\cosh\alpha - \gamma_0\gamma_1 \sinh\alpha)) \cdot (\gamma_1\gamma_2) \\
&= \langle(\gamma_0 \cosh\alpha - \gamma_1 \sinh\alpha)\, \gamma_1\gamma_2\rangle_1 \\
&= \gamma_2 \sinh\alpha.
\end{aligned} \tag{44.62}$$

By inspection the dual basis for this parameterization is

$$\begin{aligned}
\mathbf{x}^\alpha &= \gamma_1 e^{j\theta} \\
\mathbf{x}^\theta &= \frac{\gamma^2}{\sinh\alpha}
\end{aligned} \tag{44.63}$$

So, Stokes theorem, applied to a spacetime vector $\mathbf{f}$, for this subspace is

$$\int d\alpha d\theta \sinh\alpha \sin\theta \,(\gamma_1\gamma_2) \cdot \left(\left(\gamma_1 e^{j\theta}\partial_\alpha + \frac{\gamma^2}{\sinh\alpha}\partial_\theta\right) \wedge \mathbf{f}\right) = \int d\alpha \mathbf{f} \cdot \left(\gamma^1 e^{j\theta}\right)\Big|_{\theta_0}^{\theta_1} - \int d\theta \mathbf{f} \cdot (\gamma_2 \sinh\alpha)\Big|_{\alpha_0}^{\alpha_1}. \tag{44.64}$$

Since the point is to avoid the curl integral, we did not actually have to state it explicitly, nor was there any actual need to calculate the dual basis.

Example 44.4: Dual representation in three dimensions

It's clear that there is a projective nature to the differential form $d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f})$. This projective nature allows us, in three dimensions, to re-express Stokes theorem using the gradient instead of the vector derivative, and to utilize the cross product and a normal direction to the plane.

When we parameterize a normal direction to the tangent space, so that for a 2D tangent space spanned by curvilinear coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$ the vector $\mathbf{x}^3$ is normal to both, we can write our vector as

$$\mathbf{f} = f_1\mathbf{x}^1 + f_2\mathbf{x}^2 + f_3\mathbf{x}^3, \tag{44.65}$$

and express the orientation of the tangent space area element in terms of a pseudoscalar that includes this normal direction

$$\mathbf{x}_1 \wedge \mathbf{x}_2 = \mathbf{x}^3 \cdot \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) = \mathbf{x}^3 \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right). \tag{44.66}$$

Inserting this into an expansion of the curl form we have

$$\begin{aligned} d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= du^1 du^2 \left\langle \mathbf{x}^3 \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \left(\left(\sum_{i=1,2} x^i \partial_i \right) \wedge \mathbf{f} \right) \right\rangle \\ &= du^1 du^2 \mathbf{x}^3 \cdot \Big(\left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \cdot \left(\boldsymbol{\nabla} \wedge \mathbf{f} \right) - \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \cdot \left(\mathbf{x}^3 \partial_3 \wedge \mathbf{f} \right) \Big). \end{aligned} \tag{44.67}$$

Observe that this last term, the contribution of the component of the gradient perpendicular to the tangent space, has no $\mathbf{x}_3$ components

$$\begin{aligned} \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \cdot \left(\mathbf{x}^3 \partial_3 \wedge \mathbf{f} \right) &= \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \cdot \left(\mathbf{x}^3 \wedge \partial_3 \mathbf{f} \right) \\ &= \left(\left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \cdot \mathbf{x}^3 \right) \cdot \partial_3 \mathbf{f} \\ &= \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \right) \cdot \partial_3 \mathbf{f} \\ &= \mathbf{x}_1 \left(\mathbf{x}_2 \cdot \partial_3 \mathbf{f} \right) - \mathbf{x}_2 \left(\mathbf{x}_1 \cdot \partial_3 \mathbf{f} \right), \end{aligned} \tag{44.68}$$

leaving

$$d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = du^1 du^2 \mathbf{x}^3 \cdot \left(\left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \right) \cdot \left(\boldsymbol{\nabla} \wedge \mathbf{f} \right) \right). \tag{44.69}$$

Now scale the normal vector and its dual to have unit norm as follows

$$\begin{aligned} \mathbf{x}^3 &= \alpha \hat{\mathbf{x}}^3 \\ \mathbf{x}_3 &= \frac{1}{\alpha} \hat{\mathbf{x}}_3, \end{aligned} \tag{44.70}$$

so that for $\beta > 0$, the volume element can be

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \hat{\mathbf{x}}_3 = \beta I. \tag{44.71}$$

This scaling choice is illustrated in fig. 44.7, and represents the "outwards" normal. With such a scaling choice we have

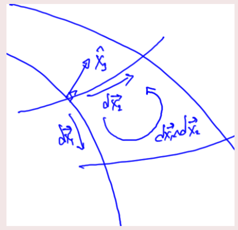

Figure 44.7: Outwards normal

$$\beta du^1 du^2 = dA, \tag{44.72}$$

and almost have the desired cross product representation

$$\begin{aligned} d^2\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= dA\hat{\mathbf{x}}^3 \cdot (I \cdot (\boldsymbol{\nabla} \wedge \mathbf{f})) \\ &= dA\hat{\mathbf{x}}^3 \cdot (I (\boldsymbol{\nabla} \wedge \mathbf{f})) \,. \end{aligned} \tag{44.73}$$

With the duality identity $\mathbf{a} \wedge \mathbf{b} = I \left(\mathbf{a} \times \mathbf{b}\right)$, we have the traditional 3D representation of Stokes theorem

$$\boxed{\int d^2\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -\int dA\hat{\mathbf{x}}^3 \cdot \left(\boldsymbol{\nabla} \times \mathbf{f}\right) = \oint \mathbf{f} \cdot d\mathbf{l}.} \tag{44.74}$$

Note that the orientation of the loop integral in the traditional statement of the 3D Stokes theorem is counterclockwise instead of clockwise, as written here.

44.4 STOKES THEOREM, THREE VARIABLE VOLUME ELEMENT PARAMETERIZATION

We can restate the identity of theorem 44.1 in an equivalent dot product form.

$$\boxed{\int_V \left(d^k\mathbf{x} \cdot \mathbf{x}^i\right) \cdot \partial_i F = \int_{\partial V} d^{k-1}\mathbf{x} \cdot F.} \tag{44.75}$$

Here $d^{k-1}\mathbf{x} = \sum_i d^k\mathbf{x} \cdot \mathbf{x}^i$, with the implicit assumption that it and the blade F that it is dotted with, are both evaluated at the end points of integration variable u^i that has been integrated against.

We've seen one specific example of this above in the expansions of eq. (44.25), and eq. (44.26), however, the equivalent result of eq. (44.75), somewhat magically, applies to any degree blade and volume element provided the degree of the blade is less than that of the volume element (i.e. $s < k$). That magic follows directly from theorem B.1.

As an expositional example, consider a three variable volume element parameterization, and a vector blade $\mathbf{f}$

$$\begin{aligned} d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= \left(d^3\mathbf{x} \cdot \mathbf{x}^i\right) \cdot \partial_i \mathbf{f} \\ &= du^1 du^2 du^3 \left((\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \mathbf{x}^i\right) \cdot \partial_i \mathbf{f} \\ &= du^1 du^2 du^3 \left((\mathbf{x}_1 \wedge \mathbf{x}_2) \delta_3{}^i - (\mathbf{x}_1 \wedge \mathbf{x}_3) \delta_2{}^i + (\mathbf{x}_2 \wedge \mathbf{x}_3) \delta_1{}^i\right) \cdot \partial_i \mathbf{f} \\ &= du^1 du^2 du^3 \left((\mathbf{x}_1 \wedge \mathbf{x}_2) \cdot \partial_3 \mathbf{f} - (\mathbf{x}_1 \wedge \mathbf{x}_3) \cdot \partial_2 \mathbf{f} + (\mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \partial_1 \mathbf{f}\right). \end{aligned} \tag{44.76}$$

It should not be surprising that this has the structure found in the theory of differential forms. Using the differentials for each of the parameterization "directions", we can write this dot product expansion as

$$d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = \left(du^3 \left(d\mathbf{x}_1 \wedge d\mathbf{x}_2\right) \cdot \partial_3 \mathbf{f} - du^2 \left(d\mathbf{x}_1 \wedge d\mathbf{x}_3\right) \cdot \partial_2 \mathbf{f} + du^1 \left(d\mathbf{x}_2 \wedge d\mathbf{x}_3\right) \cdot \partial_1 \mathbf{f}\right). \tag{44.77}$$

Observe that the sign changes with each element of $d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge d\mathbf{x}_3$ that is skipped. In differential forms, the wedge product composition of 1-forms is an abstract quantity. Here the differentials are just vectors, and their wedge product represents an oriented volume element. This interpretation is likely available in the theory of differential forms too, but is arguably less obvious.

Digression

As was the case with the loop integral, we expect that the coordinate representation has a representation that can be expressed as a number of antisymmetric terms. A bit of experimentation shows that such a sum, after dropping the parameter space volume element factor, is

$$\begin{aligned} &\mathbf{x}_1 \left(-\partial_2 f_3 + \partial_3 f_2\right) + \mathbf{x}_2 \left(-\partial_3 f_1 + \partial_1 f_3\right) + \mathbf{x}_3 \left(-\partial_1 f_2 + \partial_2 f_1\right) \\ &\quad = \mathbf{x}_1 \left(-\partial_2 \mathbf{f} \cdot \mathbf{x}_3 + \partial_3 \mathbf{f} \cdot \mathbf{x}_2\right) + \mathbf{x}_2 \left(-\partial_3 \mathbf{f} \cdot \mathbf{x}_1 + \partial_1 \mathbf{f} \cdot \mathbf{x}_3\right) + \mathbf{x}_3 \left(-\partial_1 \mathbf{f} \cdot \mathbf{x}_2 + \partial_2 \mathbf{f} \cdot \mathbf{x}_1\right) \\ &\quad = \left(\mathbf{x}_1 \partial_3 \mathbf{f} \cdot \mathbf{x}_2 - \mathbf{x}_2 \partial_3 \mathbf{f} \cdot \mathbf{x}_1\right) + \left(\mathbf{x}_3 \partial_2 \mathbf{f} \cdot \mathbf{x}_1 - \mathbf{x}_1 \partial_2 \mathbf{f} \cdot \mathbf{x}_3\right) + \left(\mathbf{x}_2 \partial_1 \mathbf{f} \cdot \mathbf{x}_3 - \mathbf{x}_3 \partial_1 \mathbf{f} \cdot \mathbf{x}_2\right) \\ &\quad = \left(\mathbf{x}_1 \wedge \mathbf{x}_2\right) \cdot \partial_3 \mathbf{f} + \left(\mathbf{x}_3 \wedge \mathbf{x}_1\right) \cdot \partial_2 \mathbf{f} + \left(\mathbf{x}_2 \wedge \mathbf{x}_3\right) \cdot \partial_1 \mathbf{f}. \end{aligned} \tag{44.78}$$

To proceed with the integration, we must again consider an infinitesimal volume element, for which the partial can be evaluated as the difference of the endpoints, with all else held constant. For this three variable parameterization, say, (u, v, w), let's delimit such an infinitesimal volume element by the parameterization ranges $[u_0, u_0 + du]$, $[v_0, v_0 + dv]$, $[w_0, w_0 + dw]$. The integral is

$$\begin{aligned}\int_{u=u_0}^{u_0+du} \int_{v=v_0}^{v_0+dv} \int_{w=w_0}^{w_0+dw} d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) = &\int_{u=u_0}^{u_0+du} du \int_{v=v_0}^{v_0+dv} dv \left((\mathbf{x}_u \wedge \mathbf{x}_v) \cdot \mathbf{f} \right)\Big|_{w=w_0}^{w_0+dw} \\ &- \int_{u=u_0}^{u_0+du} du \int_{w=w_0}^{w_0+dw} dw \left((\mathbf{x}_u \wedge \mathbf{x}_w) \cdot \mathbf{f} \right)\Big|_{v=v_0}^{v_0+dv} \\ &+ \int_{v=v_0}^{v_0+dv} dv \int_{w=w_0}^{w_0+dw} dw \left((\mathbf{x}_v \wedge \mathbf{x}_w) \cdot \mathbf{f} \right)\Big|_{u=u_0}^{u_0+du}.\end{aligned} \tag{44.79}$$

Extending this over the ranges $[u_0, u_0 + \Delta u]$, $[v_0, v_0 + \Delta v]$, $[w_0, w_0 + \Delta w]$, we have proved Stokes theorem 44.1 for vectors and a three parameter volume element, provided we have a surface element of the form

$$d^2\mathbf{x} = (d\mathbf{x}_u \wedge d\mathbf{x}_v)\Big|_{w=w_0}^{w_1} - (d\mathbf{x}_u \wedge d\mathbf{x}_w)\Big|_{v=v_0}^{v_1} + (d\mathbf{x}_v \wedge \mathbf{x}_w)\Big|_{u=u_0}^{u_1}, \tag{44.80}$$

where the evaluation of the dot products with $\mathbf{f}$ are also evaluated at the same points.

Example 44.5: Euclidean spherical polar parameterization of 3D subspace

Consider an Euclidean space where a 3D subspace is parameterized using spherical coordinates , as in

$$\mathbf{x}(x, \rho, \theta, \phi) = \mathbf{e}_1 x + \mathbf{e}_4 \rho \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right) = (x, \rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta). \tag{44.81}$$

The tangent space basis for the subspace situated at some fixed $x = x_0$, is easy to calculate, and is found to be

$$\begin{aligned}\mathbf{x}_\rho &= (0, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \mathbf{e}_4 \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right) \\ \mathbf{x}_\theta &= \rho\,(0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) = \rho\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi} \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right) \\ \mathbf{x}_\phi &= \rho\,(0, -\sin\theta\sin\phi, \sin\theta\cos\phi, 0) = \rho\sin\theta\mathbf{e}_3 e^{\mathbf{e}_2\mathbf{e}_3\phi}.\end{aligned} \tag{44.82}$$

While we can use the general relation of theorem B.7 to compute the reciprocal basis. That is

$$\mathbf{a}^* = (\mathbf{b} \wedge \mathbf{c}) \frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}}. \tag{44.83}$$

However, a naive attempt at applying this without algebraic software is a route that requires a lot of care, and is easy to make mistakes doing. In this case it is really not necessary since the tangent space basis only requires scaling to orthonormalize, satisfying for $i, j \in \{\rho, \theta, \phi\}$

$$\mathbf{x}_i \cdot \mathbf{x}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2\theta \end{bmatrix}. \tag{44.84}$$

This allows us to read off the dual basis for the tangent volume by inspection

$$\begin{aligned}
\mathbf{x}^\rho &= \mathbf{e}_4 \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right) \\
\mathbf{x}^\theta &= \frac{1}{\rho}\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi} \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right) \\
\mathbf{x}^\phi &= \frac{1}{\rho\sin\theta}\mathbf{e}_3 e^{\mathbf{e}_2\mathbf{e}_3\phi}.
\end{aligned} \tag{44.85}$$

Should we wish to explicitly calculate the curl on the tangent space, we would need these. The area and volume elements are also messy to calculate manually. This expansion can be found in the Mathematica notebook sphericalSurfaceAndVolumeElements.nb , and is

$$\begin{aligned}
\mathbf{x}_\theta \wedge \mathbf{x}_\phi &= \rho^2 \sin\theta \left(\mathbf{e}_4\mathbf{e}_2 \sin\theta\sin\phi + \mathbf{e}_2\mathbf{e}_3 \cos\theta + \mathbf{e}_3\mathbf{e}_4 \sin\theta\cos\phi\right) \\
\mathbf{x}_\phi \wedge \mathbf{x}_\rho &= \rho\sin\theta \left(-\mathbf{e}_2\mathbf{e}_3 \sin\theta - \mathbf{e}_2\mathbf{e}_4 \cos\theta\sin\phi + \mathbf{e}_3\mathbf{e}_4 \cos\theta\cos\phi\right) \\
\mathbf{x}_\rho \wedge \mathbf{x}_\theta &= -\mathbf{e}_4\rho \left(\mathbf{e}_2 \cos\phi + \mathbf{e}_3 \sin\phi\right) \\
\mathbf{x}_\rho \wedge \mathbf{x}_\theta \wedge \mathbf{x}_\phi &= \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\rho^2 \sin\theta
\end{aligned} \tag{44.86}$$

Those area elements have a Geometric algebra factorization that are perhaps useful

$$\begin{aligned}
\mathbf{x}_\theta \wedge \mathbf{x}_\phi &= -\rho^2 \sin\theta \mathbf{e}_2\mathbf{e}_3 \exp\left(-\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right) \\
\mathbf{x}_\phi \wedge \mathbf{x}_\rho &= \rho\sin\theta \mathbf{e}_3\mathbf{e}_4 e^{\mathbf{e}_2\mathbf{e}_3\phi} \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\theta\right). \\
\mathbf{x}_\rho \wedge \mathbf{x}_\theta &= -\rho\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}
\end{aligned} \tag{44.87}$$

One of the beauties of Stokes theorem is that we don't actually have to calculate the dual basis on the tangent space to proceed with the integration. For that calculation above, where we had a normal tangent basis, I still used software was used as an aid, so it is clear that this can generally get pretty messy.

To apply Stokes theorem to a vector field we can use eq. (44.80) to write down the integral directly

$$\begin{aligned}
\int_V d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= \int_{\partial V} d^2\mathbf{x} \cdot \mathbf{f} \\
&= \int \left(\mathbf{x}_\theta \wedge \mathbf{x}_\phi\right) \cdot \mathbf{f}\Big|_{\rho=\rho_0}^{\rho_1} d\theta d\phi \\
&\quad + \int \left(\mathbf{x}_\phi \wedge \mathbf{x}_\rho\right) \cdot \mathbf{f}\Big|_{\theta=\theta_0}^{\theta_1} d\phi d\rho + \int \left(\mathbf{x}_\rho \wedge \mathbf{x}_\theta\right) \cdot \mathbf{f}\Big|_{\phi=\phi_0}^{\phi_1} d\rho d\theta.
\end{aligned} \tag{44.88}$$

Observe that eq. (44.88) is a vector valued integral that expands to

$$\int \left(\mathbf{x}_\theta f_\phi - \mathbf{x}_\phi f_\theta\right)\Big|_{\rho=\rho_0}^{\rho_1} d\theta d\phi + \int \left(\mathbf{x}_\phi f_\rho - \mathbf{x}_\rho f_\phi\right)\Big|_{\theta=\theta_0}^{\theta_1} d\phi d\rho + \int \left(\mathbf{x}_\rho f_\theta - \mathbf{x}_\theta f_\rho\right)\Big|_{\phi=\phi_0}^{\phi_1} d\rho d\theta. \tag{44.89}$$

This could easily be a difficult integral to evaluate since the vectors $\mathbf{x}_i$ evaluated at the endpoints are still functions of two parameters. An easier integral would result from the application of Stokes theorem to a bivector valued field, say B, for which we have

$$\begin{aligned}
\int_V d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge B) &= \int_{\partial V} d^2\mathbf{x} \cdot B \\
&= \int \left(\mathbf{x}_\theta \wedge \mathbf{x}_\phi\right) \cdot B\Big|_{\rho=\rho_0}^{\rho_1} d\theta d\phi \\
&\quad + \int \left(\mathbf{x}_\phi \wedge \mathbf{x}_\rho\right) \cdot B\Big|_{\theta=\theta_0}^{\theta_1} d\phi d\rho + \int \left(\mathbf{x}_\rho \wedge \mathbf{x}_\theta\right) \cdot B\Big|_{\phi=\phi_0}^{\phi_1} d\rho d\theta \\
&= \int B_{\phi\theta}\Big|_{\rho=\rho_0}^{\rho_1} d\theta d\phi + \int B_{\rho\phi}\Big|_{\theta=\theta_0}^{\theta_1} d\phi d\rho + \int B_{\theta\rho}\Big|_{\phi=\phi_0}^{\phi_1} d\rho d\theta.
\end{aligned} \tag{44.90}$$

There is a geometric interpretation to these oriented area integrals, especially when written out explicitly in terms of the differentials along the parameterization directions. Pulling out a sign explicitly to match the geometry (as we had to also do for the line integrals in the two parameter volume element case), we can write this as

$$\int_{\partial V} d^2\mathbf{x} \cdot B = -\int \left(d\mathbf{x}_\phi \wedge d\mathbf{x}_\theta\right) \cdot B\Big|_{\rho=\rho_0}^{\rho_1} - \int \left(d\mathbf{x}_\rho \wedge d\mathbf{x}_\phi\right) \cdot B\Big|_{\theta=\theta_0}^{\theta_1} - \int \left(d\mathbf{x}_\theta \wedge d\mathbf{x}_\rho\right) \cdot B\Big|_{\phi=\phi_0}^{\phi_1}. \tag{44.91}$$

When written out in this differential form, each of the respective area elements is an oriented area along one of the faces of the parameterization volume, much like the line integral that results from a two parameter volume curl integral. This is visualized in fig. 44.8. In this figure, faces (1) and (3) are "top faces", those with signs matching the tops of the evaluation ranges eq. (44.91), whereas face (2) is a bottom face with a sign that is correspondingly reversed.

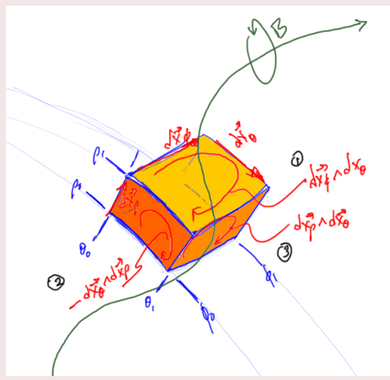

Figure 44.8: Boundary faces of a spherical parameterization region

Example 44.6: Minkowski hyperbolic-spherical polar parameterization of 3D subspace

Working with a three parameter volume element in a Minkowski space does not change much. For example in a 4D space with $(\mathbf{e}_4)^2 = -1$, we can employ a hyperbolic-spherical parameterization similar to that used above for the 4D Euclidean space

$$\mathbf{x}(x, \rho, \alpha, \phi) = \{x, \rho \sinh\alpha\cos\phi, \rho\sinh\alpha\sin\phi, \rho\cosh\alpha\} = \mathbf{e}_1 x + \mathbf{e}_4 \rho \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right). \tag{44.92}$$

This has tangent space basis elements

$$\begin{aligned}
\mathbf{x}_\rho &= \sinh\alpha\left(\cos\phi\mathbf{e}_2 + \sin\phi\mathbf{e}_3\right) + \cosh\alpha\mathbf{e}_4 = \mathbf{e}_4 \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right) \\
\mathbf{x}_\alpha &= \rho\cosh\alpha\left(\cos\phi\mathbf{e}_2 + \sin\phi\mathbf{e}_3\right) + \rho\sinh\alpha\mathbf{e}_4 = \rho\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\exp\left(-\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right) \\
\mathbf{x}_\phi &= \rho\sinh\alpha\left(\mathbf{e}_3\cos\phi - \mathbf{e}_2\sin\phi\right) = \rho\sinh\alpha\mathbf{e}_3 e^{\mathbf{e}_2\mathbf{e}_3\phi}.
\end{aligned} \tag{44.93}$$

This is a normal basis, but again not orthonormal. Specifically, for $i, j \in \{\rho, \theta, \phi\}$ we have

$$\mathbf{x}_i \cdot \mathbf{x}_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sinh^2 \alpha \end{bmatrix}, \tag{44.94}$$

where we see that the radial vector $\mathbf{x}_\rho$ is timelike. We can form the dual basis again by inspection

$$\begin{aligned} \mathbf{x}_\rho &= -\mathbf{e}_4 \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right) \\ \mathbf{x}_\alpha &= \frac{1}{\rho}\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi} \exp\left(-\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right) \\ \mathbf{x}_\phi &= \frac{1}{\rho\sinh\alpha}\mathbf{e}_3 e^{\mathbf{e}_2\mathbf{e}_3\phi}. \end{aligned} \tag{44.95}$$

The area elements are

$$\begin{aligned} \mathbf{x}_\alpha \wedge \mathbf{x}_\phi &= \rho^2 \sinh\alpha \left(-\mathbf{e}_4\mathbf{e}_3 \sinh\alpha\cos\phi + \cosh\alpha\mathbf{e}_2\mathbf{e}_3 + \sinh\alpha\sin\phi\mathbf{e}_2\mathbf{e}_4\right) \\ \mathbf{x}_\phi \wedge \mathbf{x}_\rho &= \rho\sinh\alpha\left(-\mathbf{e}_2\mathbf{e}_3\sinh\alpha - \mathbf{e}_2\mathbf{e}_4\cosh\alpha\sin\phi + \cosh\alpha\cos\phi\mathbf{e}_3\mathbf{e}_4\right) \\ \mathbf{x}_\rho \wedge \mathbf{x}_\alpha &= -\mathbf{e}_4\rho\left(\cos\phi\mathbf{e}_2 + \sin\phi\mathbf{e}_3\right), \end{aligned} \tag{44.96}$$

or

$$\begin{aligned} \mathbf{x}_\alpha \wedge \mathbf{x}_\phi &= \rho^2 \sinh\alpha\mathbf{e}_2\mathbf{e}_3 \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{-\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right) \\ \mathbf{x}_\phi \wedge \mathbf{x}_\rho &= \rho\sinh\alpha\mathbf{e}_3\mathbf{e}_4 e^{\mathbf{e}_2\mathbf{e}_3\phi} \exp\left(\mathbf{e}_4\mathbf{e}_2 e^{\mathbf{e}_2\mathbf{e}_3\phi}\alpha\right) \\ \mathbf{x}_\rho \wedge \mathbf{x}_\alpha &= -\mathbf{e}_4\mathbf{e}_2\rho e^{\mathbf{e}_2\mathbf{e}_3\phi}. \end{aligned} \tag{44.97}$$

The volume element also reduces nicely, and is

$$\mathbf{x}_\rho \wedge \mathbf{x}_\alpha \wedge \mathbf{x}_\phi = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\rho^2\sinh\alpha. \tag{44.98}$$

The area and volume element reductions were once again messy, done in software using sphericalSurfaceAndVolumeElementsMinkowski.nb . However, we really only need eq. (44.93) to perform the Stokes integration.

44.5 STOKES THEOREM, FOUR VARIABLE VOLUME ELEMENT PARAMETERIZATION

Volume elements for up to four parameters are likely of physical interest, with the four volume elements of interest for relativistic physics in $\bigwedge^{3,1}$ spaces. For example, we may wish to use a parameterization $u^1 = x, u^2 = y, u^3 = z, u^4 = \tau = ct$, with a four volume

$$d^4\mathbf{x} = d\mathbf{x}_x \wedge d\mathbf{x}_y \wedge d\mathbf{x}_z \wedge d\mathbf{x}_\tau, \tag{44.99}$$

We follow the same procedure to calculate the corresponding boundary surface "area" element (with dimensions of volume in this case). This is

$$\begin{aligned}
d^4\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{f}) &= \left(d^4\mathbf{x} \cdot \mathbf{x}^i\right) \cdot \partial_i \mathbf{f} \\
&= du^1 du^2 du^3 du^4 \left((\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4) \cdot \mathbf{x}^i\right) \cdot \partial_i \mathbf{f} \\
&= du^1 du^2 du^3 du_4 \left((\mathbf{x}_1 \wedge \mathbf{x}_2 \right. \\
&\left. \wedge \mathbf{x}_3)\, {\delta_4}^i - (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_4)\, {\delta_3}^i + (\mathbf{x}_1 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4)\, {\delta_2}^i - (\mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4)\, {\delta_1}^i\right) \cdot \partial_i \mathbf{f} \\
&= du^1 du^2 du^3 du^4 \left((\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \partial_4 \mathbf{f} \right. \\
&\left. - (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_4) \cdot \partial_3 \mathbf{f} + (\mathbf{x}_1 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4) \cdot \partial_2 \mathbf{f} - (\mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4) \cdot \partial_1 \mathbf{f}\right).
\end{aligned} \tag{44.100}$$

Our boundary value surface element is therefore

$$d^3\mathbf{x} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 - \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_4 + \mathbf{x}_1 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 - \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4. \tag{44.101}$$

where it is implied that this (and the dot products with $\mathbf{f}$) are evaluated on the boundaries of the integration ranges of the omitted index. This same boundary form can be used for vector, bivector and trivector variations of Stokes theorem.

44.6 DUALITY AND ITS RELATION TO THE PSEUDOSCALAR.

Looking to eq. (B.24) of theorem B.6, and scaling the wedge product $\mathbf{a} \wedge \mathbf{b}$ by its absolute magnitude, we can express duality using that scaled bivector as a pseudoscalar for the plane that spans $\{\mathbf{a}, \mathbf{b}\}$. Let's introduce a subscript notation for such scaled blades

$$I_{\mathbf{ab}} = \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|}. \tag{44.102}$$

This allows us to express the unit vector in the direction of $\mathbf{a}^*$ as

$$\boxed{\widehat{\mathbf{a}^*} = \hat{\mathbf{b}} \frac{|\mathbf{a} \wedge \mathbf{b}|}{\mathbf{a} \wedge \mathbf{b}} = \hat{\mathbf{b}} \frac{1}{I_{\mathbf{ab}}}.} \tag{44.103}$$

Following the pattern of eq. (B.24), it is clear how to express the dual vectors for higher dimensional subspaces. For example
or for the unit vector in the direction of $\mathbf{a}^*$,

$$\widehat{\mathbf{a}^*} = I_{\mathbf{bc}} \frac{1}{I_{\mathbf{abc}}}.$$

44.7 DIVERGENCE THEOREM.

When the curl integral is a scalar result we are able to apply duality relationships to obtain the divergence theorem for the corresponding space. We will be able to show that a relationship of the following form holds

$$\boxed{\int_V dV \boldsymbol{\nabla} \cdot \mathbf{f} = \int_{\partial V} dA_i \hat{\mathbf{n}}^i \cdot \mathbf{f}.} \tag{44.104}$$

Here $\mathbf{f}$ is a vector, $\hat{\mathbf{n}}^i$ is normal to the boundary surface, and dA_i is the area of this bounding surface element. We wish to quantify these more precisely, especially because the orientation of the normal vectors are metric dependent. Working a few specific examples will show the pattern nicely, but it is helpful to first consider some aspects of the general case.

First note that, for a scalar Stokes integral we are integrating the vector derivative curl of a blade $F \in \bigwedge^{k-1}$ over a k-parameter volume element. Because the dimension of the space matches the number of parameters, the projection of the gradient onto the tangent space is exactly that gradient

$$\int_V d^k \mathbf{x} \cdot (\boldsymbol{\partial} \wedge F) = \int_V d^k \mathbf{x} \cdot (\boldsymbol{\nabla} \wedge F). \tag{44.105}$$

Multiplication of F by the pseudoscalar will always produce a vector. With the introduction of such a dual vector, as in

$$F = I\mathbf{f}, \tag{44.106}$$

Stokes theorem takes the form

$$\int_V d^k \mathbf{x} \cdot \langle \boldsymbol{\nabla} I \mathbf{f} \rangle_k = \int_{\partial V} \left\langle d^{k-1} \mathbf{x} I \mathbf{f} \right\rangle, \tag{44.107}$$

or

$$\int_V \left\langle d^k \mathbf{x} \boldsymbol{\nabla} I \mathbf{f} \right\rangle = \int_{\partial V} \left(d^{k-1} \mathbf{x} I \right) \cdot \mathbf{f}, \tag{44.108}$$

where we will see that the vector $d^{k-1}\mathbf{x}I$ can roughly be characterized as a normal to the boundary surface. Using primes to indicate the scope of the action of the gradient, cyclic permutation within the scalar selection operator can be used to factor out the pseudoscalar

$$
\begin{aligned}
\int_V \left\langle d^k\mathbf{x}\boldsymbol{\nabla} I\mathbf{f}\right\rangle &= \int_V \left\langle \mathbf{f}' d^k\mathbf{x}\boldsymbol{\nabla}' I\right\rangle \\
&= \int_V \left\langle \mathbf{f}' d^k\mathbf{x}\boldsymbol{\nabla}'\right\rangle_k I \\
&= \int_V (-1)^{k+1} d^k\mathbf{x}\,(\boldsymbol{\nabla}\cdot\mathbf{f})\, I \\
&= (-1)^{k+1} I^2 \int_V dV\,(\boldsymbol{\nabla}\cdot\mathbf{f})\,.
\end{aligned}
\tag{44.109}
$$

The second last step uses theorem B.8, and the last writes $d^k\mathbf{x} = I^2\left|d^k\mathbf{x}\right| = I^2 dV$, where we have assumed (without loss of generality) that $d^k\mathbf{x}$ has the same orientation as the pseudoscalar for the space. We also assume that the parameterization is non-degenerate over the integration volume (i.e. no $d\mathbf{x}_i = 0$), so the sign of this product cannot change.

Let's now return to the normal vector $d^{k-1}\mathbf{x}I$. With $d^{k-1}u_i = du^1 du^2 \cdots du^{i-1} du^{i+1} \cdots du^k$ (the i indexed differential omitted), and $I_{ab\cdots c} = (\mathbf{x}_a \wedge \mathbf{x}_b \wedge \cdots \wedge \mathbf{x}_c)/|\mathbf{x}_a \wedge \mathbf{x}_b \wedge \cdots \wedge \mathbf{x}_c|$, we have

$$
\begin{aligned}
d^{k-1}\mathbf{x}I &= d^{k-1}u_i\left(\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k\right)\cdot \mathbf{x}^i I \\
&= I_{12\cdots(k-1)} I |d\mathbf{x}_1 \wedge d\mathbf{x}_2 \wedge \cdots \wedge d\mathbf{x}_{k-1}| \\
&\quad - I_{1\cdots(k-2)k} I |d\mathbf{x}_1 \wedge \cdots \wedge d\mathbf{x}_{k-2} \wedge d\mathbf{x}_k| + \cdots
\end{aligned}
\tag{44.110}
$$

We've seen in eq. (44.103) and theorem B.7 that the dual of vector $\mathbf{a}$ with respect to the unit pseudoscalar $I_{\mathbf{b}\cdots\mathbf{cd}}$ in a subspace spanned by $\{\mathbf{a}, \cdots \mathbf{c}, \mathbf{d}\}$ is

$$
\widehat{\mathbf{a}^*} = I_{\mathbf{b}\cdots\mathbf{cd}} \frac{1}{I_{\mathbf{a}\cdots\mathbf{cd}}},
\tag{44.111}
$$

or

$$
\widehat{\mathbf{a}^*} I_{\mathbf{a}\cdots\mathbf{cd}}^2 = I_{\mathbf{b}\cdots\mathbf{cd}}.
\tag{44.112}
$$

This allows us to write

$$
d^{k-1}\mathbf{x}I = I^2 \sum_i \widehat{\mathbf{x}^i} dA'_i
\tag{44.113}
$$

where $dA'_i = \pm dA_i$, and dA_i is the area of the boundary area element normal to $\mathbf{x}^i$. Note that the I^2 term will now cancel cleanly from both sides of the divergence equation, taking both the metric and the orientation specific dependencies with it.

This leaves us with

$$\int_V dV \boldsymbol{\nabla} \cdot \mathbf{f} = (-1)^{k+1} \int_{\partial V} dA'_i \widehat{\mathbf{x}^i} \cdot \mathbf{f}. \tag{44.114}$$

To spell out the details, we have to be very careful with the signs. However, that is a job best left for specific examples.

Example 44.7: 2D divergence theorem

Let's start back at

$$\int_A \left\langle d^2\mathbf{x} \boldsymbol{\nabla} I \mathbf{f} \right\rangle = \int_{\partial A} \left(d^1\mathbf{x} I \right) \cdot \mathbf{f}. \tag{44.115}$$

On the left our integral can be rewritten as

$$\begin{aligned} \int_A \left\langle d^2\mathbf{x} \boldsymbol{\nabla} I \mathbf{f} \right\rangle &= -\int_A \left\langle d^2\mathbf{x} I \boldsymbol{\nabla} \mathbf{f} \right\rangle \\ &= -\int_A d^2\mathbf{x} I \left(\boldsymbol{\nabla} \cdot \mathbf{f} \right) \\ &= -I^2 \int_A dA \boldsymbol{\nabla} \cdot \mathbf{f}, \end{aligned} \tag{44.116}$$

where $d^2\mathbf{x} = I dA$ and we pick the pseudoscalar with the same orientation as the volume (area in this case) element $I = (\mathbf{x}_1 \wedge \mathbf{x}_2)/|\mathbf{x}_1 \wedge \mathbf{x}_2|$.

For the boundary form we have

$$\begin{aligned} d^1\mathbf{x} &= du^2 \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \right) \cdot \mathbf{x}^1 + du^1 \left(\mathbf{x}_1 \wedge \mathbf{x}_2 \right) \cdot \mathbf{x}^2 \\ &= -du^2 \mathbf{x}_2 + du^1 \mathbf{x}_1. \end{aligned} \tag{44.117}$$

The duality relations for the tangent space are

$$\begin{aligned} \mathbf{x}^2 &= \mathbf{x}_1 \frac{1}{\mathbf{x}_2 \wedge \mathbf{x}_1}, \\ \mathbf{x}^1 &= \mathbf{x}_2 \frac{1}{\mathbf{x}_1 \wedge \mathbf{x}_2} \end{aligned} \tag{44.118}$$

or

$$
\begin{aligned}
\widehat{\mathbf{x}^2} &= -\widehat{\mathbf{x}_1}\frac{1}{I} \\
\widehat{\mathbf{x}^1} &= \widehat{\mathbf{x}_2}\frac{1}{I}
\end{aligned}. \tag{44.119}
$$

Back substitution into the line element gives

$$
\begin{aligned}
d^1\mathbf{x} &= -du^2|\mathbf{x}_2|\widehat{\mathbf{x}_2} + du^1|\mathbf{x}_1|\widehat{\mathbf{x}_1} \\
&= -du^2|\mathbf{x}_2|\widehat{\mathbf{x}^1}I - du^1|\mathbf{x}_1|\widehat{\mathbf{x}^2}I.
\end{aligned} \tag{44.120}
$$

Writing (no sum) $du^i|\mathbf{x}_i| = ds_i$, we have

$$
d^1\mathbf{x}I = -\left(ds_2\widehat{\mathbf{x}^1} + ds_1\widehat{\mathbf{x}^2}\right)I^2. \tag{44.121}
$$

This provides us a divergence and normal relationship, with $-I^2$ terms on each side that can be canceled. Restoring explicit range evaluation, that is

$$
\begin{aligned}
\int_A dA\boldsymbol{\nabla}\cdot\mathbf{f} &= \int_{\Delta u^2} ds_2\widehat{\mathbf{x}^1}\cdot\mathbf{f}\Big|_{\Delta u^1} + \int_{\Delta u^1} ds_1\widehat{\mathbf{x}^2}\cdot\mathbf{f}\Big|_{\Delta u^2} \\
&= \int_{\Delta u^2} ds_2\widehat{\mathbf{x}^1}\cdot\mathbf{f}\Big|_{u^1(1)} - \int_{\Delta u^2} ds_2\widehat{\mathbf{x}^1}\cdot\mathbf{f}\Big|_{u^1(0)} \\
&\quad + \int_{\Delta u^1} ds_1\widehat{\mathbf{x}^2}\cdot\mathbf{f}\Big|_{u^2(0)} - \int_{\Delta u^1} ds_1\widehat{\mathbf{x}^2}\cdot\mathbf{f}\Big|_{u^2(0)}.
\end{aligned} \tag{44.122}
$$

Let's consider this graphically for an Euclidean metric as illustrated in fig. 44.9.

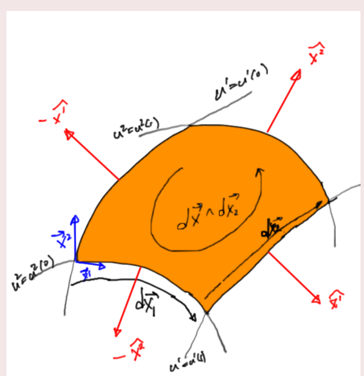

Figure 44.9: Normals on area element

We see that

- along $u^2(0)$ the outwards normal is $-\widehat{\mathbf{x}^2}$,
- along $u^2(1)$ the outwards normal is $\widehat{\mathbf{x}^2}$,
- along $u^1(0)$ the outwards normal is $-\widehat{\mathbf{x}^1}$, and
- along $u^1(1)$ the outwards normal is $\widehat{\mathbf{x}^2}$.

Writing that outwards normal as $\hat{\mathbf{n}}$, we have

$$\boxed{\int_A dA \boldsymbol{\nabla} \cdot \mathbf{f} = \oint ds \hat{\mathbf{n}} \cdot \mathbf{f}.} \tag{44.123}$$

Note that we can use the same algebraic notion of outward normal for non-Euclidean spaces, although cannot expect the geometry to look anything like that of the figure.

Example 44.8: 3D divergence theorem

As with the 2D example, let's start back with

$$\int_V \left\langle d^3\mathbf{x} \boldsymbol{\nabla} I \mathbf{f} \right\rangle = \int_{\partial V} \left(d^2\mathbf{x} I \right) \cdot \mathbf{f}. \tag{44.124}$$

In a 3D space, the pseudoscalar commutes with all grades, so we have

$$\begin{aligned} \int_V \left\langle d^3\mathbf{x} \boldsymbol{\nabla} I \mathbf{f} \right\rangle &= \int_V \left(d^3\mathbf{x} I \right) \boldsymbol{\nabla} \cdot \mathbf{f} \\ &= I^2 \int_V dV \boldsymbol{\nabla} \cdot \mathbf{f}, \end{aligned} \tag{44.125}$$

where $d^3\mathbf{x}I = dVI^2$, and we have used a pseudoscalar with the same orientation as the volume element

$$\begin{aligned} I &= \widehat{\mathbf{x}_{123}} \\ \mathbf{x}_{123} &= \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3. \end{aligned} \tag{44.126}$$

In the boundary integral our dual two form is

$$
\begin{aligned}
d^2\mathbf{x}I &= du^1 du^2 \mathbf{x}_1 \wedge \mathbf{x}_2 + du^3 du^1 \mathbf{x}_3 \wedge \mathbf{x}_1 + du^2 du^3 \mathbf{x}_2 \wedge \mathbf{x}_3 \\
&= \left(dA_3 \widehat{\mathbf{x}_{12}} \frac{1}{I} + dA_2 \widehat{\mathbf{x}_{31}} \frac{1}{I} + dA_1 \widehat{\mathbf{x}_{23}} \frac{1}{I} \right) I^2,
\end{aligned}
\tag{44.127}
$$

where $\mathbf{x}_{ij} = \mathbf{x}_i \wedge \mathbf{x}_j$, and

$$
\begin{aligned}
dA_1 &= |d\mathbf{x}_2 \wedge d\mathbf{x}_3| \\
dA_2 &= |d\mathbf{x}_3 \wedge d\mathbf{x}_1| \\
dA_3 &= |d\mathbf{x}_1 \wedge d\mathbf{x}_2|.
\end{aligned}
\tag{44.128}
$$

Observe that we can do a cyclic permutation of a 3 blade without any change of sign, for example

$$
\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = -\mathbf{x}_2 \wedge \mathbf{x}_1 \wedge \mathbf{x}_3 = \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_1. \tag{44.129}
$$

Because of this we can write the dual two form as we expressed the normals in theorem B.7

$$
\begin{aligned}
d^2\mathbf{x}I &= \left(dA_1 \widehat{\mathbf{x}_{23}} \frac{1}{\widehat{\mathbf{x}_{123}}} + dA_2 \widehat{\mathbf{x}_{31}} \frac{1}{\widehat{\mathbf{x}_{231}}} + dA_3 \widehat{\mathbf{x}_{12}} \frac{1}{\widehat{\mathbf{x}_{312}}} \right) I^2 \\
&= \left(dA_1 \widehat{\mathbf{x}^1} + dA_2 \widehat{\mathbf{x}^2} + dA_3 \widehat{\mathbf{x}^3} \right) I^2.
\end{aligned}
\tag{44.130}
$$

We can now state the 3D divergence theorem, canceling out the metric and orientation dependent term I^2 on both sides

$$
\int_V dV \boldsymbol{\nabla} \cdot \mathbf{f} = \int dA \hat{\mathbf{n}} \cdot \mathbf{f}, \tag{44.131}
$$

where (sums implied)

$$
dA \hat{\mathbf{n}} = dA_i \widehat{\mathbf{x}^i}, \tag{44.132}
$$

and

$$\begin{aligned}\hat{\mathbf{n}}|_{u^i=u^i(1)} &= \widehat{\mathbf{x}^i} \\ \hat{\mathbf{n}}|_{u^i=u^i(0)} &= -\widehat{\mathbf{x}^i}.\end{aligned} \tag{44.133}$$

The outwards normals at the upper integration ranges of a three parameter surface are depicted in fig. 44.10.

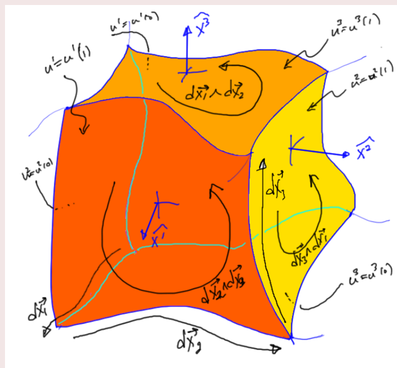

Figure 44.10: Outwards normals on volume at upper integration ranges.

This sign alternation originates with the two form elements $(d\mathbf{x}_i \wedge d\mathbf{x}_j) \cdot F$ from the Stokes boundary integral, which were explicitly evaluated at the endpoints of the integral. That is, for $k \neq i, j$,

$$\int_{\partial V} (d\mathbf{x}_i \wedge d\mathbf{x}_j) \cdot F \equiv \int_{\Delta u^i} \int_{\Delta u^j} \left((d\mathbf{x}_i \wedge d\mathbf{x}_j) \cdot F\right)\Big|_{u^k=u^k(1)} - \left((d\mathbf{x}_i \wedge d\mathbf{x}_j) \cdot F\right)\Big|_{u^k=u^k(0)} \tag{44.134}$$

In the context of the divergence theorem, this means that we are implicitly requiring the dot products $\widehat{\mathbf{x}^k} \cdot \mathbf{f}$ to be evaluated specifically at the end points of the integration where $u^k = u^k(1), u^k = u^k(0)$, accounting for the alternation of sign required to describe the normals as uniformly outwards.

Example 44.9: 4D divergence theorem

Applying Stokes theorem to a trivector $T = I\mathbf{f}$ in the 4D case we find

$$-I^2 \int_V d^4x \boldsymbol{\nabla} \cdot \mathbf{f} = \int_{\partial V} \left(d^3\mathbf{x} I\right) \cdot \mathbf{f}. \tag{44.135}$$

Here the pseudoscalar has been picked to have the same orientation as the hypervolume element $d^4\mathbf{x} = I d^4 x$. Writing $\mathbf{x}_{ij\cdots k} = \mathbf{x}_i \wedge \mathbf{x}_j \wedge \cdots \mathbf{x}_k$ the dual of the three form is

$$\begin{aligned}
d^3\mathbf{x} I &= \left(du^1 du^2 du^3 \mathbf{x}_{123} - du^1 du^2 du^4 \mathbf{x}_{124} + du^1 du^3 du^4 \mathbf{x}_{134} - du^2 du^3 du^4 \mathbf{x}_{234}\right) I \\
&= \left(dA^{123} \widehat{\mathbf{x}_{123}} - dA^{124} \widehat{\mathbf{x}_{124}} + dA^{134} \widehat{\mathbf{x}_{134}} - dA^{234} \widehat{\mathbf{x}_{234}}\right) I \\
&= \left(dA^{123} \widehat{\mathbf{x}_{123}} \frac{1}{\widehat{\mathbf{x}_{1234}}} - dA^{124} \widehat{\mathbf{x}_{124}} \frac{1}{\widehat{\mathbf{x}_{1234}}} + dA^{134} \widehat{\mathbf{x}_{134}} \frac{1}{\widehat{\mathbf{x}_{1234}}} - dA^{234} \widehat{\mathbf{x}_{234}} \frac{1}{\widehat{\mathbf{x}_{1234}}}\right) I^2 \\
&= -\left(dA^{123} \widehat{\mathbf{x}_{123}} \frac{1}{\widehat{\mathbf{x}_{4123}}} + dA^{124} \widehat{\mathbf{x}_{124}} \frac{1}{\widehat{\mathbf{x}_{3412}}} + dA^{134} \widehat{\mathbf{x}_{134}} \frac{1}{\widehat{\mathbf{x}_{2341}}} + dA^{234} \widehat{\mathbf{x}_{234}} \frac{1}{\widehat{\mathbf{x}_{1234}}}\right) I^2 \\
&= -\left(dA^{123} \widehat{\mathbf{x}_{123}} \frac{1}{\widehat{\mathbf{x}_{4123}}} + dA^{124} \widehat{\mathbf{x}_{412}} \frac{1}{\widehat{\mathbf{x}_{3412}}} + dA^{134} \widehat{\mathbf{x}_{341}} \frac{1}{\widehat{\mathbf{x}_{2341}}} + dA^{234} \widehat{\mathbf{x}_{234}} \frac{1}{\widehat{\mathbf{x}_{1234}}}\right) I^2 \\
&= -\left(dA^{123} \widehat{\mathbf{x}^4} + dA^{124} \widehat{\mathbf{x}^3} + dA^{134} \widehat{\mathbf{x}^2} + dA^{234} \widehat{\mathbf{x}^1}\right) I^2
\end{aligned} \tag{44.136}$$

Here, we've written

$$dA^{ijk} = \left| d\mathbf{x}_i \wedge d\mathbf{x}_j \wedge d\mathbf{x}_k \right|. \tag{44.137}$$

Observe that the dual representation nicely removes the alternation of sign that we had in the Stokes theorem boundary integral, since each alternation of the wedged vectors in the pseudoscalar changes the sign once.

As before, we define the outwards normals as $\hat{\mathbf{n}} = \pm \widehat{\mathbf{x}^i}$ on the upper and lower integration ranges respectively. The scalar area elements on these faces can be written in a dual form

$$\begin{aligned}
dA_4 &= dA^{123} \\
dA_3 &= dA^{124} \\
dA_2 &= dA^{134}, \\
dA_1 &= dA^{234}
\end{aligned} \tag{44.138}$$

so that the 4D divergence theorem looks just like the 2D and 3D cases

$$\int_V d^4x \boldsymbol{\nabla} \cdot \mathbf{f} = \int_{\partial V} d^3x \hat{\mathbf{n}} \cdot \mathbf{f}. \tag{44.139}$$

Here we define the volume scaled normal as

$$d^3 x\hat{\mathbf{n}} = dA_i \widehat{\mathbf{x}^i}. \tag{44.140}$$

As before, we have made use of the implicit fact that the three form (and it's dot product with $\mathbf{f}$) was evaluated on the boundaries of the integration region, with a toggling of sign on the lower limit of that evaluation that is now reflected in what we have defined as the outwards normal.

We also obtain explicit instructions from this formalism how to compute the "outwards" normal for this surface in a 4D space (unit scaling of the dual basis elements), something that we cannot compute using any sort of geometrical intuition. For free we've obtained a result that applies to both Euclidean and Minkowski (or other non-Euclidean) spaces.

44.8 VOLUME INTEGRAL COORDINATE REPRESENTATIONS

It may be useful to formulate the curl integrals in tensor form. For vectors $\mathbf{f}$, and bivectors B, the coordinate representations of those differential forms (exercise 44.1) are

$$d^2\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -d^2 u \epsilon^{ab} \partial_a f_b \tag{44.141a}$$

$$d^3\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -d^3 u \epsilon^{abc} \mathbf{x}_a \partial_b f_c \tag{44.141b}$$

$$d^4\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -\frac{1}{2} d^4 u \epsilon^{abcd} \mathbf{x}_a \wedge \mathbf{x}_b \partial_c f_d \tag{44.141c}$$

$$d^3\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge B\right) = -\frac{1}{2} d^3 u \epsilon^{abc} \partial_a B_{bc} \tag{44.141d}$$

$$d^4\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge B\right) = -\frac{1}{2} d^4 u \epsilon^{abcd} \mathbf{x}_a \partial_b B_{cd} \tag{44.141e}$$

$$d^4\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge T\right) = -d^4 u \left(\partial_4 T_{123} - \partial_3 T_{124} + \partial_2 T_{134} - \partial_1 T_{234}\right). \tag{44.141f}$$

Here the bivector B and trivector T is expressed in terms of their curvilinear components on the tangent space

$$B = \frac{1}{2}\mathbf{x}^i \wedge \mathbf{x}^j B_{ij} + B_\perp \tag{44.142a}$$

$$T = \frac{1}{3!}\mathbf{x}^i \wedge \mathbf{x}^j \wedge \mathbf{x}^k T_{ijk} + T_\perp, \tag{44.142b}$$

where

$$B_{ij} = \mathbf{x}_j \cdot \left(\mathbf{x}_i \cdot B\right) = -B_{ji}. \tag{44.143a}$$

$$T_{ijk} = \mathbf{x}_k \cdot \left(\mathbf{x}_j \cdot \left(\mathbf{x}_i \cdot B\right)\right). \tag{44.143b}$$

For the trivector components are also antisymmetric, changing sign with any interchange of indices.

Note that eq. (44.141d) and eq. (44.141f) appear much different on the surface, but both have the same structure. This can be seen by writing for former as

$$\begin{aligned} d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge B) &= -d^3u \left(\partial_1 B_{23} + \partial_2 B_{31} + \partial_3 B_{12}\right) \\ &= -d^3u \left(\partial_3 B_{12} - \partial_2 B_{13} + \partial_1 B_{23}\right). \end{aligned} \tag{44.144}$$

In both of these we have an alternation of sign, where the tensor index skips one of the volume element indices is sequence. We've seen in the 4D divergence theorem that this alternation of sign can be related to a duality transformation.

In integral form (no sum over indexes i in du^i terms), these are

$$\int d^2\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -\epsilon^{ab} \int du^b f_b\big|_{\Delta u^a} \tag{44.145a}$$

$$\int d^3\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -\epsilon^{abc} \int du^a du^c\, \mathbf{x}_a f_c|_{\Delta u^b} \tag{44.145b}$$

$$\int d^4\mathbf{x} \cdot \left(\boldsymbol{\partial} \wedge \mathbf{f}\right) = -\frac{1}{2}\epsilon^{abcd} \int du^a du^b du^d\, \mathbf{x}_a \wedge \mathbf{x}_b f_d|_{\Delta u^c} \tag{44.145c}$$

$$\int d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge B) = -\frac{1}{2}\epsilon^{abc} \int du^b du^c\, B_{bc}|_{\Delta u^a} \tag{44.145d}$$

$$\int d^4\mathbf{x} \cdot (\boldsymbol{\partial} \wedge B) = -\frac{1}{2}\epsilon^{abcd} \int du^a du^c du^d\, \mathbf{x}_a B_{cd}|_{\Delta u^b} \tag{44.145e}$$

$$\int d^4\mathbf{x} \cdot (\boldsymbol{\partial} \wedge T) = -\int \left(du^1 du^2 du^3\, T_{123}|_{\Delta u^4} - du^1 du^2 du^4\, T_{124}|_{\Delta u^3} + du^1 du^3 du^4\, T_{134}|_{\Delta u^2} - du^2 du^3 du^4\, T_{234}|_{\Delta u^1} \right). \tag{44.145f}$$

Of these, I suspect that only eq. (44.145a) and eq. (44.145d) are of use.

44.9 FINAL REMARKS

Because we have used curvilinear coordinates from the get go, we have arrived naturally at a formulation that works for both Euclidean and non-Euclidean geometries, and have demonstrated that Stokes (and the divergence theorem) holds regardless of the geometry or the parameterization. We also know explicitly how to formulate both theorems for any parameterization that we choose, something much more valuable than knowledge that this is possible.

For the divergence theorem we have introduced the concept of outwards normal (for example in 3D, eq. (44.133)), which still holds for non-Euclidean geometries. We may not be able to form intuitive geometrical interpretations for these normals, but do have an algebraic description of them.

44.10 PROBLEMS

Exercise 44.1 Expand volume elements in coordinates

Show that the coordinate representation for the volume element dotted with the curl can be represented as a sum of antisymmetric terms. That is

1. Prove eq. (44.141a)
2. Prove eq. (44.141b)
3. Prove eq. (44.141c)
4. Prove eq. (44.141d)
5. Prove eq. (44.141e)
6. Prove eq. (44.141f)

Part V

GENERAL PHYSICS

45

ANGULAR VELOCITY AND ACCELERATION (AGAIN)

A more coherent derivation of angular velocity and acceleration than my initial attempt while first learning geometric algebra.

45.1 ANGULAR VELOCITY

The goal is to take first and second derivatives of a vector expressed radially:

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{45.1}$$

The velocity is the derivative of our position vector, which in terms of radial components is:

$$\mathbf{v} = \mathbf{r}' = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}'. \tag{45.2}$$

We can also calculate the projection and rejection of the velocity by multiplication by $1 = \hat{\mathbf{r}}^2$, and expanding this product in an alternate order taking advantage of the associativity of the geometric product:

$$\begin{aligned}\mathbf{v} &= \hat{\mathbf{r}}\hat{\mathbf{r}}\mathbf{v} \\ &= \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \mathbf{v} + \hat{\mathbf{r}} \wedge \mathbf{v}\right)\end{aligned} \tag{45.3}$$

Since $\hat{\mathbf{r}} \wedge (\hat{\mathbf{r}} \wedge \mathbf{v}) = 0$, the total velocity in terms of radial components is:

$$\mathbf{v} = \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \mathbf{v}\right) + \hat{\mathbf{r}} \cdot \left(\hat{\mathbf{r}} \wedge \mathbf{v}\right). \tag{45.4}$$

Here the first component above is the projection of the vector in the radial direction:

$$\mathrm{Proj}_{\mathbf{r}}(\mathbf{v}) = \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \mathbf{v}\right) \tag{45.5}$$

This projective term can also be rewritten in terms of magnitude:

$$\left(r^2\right)' = 2rr' = \left(\mathbf{r} \cdot \mathbf{r}\right)' = 2\mathbf{r} \cdot \mathbf{v}. \tag{45.6}$$

So the magnitude variation can be expressed the radial coordinate of the velocity:

$$r' = \hat{\mathbf{r}} \cdot \mathbf{v} \tag{45.7}$$

The remainder is the rejection of the radial component from the velocity, leaving just the part portion perpendicular to the radial direction.

$$\mathrm{Rej}_{\mathbf{r}}(\mathbf{v}) = \hat{\mathbf{r}} \cdot \left(\hat{\mathbf{r}} \wedge \mathbf{v}\right) \tag{45.8}$$

It is traditional to introduce an angular velocity vector normal to the plane of rotation that describes this rejective component using a triple cross product. With the formulation above, one can see that it is more natural to directly use an angular velocity bivector instead:

$$\boldsymbol{\Omega} = \frac{\mathbf{r} \wedge \mathbf{v}}{r^2} \tag{45.9}$$

This bivector encodes the angular velocity as a plane directly. The product of a vector with the bivector that contains it produces another vector in the plane. That product is a scaled and rotated by 90 degrees, much like the multiplication by a unit complex imaginary. That is no coincidence since the square of a bivector is negative and directly encodes this complex structure of an arbitrarily oriented plane.

Using this angular velocity bivector we have the following radial expression for velocity:

$$\mathbf{v} = \hat{\mathbf{r}} r' + \mathbf{r} \cdot \boldsymbol{\Omega}. \tag{45.10}$$

A little thought will show that $\hat{\mathbf{r}}'$ is also entirely perpendicular to $\hat{\mathbf{r}}$. The $\hat{\mathbf{r}}$ vector describes a path traced out on the unit sphere, and any variation of that vector must be tangential to the sphere. It is thus not surprising that we can also express $\hat{\mathbf{r}}'$ using the rejective term of equation eq. (45.4). Using the angular velocity bivector this is:

$$\hat{\mathbf{r}}' = \hat{\mathbf{r}} \cdot \boldsymbol{\Omega}. \tag{45.11}$$

This identity will be useful below for the calculation of angular acceleration.

45.2 ANGULAR ACCELERATION

Next we want the second derivatives of position

$$
\begin{aligned}
\mathbf{a} &= \mathbf{r}'' \\
&= r''\hat{\mathbf{r}} + 2r'\hat{\mathbf{r}}' + r\hat{\mathbf{r}}'' \\
&= r''\hat{\mathbf{r}} + \frac{1}{r}\left(r^2\hat{\mathbf{r}}'\right)'
\end{aligned}
\tag{45.12}
$$

This last step I found scribbled in a margin note in my old mechanics book. It is a trick that somebody clever once noticed and it simplifies this derivation to use it since it avoids the generation of a number of terms that will just cancel out anyways after more tedious manipulation (see examples section).

Expanding just this last derivative:

$$
\begin{aligned}
\left(r^2\hat{\mathbf{r}}'\right)' &= \left(r^2\hat{\mathbf{r}}\cdot\mathbf{\Omega}\right)' \\
&= \left(\hat{\mathbf{r}}\cdot\left(\mathbf{r}\wedge\mathbf{v}\right)\right)' \\
&= \left(\hat{\mathbf{r}}\cdot\left(\mathbf{r}\wedge\mathbf{v}\right)\right)' \\
&= \hat{\mathbf{r}}'\cdot\left(\mathbf{r}\wedge\mathbf{v}\right) + \hat{\mathbf{r}}\cdot(\overbrace{\boxed{\mathbf{v}\wedge\mathbf{v}}}^{=0}) + \hat{\mathbf{r}}\cdot\left(\mathbf{r}\wedge\mathbf{a}\right)
\end{aligned}
\tag{45.13}
$$

Thus the acceleration is:

$$
\mathbf{a} = r''\hat{\mathbf{r}} + \left(\mathbf{r}\cdot\mathbf{\Omega}\right)\cdot\mathbf{\Omega} + \hat{\mathbf{r}}\cdot\left(\hat{\mathbf{r}}\wedge\mathbf{a}\right)
$$

Note that the action of taking two successive dot products with the plane bivector $\mathbf{\Omega}$ just acts to rotate the vector by 180 degrees (as well as scale it).

One can verify this explicitly using grade selection operators. This allows the total acceleration to be expressed in the final form:

$$
\mathbf{a} = r''\hat{\mathbf{r}} + \mathbf{r}\mathbf{\Omega}^2 + \hat{\mathbf{r}}\cdot\left(\hat{\mathbf{r}}\wedge\mathbf{a}\right)
$$

Note that the squared bivector $\mathbf{\Omega}^2$ is a negative scalar, so the first two terms are radially directed. The last term is perpendicular to the acceleration, in the plane formed by the vector and its second derivative.

Given the acceleration, the force on a particle is thus:

$$\mathbf{F} = m\mathbf{a} = m\hat{\mathbf{r}}r'' + m\mathbf{r}\mathbf{\Omega}^2 + \frac{\mathbf{r}}{r^2}\left(\mathbf{r} \wedge \mathbf{p}\right)' \tag{45.14}$$

Writing the angular momentum as:

$$\mathbf{L} = \mathbf{r} \wedge \mathbf{p} = mr^2\mathbf{\Omega} \tag{45.15}$$

the force is thus:

$$\mathbf{F} = m\mathbf{a} = m\hat{\mathbf{r}}r'' + m\mathbf{r}\mathbf{\Omega}^2 + \frac{1}{\mathbf{r}} \cdot \frac{d\mathbf{L}}{dt} \tag{45.16}$$

The derivative of the angular momentum is called the torque $\boldsymbol{\tau}$, also a bivector:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{45.17}$$

When $\mathbf{r}$ is constant this has the radial arm times force form that we expect of torque:

$$\boldsymbol{\tau} = \mathbf{r} \wedge \frac{d\mathbf{p}}{dt} = \mathbf{r} \wedge \mathbf{F} \tag{45.18}$$

We can also write the equation of motion in terms of torque, in which case we have:

$$\mathbf{F} = m\hat{\mathbf{r}}r'' + m\mathbf{r}\Omega^2 + \frac{1}{\mathbf{r}} \cdot \boldsymbol{\tau} \tag{45.19}$$

As with all these plane quantities (angular velocity, momentum, acceleration), the torque as well is a bivector as it is natural to express this as a planar quantity. This makes more sense in many ways than a cross product, since all of these quantities should be perfectly well defined in a plane (or in spaces of degree greater than three), whereas the cross product is a strictly three dimensional entity.

45.3 EXPRESSING THESE USING TRADITIONAL FORM (CROSS PRODUCT)

To compare with traditional results to see if I got things right, remove the geometric algebra constructs (wedge products and bivector/vector products) in favor of cross products. Do this by using the duality relationships, multiplication by the three dimensional pseudoscalar $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, to convert bivectors to vectors and wedge products to cross and dot products ($\mathbf{u} \wedge \mathbf{v} = \mathbf{u} \times \mathbf{v} i$).

First define some vector quantities in terms of the corresponding bivectors:

$$\omega = \boldsymbol{\Omega}/i = \frac{\mathbf{r} \wedge \mathbf{v}}{r^2 i} = \frac{\mathbf{r} \times \mathbf{v}}{r^2} \tag{45.20}$$

$$\mathbf{r} \cdot \boldsymbol{\Omega} = \frac{1}{2}(\mathbf{r}\omega i - \omega i \mathbf{r}) = \mathbf{r} \wedge \omega i = \omega \times \mathbf{r} \tag{45.21}$$

Thus the velocity is:

$$\mathbf{v} = \hat{\mathbf{r}} r' + \omega \times \mathbf{r}. \tag{45.22}$$

In the same way, write the angular momentum vector as the dual of the angular momentum bivector:

$$\mathbf{l} = \mathbf{L}/i = \mathbf{r} \times \mathbf{p} = m r^2 \omega \tag{45.23}$$

And the torque vector $\mathbf{N}$ as the dual of the torque bivector $\boldsymbol{\tau}$

$$\mathbf{N} = \boldsymbol{\tau}/i = \frac{d\mathbf{l}}{dt} = \frac{d}{dt}\left(\mathbf{r} \times \mathbf{p}\right) \tag{45.24}$$

The equation of motion for a single particle then becomes:

$$\mathbf{F} = m\hat{\mathbf{r}} r'' - m\mathbf{r} \|\omega\|^2 + \mathbf{N} \times \frac{\mathbf{r}}{r^2} \tag{45.25}$$

45.4 EXAMPLES (PERHAPS FUTURE EXERCISES?)

45.4.1 *Unit vector derivative*

Demonstrate by direct calculation the result of eq. (45.11).

$$
\begin{aligned}
\hat{\mathbf{r}}' &= \left(\frac{\mathbf{r}}{r}\right)' \\
&= \frac{\mathbf{r}'}{r} - \frac{\mathbf{r}r'}{r^2} \\
&= \frac{1}{r}\left(\mathbf{v} - \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \cdot \mathbf{v}\right)\right) \\
&= \frac{\hat{\mathbf{r}}}{r}\left(\hat{\mathbf{r}}\mathbf{v} - \hat{\mathbf{r}} \cdot \mathbf{v}\right) \\
&= \frac{\hat{\mathbf{r}}}{r}\left(\hat{\mathbf{r}} \wedge \mathbf{v}\right)
\end{aligned}
\tag{45.26}
$$

45.4.2 *Direct calculation of acceleration*

It is more natural to calculate this acceleration directly by taking derivatives of eq. (45.10), but as noted above this is messier. Here is exactly that calculation for comparison.

Taking second derivatives of the velocity we have:

$$
\mathbf{v}' = \mathbf{a} = \left(\hat{\mathbf{r}}r' + \frac{\mathbf{r}}{r^2}\left(\mathbf{r} \wedge \mathbf{v}\right)\right)'
\tag{45.27}
$$

$$
\begin{aligned}
\mathbf{a} &= \hat{\mathbf{r}}'r' + \hat{\mathbf{r}}r'' + \frac{\mathbf{r}}{r^2}\underbrace{\boxed{\left(\mathbf{v} \wedge \mathbf{v}\right)}}_{=0} + \frac{\mathbf{r}}{r^2}\left(\mathbf{r} \wedge \mathbf{a}\right) + \left(\frac{\hat{\mathbf{r}}}{r}\right)'\left(\mathbf{r} \wedge \mathbf{v}\right) \\
&= \hat{\mathbf{r}}r'' + \hat{\mathbf{r}}'\left(r' + \frac{1}{r}\mathbf{r} \wedge \mathbf{v}\right) - r'\frac{\hat{\mathbf{r}}}{r^2}\left(\mathbf{r} \wedge \mathbf{v}\right) + \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \wedge \mathbf{a}\right) \\
&= \hat{\mathbf{r}}r'' + \frac{1}{r^3}\mathbf{r}\left(\mathbf{r} \wedge \mathbf{v}\right)\left(r' + \frac{1}{r}\mathbf{r} \wedge \mathbf{v}\right) - r'\frac{\hat{\mathbf{r}}}{r^2}\left(\mathbf{r} \wedge \mathbf{v}\right) + \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \wedge \mathbf{a}\right)
\end{aligned}
\tag{45.28}
$$

The r' terms cancel out, leaving just:

$$
\mathbf{a} = \hat{\mathbf{r}}r'' + \mathbf{r}\boldsymbol{\Omega}^2 + \hat{\mathbf{r}}\left(\hat{\mathbf{r}} \wedge \mathbf{a}\right)
\tag{45.29}
$$

45.4.3 *Expand the omega omega triple product*

$$
\begin{aligned}
(\mathbf{r} \cdot \boldsymbol{\Omega}) \cdot \boldsymbol{\Omega} &= \langle (\mathbf{r} \cdot \boldsymbol{\Omega})\, \boldsymbol{\Omega} \rangle_1 \\
&= \frac{1}{2} \left\langle \mathbf{r}\boldsymbol{\Omega}^2 - \boldsymbol{\Omega}\mathbf{r}\boldsymbol{\Omega} \right\rangle_1 \\
&= \frac{1}{2}\mathbf{r}\boldsymbol{\Omega}^2 - \frac{1}{2}\langle \boldsymbol{\Omega}\mathbf{r}\boldsymbol{\Omega} \rangle_1 \\
&= \frac{1}{2}\mathbf{r}\boldsymbol{\Omega}^2 + \frac{1}{2}\langle \mathbf{r}\boldsymbol{\Omega}\boldsymbol{\Omega} \rangle_1 \\
&= \frac{1}{2}\mathbf{r}\boldsymbol{\Omega}^2 + \frac{1}{2}\mathbf{r}\boldsymbol{\Omega}^2 \\
&= \mathbf{r}\boldsymbol{\Omega}^2
\end{aligned}
\tag{45.30}
$$

Also used above implicitly was the following:

$$
\mathbf{r}\boldsymbol{\Omega} = \mathbf{r} \cdot \boldsymbol{\Omega} + \overbrace{\boxed{\mathbf{r} \wedge \boldsymbol{\Omega}}}^{=0} = -\boldsymbol{\Omega} \cdot \mathbf{r} = -\boldsymbol{\Omega}\mathbf{r}
\tag{45.31}
$$

(ie: a vector anticommutes with a bivector describing a plane that contains it).

46

CROSS PRODUCT RADIAL DECOMPOSITION

We have seen how to use GA constructs to perform a radial decomposition of a velocity and acceleration vector. Is it that much harder to do this with straight vector algebra. This shows that the answer is no, but we need to at least assume some additional identities that can take work to separately prove. Here is a quick demonstration for comparision purposes how a radial decomposition can be performed entirely without any GA usage.

46.1 STARTING POINT

Starting point is taking derivatives of:

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{46.1}$$

$$\mathbf{v} = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}' \tag{46.2}$$

It can be shown without any Geometric Algebra use (see for example [38]) that the unit vector derivative can be expressed using the cross product:

$$\hat{\mathbf{r}}' = \frac{1}{r}\left(\hat{\mathbf{r}} \times \frac{d\mathbf{r}}{dt}\right) \times \hat{\mathbf{r}}. \tag{46.3}$$

Now, one can express r' in terms of $\mathbf{r}$ as well as follows:

$$(\mathbf{r} \cdot \mathbf{r})' = 2\mathbf{v} \cdot \mathbf{r} = 2rr'. \tag{46.4}$$

Thus the derivative of the vector magnitude is part of a projective term:

$$r' = \hat{\mathbf{r}} \cdot \mathbf{v}. \tag{46.5}$$

Putting this together one has velocity in terms of projective and rejective components along a radial direction:

$$\mathbf{v} = (\hat{\mathbf{r}} \cdot \mathbf{v})\,\hat{\mathbf{r}} + \left(\hat{\mathbf{r}} \times \frac{d\mathbf{r}}{dt}\right) \times \hat{\mathbf{r}}. \tag{46.6}$$

Now $\boldsymbol{\omega} = \frac{\mathbf{r}\times\mathbf{v}}{r^2}$ term is what we call the angular velocity. The magnitude of this is the rate of change of the angle between the radial arm and the direction of rotation. The direction of this cross product is normal to the plane of rotation and encodes both the rotational plane and the direction of the rotation. Putting these together one has the total velocity expressed radially:

$$\mathbf{v} = \left(\hat{\mathbf{r}} \cdot \mathbf{v}\right) \hat{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}. \tag{46.7}$$

46.2 ACCELERATION

Acceleration follows in the same fashion.

$$\begin{aligned}
\mathbf{v}' &= \Bigl(\overset{r'\hat{\mathbf{r}}}{\boxed{\left(\hat{\mathbf{r}} \cdot \mathbf{v}\right) \hat{\mathbf{r}}}} \Bigr)' + \Bigl(\overset{(\mathbf{r}\times\mathbf{v})\times\frac{\mathbf{r}}{r^2}}{\boxed{\boldsymbol{\omega}\times\mathbf{r}}} \Bigr)' \\
&= r''\hat{\mathbf{r}} + r'\frac{\boldsymbol{\omega}\times\mathbf{r}}{r} + (\hat{\mathbf{r}}\times\mathbf{a})\times\hat{\mathbf{r}} + \Bigl(\overset{=0}{\boxed{\mathbf{v}\times\mathbf{v}}} \Bigr) \times \frac{\mathbf{r}}{r^2} + (\mathbf{r}\times\mathbf{v})\times\left(\frac{\mathbf{r}}{r^2}\right)'
\end{aligned} \tag{46.8}$$

That last derivative is

$$\begin{aligned}
\left(\frac{\mathbf{r}}{r^2}\right)' &= \left(\frac{\hat{\mathbf{r}}}{r}\right)' \\
&= \frac{\hat{\mathbf{r}}'}{r} - \frac{\hat{\mathbf{r}}r'}{r^2} \\
&= \frac{\boldsymbol{\omega}\times\mathbf{r}}{r^2} - \frac{\hat{\mathbf{r}}r'}{r^2},
\end{aligned} \tag{46.9}$$

and back substitution gives:

$$\mathbf{v}' = r''\hat{\mathbf{r}} + r'\frac{\boldsymbol{\omega}\times\mathbf{r}}{r} + (\hat{\mathbf{r}}\times\mathbf{a})\times\hat{\mathbf{r}} + (\mathbf{r}\times\mathbf{v})\times\left(\frac{\boldsymbol{\omega}\times\mathbf{r}}{r^2} - \frac{\hat{\mathbf{r}}r'}{r^2}\right). \tag{46.10}$$

Canceling terms and collecting we have the final result for acceleration expressed radially:

$$\mathbf{v}' = \mathbf{a} = r''\hat{\mathbf{r}} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}\right) + (\hat{\mathbf{r}}\times\mathbf{a})\times\hat{\mathbf{r}} \tag{46.11}$$

Now, applying the angular velocity via cross product takes the vector back to the original plane, but inverts it. Thus we can write the acceleration completely in terms of the radially directed components, and the perpendicular component.

$$\mathbf{a} = r''\hat{\mathbf{r}} - \mathbf{r}\omega^2 + (\hat{\mathbf{r}} \times \mathbf{a}) \times \hat{\mathbf{r}} \tag{46.12}$$

An alternate way to express this is in terms of radial scalar acceleration:

$$\mathbf{a} \cdot \hat{\mathbf{r}} = r'' - r\omega^2. \tag{46.13}$$

This is the acceleration analogue of the scalar radial velocity component demonstrated above:

$$\mathbf{v} \cdot \hat{\mathbf{r}} = r'. \tag{46.14}$$

47 KINETIC ENERGY IN ROTATIONAL FRAME

47.1 MOTIVATION

Fill in the missing details of the rotational Kinetic Energy derivation in [42] and contrast matrix and GA approach.

Generalize acceleration in terms of rotating frame coordinates without unproved extrapolation that the z axis result of Tong's paper is good unconditionally (his cross products are kind of pulled out of a magic hat and this write up will show a couple ways to see where they come from).

Given coordinates for a point in a rotating frame $\mathbf{r}'$, the coordinate vector for that point in a rest frame is:

$$\mathbf{r} = R\mathbf{r}' \tag{47.1}$$

Where the rotating frame moves according to the following z-axis rotation matrix:

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{47.2}$$

To compute the Lagrangian we want to re-express the kinetic energy of a particle:

$$K = \frac{1}{2}m\dot{\mathbf{r}}^2 \tag{47.3}$$

in terms of the rotating frame coordinate system.

47.2 WITH MATRIX FORMULATION

The Tong paper does this for a z axis rotation with $\theta = \omega t$. Constant angular frequency is assumed.

First we calculate our position vector in terms of the rotational frame

$$\mathbf{r} = R\mathbf{r}' \tag{47.4}$$

The rest frame velocity is:

$$\dot{\mathbf{r}} = \dot{R}_\theta \mathbf{r}' + R_\theta \dot{\mathbf{r}'}. \tag{47.5}$$

Taking the matrix time derivative we have:

$$\dot{R}_\theta = -\dot{\theta} \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{47.6}$$

Taking magnitudes of the velocity we have three terms

$$\begin{aligned} \dot{\mathbf{r}}^2 &= (\dot{R}_\theta \mathbf{r}') \cdot (\dot{R}_\theta \mathbf{r}') + 2(\dot{R}_\theta \mathbf{r}') \cdot (R_\theta \dot{\mathbf{r}'}) + (R_\theta \dot{\mathbf{r}'}) \cdot (R_\theta \dot{\mathbf{r}'}) \\ &= \mathbf{r}'^{\mathrm{T}} \dot{R}_\theta^{\mathrm{T}} \dot{R}_\theta \mathbf{r}' + 2\mathbf{r}'^{\mathrm{T}} \dot{R}_\theta^{\mathrm{T}} R_\theta \dot{\mathbf{r}'} + \dot{\mathbf{r}'}^2 \end{aligned} \tag{47.7}$$

We need to calculate all the intermediate matrix products. The last was identity, and the first is:

$$\dot{R}_\theta^{\mathrm{T}} \dot{R}_\theta = \dot{\theta}^2 \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{47.8}$$

$$= \dot{\theta}^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{47.9}$$

This leaves just the mixed term

$$\dot{R}_\theta^{\mathrm{T}} R_\theta = -\dot{\theta} \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{47.10}$$

$$= -\dot{\theta} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{47.11}$$

With $\dot{\theta} = \omega$, the total magnitude of the velocity is thus

$$\dot{\mathbf{r}}^2 = \mathbf{r}'^{\mathrm{T}} \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{r}' - 2\omega \mathbf{r}'^{\mathrm{T}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{r}}' + \dot{\mathbf{r}}'^2 \tag{47.12}$$

Tong's paper presents this expanded out in terms of coordinates:

$$\dot{\mathbf{r}}^2 = \omega^2 \left(x'^2 + y'^2 \right) + 2\omega \left(x'\dot{y}' - y'\dot{x}' \right) + \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 \right) \tag{47.13}$$

Or,

$$\dot{\mathbf{r}}^2 = \left(-\omega y' + \dot{x}' \right)^2 + \left(\omega x' + \dot{y}' \right)^2 + \dot{z}'^2 \tag{47.14}$$

He also then goes on to show that this can be written, with $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$, as

$$\dot{\mathbf{r}}^2 = (\dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}')^2 \tag{47.15}$$

The implication here is that this is a valid result for any rotating coordinate system. How to prove this in the general rotation case, is shown much later in his treatment of rigid bodies.

47.3 WITH ROTOR

The equivalent to eq. (47.1) using a rotor is:

$$\mathbf{r}' = R^\dagger \mathbf{r} R \tag{47.16}$$

Where $R = \exp(i\theta/2)$.

Unlike the matrix formulation above we are free to pick any constant unit bivector for i if we want to generalize this to any rotational axis, but if we want an equivalent to the above rotation matrix we just have to take $i = \mathbf{e}_1 \wedge \mathbf{e}_2$.

We need a double sided inversion to get our unprimed vector:

$$\mathbf{r} = R\mathbf{r}'R^\dagger \tag{47.17}$$

and can then take derivatives:

$$\dot{\mathbf{r}} = \dot{R}\mathbf{r}'R^\dagger + R\mathbf{r}'\dot{R}^\dagger + R\dot{\mathbf{r}}'R^\dagger \tag{47.18}$$

$$= i\omega\frac{1}{2}R\mathbf{r}'R^\dagger - R\mathbf{r}'R^\dagger i\omega\frac{1}{2} + R\dot{\mathbf{r}}'R^\dagger \tag{47.19}$$

$$\implies \dot{\mathbf{r}} = \omega i \cdot (R\mathbf{r}'R^\dagger) + R\dot{\mathbf{r}}'R^\dagger \tag{47.20}$$

One can put this into the traditional cross product form by introducing a normal vector for the rotational axis in the usual way:

$$\boldsymbol{\Omega} = \omega i \tag{47.21}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega}/\mathbf{I}_3 \tag{47.22}$$

We can describe the angular velocity by a scaled normal vector ($\boldsymbol{\omega}$) to the rotational plane, or by a scaled bivector for the plane itself ($\boldsymbol{\Omega}$).

$$\begin{aligned}
\boldsymbol{\Omega} \cdot (R\mathbf{r}'R^\dagger) &= \left\langle \boldsymbol{\Omega} R\mathbf{r}'R^\dagger \right\rangle_1 \\
&= \left\langle R\boldsymbol{\Omega}\mathbf{r}'R^\dagger \right\rangle_1 \\
&= R\boldsymbol{\Omega} \cdot \mathbf{r}'R^\dagger \\
&= R(\boldsymbol{\omega}\mathbf{I}_3) \cdot \mathbf{r}'R^\dagger \\
&= R(\boldsymbol{\omega} \times \mathbf{r}')R^\dagger
\end{aligned} \tag{47.23}$$

Note that here as before this is valid only when the rotational plane orientation is constant (ie: no wobble), since only then can we assume i, and thus $\boldsymbol{\Omega}$ will commute with the rotor R.

Summarizing, we can write our velocity using rotational frame components as:

$$\dot{\mathbf{r}} = R\left(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{r}}'\right)R^\dagger \tag{47.24}$$

Or

$$\dot{\mathbf{r}} = R\left(\boldsymbol{\Omega} \cdot \mathbf{r}' + \dot{\mathbf{r}}'\right)R^\dagger \tag{47.25}$$

Using the result above from eq. (47.24), we can calculate the squared magnitude directly:

$$\begin{aligned}
\dot{\mathbf{r}}^2 &= \left\langle R\left(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{r}}'\right)R^\dagger R\left(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{r}}'\right)R^\dagger \right\rangle \\
&= \left\langle R(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{r}}')^2 R^\dagger \right\rangle \\
&= (\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{r}}')^2
\end{aligned} \tag{47.26}$$

We are able to go straight to the end result this way without the mess of sine and cosine terms in the rotation matrix. This is something that we can expand by components if desired:

$$
\begin{aligned}
\omega \times \mathbf{r}' + \dot{\mathbf{r}}' &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & \omega \\ x' & y' & z' \end{vmatrix} + \dot{\mathbf{r}}' \\
&= \begin{bmatrix} -\omega y' + \dot{x}' \\ \omega x' + \dot{y}' \\ \dot{z}' \end{bmatrix}
\end{aligned}
\tag{47.27}
$$

This verifies the second part of Tong's equation 2.19, also consistent with the derivation of eq. (47.14).

47.4 ACCELERATION IN ROTATING COORDINATES

Having calculated velocity in terms of rotational frame coordinates, acceleration is the next logical step.

The starting point is the velocity

$$
\dot{\mathbf{r}} = R(\boldsymbol{\Omega} \cdot \mathbf{r}' + \dot{\mathbf{r}}')R^\dagger \tag{47.28}
$$

Taking derivatives we have

$$
\ddot{\mathbf{r}} = i\omega/2\dot{\mathbf{r}} - \dot{\mathbf{r}}i\omega/2 + R\left(\dot{\boldsymbol{\Omega}} \cdot \mathbf{r}' + \boldsymbol{\Omega} \cdot \dot{\mathbf{r}}' + \ddot{\mathbf{r}}'\right)R^\dagger \tag{47.29}
$$

The first two terms are a bivector vector dot product and we can simplify this as follows

$$
\begin{aligned}
i\omega/2\dot{\mathbf{r}} - \dot{\mathbf{r}}i\omega/2 &= \boldsymbol{\Omega}/2\dot{\mathbf{r}} - \dot{\mathbf{r}}\boldsymbol{\Omega} \\
&= \boldsymbol{\Omega} \cdot \dot{\mathbf{r}} \\
&= \left\langle \boldsymbol{\Omega} R(\boldsymbol{\Omega} \cdot \mathbf{r}' + \dot{\mathbf{r}}')R^\dagger \right\rangle_1 \\
&= \left\langle R(\boldsymbol{\Omega}(\boldsymbol{\Omega} \cdot \mathbf{r}' + \dot{\mathbf{r}}'))R^\dagger \right\rangle_1 \\
&= R(\boldsymbol{\Omega} \cdot (\boldsymbol{\Omega} \cdot \mathbf{r}') + \boldsymbol{\Omega} \cdot \dot{\mathbf{r}}')R^\dagger
\end{aligned}
\tag{47.30}
$$

Thus the total acceleration is

$$
\ddot{\mathbf{r}} = R\left(\boldsymbol{\Omega} \cdot (\boldsymbol{\Omega} \cdot \mathbf{r}') + \dot{\boldsymbol{\Omega}} \cdot \mathbf{r}' + 2\boldsymbol{\Omega} \cdot \dot{\mathbf{r}}' + \ddot{\mathbf{r}}'\right)R^\dagger \tag{47.31}
$$

Or, in terms of cross products, and angular velocity and acceleration vectors $\boldsymbol{\omega}$, and $\boldsymbol{\alpha}$ respectively, this is

$$\ddot{\mathbf{r}} = R\left(\omega \times (\omega \times \mathbf{r}') + \alpha \times \mathbf{r}' + 2\omega \times \dot{\mathbf{r}}' + \ddot{\mathbf{r}}'\right) R^\dagger \tag{47.32}$$

47.5 ALLOW FOR A WOBBLE IN ROTATIONAL PLANE

A calculation similar to this can be found in GAFP, but for strictly rigid motion. It does not take too much to combine the two for a generalized result that expresses the total acceleration expressed in rotating frame coordinates, but also allowing for general rotation where the frame rotation and the angular velocity bivector do not have to be coplanar (ie: commute as above).

Since the primes and dots are kind of cumbersome switch to the GAFP notation where the position of a particle is expressed in terns of a rotational component $\mathbf{x}$ and origin translation $\mathbf{x}_0$:

$$\mathbf{y} = R\mathbf{x}R^\dagger + \mathbf{x}_0 \tag{47.33}$$

Taking derivatives for velocity

$$\dot{\mathbf{y}} = \dot{R}\mathbf{x}R^\dagger + R\mathbf{x}\dot{R}^\dagger + R\dot{\mathbf{x}}R^\dagger + \dot{\mathbf{x}}_0 \tag{47.34}$$

Now use the same observation that the derivative of $RR^\dagger = 1$ is zero:

$$\frac{d(RR^\dagger)}{dt} = \dot{R}R^\dagger + R\dot{R}^\dagger = 0$$

$$\implies \dot{R}R^\dagger = -R\dot{R}^\dagger = -\left(\dot{R}R^\dagger\right)^\dagger \tag{47.35}$$

Since R has only grade 0 and 2 terms, so does its derivative. Thus the product of the two has grade 0, 2, and 4 terms, but eq. (47.35) shows that the product $\dot{R}R^\dagger$ has a value that is the negative of its reverse, so it must have only grade 2 terms (the reverse of the grade 0 and 4 terms would not change sign).

As in eq. (47.20) we want to write $\dot{R}$ as a bivector/rotor product and eq. (47.35) gives us a means to do so. This would have been clearer in GAFP if they had done the simple example first with the orientation of the rotational plane fixed.

So, write:

$$\dot{R}R^\dagger = \frac{1}{2}\mathbf{\Omega} \tag{47.36}$$

$$\dot{R} = \frac{1}{2}\mathbf{\Omega}R \tag{47.37}$$

$$\dot{R}^\dagger = -\frac{1}{2}R^\dagger\mathbf{\Omega} \tag{47.38}$$

(including the $1/2$ here is a bit of a cheat ... it is here because having done the calculation on paper first one sees that it is natural to do so).

With this we can substitute back into eq. (47.34), writing $\mathbf{y}_0 = \mathbf{y} - \mathbf{x}_0$:

$$\begin{aligned}
\dot{\mathbf{y}} &= \frac{1}{2}\mathbf{\Omega}R\mathbf{x}R^\dagger - \frac{1}{2}R\mathbf{x}R^\dagger\mathbf{\Omega} + R\dot{\mathbf{x}}R^\dagger + \dot{\mathbf{x}}_0 \\
&= \frac{1}{2}\left(\mathbf{\Omega}\mathbf{y}_{-}\mathbf{y}_0\mathbf{\Omega}\right) + R\dot{\mathbf{x}}R^\dagger + \dot{\mathbf{x}}_0 \\
&= \mathbf{\Omega}\cdot\mathbf{y}_0 + R\dot{\mathbf{x}}R^\dagger + \dot{\mathbf{x}}_0
\end{aligned} \tag{47.39}$$

We also want to pull in this $\mathbf{\Omega}$ into the rotor as in the fixed orientation case, but cannot use commutativity this time since the rotor and angular velocity bivector are not necessarily in the same plane.

This is where GAFP introduces their body angular velocity, which applies an inverse rotation to the angular velocity.

Let:

$$\mathbf{\Omega} = R\mathbf{\Omega}_B R^\dagger \tag{47.40}$$

Computing this bivector dot product with $\mathbf{y}$ we have

$$\begin{aligned}
\mathbf{\Omega}\cdot\mathbf{y}_0 &= (R\mathbf{\Omega}_B R^\dagger)\cdot(R\mathbf{x}R^\dagger) \\
&= \left\langle R\mathbf{\Omega}_B R^\dagger R\mathbf{x}R^\dagger\right\rangle_1 \\
&= \left\langle R\mathbf{\Omega}_B\mathbf{x}R^\dagger\right\rangle_1 \\
&= \left\langle R(\mathbf{\Omega}_B\cdot\mathbf{x} + \mathbf{\Omega}_B\wedge\mathbf{x})R^\dagger\right\rangle_1 \\
&= R\mathbf{\Omega}_B\cdot\mathbf{x}R^\dagger
\end{aligned} \tag{47.41}$$

Thus the total velocity is:

$$\dot{\mathbf{y}} = R(\mathbf{\Omega}_B\cdot\mathbf{x} + \dot{\mathbf{x}})R^\dagger + \dot{\mathbf{x}}_0 \tag{47.42}$$

Thus given any vector $\mathbf{x}$ in the rotating frame coordinate system, we have the relationship for the inertial frame velocity. We can apply this a second time to compute the inertial (rest frame) acceleration in terms of rotating coordinates. Write $\mathbf{v} = \mathbf{\Omega}_B \cdot \mathbf{x} + \dot{\mathbf{x}}$,

$$\begin{aligned}
\dot{\mathbf{y}} &= R\mathbf{v}R^\dagger + \dot{\mathbf{x}}_0 \\
\implies \ddot{\mathbf{y}} &= R(\mathbf{\Omega}_B \cdot \mathbf{v} + \dot{\mathbf{v}})R^\dagger + \ddot{\mathbf{x}}_0
\end{aligned}$$

$$\dot{\mathbf{v}} = \dot{\mathbf{\Omega}}_B \cdot \mathbf{x} + \mathbf{\Omega}_B \cdot \dot{\mathbf{x}} + \ddot{\mathbf{x}} \tag{47.43}$$

Combining these we have:

$$\ddot{\mathbf{y}} = R(\mathbf{\Omega}_B \cdot (\mathbf{\Omega}_B \cdot \mathbf{x} + \dot{\mathbf{x}}) + \dot{\mathbf{\Omega}}_B \cdot \mathbf{x} + \mathbf{\Omega}_B \cdot \dot{\mathbf{x}} + \ddot{\mathbf{x}})R^\dagger + \ddot{\mathbf{x}}_0 \tag{47.44}$$

$$\implies \ddot{\mathbf{y}} = R(\mathbf{\Omega}_B \cdot (\mathbf{\Omega}_B \cdot \mathbf{x}) + \dot{\mathbf{\Omega}}_B \cdot \mathbf{x} + 2\mathbf{\Omega}_B \cdot \dot{\mathbf{x}} + \ddot{\mathbf{x}})R^\dagger + \ddot{\mathbf{x}}_0 \tag{47.45}$$

This generalizes eq. (47.32), providing the rest frame acceleration in terms of rotational frame coordinates, with centrifugal acceleration, Euler force acceleration, and Coriolis force acceleration terms that accompany the plain old acceleration term $\ddot{\mathbf{x}}$. The only requirement for the generality of allowing the orientation of the rotational plane to potentially vary is the use of the "body angular velocity" $\mathbf{\Omega}_B$, replacing the angular velocity as seen from the rest frame $\mathbf{\Omega}$.

47.5.1 *Body angular acceleration in terms of rest frame*

Since we know the relationship between the body angular velocity $\mathbf{\Omega}_B$ with the Rotor (rest frame) angular velocity bivector, for completeness, lets compute the body angular acceleration bivector $\dot{\mathbf{\Omega}}_B$ in terms of the rest frame angular acceleration $\dot{\mathbf{\Omega}}$.

$$\mathbf{\Omega}_B = R^\dagger \mathbf{\Omega} R \tag{47.46}$$

$$\begin{aligned}
\implies \dot{\mathbf{\Omega}}_B &= \dot{R}^\dagger \mathbf{\Omega} R + R^\dagger \dot{\mathbf{\Omega}} R + R^\dagger \mathbf{\Omega} \dot{R} \\
&= -\frac{1}{2} R^\dagger \mathbf{\Omega}^2 R + R^\dagger \dot{\mathbf{\Omega}} R + R^\dagger \mathbf{\Omega}^2 R \frac{1}{2} \\
&= \frac{1}{2}\left(R^\dagger \mathbf{\Omega}^2 R - R^\dagger \mathbf{\Omega}^2 R\right) + R^\dagger \dot{\mathbf{\Omega}} R \\
&= R^\dagger \dot{\mathbf{\Omega}} R
\end{aligned} \tag{47.47}$$

This shows that the body angular acceleration is just an inverse rotation of the rest frame angular acceleration like the angular velocities are.

47.6 REVISIT GENERAL ROTATION USING MATRICES

Having fully calculated velocity and acceleration in terms of rotating frame coordinates, lets go back and revisit this with matrices and see how one would do the same for a general rotation.

Following GAFP express the rest frame coordinates for a point **y** in terms of a rotation applied to a rotating frame position **x** (this is easier than the mess of primes and dots used in Tong's paper). Also omit the origin translation (that can be added in later if desired easily enough)

$$\mathbf{y} = R\mathbf{x} \tag{47.48}$$

Thus the derivative is:

$$\dot{\mathbf{y}} = \dot{R}\mathbf{x} + R\dot{\mathbf{x}}. \tag{47.49}$$

As in the GA case we want to factor this so that we have a rotation applied to a something that is completely specified in the rotating frame. This is quite easy with matrices, as we just have to factor out a rotation matrix from $\dot{R}$:

$$\begin{aligned}\dot{\mathbf{y}} &= RR^{\mathrm{T}}\dot{R}\mathbf{x} + R\dot{\mathbf{x}} \\ &= R\left(R^{\mathrm{T}}\dot{R}\mathbf{x} + \dot{\mathbf{x}}\right)\end{aligned} \tag{47.50}$$

This new product $R^{\mathrm{T}}\dot{R}\mathbf{x}$ we have seen above in the special case of z-axis rotation as a cross product. In the GA general rotation case, we have seen that this as a bivector-vector dot product. Both of these are fundamentally antisymmetric operations, so we expect this of the matrix operator too. Verification of this antisymmetry follows in almost the same fashion as the GA case, by observing that the derivative of an identity matrix $I = R^{\mathrm{T}}R$ is zero:

$$\dot{I} = 0 \tag{47.51}$$

$$\implies \dot{R}^{\mathrm{T}}R + R^{\mathrm{T}}\dot{R} = 0 \tag{47.52}$$

$$\implies R^{\mathrm{T}}\dot{R} = -\dot{R}^{\mathrm{T}}R = -R^{\mathrm{T}}\dot{R}^{\mathrm{T}} \tag{47.53}$$

Thus if one writes:

$$\mathbf{\Omega} = R^{\mathrm{T}}\dot{R} \tag{47.54}$$

the antisymmetric property of this matrix can be summarized as:

$$\mathbf{\Omega} = -\mathbf{\Omega}^{\mathrm{T}}. \tag{47.55}$$

Let us write out the form of this matrix in the first few dimensions:

- $\mathbb{R}^2$

  $$\mathbf{\Omega} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \tag{47.56}$$

  For some a.

- $\mathbb{R}^3$

  $$\mathbf{\Omega} = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \tag{47.57}$$

  For some a, b, c.

- $\mathbb{R}^4$

  $$\mathbf{\Omega} = \begin{bmatrix} 0 & -a & -b & -d \\ a & 0 & -c & -e \\ b & c & 0 & -f \\ d & e & f & 0 \end{bmatrix} \tag{47.58}$$

  For some a, b, c, d, e, f.

For $\mathbb{R}^N$ we have $(N^2 - N)/2$ degrees of freedom. It is noteworthy to observe that this is exactly the number of basis elements of a bivector. For example, in $\mathbb{R}^4$, such a bivector basis is $\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}$.

For $\mathbb{R}^3$ we have three degrees of freedom and because of the antisymmetry can express this matrix-vector product using the cross product. Let

$$(a, b, c) = (\omega_3, -\omega_2, \omega_1) \tag{47.59}$$

One has:

$$\mathbf{\Omega x} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\omega_3 x_2 + \omega_2 x_3 \\ +\omega_3 x_1 - \omega_1 x_3 \\ -\omega_2 x_1 + \omega_1 x_2 \end{bmatrix} = \boldsymbol{\omega} \times \mathbf{x} \tag{47.60}$$

Summarizing the velocity result we have, using $\mathbf{\Omega}$ from eq. (47.54):

$$\dot{\mathbf{y}} = R\left(\mathbf{\Omega x} + \dot{\mathbf{x}}\right) \tag{47.61}$$

Or, for $\mathbb{R}^3$, we can define a body angular velocity vector

$$\omega = \begin{bmatrix} \mathbf{\Omega}_{32} \\ \mathbf{\Omega}_{13} \\ \mathbf{\Omega}_{21} \end{bmatrix} \tag{47.62}$$

and thus write the velocity as:

$$\dot{\mathbf{y}} = R\left(\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}\right) \tag{47.63}$$

This, like the GA result is good for general rotations. Then do not have to be constant rotation rates, and it allows for arbitrarily oriented as well as wobbly motion of the rotating frame.

As with the GA general velocity calculation, this general form also allows us to calculate the squared velocity easily, since the rotation matrices will cancel after transposition:

$$\dot{\mathbf{y}}^2 = \left(R\left(\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}\right)\right) \cdot \left(R\left(\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}\right)\right) = \left(\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}\right)^{\mathrm{T}} R^{\mathrm{T}} R\left(\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}\right) \tag{47.64}$$

$$\implies \dot{\mathbf{y}}^2 = \left(\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}\right)^2 \tag{47.65}$$

47.7 EQUATIONS OF MOTION FROM LAGRANGE PARTIALS

TBD. Do this using the Rotor formulation. How?

48

POLAR VELOCITY AND ACCELERATION

48.1 MOTIVATION

Have previously worked out the radial velocity and acceleration components a pile of different ways in 45, 46, 9, 47, 8, and 7.

So, what is a couple more?

When the motion is strictly restricted to a plane we can get away with doing this either in complex numbers (used in a number of the Tong Lagrangian solutions), or with a polar form $\mathbb{R}^2$ vector (a polar representation I have not seen since High School).

48.2 WITH COMPLEX NUMBERS

Let

$$z = re^{i\theta} \tag{48.1}$$

So our velocity is

$$\dot{z} = \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta} \tag{48.2}$$

and the acceleration is

$$\begin{aligned}\ddot{z} &= \ddot{r}e^{i\theta} + i\dot{r}\dot{\theta}e^{i\theta} + i\dot{r}\dot{\theta}e^{i\theta} + ir\ddot{\theta}e^{i\theta} - r\dot{\theta}^2 e^{i\theta} \\ &= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (2\dot{r}\dot{\theta} + r\ddot{\theta})ie^{i\theta}\end{aligned} \tag{48.3}$$

48.3 PLANE VECTOR REPRESENTATION

Also can do this with polar vector representation directly (without involving the complexity of rotation matrices or anything fancy)

$$\mathbf{r} = r\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \tag{48.4}$$

Velocity is then

$$\mathbf{v} = \dot{r}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} + r\dot{\theta}\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} \tag{48.5}$$

and for acceleration we have

$$\begin{aligned}\mathbf{a} &= \ddot{r}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} + \dot{r}\dot{\theta}\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} + \dot{r}\dot{\theta}\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} + r\ddot{\theta}\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} - r\dot{\theta}^2\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \\ &= (\ddot{r} - r\dot{\theta}^2)\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}\end{aligned} \tag{48.6}$$

49

GRADIENT AND TENSOR NOTES

49.1 MOTIVATION

Some notes on tensors filling in assumed details covered in [10].

Conclude with the solution of problem 6.1 to demonstrate the frame independence of the vector derivative.

Despite being notes associated with a Geometric Algebra text, there is no GA content. Outside of the eventual GA application of the gradient as described in this form, the only GA connection is the fact that the the reciprocal frame vectors can be thought of as a result of a duality calculation. That connection is not necessary though since one can just as easily define the reciprocal frame in terms of matrix operations. As an example, for a Euclidean metric the reciprocal frame vectors are the columns of $F(F^{\mathrm{T}}F)^{-1}$ where the columns of F are the vectors in question.

These notes may not stand well on their own without the text, at least as learning material.

49.1.1 *Raised and lowered indices. Coordinates of vectors with non-orthonormal frames*

Let $\{e_i\}$ represent a frame of not necessarily orthonormal basis vectors for a metric space, and $\{e^i\}$ represent the reciprocal frame.

The reciprocal frame vectors are defined by the relation:

$$e_i \cdot e^j = {\delta_i}^j. \tag{49.1}$$

Lets compute the coordinates of a vector x in terms of both frames:

$$x = \sum \alpha_j e_j = \sum \beta_j e^j \tag{49.2}$$

Forming $x \cdot e^i$, and $x \cdot e_i$ respectively solves for the α, and β coefficients

$$x \cdot e^i = \sum \alpha_j e_j \cdot e^i = \sum \alpha_j {\delta_j}^i = \alpha_i \tag{49.3}$$

$$x \cdot e_i = \sum \beta_j e^j \cdot e_i = \sum \beta_j {\delta_i}^j = \beta_i \tag{49.4}$$

Thus, the reciprocal frame vectors allow for simple determination of coordinates for an arbitrary frame. We can summarize this as follows:

$$x = \sum (x \cdot e^i) e_i = \sum (x \cdot e_i) e^i \tag{49.5}$$

Now, for orthonormal frames, where $e_i = e^i$ we are used to writing:

$$x = \sum x_i e_i, \tag{49.6}$$

however for non-orthonormal frames the convention is to mix raised and lowered indices as follows:

$$x = \sum x^i e_i = \sum x_i e^i. \tag{49.7}$$

Where, as demonstrated above these generalized coordinates have the values, $x^i = x \cdot e^i$, and $x_i = x \cdot e_i$. This is a strange seeming notation at first especially since most of linear algebra is done with always lowered (or always upper for some authors) indices. However one quickly gets used to it, especially after seeing how powerful the reciprocal frame concept is for dealing with non-orthonormal frames. The alternative is probably the use of matrices and their inverses to express the same vector decompositions.

49.1.2 *Metric tensor*

It is customary in tensor formulations of physics to utilize a metric tensor to express the dot product.

Compute the dot product using the coordinate vectors

$$x \cdot y = \left(\sum x^i e_i\right)\left(\sum y^j e_j\right) = \sum x^i y^j \left(e_i \cdot e_j\right) \tag{49.8}$$

$$x \cdot y = \left(\sum x_i e^i\right)\left(\sum y_j e^j\right) = \sum x_i y_j \left(e^i \cdot e^j\right) \tag{49.9}$$

Introducing second rank (symmetric) tensors for the dot product pairs $e_i \cdot e_j = g_{ij}$, and $g^{ij} = e^i \cdot e^j$ we have

$$x \cdot y = \sum x_i y_j g^{ij} = \sum x^i y^j g_{ij} = \sum x_i y^i = \sum x^i y_i \tag{49.10}$$

We see that the metric tensor provides a way to specify the dot product in index notation, and removes the explicit references to the original frame vectors. Mixed indices also removes the references to the original frame vectors, but additionally eliminates the need for either of the metric tensors.

Note that it is also common to see Einstein summation convention employed, which omits the $\sum$:

$$x \cdot y = x_i y_j g^{ij} = x^i y^j g_{ij} = x^i y_i = x_i y^i \tag{49.11}$$

Here summation over all matched upper, lower index pairs is implied.

49.1.3 *Metric tensor relations to coordinates*

Given a coordinate expression of a vector, we dot that with the frame vectors to observe the relation between coordinates and the metric tensor:

$$x \cdot e_i = \sum x^j e_j \cdot e_i = \sum x^j g_{ij} \tag{49.12}$$

$$x \cdot e^i = \sum x_j e^j \cdot e^i = \sum x_j g^{ij} \tag{49.13}$$

The metric tensors can therefore be used be used to express the relations between the upper and lower index coordinates:

$$x_i = \sum g_{ij} x^j \tag{49.14}$$
$$x^i = \sum g^{ij} x_j \tag{49.15}$$

It is therefore apparent that the matrix of the index lowered metric tensor g_{ij} is the inverse of the matrix for the raised index metric tensor g^{ij}.

Expressed more exactly,

$$\begin{aligned} x_i &= \sum_{ij} g_{ij} x^j \\ &= \sum_{ijk} g_{ij} g^{jk} x_k \\ &= \sum_{ik} x_k \sum_j g_{ij} g^{jk} \end{aligned} \tag{49.16}$$

Since the left and right hand sides are equal for any x_i, x_k, we have:

$$\delta_i{}^j = \sum_m g_{im} g^{mj} \tag{49.17}$$

Demonstration of the inverse property required for summation on other set of indices too for completeness, but since these functions are symmetric, there is no potential that this would have a “left” or “right” inverse type of action.

49.1.4 *Metric tensor as a Jacobian*

The relations of equations eq. (49.14), and eq. (49.15) show that the metric tensor can be expressed in terms of partial derivatives:

$$\begin{aligned} \frac{\partial x_i}{\partial x^j} &= g_{ij} \\ \frac{\partial x^i}{\partial x_j} &= g^{ij} \end{aligned} \tag{49.18}$$

Therefore the metric tensors can also be expressed as Jacobian matrices (not Jacobian determinants) :

$$\begin{aligned} g_{ij} &= \frac{\partial(x_1, \cdots, x_n)}{\partial(x^1, \cdots, x^n)} \\ g^{ij} &= \frac{\partial(x^1, \cdots, x^n)}{\partial(x_1, \cdots, x_n)} \end{aligned} \tag{49.19}$$

Will this be useful in any way?

49.1.5 *Change of basis*

To perform a change of basis from one non-orthonormal basis $\{e_i\}$ to a second $\{f_i\}$, relations between the sets of vectors are required. Using Greek indices for the f frame, and English for the e frame, those are:

$$\begin{aligned} e_i &= \sum f^\mu e_i \cdot f_\mu = \sum f_\mu e_i \cdot f^\mu \\ f_\alpha &= \sum e^k f_\alpha \cdot e_k = \sum e_k f_\alpha \cdot e^k \\ e^i &= \sum f^\mu e^i \cdot f_\mu = \sum f_\mu e^i \cdot f^\mu \\ f^\alpha &= \sum e^k f^\alpha \cdot e_k = \sum e_k f^\alpha \cdot e^k \end{aligned} \tag{49.20}$$

Following GAFP we can write the dot product terms as a second order tensors f (ie: matrix relation) :

$$
\begin{aligned}
e_i &= \sum f^\mu f_{i\mu} = \sum f_\mu f_i^\mu \\
f_\alpha &= \sum e^k f_{k\alpha} = \sum e_k f^k{}_\alpha \\
e^i &= \sum f^\mu f^i{}_\mu = \sum f_\mu f^{i\mu} \\
f^\alpha &= \sum e^k f_k{}^\alpha = \sum e_k f^{k\alpha}
\end{aligned}
\tag{49.21}
$$

Note that all these various tensors are related to each other using the metric tensors for f and e. FIXME: show example. Also note that using this notation the metric tensors g_{ij} and $g_{\alpha\beta}$ are two completely different linear functions, and careful use of the index conventions are required to keep these straight.

49.1.6 *Inverse relationships*

Looking at these relations in pairs, such as

$$
\begin{aligned}
f_\alpha &= \sum e^k f_{k\alpha} \\
e^i &= \sum f_\mu f^{i\mu}
\end{aligned}
\tag{49.22}
$$

and

$$
\begin{aligned}
e_i &= \sum f^\mu f_{i\mu} \\
f^\alpha &= \sum e_k f^{k\alpha}
\end{aligned}
\tag{49.23}
$$

It is clear that $f_{i\alpha}$ is the inverse of $f^{i\alpha}$.
To be more precise

$$
\begin{aligned}
f_\alpha &= \sum e^k f_{k\alpha} \\
&= \sum f_\mu f^{k\mu} f_{k\alpha}
\end{aligned}
\tag{49.24}
$$

Thus

$$
\sum_k f^{k\beta} f_{k\alpha} = \delta_\alpha^\beta
\tag{49.25}
$$

To verify both "left" and "right" inverse properties we also need:

$$\begin{aligned} e^i &= \sum f_\mu f^{i\mu} \\ &= \sum e^k f_{k\mu} f^{i\mu} \end{aligned} \tag{49.26}$$

which shows that summation on the Greek indices also yields an inverse:

$$\sum_\mu f_{i\mu} f^{j\mu} = \delta_i^j \tag{49.27}$$

There are also inverse relationships for the mixed index tensors above. Specifically,

$$\begin{aligned} x^\alpha &= \sum x^i f_i^\alpha \\ &= \sum x^\beta f_\beta^i f_i^\alpha \end{aligned} \tag{49.28}$$

Thus,

$$\sum_i f_\beta^i f_i^\alpha = \delta_\beta^\alpha \tag{49.29}$$

And,

$$\begin{aligned} x^i &= \sum x^\beta f_\beta^i \\ &= \sum x^j f_j^\beta f_\beta^i \end{aligned} \tag{49.30}$$

Thus,

$$\sum_\alpha f_j^\alpha f_\alpha^i = \delta_j^i \tag{49.31}$$

This completely demonstrates the inverse relationship.

49.1.7 *Vector derivative*

GAFP exercise 6.1. Show that the vector derivative:

$$\nabla = \sum e^i \frac{\partial}{\partial x^i} \tag{49.32}$$

is not frame dependent.

To show this we will need to utilize the chain rule to rewrite the partials in terms of the alternate frame:

$$\frac{\partial}{\partial x^i} = \sum \frac{\partial x^\alpha}{\partial x^i} \frac{\partial}{\partial x^\alpha} \tag{49.33}$$

To evaluate the first partial here, we write the coordinates of a vector in terms of both, and take dot products:

$$\left(\sum x^\gamma f_\gamma\right) \cdot f^\alpha = \left(\sum x^i e_i\right) \cdot f^\alpha \tag{49.34}$$

$$x^\alpha = \sum x^i f_i^\alpha \tag{49.35}$$

$$\frac{\partial x^\alpha}{\partial x^i} = f_i^\alpha \tag{49.36}$$

Similar expressions for the other change of basis tensors is also possible, but not required for this problem.

With this result we have the partial re-expressed in terms of coordinates in the new frame.

$$\frac{\partial}{\partial x^i} = \sum f_i^\alpha \frac{\partial}{\partial x^\alpha} \tag{49.37}$$

Combine this with the alternate contra-variant frame vector as calculated above:

$$e^i = \sum f^\mu f^i{}_\mu \tag{49.38}$$

and we have:

$$\begin{aligned}
\sum_i e^i \frac{\partial}{\partial x^i} &= \sum_i \left(\sum_\mu f^\mu f^i{}_\mu\right)\left(\sum_\alpha f_i^\alpha \frac{\partial}{\partial x^\alpha}\right) \\
&= \sum_{\mu\alpha} \left(f^\mu \frac{\partial}{\partial x^\alpha}\right) \sum_i f^i{}_\mu f_i^\alpha \\
&= \sum_{\mu\alpha} \left(f^\mu \frac{\partial}{\partial x^\alpha}\right) \delta_\mu{}^\alpha \\
&= \sum_\alpha f^\alpha \frac{\partial}{\partial x^\alpha}
\end{aligned} \tag{49.39}$$

49.1.8 *Why a preference for index upper vector and coordinates in the gradient?*

We can express the gradient in terms of index lower variables and vectors too, as follows:

$$\frac{\partial}{\partial x^i} = \sum \frac{\partial x_j}{\partial x^i} \frac{\partial}{\partial x_j}$$

Employing the coordinate relations we have:

$$\sum x_i e^i \cdot e_j = x_j = \sum x^i e_i \cdot e_j = \sum x^i g_{ij} \tag{49.40}$$

and can thus calculate the partials:

$$\frac{\partial x_j}{\partial x^i} = g_{ij}, \tag{49.41}$$

and can use that to do the change of variables to index lower coordinates:

$$\frac{\partial}{\partial x^i} = \sum g_{ij} \frac{\partial}{\partial x_j}$$

Now we also can write the reciprocal frame vectors:

$$e^i = \sum e_j e^i \cdot e^j = \sum e_j g^{ij}$$

Thus the gradient is:

$$\begin{aligned}
\sum_i e^i \frac{\partial}{\partial x^i} &= \sum_{ijk} e_j g^{ij} g_{ik} \frac{\partial}{\partial x_k} \\
&= \sum_{jk} \left(e_j \frac{\partial}{\partial x_k} \right) \sum_i g^{ij} g_{ik} \\
&= \sum_{jk} \left(e_j \frac{\partial}{\partial x_k} \right) \delta^j_k \\
&= \sum_i e_i \frac{\partial}{\partial x_i}
\end{aligned} \tag{49.42}$$

My conclusion is that there is not any preference for the index upper form of the gradient in GAFP. Both should be equivalent. That said consistency is likely required. FIXME: To truly get a feel for why index upper is used in this definition one likely needs to step back and look at the defining directional derivative relation for the gradient.

50

INERTIAL TENSOR

[10] derives the angular momentum for rotational motion in the following form

$$L = R\left(\int \mathbf{x} \wedge (\mathbf{x} \cdot \Omega_B) dm\right) R^\dagger \tag{50.1}$$

and calls the integral part, the inertia tensor

$$I(B) = \int \mathbf{x} \wedge (\mathbf{x} \cdot \Omega_B) dm \tag{50.2}$$

which is a linear mapping from bivectors to bivectors. To understand the form of this I found it helpful to expanding the wedge product part of this explicitly for the $\mathbb{R}^3$ case.

Ignoring the sum in this expansion write

$$f(B) = \mathbf{x} \wedge (\mathbf{x} \cdot B) \tag{50.3}$$

And writing $\mathbf{e}_{ij} = \mathbf{e}_i \mathbf{e}_j$ introduce a basis

$$b = \{\mathbf{e}_1 I, \mathbf{e}_2 I, \mathbf{e}_3 I\} = \{\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\} \tag{50.4}$$

for the $\mathbb{R}^3$ bivector product space.

Now calculate $f(B)$ for each of the basis vectors

$$\begin{aligned} f(\mathbf{e}_1 I) &= \mathbf{x} \wedge (\mathbf{x} \cdot \mathbf{e}_{23}) \\ &= (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3) \wedge (x_2 \mathbf{e}_3 - x_3 \mathbf{e}_2) \end{aligned} \tag{50.5}$$

Completing this calculation for each of the unit basic bivectors, we have

$$\begin{aligned} f(\mathbf{e}_1 I) &= (x_2^2 + x_3^2)\mathbf{e}_{23} - (x_1 x_2)\mathbf{e}_{31} - (x_1 x_3)\mathbf{e}_{12} \\ f(\mathbf{e}_2 I) &= -(x_1 x_2)\mathbf{e}_{23} + (x_1^2 + x_3^2)\mathbf{e}_{31} - (x_2 x_3)\mathbf{e}_{12} \\ f(\mathbf{e}_3 I) &= -(x_1 x_3)\mathbf{e}_{23} - (x_2 x_3)\mathbf{e}_{31} + (x_1^2 + x_2^2)\mathbf{e}_{12} \end{aligned} \tag{50.6}$$

Observe that taking dot products with $(\mathbf{e}_i I)^\dagger$ will select just the $\mathbf{e}_i I$ term of the result, so one can form the matrix of this linear transformation that maps bivectors in basis b to image vectors also in basis b as follows

$$\left[I(B)\right]_b^b = \left[I(\mathbf{e}_i I) \cdot (\mathbf{e}_j I)^\dagger\right]_{ij} = \int \begin{bmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_1 x_2 & x_1^2 + x_3^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & x_1^2 + x_2^2 \end{bmatrix} dm \tag{50.7}$$

Here the notation $\left[A\right]_b^c$ is borrowed from [6] for the matrix of a linear transformation that takes one from basis b to c.

Observe that this ($\mathbb{R}^3$ specific expansion) can also be written in a more typical tensor notation with $\left[I\right]_b^b = \left[I_{ij}\right]_{ij}$

$$I_{ij} = I(\mathbf{e}_i I) \cdot (\mathbf{e}_j I)^\dagger = \int (\delta_{ij} \mathbf{x}^2 - x_i x_j) dm \tag{50.8}$$

Where, as usual for tensors, the meaning of the indices and whether summation is required is implied. In this case the coordinate transformation matrix for this linear transformation has components I_{ij} (and no summation).

50.1 ORTHOGONAL DECOMPOSITION OF A FUNCTION MAPPING A BLADE TO A BLADE

Arriving at this result without explicit expansion is also possible by observing that an orthonormal decomposition of a function can be written in terms of an orthogonal basis $\{\sigma_i\}$ as follows:

$$f(B) = \sum_i (f(B) \cdot \sigma_i) \cdot \frac{1}{\sigma_i} \tag{50.9}$$

The dot product is required since the general product of two bivectors has grade-0, grade-2, and grade-4 terms (with a similar mix of higher grade terms for k-blades).

Perhaps unobviously since one is not normally used to seeing a scalar-vector dot product, this formula is not only true for bivectors, but any grade blade, including vectors. To verify this recall that the general definition of the dot product is the lowest grade term of the geometric product of two blades. For example with grade i, j blades a, and b respectively the dot product is:

$$a \cdot b = \langle ab \rangle_{|i-j|} \tag{50.10}$$

So, for a scalar-vector dot product is just the scalar product of the two

$$a \cdot \mathbf{x} = \langle a\mathbf{x} \rangle_1 = a\mathbf{x} \tag{50.11}$$

The inverse in section 50.1 can be removed by reversion, and for a grade-r blade this sum of projective terms then becomes:

$$f(B) = (-1)^{r(r-1)/2} \frac{1}{|\sigma_i|^2} \sum_i (f(B) \cdot \sigma_i) \cdot \sigma_i \tag{50.12}$$

For an orthonormal basis we have

$$\sigma_i \sigma_i^\dagger = |\sigma_i|^2 = 1 \tag{50.13}$$

Which allows for a slightly simpler set of projective terms:

$$f(B) = (-1)^{r(r-1)/2} \sum_i (f(B) \cdot \sigma_i) \cdot \sigma_i \tag{50.14}$$

50.2 COORDINATE TRANSFORMATION MATRIX FOR A COUPLE OTHER LINEAR TRANSFORMATIONS

Seeing a function of a bivector for the first time is kind of intriguing. We can form the matrix of such a linear transformation from a basis of the bivector space to the space spanned by function. For fun, let us calculate that matrix for the basis b above for the following function:

$$f(B) = \mathbf{e}_1 \wedge (\mathbf{e}_2 \cdot B) \tag{50.15}$$

For this function operating on $\mathbb{R}^3$ bivectors we have:

$$\begin{aligned}
f(\mathbf{e}_{23}) &= \mathbf{e}_1 \wedge (\mathbf{e}_2 \cdot \mathbf{e}_{23}) = -\mathbf{e}_{31} \\
f(\mathbf{e}_{31}) &= \mathbf{e}_1 \wedge (\mathbf{e}_2 \cdot \mathbf{e}_{31}) = 0 \\
f(\mathbf{e}_{12}) &= \mathbf{e}_1 \wedge (\mathbf{e}_2 \cdot \mathbf{e}_{12}) = 0
\end{aligned} \tag{50.16}$$

So

$$\left[f\right]_b^b = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{50.17}$$

For $\mathbb{R}^4$ one orthonormal basis is

$$b = \{\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}\} \tag{50.18}$$

A basis for the span of f is $b' = \{\mathbf{e}_{13}, \mathbf{e}_{14}\}$. Like any other coordinate transformation associated with a linear transformation we can write the matrix of the transformation that takes a coordinate vector in one basis into a coordinate vector for the basis for the image:

$$\left[f(x)\right]_{b'} = \left[f\right]_b^{b'} \left[x\right]_b \tag{50.19}$$

For this function f and these pair of basis bivectors we have:

$$\left[f\right]_b^{b'} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{50.20}$$

50.3 EQUATION 3.126 DETAILS

This statement from GAFP deserves expansion (or at least an exercise):

$$A \cdot (\mathbf{x} \wedge (\mathbf{x} \cdot B)) = \langle A\mathbf{x}(\mathbf{x} \cdot B)\rangle = \langle (A \cdot \mathbf{x})\mathbf{x}B\rangle = B \cdot (\mathbf{x} \wedge (\mathbf{x} \cdot A)) \tag{50.21}$$

Perhaps this is obvious to the author, but was not to me. To clarify this observe the following product

$$\mathbf{x}(\mathbf{x} \cdot B) = \mathbf{x} \cdot (\mathbf{x} \cdot B) + \mathbf{x} \wedge (\mathbf{x} \cdot B) \tag{50.22}$$

By writing $B = \mathbf{b} \wedge \mathbf{c}$ we can show that the dot product part of this product is zero:

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{x} \cdot B) &= \mathbf{x} \cdot ((\mathbf{x} \cdot \mathbf{b})\mathbf{c} - (\mathbf{x} \cdot \mathbf{c})\mathbf{b}) \\ &= (\mathbf{x} \cdot \mathbf{c})(\mathbf{x} \cdot \mathbf{b}) - (\mathbf{x} \cdot \mathbf{b})(\mathbf{x} \cdot \mathbf{c})) \\ &= 0 \end{aligned} \tag{50.23}$$

This provides the justification for the wedge product removal in the text, since one can write

$$\mathbf{x} \wedge (\mathbf{x} \cdot B) = \mathbf{x}(\mathbf{x} \cdot B) \tag{50.24}$$

Although it was not stated in the text section 50.3, can be used to put this inertia product in a pure dot product form

$$\begin{aligned} A^\dagger \cdot (\mathbf{x} \wedge (\mathbf{x} \cdot B)) &= -\langle A\mathbf{x}(\mathbf{x} \cdot B)\rangle \\ &= \langle (\mathbf{x} \cdot A - A \wedge \mathbf{x})(\mathbf{x} \cdot B)\rangle \end{aligned} \tag{50.25}$$

The trivector-vector part of this product has only vector and trivector components

$$(A \wedge \mathbf{x})(\mathbf{x} \cdot B) = \langle (A \wedge \mathbf{x})(\mathbf{x} \cdot B)\rangle_1 + \langle (A \wedge \mathbf{x})(\mathbf{x} \cdot B)\rangle_3 \tag{50.26}$$

So $\langle (A \wedge \mathbf{x})(\mathbf{x} \cdot B)\rangle_0 = 0$, and one can write

$$A^\dagger \cdot (\mathbf{x} \wedge (\mathbf{x} \cdot B)) = (\mathbf{x} \cdot A) \cdot (\mathbf{x} \cdot B) \tag{50.27}$$

As pointed out in the text this is symmetric. That can not be more clear than in section 50.3.

50.4 JUST FOR FUN. GENERAL DIMENSION COMPONENT EXPANSION OF INERTIA TENSOR TERMS

This triple dot product expansion allows for a more direct component expansion of the component form of the inertia tensor. There are three general cases to consider.

- The diagonal terms:

  $$(\mathbf{x} \cdot \sigma_i) \cdot (\mathbf{x} \cdot \sigma_i) = (\mathbf{x} \cdot \sigma_i)^2 \tag{50.28}$$

  Writing $\sigma_i = \mathbf{e}_{st}$ where $s \neq t$, we have

  $$\begin{aligned} (\mathbf{x} \cdot \mathbf{e}_{st})^2 &= ((\mathbf{x} \cdot \mathbf{e}_s)\mathbf{e}_t - (\mathbf{x} \cdot \mathbf{e}_t)\mathbf{e}_s)^2 \\ &= x_s^2 + x_t^2 - 2x_s x_t \mathbf{e}_t \cdot \mathbf{e}_s \\ &= x_s^2 + x_t^2 \end{aligned} \tag{50.29}$$

- Off diagonal terms where basis bivectors have a line of intersection (always true for $\mathbb{R}^3$).

  Here, ignoring the potential variation in sign, we can write the two basis bivectors as $\sigma_i = \mathbf{e}_{si}$ and $\sigma_j = \mathbf{e}_{ti}$, where $s \neq t \neq i$. Computing the products we have

$$\begin{aligned}
(\mathbf{x} \cdot \sigma_i) \cdot (\mathbf{x} \cdot \sigma_j) &= (\mathbf{x} \cdot \mathbf{e}_{si}) \cdot (\mathbf{x} \cdot \mathbf{e}_{ti}) \\
&= ((\mathbf{x} \cdot \mathbf{e}_s)\mathbf{e}_i - (\mathbf{x} \cdot \mathbf{e}_i)\mathbf{e}_s) \cdot ((\mathbf{x} \cdot \mathbf{e}_t)\mathbf{e}_i - (\mathbf{x} \cdot \mathbf{e}_i)\mathbf{e}_t) \\
&= (x_s \mathbf{e}_i - x_i \mathbf{e}_s) \cdot (x_t \mathbf{e}_i - x_i \mathbf{e}_t) \\
&= x_s x_t
\end{aligned} \tag{50.30}$$

- Off diagonal terms where basis bivectors have no intersection.

  An example from $\mathbb{R}^4$ are the two bivectors $\mathbf{e}_1 \wedge \mathbf{e}_2$ and $\mathbf{e}_3 \wedge \mathbf{e}_4$

  In general, again ignoring the potential variation in sign, we can write the two basis bivectors as $\sigma_i = \mathbf{e}_{su}$ and $\sigma_j = \mathbf{e}_{tv}$, where $s \neq t \neq u \neq v$. Computing the products we have

$$\begin{aligned}
(\mathbf{x} \cdot \sigma_i) \cdot (\mathbf{x} \cdot \sigma_j) &= (\mathbf{x} \cdot \mathbf{e}_{su}) \cdot (\mathbf{x} \cdot \mathbf{e}_{tv}) \\
&= ((\mathbf{x} \cdot \mathbf{e}_s)\mathbf{e}_u - (\mathbf{x} \cdot \mathbf{e}_u)\mathbf{e}_s) \cdot ((\mathbf{x} \cdot \mathbf{e}_t)\mathbf{e}_v - (\mathbf{x} \cdot \mathbf{e}_v)\mathbf{e}_t) \\
&= 0
\end{aligned} \tag{50.31}$$

For example, choosing basis $\sigma = \{\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}\}$ the coordinate transformation matrix can be written out

$$\left[f \right]_\sigma^\sigma = \begin{bmatrix}
x_1^2 + x_2^2 & x_2 x_3 & x_2 x_4 & -x_1 x_3 & -x_1 x_4 & 0 \\
x_2 x_3 & x_1^2 + x_3^2 & x_3 x_4 & x_1 x_2 & 0 & -x_1 x_4 \\
x_2 x_4 & x_3 x_4 & x_1^2 + x_4^2 & 0 & x_1 x_2 & x_1 x_3 \\
-x_1 x_3 & x_1 x_2 & 0 & x_2^2 + x_3^2 & x_3 x_4 & -x_2 x_4 \\
-x_1 x_4 & 0 & x_1 x_2 & x_3 x_4 & x_1^2 + x_4^2 & x_2 x_3 \\
0 & -x_1 x_4 & x_1 x_3 & -x_2 x_4 & x_2 x_3 & x_3^2 + x_4^2
\end{bmatrix} \tag{50.32}$$

50.5 EXAMPLE CALCULATION. MASSES IN A LINE

Pick some points on the x-axis, $\mathbf{r}^{(i)}$ with masses m_i. The ($\mathbb{R}^3$) inertia tensor with respect to basis $\{\mathbf{e}_i I\}$, is

$$\sum_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & (r_1^{(i)})^2 & 0 \\ 0 & 0 & (r_1^{(i)})^2 \end{bmatrix} m_i = \sum m_i \mathbf{r}_i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{50.33}$$

Observe that in this case the inertia tensor here only has components in the zx and xy planes (no component in the yz plane that is perpendicular to the line).

50.6 EXAMPLE CALCULATION. MASSES IN A PLANE

Let $x = re^{i\theta}\mathbf{e}_1$, where $i = \mathbf{e}_1 \wedge \mathbf{e}_2$ be a set of points in the xy plane, and use $\sigma = \{\sigma_i = \mathbf{e}_i I\}$ as the basis for the $\mathbb{R}^3$ bivector space.

We need to compute

$$\begin{aligned} \mathbf{x} \cdot \sigma_i &= r(e^{i\theta}\mathbf{e}_1) \cdot (\mathbf{e}_i I) \\ &= r\langle e^{i\theta}\mathbf{e}_1\mathbf{e}_i I\rangle \end{aligned} \tag{50.34}$$

Calculation of the inertia tensor components has three cases, depending on the value of i

- $i = 1$

$$\begin{aligned} \frac{1}{r}(\mathbf{x} \cdot \sigma_i) &= \langle e^{i\theta} I\rangle_1 \\ &= i \sin\theta I \\ &= -\mathbf{e}_3 \sin\theta \end{aligned} \tag{50.35}$$

- $i = 2$

$$\begin{aligned} \frac{1}{r}(\mathbf{x} \cdot \sigma_i) &= \langle e^{i\theta}\mathbf{e}_1\mathbf{e}_2 I\rangle_1 \\ &= -\langle e^{i\theta}\mathbf{e}_3\rangle_1 \\ &= -\mathbf{e}_3 \cos\theta \end{aligned} \tag{50.36}$$

- $i = 3$

$$
\begin{aligned}
\frac{1}{r}(\mathbf{x} \cdot \sigma_i) &= \langle e^{i\theta} \mathbf{e}_1 \mathbf{e}_3 (\mathbf{e}_3 \mathbf{e}_1 \mathbf{e}_2) \rangle_1 \\
&= \langle e^{i\theta} \mathbf{e}_2 \rangle_1 \\
&= e^{i\theta} \mathbf{e}_2
\end{aligned}
\tag{50.37}
$$

Thus for $i = \{1, 2, 3\}$, the diagonal terms are

$$
(\mathbf{x} \cdot \sigma_i)^2 = r^2 \{\sin^2\theta, \cos^2\theta, 1\}
\tag{50.38}
$$

and the non-diagonal terms are

$$
(\mathbf{x} \cdot \sigma_1) \cdot (\mathbf{x} \cdot \sigma_2) = r^2 \sin\theta \cos\theta
\tag{50.39}
$$

$$
(\mathbf{x} \cdot \sigma_1) \cdot (\mathbf{x} \cdot \sigma_3) = 0
\tag{50.40}
$$

$$
(\mathbf{x} \cdot \sigma_2) \cdot (\mathbf{x} \cdot \sigma_3) = 0
\tag{50.41}
$$

Thus, with indices implied ($r = \mathbf{r}_i$, $\theta = \theta_i$, and $m = m_i$, the inertia tensor is

$$
\left[I\right]_\sigma^\sigma = \sum m r^2 \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
\tag{50.42}
$$

It is notable that this can be put into double angle form

$$
\begin{aligned}
\left[I\right]_\sigma^\sigma &= \sum m r^2 \begin{bmatrix} \frac{1}{2}(1 - \cos 2\theta) & \frac{1}{2}\sin 2\theta & 0 \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1 + \cos 2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{2} \sum m r^2 \left(I + \begin{bmatrix} -\cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)
\end{aligned}
\tag{50.43}
$$

So if grouping masses along each distinct line in the plane, those components of the inertia tensor can be thought of as functions of twice the angle. This is natural in terms of a rotor interpretation, which is likely possible since each of these groups of masses in a line can be diagonalized with a rotation.

It can be verified that the following xy plane rotation diagonalizes all the terms of constant angle. Writing

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{50.44}$$

We have

$$\left[I\right]_\sigma^\sigma = \sum mr^2 R_{-\theta} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_\theta \tag{50.45}$$

51

SATELLITE TRIANGULATION OVER SPHERE

51.1 MOTIVATION AND PREPARATION

Was playing around with what is probably traditionally a spherical trig type problem using geometric algebra (locate satellite position using angle measurements from two well separated points). Origin of the problem was just me looking at my Feynman Lectures introduction where there is a diagram illustrating how triangulation could be used to locate "Sputnik" and thought I had try such a calculation, but in a way that I thought was more realistic.

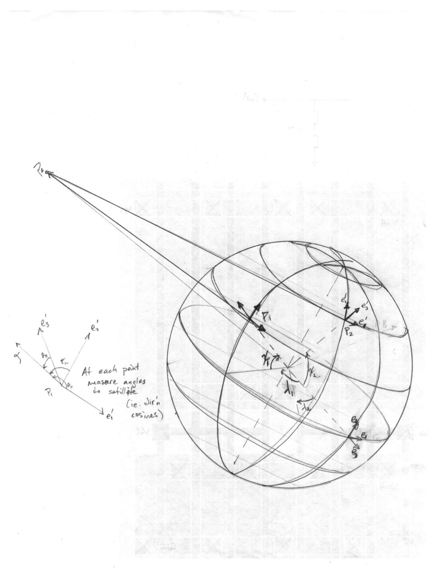

Figure 51.1: Satellite location by measuring direction from two points

Figure 51.1 illustrates the problem I attempted to solve. Pick two arbitrary points P_1, and P_2 on the globe, separated far enough that the curvature of the earth may be a factor. For this problem it is assumed that the angles to the satellite will be measured concurrently.

Place a fixed reference frame at the center of the earth. In the figure this is shown translated to the $(0,0)$ point (equator and prime meridian intersection). I have picked $\mathbf{e}_1$ facing east, $\mathbf{e}_2$ facing north, and $\mathbf{e}_3$ facing outwards from the core.

Each point P_i can be located by a rotation along the equatorial plane by angle λ_i (measured with an east facing orientation (direction of $\mathbf{e}_1$), and a rotation ψ_i towards the north (directed towards $\mathbf{e}_2$).

To identify a point on the surface we translate our $(0, 0)$ reference frame to that point using the following rotor equation:

$$R_\psi = \exp(-\mathbf{e}_{32}\psi/2) = \cos(\psi/2) - \mathbf{e}_{32}\sin(\psi/2) \tag{51.1}$$

$$R_\lambda = \exp(-\mathbf{e}_{31}\lambda/2) = \cos(\lambda/2) - \mathbf{e}_{31}\sin(\lambda/2) \tag{51.2}$$

$$R(x) = R_\psi R_\lambda x R_\lambda^\dagger R_\psi^\dagger \tag{51.3}$$

To verify that I got the sign of these rotations right, I applied them to the unit vectors using a $\pi/2$ rotation. We want the following for the equatorial plane rotation:

$$R_\lambda(\pi/2)\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}R_\lambda(\pi/2)^\dagger = \begin{bmatrix}-\mathbf{e}_3\\ \mathbf{e}_2\\ \mathbf{e}_1\end{bmatrix}$$

And for the northwards rotation:

$$R_\psi(\pi/2)\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}R_\psi(\pi/2)^\dagger = \begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ -\mathbf{e}_3\end{bmatrix}$$

Verifying this is simple enough using the explicit sine and cosine expansion of the rotors in eq. (51.1) and eq. (51.2).

Once we have the ability to translate our reference frame to each point on the Earth, we can use the inverse rotation to translate our measured unit vector to the satellite at that point back to the reference frame.

Suppose one calculates a local unit vector α' towards the satellite by measuring direction cosines in our local reference frame (ie: angle from gravity opposing (up facing) direction, east, and north directions at that point). Once that is done, that unit vector α in our reference frame is obtained by inverse rotation:

$$\alpha = R_{\lambda_i}^\dagger R_{\psi_i}^\dagger \alpha' R_{\psi_i} R_{\lambda_i} \tag{51.4}$$

The other place we need this rotation for is to calculate the points P_i in our reference from (treating this now as being at the core of the earth). This is just:

$$P_i = R_\psi R_\lambda A_i \mathbf{e}_3 R_\lambda^\dagger R_\psi^\dagger \tag{51.5}$$

Where A_i is the altitude (relative to the center of the earth) at the point of interest.

51.2 SOLUTION

Solving for the position of the satellite P_s we have:

$$P_s = a_1\alpha_1 + P_1 = a_2\alpha_2 + P_2 \tag{51.6}$$

Solution of this follows directly by taking wedge products. Solve for a_1 for example, we wedge with α_2 :

$$a_1\alpha_1 \wedge \alpha_2 + P_1 \wedge \alpha_2 = a_2 \overset{= 0}{\boxed{\alpha_2 \wedge \alpha_2}} + P_2 \wedge \alpha_2 \tag{51.7}$$

Provided the points are far enough apart to get distinct α_i measurements, then we have:

$$a_1 = \frac{(P_2 - P_1) \wedge \alpha_2}{\alpha_1 \wedge \alpha_2}. \tag{51.8}$$

Thus the position vector from the core of earth reference frame to the satellite is:

$$P_s = \left(\frac{(P_2 - P_1) \wedge \alpha_2}{\alpha_1 \wedge \alpha_2}\right)\alpha_1 + P_1 \tag{51.9}$$

Notice how all the trigonometry is encoded directly in the rotor equations. If one had to calculate all this using the spherical trigonometry generalized triangle relations I expect that you would have an ungodly mess of sine and cosines here.

This demonstrates two very distinct applications of the wedge product. The first was to define an oriented plane, and was used as a generator of rotations (very much like the unit imaginary). This second application, to solve linear equations takes advantage of $a \wedge a = 0$ property of the wedge product. It was convenient as it allowed simple simultaneous solution of the three equations (one for each component) and two unknowns problem in this particular case.

51.3 MATRIX FORMULATION

Instead of solving with the wedge product one could formulate this as a matrix equation:

$$\begin{bmatrix} \alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} P_2 - P_1 \end{bmatrix} \tag{51.10}$$

This highlights the fact that the equations are over-specified, which is more obvious still when this is written out in component form:

$$\begin{bmatrix} \alpha_{11} & -\alpha_{21} \\ \alpha_{12} & -\alpha_{22} \\ \alpha_{13} & -\alpha_{23} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} P_{21} - P_{11} \\ P_{22} - P_{12} \\ P_{23} - P_{13} \end{bmatrix} \tag{51.11}$$

We have one more equation than we need to actually solve it, and cannot use matrix inversion directly (Gaussian elimination or a generalized inverse is required).

Recall the figure in the Feynman lectures when the observation points and the satellite are all in the same plane. For that all that was needed was two angles, whereas we have measured six for each of the direction cosines used above, so the fact that our equations can include more info than required to solve the problem is not unexpected.

We could also generalize this, perhaps to remove measurement error, by utilizing more than two observation points. This will compound the over-specification of the equations, and makes it clear that we likely want a least squares approach to solve it. Here is an example of the matrix to solve for three points:

$$\begin{bmatrix} \alpha_1 & -\alpha_2 & 0 \\ -\alpha_1 & 0 & \alpha_3 \\ 0 & \alpha_2 & -\alpha_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} P_2 - P_1 \\ P_1 - P_3 \\ P_3 - P_2 \end{bmatrix} \tag{51.12}$$

Since the α_i are vectors, this matrix of rotated direction cosines has dimensions 9 by 3 (just as eq. (51.10) is a 3 by 2 matrix).

51.4 QUESTION. ORDER OF LATITUDE AND LONGITUDE ROTORS?

Looking at a globe, it initially seemed clear to me that these "perpendicular" (abusing the word) rotations could be applied in either order, but their rotors definitely do not commute, so I assume that together the non-commutative bits of the rotors "cancel out".

Question, is it actually true that the end effect of applying these rotors in either order is the same?

$$x' = R_\psi R_\lambda x R_\lambda^\dagger R_\psi^\dagger = R_\lambda R_\psi x R_\psi^\dagger R_\lambda^\dagger \tag{51.13}$$

Attempting to show this is true or false by direct brute force expansion is not productive (perhaps would be okay with a symbolic GA calculator). However, such a direct expansion of just the rotor products in either order allows for a comparison:

$$\begin{aligned}R_\psi R_\lambda &= (\cos(\psi/2) - \mathbf{e}_{32}\sin(\psi/2))(\cos(\lambda/2) - \mathbf{e}_{31}\sin(\lambda/2)) \\ &= \cos(\psi/2)\cos(\lambda/2) - \mathbf{e}_{32}\sin(\psi/2)\cos(\lambda/2) - \mathbf{e}_{31}\cos(\psi/2)\sin(\lambda/2) - \mathbf{e}_{21}\sin(\psi/2)\sin(\lambda/2)\end{aligned} \tag{51.14}$$

$$\begin{aligned}R_\lambda R_\psi &= (\cos(\lambda/2) - \mathbf{e}_{31}\sin(\lambda/2))(\cos(\psi/2) - \mathbf{e}_{32}\sin(\psi/2)) \\ &= \overbrace{\boxed{\cos(\psi/2)\cos(\lambda/2)}}^{a_0} \\ &\quad + \overbrace{\boxed{-\mathbf{e}_{32}\sin(\psi/2)\cos(\lambda/2) - \mathbf{e}_{31}\cos(\psi/2)\sin(\lambda/2)}}^{A} \\ &\quad + \overbrace{\boxed{\mathbf{e}_{21}\sin(\psi/2)\sin(\lambda/2)}}^{B}\end{aligned} \tag{51.15}$$

Observe that these are identical except for an inversion of sign of the $\mathbf{e}_{21}$ term. Using the shorthand above the respective rotations are:

$$R_{\lambda,\psi}(x) = R_\psi R_\lambda x R_\lambda^\dagger R_\psi^\dagger = (a_0 + A - B)x(a_0 - A + B) \tag{51.16}$$

And

$$R_{\psi,\lambda}(x) = R_\lambda R_\psi x R_\psi^\dagger R_\lambda^\dagger = (a_0 + A + B)x(a_0 - A - B) \tag{51.17}$$

And this can be used to disprove the general rotation commutativity. We take the difference between these two rotation results, and see if it can be shown to equal zero. Taking differences, also temporarily writing $a = a_0 + A$, and exploiting a grade one filter since the final result must be a vector we have:

$$
\begin{aligned}
R_{\lambda,\psi}(x) - R_{\psi,\lambda}(x) &= \left\langle R_{\lambda,\psi}(x) - R_{\psi,\lambda}(x) \right\rangle_1 \\
&= \left\langle (a - B)x(a^\dagger + B) - (a + B)x(a^\dagger - B) \right\rangle_1 \\
&= \left\langle (axa^\dagger - BxB - Bxa^\dagger + axB) + (-axa^\dagger + BxB + axB - Bxa^\dagger) \right\rangle_1 \\
&= \left\langle (-Bxa^\dagger + axB) + (+axB - Bxa^\dagger) \right\rangle_1 \\
&= 2\left\langle -Bxa^\dagger + axB \right\rangle_1 \\
&= 2\langle -Bx(a_0 - A) + (a_0 + A)xB \rangle_1 \\
&= 2a_0(-Bx + xB) + 2\langle BxA + AxB \rangle_1 \\
&= 4a_0 x \cdot B + 2\langle BxA + AxB \rangle_1 \\
&= 4a_0 x \cdot B + 2\langle B \cdot xA - AB \cdot x \rangle_1 + 2\langle B \wedge xA + Ax \wedge B \rangle_1 \\
&= 4a_0 x \cdot B + 4(B \cdot x) \cdot A + 2(B \wedge x) \cdot A + 2A \cdot (B \wedge x) \\
&= 4a_0 x \cdot B + 4(B \cdot x) \cdot A + 2(B \wedge x) \cdot A - 2(B \wedge x) \cdot A \\
&= 4a_0 x \cdot B + 4(B \cdot x) \cdot A \\
&= 4(B \cdot x) \cdot (-a_0 + A) \\
&= -4(B \cdot x) \cdot a^\dagger
\end{aligned}
\tag{51.18}
$$

Evaluate this for $x = \mathbf{e}_1$ we do not have zero (a vector with $\mathbf{e}_2$ and $\mathbf{e}_3$ components), and for $x = \mathbf{e}_2$ this difference has $\mathbf{e}_1$, and $\mathbf{e}_3$ components. However, for $x = \mathbf{e}_3$ this is zero. Thus these rotations only commute when applied to a vector that is completely normal to the sphere. This is what messes up the intuition. Rotating a point (represented by a vector) in either order works fine, but rotating a frame located at the surface back to a different point on the surface, and maintaining the east and north orientations we have to be careful which orientation to use.

So which order is right? It has to be rotate first in the equatorial plane (λ), then the northwards rotation, where both are great circle rotations.

A numeric confirmation of this is likely prudent.

52

EXPONENTIAL SOLUTIONS TO LAPLACE EQUATION IN $\mathbb{R}^N$

52.1 THE PROBLEM

Want solutions of

$$\nabla^2 f = \sum_k \frac{\partial^2 f}{\partial {x_k}^2} = 0 \tag{52.1}$$

For real f.

52.1.1 *One dimension*

Here the problem is easy, just integrate twice:

$$f = cx + d. \tag{52.2}$$

52.1.2 *Two dimensions*

For the two dimensional case we want to solve:

$$\frac{\partial^2 f}{\partial {x_1}^2} + \frac{\partial^2 f}{\partial {x_2}^2} = 0 \tag{52.3}$$

Using separation of variables one can find solutions of the form $f = X(x_1)Y(x_2)$. Differentiating we have:

$$X''Y + XY'' = 0 \tag{52.4}$$

So, for $X \neq 0$, and $Y \neq 0$:

$$\frac{X''}{X} = -\frac{Y''}{Y} = k^2 \tag{52.5}$$

$$\implies X = e^{kx} \tag{52.6}$$

$$Y = e^{k\mathbf{i}y} \tag{52.7}$$

$$\implies f = XY = e^{k(x+\mathbf{i}y)} \tag{52.8}$$

Here **i** is anything that squares to -1. Traditionally this is the complex unit imaginary, but we are also free to use a geometric product unit bivector such as $\mathbf{i} = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_{12}$, or $\mathbf{i} = \mathbf{e}_{21}$.
With $\mathbf{i} = \mathbf{e}_{12}$ for example we have:

$$\begin{aligned} f = XY = e^{k(x+\mathbf{i}y)} &= e^{k(x+\mathbf{e}_{12}y)} \\ &= e^{k(x\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_{12}y)} \\ &= e^{k\mathbf{e}_1(x\mathbf{e}_1+\mathbf{e}_2y)} \end{aligned} \tag{52.9}$$

Writing $\mathbf{x} = \sum x_i\mathbf{e}_i$, all of the following are solutions of the Laplacian

$$\begin{aligned} &e^{k\mathbf{e}_1\mathbf{x}} \\ &e^{\mathbf{x}k\mathbf{e}_1} \\ &e^{k\mathbf{e}_2\mathbf{x}} \\ &e^{\mathbf{x}k\mathbf{e}_2} \end{aligned} \tag{52.10}$$

Now there is not anything special about the use of the x and y axis so it is reasonable to expect that, given any constant vector **k**, the following may also be solutions to the two dimensional Laplacian problem

$$e^{\mathbf{x}\mathbf{k}} = e^{\mathbf{x}\cdot\mathbf{k}+\mathbf{x}\wedge\mathbf{k}} \tag{52.11}$$

$$e^{\mathbf{k}\mathbf{x}} = e^{\mathbf{x}\cdot\mathbf{k}-\mathbf{x}\wedge\mathbf{k}} \tag{52.12}$$

52.1.3 *Verifying it is a solution*

To verify that equations eq. (52.11) and eq. (52.12) are Laplacian solutions, start with taking the first order partial with one of the coordinates. Since there are conditions where this form of solution works in $\mathbb{R}^N$, a two dimensional Laplacian will not be assumed here.

$$\frac{\partial}{\partial x_j} e^{\mathbf{x}\mathbf{k}} \tag{52.13}$$

This can be evaluated without any restrictions, but introducing the restriction that the bivector part of $\mathbf{x}\mathbf{k}$ is coplanar with its derivative simplifies the result considerably. That is introduce a restriction:

$$\left\langle \mathbf{x} \wedge \mathbf{k} \frac{\partial \mathbf{x} \wedge \mathbf{k}}{\partial x_j} \right\rangle_2 = \left\langle \mathbf{x} \wedge \mathbf{k} \mathbf{e}_j \wedge \mathbf{k} \right\rangle_2 = 0 \tag{52.14}$$

With such a restriction we have

$$\frac{\partial}{\partial x_j} e^{\mathbf{x}\mathbf{k}} = \mathbf{e}_j \mathbf{k} e^{\mathbf{x}\mathbf{k}} = e^{\mathbf{x}\mathbf{k}} \mathbf{e}_j \mathbf{k} \tag{52.15}$$

Now, how does one enforce a restriction of this form in general? Some thought will show that one way to do so is to require that both $\mathbf{x}$ and $\mathbf{k}$ have only two components. Say, components j, and m. Then, summing second partials we have:

$$\begin{aligned}
\sum_{u=j,m} \frac{\partial^2}{\partial {x_u}^2} e^{\mathbf{x}\mathbf{k}} &= \left(\mathbf{e}_j \mathbf{k} \mathbf{e}_j \mathbf{k} + \mathbf{e}_m \mathbf{k} \mathbf{e}_m \mathbf{k} \right) e^{\mathbf{x}\mathbf{k}} \\
&= \left(\mathbf{e}_j \mathbf{k} (-\mathbf{k}\mathbf{e}_j + 2\mathbf{k} \cdot \mathbf{e}_j) + \mathbf{e}_m \mathbf{k} (-\mathbf{k}\mathbf{e}_m + 2\mathbf{e}_m \cdot \mathbf{k}) \right) e^{\mathbf{x}\mathbf{k}} \\
&= \left(-2\mathbf{k}^2 + 2k_j^2 + 2k_m k_j \mathbf{e}_{jm} + 2k_m^2 + 2k_j k_m \mathbf{e}_{mj} \right) e^{\mathbf{x}\mathbf{k}} \\
&= \left(-2\mathbf{k}^2 + 2\mathbf{k}^2 + 2k_j k_m (\mathbf{e}_{mj} + \mathbf{e}_{jm}) \right) e^{\mathbf{x}\mathbf{k}} \\
&= 0
\end{aligned} \tag{52.16}$$

This proves the result, but essentially just says that this form of solution is only valid when the constant parametrization vector $\mathbf{k}$ and $\mathbf{x}$ and its variation are restricted to a specific plane. That result could have been obtained in much simpler ways, but I learned a lot about bivector geometry in the approach! (not all listed here since it caused serious digressions)

52.1.4 *Solution for an arbitrarily oriented plane*

Because the solution above is coordinate free, one would expect that this works for any solution that is restricted to the plane with bivector $\mathbf{i}$ even when those do not line up with any specific pair of two coordinates. This can be verified by performing a rotational coordinate transformation of the Laplacian operator, since one can always pick a pair of mutually orthogonal basis vectors with corresponding coordinate vectors that lie in the plane defined by such a bivector.

Given two arbitrary vectors in the space when both are projected onto the plane with constant bivector $\mathbf{i}$ their product is:

$$\left(\mathbf{x} \cdot \mathbf{i}\frac{1}{\mathbf{i}}\right)\left(\frac{1}{\mathbf{i}}\mathbf{i} \cdot \mathbf{k}\right) = (\mathbf{x} \cdot \mathbf{i})(\mathbf{k} \cdot \mathbf{i}) \tag{52.17}$$

Thus one can express the general equation for a planar solution to the homogeneous Laplace equation in the form

$$\exp((\mathbf{x} \cdot \mathbf{i})(\mathbf{k} \cdot \mathbf{i})) = \exp((\mathbf{x} \cdot \mathbf{i}) \cdot (\mathbf{k} \cdot \mathbf{i}) + (\mathbf{x} \cdot \mathbf{i}) \wedge (\mathbf{k} \cdot \mathbf{i})) \tag{52.18}$$

52.1.5 *Characterization in real numbers*

Now that it has been verified that equations eq. (52.11) and eq. (52.12) are solutions of eq. (52.1) let us characterize this in terms of real numbers.

If $\mathbf{x}$, and $\mathbf{k}$ are colinear, the solution has the form

$$e^{\pm \mathbf{x} \cdot \mathbf{k}} \tag{52.19}$$

(ie: purely hyperbolic solutions).

Whereas with $\mathbf{x}$ and $\mathbf{k}$ orthogonal we have can employ the unit bivector for the plane spanned by these vectors $\mathbf{i} = \frac{\mathbf{x} \wedge \mathbf{k}}{|\mathbf{x} \wedge \mathbf{k}|}$:

$$e^{\pm \mathbf{x} \wedge \mathbf{k}} = \cos|\mathbf{x} \wedge \mathbf{k}| \pm \mathbf{i} \sin|\mathbf{x} \wedge \mathbf{k}| \tag{52.20}$$

Or:

$$e^{\pm \mathbf{x} \wedge \mathbf{k}} = \cos\left(\frac{\mathbf{x} \wedge \mathbf{k}}{\mathbf{i}}\right) \pm \mathbf{i} \sin\left(\frac{\mathbf{x} \wedge \mathbf{k}}{\mathbf{i}}\right) \tag{52.21}$$

(ie: purely trigonometric solutions)

Provided $\mathbf{x}$, and $\mathbf{k}$ are not colinear, the wedge product component of the above can be written in terms of a unit bivector $\mathbf{i} = \frac{\mathbf{x}\wedge\mathbf{k}}{|\mathbf{x}\wedge\mathbf{k}|}$:

$$\begin{aligned} e^{\mathbf{xk}} &= e^{\mathbf{x}\cdot\mathbf{k}+\mathbf{x}\wedge\mathbf{k}} \\ &= e^{\mathbf{x}\cdot\mathbf{k}} \left(\cos|\mathbf{x}\wedge\mathbf{k}| + \mathbf{i}\sin|\mathbf{x}\wedge\mathbf{k}|\right) \\ &= e^{\mathbf{x}\cdot\mathbf{k}} \left(\cos\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) + \mathbf{i}\sin\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right)\right) \end{aligned} \tag{52.22}$$

And, for the reverse:

$$\begin{aligned} (e^{\mathbf{xk}})^\dagger = e^{\mathbf{kx}} &= e^{\mathbf{x}\cdot\mathbf{k}} \left(\cos|\mathbf{x}\wedge\mathbf{k}| - \mathbf{i}\sin\left(|\mathbf{x}\wedge\mathbf{k}|\right)\right) \\ &= e^{\mathbf{x}\cdot\mathbf{k}} \left(\cos\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) - \mathbf{i}\sin\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right)\right) \end{aligned} \tag{52.23}$$

This exponential however has both scalar and bivector parts, and we are looking for a strictly scalar result, so we can use linear combinations of the exponential and its reverse to form a strictly real sum for the $\mathbf{x}\wedge\mathbf{k} \neq 0$ cases:

$$\begin{aligned} \frac{1}{2}\left(e^{\mathbf{xk}} + e^{\mathbf{kx}}\right) &= e^{\mathbf{x}\cdot\mathbf{k}} \cos\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) \\ \frac{1}{2\mathbf{i}}\left(e^{\mathbf{xk}} - e^{\mathbf{kx}}\right) &= e^{\mathbf{x}\cdot\mathbf{k}} \sin\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}} \end{aligned} \tag{52.24}$$

Also note that further linear combinations (with positive and negative variations of $\mathbf{k}$) can be taken, so we can combine equations eq. (52.11) and eq. (52.12) into the following real valued, coordinate free, form:

$$\begin{aligned} &\cosh(\mathbf{x}\cdot\mathbf{k})\cos\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) \\ &\sinh(\mathbf{x}\cdot\mathbf{k})\cos\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) \\ &\cosh(\mathbf{x}\cdot\mathbf{k})\sin\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) \\ &\sinh(\mathbf{x}\cdot\mathbf{k})\sin\left(\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}\right) \end{aligned} \tag{52.25}$$

Observe that the ratio $\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}}$ is just a scalar determinant

$$\frac{\mathbf{x}\wedge\mathbf{k}}{\mathbf{i}} = x_j k_m - x_m k_j \tag{52.26}$$

So one is free to choose $k' = k_m \mathbf{e}_j - k_j \mathbf{e}_m$, in which case the solution takes the alternate form:

$$
\begin{aligned}
\cos(\mathbf{x} \cdot \mathbf{k}') \cosh\left(\frac{\mathbf{x} \wedge \mathbf{k}'}{\mathbf{i}}\right) \\
\sin(\mathbf{x} \cdot \mathbf{k}') \cosh\left(\frac{\mathbf{x} \wedge \mathbf{k}'}{\mathbf{i}}\right) \\
\cos(\mathbf{x} \cdot \mathbf{k}') \sinh\left(\frac{\mathbf{x} \wedge \mathbf{k}'}{\mathbf{i}}\right) \\
\sin(\mathbf{x} \cdot \mathbf{k}') \sinh\left(\frac{\mathbf{x} \wedge \mathbf{k}'}{\mathbf{i}}\right)
\end{aligned}
\tag{52.27}
$$

These sets of equations and the exponential form both remove the explicit reference to the pair of coordinates used in the original restriction

$$
\left\langle \mathbf{x} \wedge \mathbf{k} \mathbf{e}_j \wedge \mathbf{k} \right\rangle_2 = 0
\tag{52.28}
$$

that was used in the proof that $e^{\mathbf{x}\mathbf{k}}$ was a solution.

53 HYPER COMPLEX NUMBERS AND SYMPLECTIC STRUCTURE

53.1 ON 4.2 HERMITIAN NORMS AND UNITARY GROUPS

These are some rather rough notes filling in some details on the treatment of [28].

Expanding equation 4.17

$$\begin{aligned}
J &= e_i \wedge f_i \\
a &= u_i e_i + v_i f_i \\
b &= x_i e_i + y_i f_i \\
B &= a \wedge b + (a \cdot J) \wedge (b \cdot J)
\end{aligned} \tag{53.1}$$

$$\begin{aligned}
a \wedge b &= (u_i e_i + v_i f_i) \wedge (x_j e_j + y_j f_j) \\
&= u_i x_j e_i \wedge e_j + u_i y_j e_i \wedge f_j + v_i x_j f_i \wedge e_j + v_i y_j f_i \wedge f_j
\end{aligned} \tag{53.2}$$

$$\begin{aligned}
a \cdot J &= u_i e_i \cdot (e_j \wedge f_j) + v_i f_i \cdot (e_j \wedge f_j) \\
&= u_j f_j - v_j e_j
\end{aligned} \tag{53.3}$$

Search and replace for $b \cdot J$ gives

$$\begin{aligned}
b \cdot J &= x_i e_i \cdot (e_j \wedge f_j) + y_i f_i \cdot (e_j \wedge f_j) \\
&= x_j f_j - y_j e_j
\end{aligned} \tag{53.4}$$

So we have

$$\begin{aligned}
(a \cdot J) \wedge (b \cdot J) &= (u_i f_i - v_i e_i) \wedge (x_j f_j - y_j e_j) \\
&= u_i x_j f_i \wedge f_j - u_i y_j f_i \wedge e_j - v_i x_j e_i \wedge f_j + v_i y_j e_i \wedge e_j
\end{aligned} \tag{53.5}$$

For

$$a \wedge b + (a \cdot J) \wedge (b \cdot J) = (u_i y_j - v_i x_j)(e_i \wedge f_j - f_i \wedge e_j) + (u_i x_j + v_i y_j)(e_i \wedge e_j + f_i \wedge f_j) \tag{53.6}$$

This shows why the elements were picked as a basis

$$e_i \wedge f_j - f_i \wedge e_j \tag{53.7}$$

$$e_i \wedge e_j + f_i \wedge f_j \tag{53.8}$$

The first of which is a multiple of $J_i = e_i \wedge f_i$ when $i = j$, and the second of which is zero if $i = j$.

53.2 5.1 CONSERVATION THEOREMS AND FLOWS

equation 5.10 is

$$\dot{f} = \dot{x} \cdot \nabla f = (\nabla f \wedge \nabla H) \cdot J \tag{53.9}$$

This one is not obvious to me. For $\dot{f}$ we have

$$\dot{f} = \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial q_i}\dot{q}_i + \overbrace{\boxed{\frac{\partial f}{\partial t}}}^{=0} \tag{53.10}$$

compare to

$$\begin{aligned} \dot{x} \cdot \nabla f &= (\dot{p}_i e_i + \dot{q}_i f_i) \cdot (e_j \frac{\partial f}{\partial p_j} + f_j \frac{\partial f}{\partial q_j}) \\ &= \dot{p}_i \frac{\partial f}{\partial p_i} + \dot{q}_i \frac{\partial f}{\partial q_i} \end{aligned} \tag{53.11}$$

Okay, this part matches the first part of (5.10). Writing this in terms of the Hamiltonian relation (5.9) $\dot{x} = \nabla H \cdot J$ we have

$$\begin{aligned} \dot{f} &= (\nabla H \cdot J) \cdot \nabla f \\ &= \nabla f \cdot (\nabla H \cdot J) \end{aligned} \tag{53.12}$$

The relation $a \cdot (b \cdot (c \wedge d)) = (a \wedge b) \cdot (c \wedge d)$, can be used here to factor out the J, we have

$$\begin{aligned} \dot{f} &= \nabla f \cdot (\nabla H \cdot J) \\ &= (\nabla f \wedge \nabla H) \cdot J \end{aligned} \tag{53.13}$$

which completes (5.10).
Also with $f = H$ since H was also specified as having no explicit time dependence, one has

$$\dot{H} = (\nabla H \wedge \nabla H) \cdot J = 0 \cdot J = 0 \tag{53.14}$$

54

NEWTON'S METHOD FOR INTERSECTION OF CURVES IN A PLANE

54.1 MOTIVATION

Reading the blog post Problem solving, artificial intelligence and computational linear algebra some variations of Newton's method for finding local minimums and maximums are given.

While I had seen the Hessian matrix eons ago in the context of back propagation feedback methods, Newton's method itself I remember as a first order root finding method. Here I refresh my memory what that simpler Newton's method was about, and build on that slightly to find the form of the solution for the intersection of an arbitrarily oriented line with a curve, and finally the problem of refining an approximation for the intersection of two curves using the same technique.

54.2 ROOT FINDING AS THE INTERSECTION WITH A HORIZONTAL

The essence of Newton's method for finding roots is following the tangent from the point of first guess down to the line that one wants to intersect with the curve. This is illustrated in fig. 54.1.

Algebraically, the problem is that of finding the point x_1, which is given by the tangent

$$\frac{f(x_0) - b}{x_0 - x_1} = f'(x_0). \tag{54.1}$$

Rearranging and solving for x_1, we have

$$x_1 = x_0 - \frac{f(x_0) - b}{f'(x_0)} \tag{54.2}$$

If one presumes convergence, something not guaranteed, then a first guess, if good enough, will get closer and closer to the target with each iteration. If this first guess is far from the target, following the tangent line could ping pong you to some other part of the curve, and it is possible not to find the root, or to find some other one.

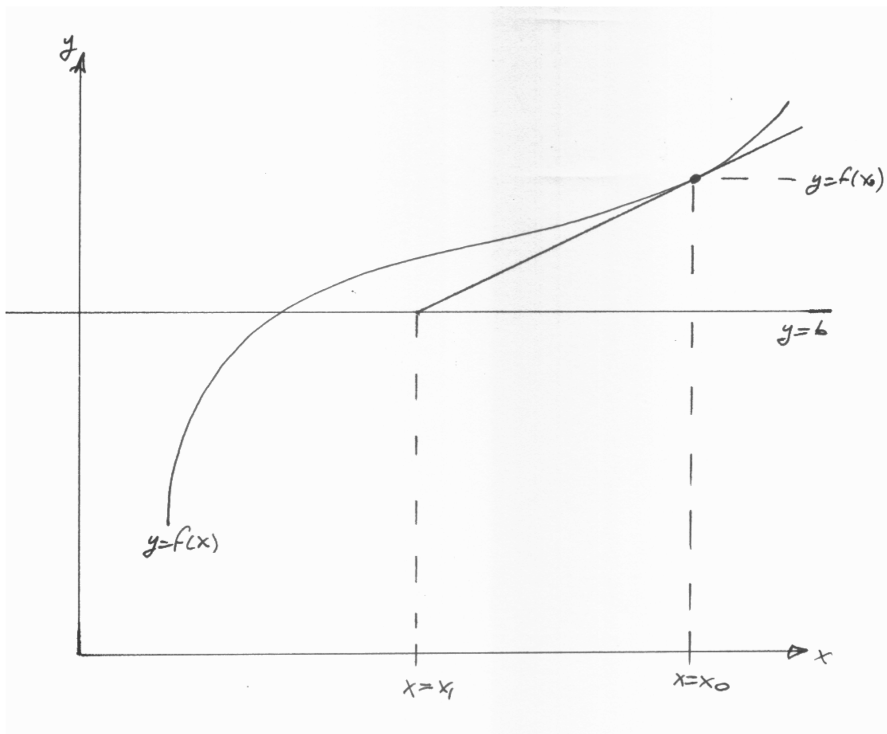

Figure 54.1: Refining an approximate horizontal intersection

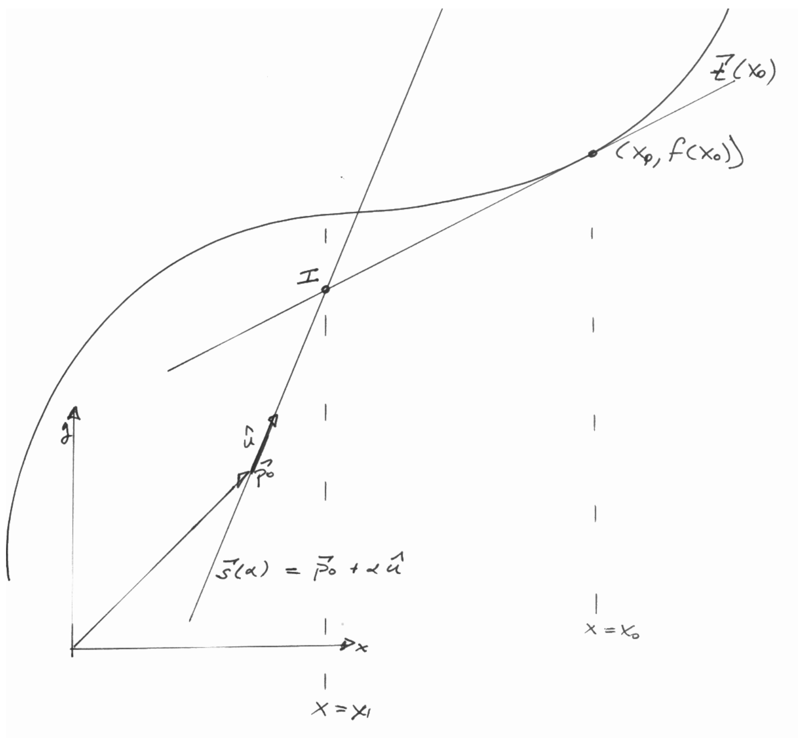

Figure 54.2: Refining an approximation for the intersection with an arbitrarily oriented line

54.3 INTERSECTION WITH A LINE

The above pictorial treatment works nicely for the intersection of a horizontal line with a curve. Now consider the intersection of an arbitrarily oriented line with a curve, as illustrated in fig. 54.2. Here it is useful to setup the problem algebraically from the beginning. Our problem is really still just that of finding the intersection of two lines. The curve itself can be considered the set of end points of the vector

$$\mathbf{r}(x) = x\mathbf{e}_1 + f(x)\mathbf{e}_2, \tag{54.3}$$

for which the tangent direction vector is

$$\mathbf{t}(x) = \frac{d\mathbf{r}}{dx} = \mathbf{e}_1 + f'(x)\mathbf{e}_2. \tag{54.4}$$

The set of points on this tangent, taken at the point x_0, can also be written as a vector, namely

$$(x_0, f(x)) + \alpha\mathbf{t}(x_0). \tag{54.5}$$

For the line to intersect this, suppose we have one point on the line $\mathbf{p}_0$, and a direction vector for that line $\hat{\mathbf{u}}$. The points on this line are therefore all the endpoints of

$$\mathbf{p}_0 + \beta\hat{\mathbf{u}}. \tag{54.6}$$

Provided that the tangent and the line of intersection do in fact intersect then our problem becomes finding α or β after equating eq. (54.5) and eq. (54.6). This is the solution of

$$(x_0, f(x_0)) + \alpha\mathbf{t}(x_0) = \mathbf{p}_0 + \beta\hat{\mathbf{u}}. \tag{54.7}$$

Since we do not care which of α or β we solve for, setting this up as a matrix equation in two variables is not the best approach. Instead we wedge both sides with $\mathbf{t}(x_0)$ (or $\hat{\mathbf{u}}$), essentially using Cramer's method. This gives

$$\left((x_0, f(x_0)) - \mathbf{p}_0\right) \wedge \mathbf{t}(x_0) = \beta\hat{\mathbf{u}} \wedge \mathbf{t}(x_0). \tag{54.8}$$

If the lines are not parallel, then both sides are scalar multiples of $\mathbf{e}_1 \wedge \mathbf{e}_2$, and dividing out one gets

$$\beta = \frac{\big((x_0, f(x_0)) - \mathbf{p}_0\big) \wedge \mathbf{t}(x_0)}{\hat{\mathbf{u}} \wedge \mathbf{t}(x_0)}. \tag{54.9}$$

Writing out $\mathbf{t}(x_0) = \mathbf{e}_1 + f'(x_0)\mathbf{e}_2$, explicitly, this is

$$\beta = \frac{\big((x_0, f(x_0)) - \mathbf{p}_0\big) \wedge \big(\mathbf{e}_1 + f'(x_0)\mathbf{e}_2\big)}{\hat{\mathbf{u}} \wedge \big(\mathbf{e}_1 + f'(x_0)\mathbf{e}_2\big)}. \tag{54.10}$$

Further, dividing out the common $\mathbf{e}_1 \wedge \mathbf{e}_2$ bivector, we have a ratio of determinants

$$\beta = \frac{\begin{vmatrix} x_0 - \mathbf{p}_0 \cdot \mathbf{e}_1 & f(x_0) - \mathbf{p}_0 \cdot \mathbf{e}_2 \\ 1 & f'(x_0) \end{vmatrix}}{\begin{vmatrix} \hat{\mathbf{u}} \cdot \mathbf{e}_1 & \hat{\mathbf{u}} \cdot \mathbf{e}_2 \\ 1 & f'(x_0) \end{vmatrix}}. \tag{54.11}$$

The final step in the solution is noting that the point of intersection is just

$$\mathbf{p}_0 + \beta\hat{\mathbf{u}}, \tag{54.12}$$

and in particular, the x coordinate of this is the desired result of one step of iteration

$$x_1 = \mathbf{p}_0 \cdot \mathbf{e}_1 + (\hat{\mathbf{u}} \cdot \mathbf{e}_1)\frac{\begin{vmatrix} x_0 - \mathbf{p}_0 \cdot \mathbf{e}_1 & f(x_0) - \mathbf{p}_0 \cdot \mathbf{e}_2 \\ 1 & f'(x_0) \end{vmatrix}}{\begin{vmatrix} \hat{\mathbf{u}} \cdot \mathbf{e}_1 & \hat{\mathbf{u}} \cdot \mathbf{e}_2 \\ 1 & f'(x_0) \end{vmatrix}}. \tag{54.13}$$

This looks a whole lot different than the original x_1 for the horizontal from back at eq. (54.2), but substitution of $\hat{\mathbf{u}} = \mathbf{e}_1$, and $\mathbf{p}_0 = b\mathbf{e}_2$, shows that these are identical.

54.4 INTERSECTION OF TWO CURVES

Can we generalize this any further? It seems reasonable that we would be able to use this Newton's method technique of following the tangent to refine an approximation for the intersection point of two general curves. This is not expected to be much harder, and the geometric idea is illustrated in fig. 54.3

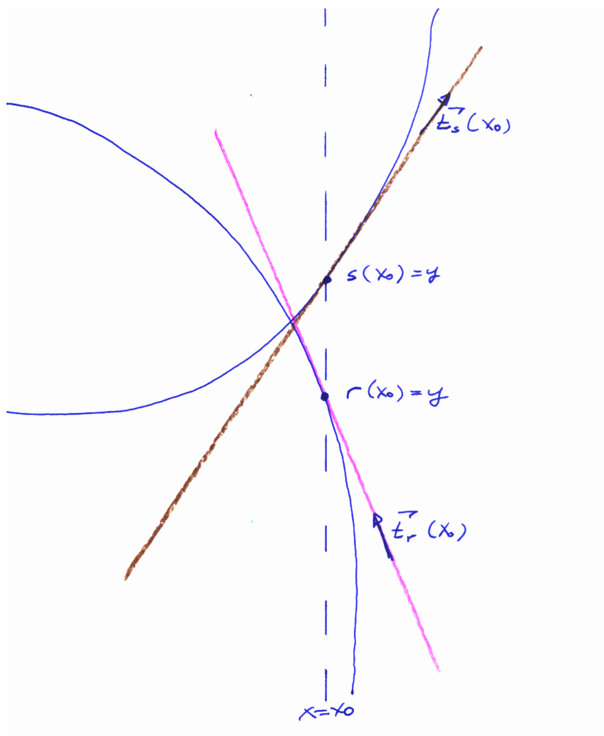

Figure 54.3: Refining an approximation for the intersection of two curves in a plane

The task at hand is to setup this problem algebraically. Suppose the two curves $s(x)$, and $r(x)$ are parametrized as vectors

$$\mathbf{s}(x) = x\mathbf{e}_1 + s(x)\mathbf{e}_2 \tag{54.14}$$

$$\mathbf{r}(x) = x\mathbf{e}_1 + r(x)\mathbf{e}_2. \tag{54.15}$$

Tangent direction vectors at the point x_0 are then

$$\mathbf{s}'(x_0) = \mathbf{e}_1 + s'(x_0)\mathbf{e}_2 \tag{54.16}$$

$$\mathbf{r}'(x_0) = \mathbf{e}_1 + r'(x_0)\mathbf{e}_2. \tag{54.17}$$

The intersection of interest is therefore the solution of

$$(x_0, s(x_0)) + \alpha\mathbf{s}' = (x_0, r(x_0)) + \beta\mathbf{r}'. \tag{54.18}$$

Wedging with one of tangent vectors $\mathbf{s}'$ or $\mathbf{r}'$ provides our solution. Solving for α this is

$$\alpha = \frac{(0, r(x_0) - s(x_0)) \wedge \mathbf{r}'}{\mathbf{s}' \wedge \mathbf{r}'} = \frac{\begin{vmatrix} 0 & r(x_0) - s(x_0) \\ \mathbf{r}' \cdot \mathbf{e}_1 & \mathbf{r}' \cdot \mathbf{e}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{s}' \cdot \mathbf{e}_1 & \mathbf{s}' \cdot \mathbf{e}_2 \\ \mathbf{r}' \cdot \mathbf{e}_1 & \mathbf{r}' \cdot \mathbf{e}_2 \end{vmatrix}} = -\frac{r(x_0) - s(x_0)}{r'(x_0) - s'(x_0)}. \tag{54.19}$$

To finish things off, we just have to calculate the new x coordinate on the line for this value of α, which gives us

$$x_1 = x_0 - \frac{r(x_0) - s(x_0)}{r'(x_0) - s'(x_0)}. \tag{54.20}$$

It is ironic that generalizing things to two curves leads to a tidier result than the more specific line and curve result from eq. (54.13). With a substitution of $r(x) = f(x)$, and $s(x) = b$, we once again recover the result eq. (54.2), for the horizontal line intersecting a curve.

54.5 FOLLOWUP

Having completed the play that I set out to do, the next logical step would be to try the min/max problem that leads to the Hessian. That can be for another day.

55

CENTER OF MASS OF A TOROIDAL SEGMENT

55.1 MOTIVATION

In I love when kids stump me, the center of mass of a toroidal segment is desired, and the simpler problem of a circular ring segment is considered.

Let us try the solid torus problem for fun using the geometric algebra toolbox. To setup the problem, it seems reasonable to introduce two angle, plus radius, toroidal parametrization as shown in fig. 55.1.

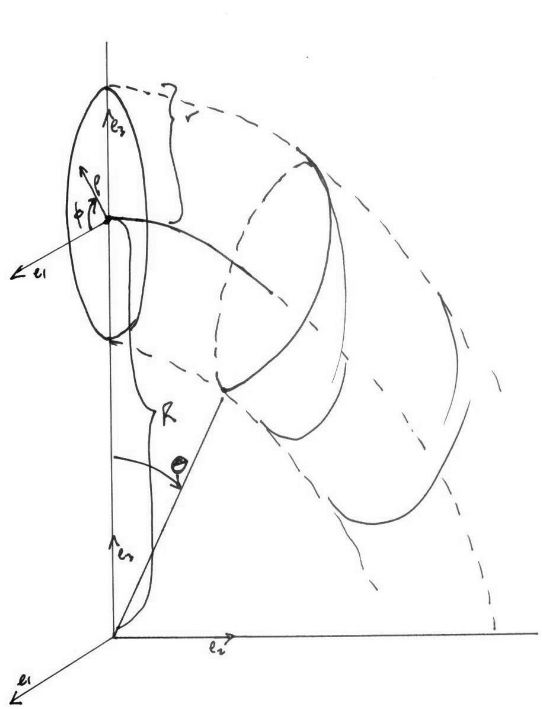

Figure 55.1: Toroidal parametrization

Our position vector to a point within the torus is then

$$\mathbf{r}(\rho, \theta, \phi) = e^{-j\theta/2} \left(\rho \mathbf{e}_1 e^{i\phi} + R\mathbf{e}_3\right) e^{j\theta/2} \tag{55.1a}$$

$$i = \mathbf{e}_1 \mathbf{e}_3 \tag{55.1b}$$

$$j = \mathbf{e}_3 \mathbf{e}_2 \tag{55.1c}$$

Here i and j for the bivectors are labels picked at random. They happen to have the quaternionic properties $ij = -ji$, and $i^2 = j^2 = -1$ which can be verified easily.

55.2 VOLUME ELEMENT

Before we can calculate the center of mass, we will need the volume element. I do not recall having ever seen such a volume element, so let us calculate it from scratch.

We want

$$dV = \pm \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \left(\frac{\partial \mathbf{r}}{\partial \rho} \wedge \frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} \right) d\rho d\theta d\phi, \tag{55.2}$$

so the first order of business is calculation of the partials. After some regrouping those are

$$\frac{\partial \mathbf{r}}{\partial \rho} = e^{-j\theta/2} \mathbf{e}_1 e^{i\phi} e^{j\theta/2} \tag{55.3a}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = e^{-j\theta/2} \left(R + \rho \sin\phi\right) \mathbf{e}_2 e^{j\theta/2} \tag{55.3b}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = e^{-j\theta/2} \rho \mathbf{e}_3 e^{i\phi} e^{j\theta/2}. \tag{55.3c}$$

For the volume element we want the wedge of each of these, and can instead select the trivector grades of the products, which conveniently wipes out a number of the interior exponentials

$$\frac{\partial \mathbf{r}}{\partial \rho} \wedge \frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} = \rho \left(R + \rho \sin\phi\right) \left\langle e^{-j\theta/2} \mathbf{e}_1 e^{i\phi} \mathbf{e}_2 \mathbf{e}_3 e^{i\phi} e^{j\theta/2} \right\rangle_3 \tag{55.4}$$

Note that $\mathbf{e}_1$ commutes with $j = \mathbf{e}_3\mathbf{e}_2$, so also with $e^{-j\theta/2}$. Also $\mathbf{e}_2\mathbf{e}_3 = -j$ anticommutes with i, so we have a conjugate commutation effect $e^{i\phi} j = j e^{-i\phi}$. Together the trivector grade selection reduces almost magically to just

$$\frac{\partial \mathbf{r}}{\partial \rho} \wedge \frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} = \rho \left(R + \rho \sin\phi\right) \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \tag{55.5}$$

Thus the volume element, after taking the positive sign, is

$$dV = \rho\left(R + \rho \sin\phi\right) d\rho d\theta d\phi. \tag{55.6}$$

As a check we should find that we can use this to calculate the volume of the complete torus, and obtain the expected $V = (2\pi R)(\pi r^2)$ result. That volume is

$$V = \int_{\rho=0}^{r}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} \rho\left(R + \rho \sin\phi\right) d\rho d\theta d\phi. \tag{55.7}$$

The sine term conveniently vanishes over the 2π interval, leaving just

$$V = \frac{1}{2} r^2 R(2\pi)(2\pi), \tag{55.8}$$

as expected.

55.3 CENTER OF MASS

With the prep done, we are ready to move on to the original problem. Given a toroidal segment over angle $\theta \in [-\Delta\theta/2, \Delta\theta/2]$, then the volume of that segment is

$$\Delta V = r^2 R \pi \Delta\theta. \tag{55.9}$$

Our center of mass position vector is then located at

$$\mathbf{R}\Delta V = \int_{\rho=0}^{r}\int_{\theta=-\Delta\theta/2}^{\Delta\theta/2}\int_{\phi=0}^{2\pi} e^{-j\theta/2}\left(\rho \mathbf{e}_1 e^{i\phi} + R\mathbf{e}_3\right) e^{j\theta/2} \rho\left(R + \rho \sin\phi\right) d\rho d\theta d\phi. \tag{55.10}$$

Evaluating the ϕ integrals we loose the $\int_0^{2\pi} e^{i\phi}$ and $\int_0^{2\pi} \sin\phi$ terms and are left with $\int_0^{2\pi} e^{i\phi} \sin\phi d\phi = i\pi/2$ and $\int_0^{2\pi} d\phi = 2\pi$. This leaves us with

$$\begin{aligned}
\mathbf{R}\Delta V &= \int_{\rho=0}^{r}\int_{\theta=-\Delta\theta/2}^{\Delta\theta/2} \left(e^{-j\theta/2}\rho^3 \mathbf{e}_3 \frac{\pi}{2} e^{j\theta/2} + 2\pi\rho R^2 \mathbf{e}_3 e^{j\theta}\right) d\rho d\theta && (55.11)\\
&= \int_{\theta=-\Delta\theta/2}^{\Delta\theta/2} \left(e^{-j\theta/2} r^4 \mathbf{e}_3 \frac{\pi}{8} e^{j\theta/2} + 2\pi\frac{1}{2} r^2 R^2 \mathbf{e}_3 e^{j\theta}\right) d\theta && (55.12)\\
&= \int_{\theta=-\Delta\theta/2}^{\Delta\theta/2} \left(e^{-j\theta/2} r^4 \mathbf{e}_3 \frac{\pi}{8} e^{j\theta/2} + \pi r^2 R^2 \mathbf{e}_3 e^{j\theta}\right) d\theta. && (55.13)
\end{aligned}$$

Since $\mathbf{e}_3 j = -j\mathbf{e}_3$, we have a conjugate commutation with the $e^{-j\theta/2}$ for just

$$\mathbf{R}\Delta V = \pi r^2 \left(\frac{r^2}{8} + R^2\right) \mathbf{e}_3 \int_{\theta=-\Delta\theta/2}^{\Delta\theta/2} e^{j\theta} d\theta \tag{55.14}$$

$$= \pi r^2 \left(\frac{r^2}{8} + R^2\right) \mathbf{e}_3 2 \sin(\Delta\theta/2). \tag{55.15}$$

A final reassembly, provides the desired final result for the center of mass vector

$$\mathbf{R} = \mathbf{e}_3 \frac{1}{R} \left(\frac{r^2}{8} + R^2\right) \frac{\sin(\Delta\theta/2)}{\Delta\theta/2}. \tag{55.16}$$

Presuming no algebraic errors have been made, how about a couple of sanity checks to see if the correctness of this seems plausible.

We are pointing in the z-axis direction as expected by symmetry. Good. For $\Delta\theta = 2\pi$, our center of mass vector is at the origin. Good, that is also what we expected. If we let $r \rightarrow 0$, and $\Delta\theta \rightarrow 0$, we have $\mathbf{R} = R\mathbf{e}_3$ as also expected for a tiny segment of "wire" at that position. Also good.

55.4 CENTER OF MASS FOR A CIRCULAR WIRE SEGMENT

As an additional check for the correctness of the result above, we should be able to compare with the center of mass of a circular wire segment, and get the same result in the limit $r \rightarrow 0$.

For that we have

$$Z(R\Delta\theta) = \int_{\theta=-\Delta\theta/2}^{\Delta\theta/2} Rie^{-i\theta} R d\theta \tag{55.17}$$

So we have

$$Z = \frac{1}{\Delta\theta} Ri \frac{1}{-i} (e^{-i\Delta\theta/2} - e^{i\Delta\theta/2}). \tag{55.18}$$

Observe that this is

$$Z = Ri \frac{\sin(\Delta\theta/2)}{\Delta\theta/2}, \tag{55.19}$$

which is consistent with the previous calculation for the solid torus when we let that solid diameter shrink to zero.

In particular, for 3/4 of the torus, we have $\Delta\theta = 2\pi(3/4) = 3\pi/2$, and

$$Z = Ri \frac{4 \sin(3\pi/4)}{3\pi} = Ri \frac{2\sqrt{2}}{3\pi} \approx 0.3Ri. \tag{55.20}$$

We are a little bit up the imaginary axis as expected.

I had initially somehow thought I had been off by a factor of two compared to the result by The Virtuosi, without seeing a mistake in either. But that now appears not to be the case, and I just screwed up plugging in the numbers. Once again, I should go to my eight year old son when I have arithmetic problems, and restrict myself to just the calculus and algebra bits.

Part VI

RELATIVITY

56

WAVE EQUATION BASED LORENTZ TRANSFORMATION DERIVATION

56.1 INTRO

My old electrodynamics book did a Lorentz transformation derivation using a requirement for invariance of a spherical light shell. ie:

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \tag{56.1}$$

Such an approach does not require any sophisticated math, but I never understood why that invariance condition could be assumed. To understand that intuitively, requires that you understand how the speed of light is constant. There are some subtleties involved in understanding that which are not necessarily obvious to me. A good illustration of this is Feynman's question about what speed to expect light to be going from a rocket ship going 100000 miles per second is a good example (ref: book: Six not so easy parts). Many people who would say "yes, the speed of light is constant" would still answer 280000 miles per second for that question.

I present below an alternate approach to deriving the Lorentz transformation. This has a bit more math (ie: partial differentials for change of variables in the wave equation). However, compared to really understanding that the speed of light is constant, I think it is easier to to conceptualize the idea that light is wavelike regardless of the motion of the observer since it (ie: an electrodynamic field) must satisfy the wave equation (ie: Maxwell's equations) regardless of the parametrization. I am curious if somebody else also new to the subject of relativity would agree?

The motivation for this is the fact that many introductory relativity texts mention how Lorentz observed that while Maxwell's equations were not invariant with respect to Galilean transformation, they were with his modified transformation.

I found it interesting to consider this statement with a bit of detail. The result is what I think is an interesting approach to introducing the Lorentz transformation.

56.2 THE WAVE EQUATION FOR ELECTRODYNAMIC FIELDS (LIGHT)

From Maxwell's equations one can show that in a charge and current free region the electric field and magnetic field both satisfy the wave equation:

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 0 \tag{56.2}$$

I believe this is the specific case where there are the light contains enough photons that the bulk (wavelike) phenomena dominate and quantum effects do not have to be considered.

The wikipedia article Electromagnetic radiation (under Derivation)

goes over this nicely.

Although this can be solved separately for either **E** or **B** the two are not independent. This dependence is nicely expressed by writing the electromagnetic field as a complete bivector $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$, and in that form the general solution to this equation for the combined electromagnetic field is:

$$\mathbf{F} = (\mathbf{E}_0 + \hat{\mathbf{k}} \wedge \mathbf{E}_0) f(\hat{\mathbf{k}} \cdot \mathbf{r} \pm ct) \tag{56.3}$$

Here f is any function, and represents the amplitude of the waveform.

56.3 VERIFYING LORENTZ INVARIANCE

The Lorentz transform for a moving (primed) frame where the motion is along the x axis is ($\beta = v/c, \gamma^{-2} = 1 - \beta^2$).

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix}$$

Or,

$$\begin{bmatrix} x \\ ct \end{bmatrix} = \gamma \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix}$$

Using this we can express the partials of the wave equation in the primed frame. Starting with the first derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial ct'}{\partial x}\frac{\partial}{\partial ct'} \\ &= \gamma\frac{\partial}{\partial x'} - \gamma\beta\frac{\partial}{\partial ct'} \end{aligned} \tag{56.4}$$

And:

$$\begin{aligned}\frac{\partial}{\partial ct} &= \frac{\partial x'}{\partial ct}\frac{\partial}{\partial x'} + \frac{\partial ct'}{\partial ct}\frac{\partial}{\partial ct'} \\ &= -\beta\gamma\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial ct'}\end{aligned} \tag{56.5}$$

Thus the second partials in terms of the primed frame are:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \gamma^2\left(\frac{\partial}{\partial x'} - \beta\frac{\partial}{\partial ct'}\right)\left(\frac{\partial}{\partial x'} - \beta\frac{\partial}{\partial ct'}\right) \\ &= \gamma^2\left(\frac{\partial^2}{\partial x'\partial x'} + \beta^2\frac{\partial^2}{\partial ct'\partial ct'} - \beta\left(\frac{\partial^2}{\partial x'\partial ct'}\frac{\partial^2}{\partial ct'\partial x'}\right)\right)\end{aligned} \tag{56.6}$$

$$\frac{\partial^2}{\partial ct\partial ct} = \gamma^2\left(\beta^2\frac{\partial^2}{\partial x'\partial x'} + \frac{\partial^2}{\partial ct'\partial ct'} - \beta\left(\frac{\partial^2}{\partial x'\partial ct'}\frac{\partial^2}{\partial ct'\partial x'}\right)\right) \tag{56.7}$$

Thus the wave equation transforms as:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial ct\partial ct} &= \gamma^2\left((1-\beta^2)\frac{\partial^2}{\partial x'\partial x'} + (\beta^2 - 1)\frac{\partial^2}{\partial ct'\partial ct'}\right) \\ &= \frac{\partial^2}{\partial x'\partial x'} - \frac{\partial^2}{\partial ct'\partial ct'}\end{aligned} \tag{56.8}$$

which is what we expect but nice to see written out in full without having to introduce Minkowski space, and its invariant norm, or use Einstein's subtle arguments from his "Relativity, the special and general theory" (the latter requires actual understanding whereas the former and this just require math).

56.4 DERIVE LORENTZ TRANSFORMATION REQUIRING INVARIANCE OF THE WAVE EQUATION

Now, lets look at a general change of variables for the wave equation for the electromagnetic field. This will include the Galilean transformation, as well as the Lorentz transformation above, as special cases.

Consider a two variable, scaled Laplacian:

$$\nabla^2 = m\frac{\partial^2}{\partial u^2} + n\frac{\partial^2}{\partial v^2} \tag{56.9}$$

and a linear change of variables defined by:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} \tag{56.10}$$

To perform the change of variables we need to evaluate the following:

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial v} &= \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} \end{aligned} \tag{56.11}$$

To compute the partials we must invert A. Writing

$$J = \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}, \tag{56.12}$$

that inverse is

$$A^{-1} = \frac{1}{\begin{vmatrix} e & f \\ g & h \end{vmatrix}} \begin{bmatrix} h & -f \\ -g & e \end{bmatrix}. \tag{56.13}$$

The first partials are therefore:

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{1}{J}\left(h\frac{\partial}{\partial x} - g\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial v} &= \frac{1}{J}\left(-f\frac{\partial}{\partial x} + e\frac{\partial}{\partial y}\right). \end{aligned} \tag{56.14}$$

Repeating for the second partials yields:

$$\begin{aligned} \frac{\partial^2}{\partial u^2} &= \frac{1}{J^2}\left(h^2\frac{\partial^2}{\partial x^2} + g^2\frac{\partial^2}{\partial y^2} - gh\left(\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y\partial x}\right)\right) \\ \frac{\partial^2}{\partial v^2} &= \frac{1}{J^2}\left(f^2\frac{\partial^2}{\partial x^2} + e^2\frac{\partial^2}{\partial y^2} - ef\left(\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y\partial x}\right)\right) \end{aligned} \tag{56.15}$$

That is the last calculation required to compute the transformed Laplacian:

$$\nabla^2 = \frac{1}{J^2}\Big((mh^2 + nf^2)\partial_{xx} + (mg^2 + ne^2)\partial_{yy} - (mgh + nef)(\partial_{xy} + \partial_{yx})\Big) \tag{56.16}$$

56.4.1 *Galilean transformation*

Lets apply this to the electrodynamics wave equation, first using a Galilean transformation $x = x' + vt$, $t = t'$, $\beta = v/c$.

$$\begin{bmatrix} x \\ ct \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ ct' \end{bmatrix} \tag{56.17}$$

$$\partial_{xx} - \frac{1}{c^2}\partial_{tt} = (1 - \beta^2)\partial_{x'x'} - \frac{1}{c^2}\partial_{t't'} + \frac{1}{c}\beta(\partial_{x't'} + \partial_{t'x'}) \tag{56.18}$$

Thus we see that the equations of light when subjected to a Galilean transformation have a different form after such a transformation. If this was correct we should see the effects of the mixed product terms and the reduced effect of the spatial component when there is any motion. However, light comes in a wave form regardless of the motion, so there is something wrong with application of this transformation to the equations of light. This was the big problem of physics over a hundred years ago before Einstein introduced relativity to explain all this.

56.4.2 *Determine the transformation of coordinates that retains the form of the equations of light*

Before Einstein, Lorentz worked out the transformation that left Maxwell's equation "invariant". I have not seen any text that actually showed this. Lorentz may have showed that his transformations left Maxwell's equations invariant in their full generality, however that complexity is not required to derive the transformation itself. Instead this can be done considering only the wave equation for light in source free space.

Let us define the matrix A for a general change of space and time variables in one spatial dimension:

$$\begin{bmatrix} x \\ ct \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x' \\ ct' \end{bmatrix} \tag{56.19}$$

Application of this to eq. (56.16) gives:

$$\partial_{xx} - \partial_{ct,ct} = \frac{1}{J^2}\Big((h^2 - f^2)\partial_{x'x'} + (g^2 - e^2)\partial_{ct',ct'} - (gh - ef)(\partial_{x',ct'} + \partial_{ct',x'})\Big) \tag{56.20}$$

Now, we observe that light has wavelike behavior regardless of our velocity (we do observe frequency variation with velocity but the fundamental waviness does not change). Once that is accepted as a requirement for a transformation of coordinates of the wave equation for light we get the Lorentz transformation.

Expressed mathematically, this means that we want eq. (56.20) to have the form:

$$\partial_{xx} - \partial_{ct,ct} = \partial_{x'x'} - \partial_{ct',ct'} \tag{56.21}$$

This requirement is equivalent to the following system of equations:

$$\begin{aligned} J &= eh - fg \\ h^2 - f^2 &= J^2 \\ g^2 - e^2 &= -J^2 \\ gh &= ef. \end{aligned} \tag{56.22}$$

Attempting to solve this in full generality for any J gets messy (ie: non-linear). To simplify things, it is not unreasonable to require $J = 1$, which is consistent with Galilean transformation, in particular for the limiting case as $v \rightarrow 0$.

Additionally, we want to give physical significance to these values e, f, g, h. Following Einstein's simple derivation of the Lorentz transformation, we do this by defining $x' = 0$ as the origin of the moving frame:

$$x' = \frac{1}{J}\begin{bmatrix} h & -f \end{bmatrix}\begin{bmatrix} x \\ ct \end{bmatrix} = 0 \tag{56.23}$$

This allows us to relate f, h to the velocity:

$$xh = fct \tag{56.24}$$

$$\implies \frac{dx}{dt} = \frac{fc}{h} = v, \tag{56.25}$$

and provides physical meaning to the first of the elements of the linear transformation:

$$f = h\frac{v}{c} = h\beta. \tag{56.26}$$

The significance and values of e, g, h remain to be determined. Substituting eq. (56.26) into our system of equations we have:

$$\begin{aligned}
h^2 - h^2\beta^2 &= 1 \\
g^2 - e^2 &= -1 \\
gh &= eh\beta.
\end{aligned} \tag{56.27}$$

From the first equation we have $h^2 = \frac{1}{1-\beta^2}$, which is what is usually designated γ^2. Considering the limiting case again of $v \to 0$, we want to take the positive root. Summarizing what has been found so far we have:

$$\begin{aligned}
h &= \frac{1}{\sqrt{1-\beta^2}} = \gamma \\
f &= \gamma\beta \\
g^2 - e^2 &= -1 \\
g &= e\beta.
\end{aligned} \tag{56.28}$$

Substitution of the last yields

$$e^2(\beta^2 - 1) = -1 \tag{56.29}$$

which means that $e^2 = \gamma^2$, or $e = \gamma$, and $g = \gamma\beta$ (again taking the positive root to avoid a reflective transformation in the limiting case). This completely specifies the linear transformation required to maintain the wave equation in wave equation form after a change of variables that includes a velocity transformation in one direction:

$$\begin{bmatrix} x \\ ct \end{bmatrix} = \gamma \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \begin{bmatrix} x' \\ ct' \end{bmatrix} \tag{56.30}$$

Inversion of this yields the typical one dimensional Lorentz transformation where the position and time of a moving frame is specified in terms of the inertial frame:

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix}. \tag{56.31}$$

That is perhaps more evident when this is written out explicitly in terms of velocity:

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ t' &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}. \end{aligned} \tag{56.32}$$

56.5 LIGHT SPHERE, AND RELATIVISTIC METRIC

TBD.

My old E&M book did this derivation using a requirement for invariance of a spherical light shell. ie:

$x^2 - c^2t^2 = x'^2 - c^2t'^2$.

That approach requires less math (ie: to partial derivatives or change of variables), but I never understood why that invariance condition could be assumed (to understand that intuitively, you have to understand the constancy of light phenomena, which has a few subtleties that are not obvious in my opinion).

I like my approach, which has a bit more math, but I think is easier (vs. light constancy) to conceptualize the idea that light is wavelike regardless of the motion of the observer since it (ie: an electrodynamic field) must satisfy the wave equation (ie: Maxwell's equations). I am curious if somebody else also new to the subject of relativity would agree?

56.6 DERIVE RELATIVISTIC DOPPLER SHIFT

TBD.

This is something I think would make sense to do considering solutions to the wave equation instead of utilizing more abstract wave number, and frequency four vector concepts. Have not yet done the calculations for this part.

57

EQUATIONS OF MOTION GIVEN MASS VARIATION WITH SPACETIME POSITION

57.1

Let

$$
\begin{aligned}
x &= \sum \gamma_\mu x^\mu \\
v &= \frac{dx}{d\tau} = \sum \gamma_\mu \dot{x}^\mu
\end{aligned}
\tag{57.1}
$$

Where whatever spacetime basis you pick has a corresponding reciprocal frame defined implicitly by:

$$
\gamma^\mu \cdot \gamma_\nu = \delta^\mu{}_\nu
$$

You could for example pick these so that these are orthonormal with:

$$
\begin{aligned}
\gamma_i^2 &= \gamma_i \cdot \gamma_i = -1 \\
\gamma^i &= -\gamma_i \\
\gamma^0 &= \gamma_0 \\
\gamma_0^2 &= 1 \\
\gamma_i \cdot \gamma_0 &= 0
\end{aligned}
\tag{57.2}
$$

ie: the frame vectors define the metric tensor implicitly:

$$
g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\tag{57.3}
$$

Now, my assumption is that given a Lagrangian of the form:

$$
\mathcal{L} = \frac{1}{2} m v^2 + \phi
\tag{57.4}
$$

That the equations of motion follow by computation of:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \tag{57.5}$$

I do not have any proof of this (I do not yet know any calculus of variations, and this is a guess based on intuition). It does however work out to get the covariant form of the Lorentz force law, so I think it is right.

To get the EOM we need the squared proper velocity. This is just c^2. Example: for an orthonormal spacetime frame one has:

$$\begin{aligned} v^2 &= \left(\gamma^0 c dt/d\tau + \sum \gamma_i dx/d\tau\right)^2 \\ &= \gamma\left(\gamma_0 c + \sum \gamma_i dx/dt\right)^2 \\ &= \gamma^2\left(c^2 - \mathbf{v}^2\right) = c^2 \end{aligned} \tag{57.6}$$

but if we leave this expressed in terms of coordinates (also do not have to assume the diagonal metric tensor, since we can use non-orthonormal basis vectors if desired) we have:

$$\begin{aligned} v^2 &= \left(\sum \gamma_\mu \dot{x}^\mu\right) \cdot \left(\sum \gamma_\nu \dot{x}^\nu\right) \\ &= \sum \gamma_\mu \cdot \gamma_\nu \dot{x}^\mu \dot{x}^\nu \\ &= \sum g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \end{aligned} \tag{57.7}$$

Therefore the Lagrangian to minimize is:

$$\mathcal{L} = \frac{1}{2} m \sum g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \phi. \tag{57.8}$$

Performing the calculations for the EOM, and in this case, also allowing mass to be a function of space or time position ($m = m(x^\mu)$)

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \\
\frac{\partial \phi}{\partial x^\mu} + \frac{1}{2}\frac{\partial m}{\partial x^\mu}\sum g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta &= \\
\frac{\partial \phi}{\partial x^\mu} + \frac{1}{2}\frac{\partial m}{\partial x^\mu} v^2 &= \\
&= \frac{1}{2}\frac{d}{d\tau} m \sum g_{\alpha\beta}\frac{\partial}{\partial x^\mu}\left(\dot{x}^\alpha \dot{x}^\beta\right) \\
&= \frac{1}{2}\frac{d}{d\tau} m \sum g_{\alpha\beta}\left(\delta^{\mu\alpha}\dot{x}^\beta + \dot{x}^\alpha \delta^{\mu\beta}\right) \\
&= \frac{d}{d\tau} m \sum g_{\alpha\mu}\dot{x}^\alpha \\
&= \sum \frac{\partial m}{\partial x^\beta}\dot{x}^\beta g_{\alpha\mu}\dot{x}^\alpha + m g_{\alpha\mu}\ddot{x}^\alpha
\end{aligned}
\tag{57.9}
$$

Now, the metric tensor values can be removed by summing since they can be used to switch upper and lower indices of the frame vectors:

$$
\begin{aligned}
\gamma_\mu &= \sum a^\nu \gamma^\nu \\
\gamma_\mu \cdot \gamma_\beta &= \sum a^\nu \gamma^\nu \cdot \gamma_\beta \\
&= \sum a^\nu \delta^\nu{}_\beta \\
&= a^\beta \\
\implies & \\
\gamma_\mu &= \sum \gamma_\mu \cdot \gamma_\nu \gamma^\nu \\
&= \sum g_{\mu\nu}\gamma^\nu
\end{aligned}
\tag{57.10}
$$

If you are already familiar with tensors then this may be obvious to you (but was not to me with only vector background).

Multiplying throughout by γ^μ, and summing over μ one has:

$$\begin{aligned}
\sum \gamma^\mu \left(\frac{\partial \phi}{\partial x^\mu} + \frac{1}{2} \frac{\partial m}{\partial x^\mu} v^2 \right) &= \sum \gamma^\mu \left(\frac{\partial m}{\partial x^\beta} \dot{x}^\beta g_{\alpha\mu} \dot{x}^\alpha + m g_{\alpha\mu} \ddot{x}^\alpha \right) \\
+ \left(\sum \gamma^\mu \frac{\partial}{\partial x^\mu} \right) \phi + \frac{1}{2} v^2 \left(\sum \gamma^\mu \frac{\partial}{\partial x^\mu} \right) m = & \\
&= \sum \frac{\partial m}{\partial x^\beta} \dot{x}^\beta \gamma^\mu \gamma_\alpha \cdot \gamma_\mu \dot{x}^\alpha + m \gamma^\mu \gamma_\alpha \cdot \gamma_\mu \ddot{x}^\alpha \\
&= \sum \frac{\partial m}{\partial x^\beta} \dot{x}^\beta \gamma_\alpha \dot{x}^\alpha + m \gamma_\alpha \ddot{x}^\alpha
\end{aligned} \tag{57.11}$$

Writing:

$$\nabla = \sum \gamma^\mu \frac{\partial}{\partial x^\mu}$$

This is:

$$\nabla \phi + \frac{1}{2} v^2 \nabla m = v \sum \frac{\partial m}{\partial x^\beta} \dot{x}^\beta + m \dot{v}$$

However,

$$\begin{aligned}
(\nabla m) \cdot v &= \left(\sum \gamma^\mu \frac{\partial m}{\partial x^\mu} \right) \cdot \left(\sum \gamma_\nu \dot{x}^\nu \right) \\
&= \sum \gamma^\mu \cdot \gamma_\nu \frac{\partial m}{\partial x^\mu} \dot{x}^\nu \\
&= \sum \delta^\mu{}_\nu \frac{\partial m}{\partial x^\mu} \dot{x}^\nu \\
&= \sum \frac{\partial m}{\partial x^\mu} \dot{x}^\mu = \frac{dm}{d\tau}
\end{aligned} \tag{57.12}$$

That allows for expressing the EOM in strict vector form:

$$\nabla \phi + \frac{1}{2} v^2 \nabla m = v \nabla m \cdot v + m \dot{v}. \tag{57.13}$$

However, there is still an asymmetry here, as one would expect a $\dot{m}v$ term. Regrouping slightly, and using some algebraic vector manipulation we have:

$$
\begin{aligned}
m\dot{v} + v\nabla m \cdot v - \frac{1}{2}v^2\nabla m &= \nabla\phi \\
2a \cdot b - ba &= ab \\
m\dot{v} + \frac{1}{2}v(\boxed{2\nabla m \cdot v - v\nabla m}) &= \\
m\dot{v} + \frac{1}{2}v(\nabla m)v &= \\
m\dot{v} + \frac{1}{2}(v\nabla m)v &= \\
m\dot{v} + \frac{1}{2}(2v \cdot \nabla m - \nabla m v)v &= \\
m\dot{v} + (v \cdot \nabla m)v - \frac{1}{2}(\nabla m v)v &= \\
m\dot{v} + \dot{m}v - \frac{1}{2}\nabla m(vv) &= \\
&\implies \\
\frac{d(mv)}{d\tau} = m\dot{v} + \dot{m}v &= \frac{1}{2}\nabla mc^2 + \nabla\phi \\
&= \nabla\left(\phi - \frac{1}{2}mc^2\right) \\
&= \nabla\left(\phi - \frac{1}{2}mv^2\right)
\end{aligned}
\tag{57.14}
$$

So, after a whole wack of algebra, the end result is to show the proper time variant of the Lagrangian equations imply that our proper force can be expressed as a (spacetime) gradient.

The caveat is that if the mass is allowed to vary, it also needs to be included in the generalized potential associated with the equation of motion.

57.1.1 *Summarizing*

We took this Lagrangian with kinetic energy and non-velocity dependent potential terms, where the mass in the kinetic energy term is allowed to vary with position or time. That plus the presumed proper-time Lagrange equations:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}mv^2 + \phi \\ \frac{\partial \mathcal{L}}{\partial x^\mu} &= \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu},\end{aligned} \tag{57.15}$$

when followed to their algebraic conclusion together imply that the equation of motion is:

$$\frac{d(mv)}{d\tau} = \nabla \mathcal{L}, \tag{57.16}$$

57.2 EXAMINE SPATIAL COMPONENTS FOR COMPARISON WITH NEWTONIAN LIMIT

Now, in the original version of this document, the signs for all the ϕ terms were inverted. This was changed since we want agreement with the Newtonian limit, and there is an implied sign change hiding in the above equations.

Consider, the constant mass case, where the Lagrangian is specified in terms of spatial quantities:

$$\mathcal{L} = \frac{1}{2}mv^2 + \phi = \frac{1}{2}m\gamma^2(c^2 - \mathbf{v}^2) = \frac{1}{2}m\gamma^2 c^2 - \gamma^2\left(\frac{1}{2}m\mathbf{v}^2 - \phi\right)$$

For $|\mathbf{v}| << c$, $\gamma \approx 1$, so we have a constant term in the Lagrangian of $\frac{1}{2}mc^2$ which will not change the EOM and can be removed. The remainder is our normal kinetic minus potential Lagrangian (the sign inversion on the entire remaining Lagrangian also will not change the EOM result).

Suppose one picks an orthonormal spacetime frame as given in the example metric tensor of eq. (57.3). To select our spatial quantities we wedge with γ_0.

For the left hand side of our equation of motion eq. (57.16) we have:

$$
\begin{aligned}
\frac{d(mv)}{d\tau} \wedge \gamma_0 &= \frac{d(mv) \wedge \gamma_0}{dt} \frac{dt}{d\tau} \\
&= \frac{dp \wedge \gamma_0}{dt} \frac{dt}{d\tau} \\
&= \frac{dt}{d\tau} \frac{d}{dt} m(c\gamma_0 + \sum \gamma_i \dot{x}^i) \wedge \gamma_0 \\
&= \frac{dt}{d\tau} \frac{d}{dt} m \sum (\gamma_i \wedge \gamma_0) \dot{x}^i \\
&= \frac{dt}{d\tau} \frac{d}{dt} m \sum \sigma_i \dot{x}^i \\
&= \frac{dt}{d\tau} \frac{d}{dt} (m\mathbf{v}\gamma) \\
&= \gamma \frac{d(\gamma \mathbf{p})}{dt}
\end{aligned}
\tag{57.17}
$$

Now, looking at the right hand side of the EOM we have (again for the constant mass case where we expect agreement with our familiar Newtonian EOM):

$$
\begin{aligned}
\nabla \left(\phi - \frac{1}{2} m v^2 \right) \wedge \gamma_0 &= (\nabla \phi) \wedge \gamma_0 \\
&= \sum \gamma^\mu \wedge \gamma_0 \frac{\partial \phi}{\partial x^\mu} \\
&= \sum \gamma^i \wedge \gamma_0 \frac{\partial \phi}{\partial x^i} \\
&= -\sum \gamma_i \wedge \gamma_0 \frac{\partial \phi}{\partial x^i} \\
&= -\sum \sigma_i \frac{\partial \phi}{\partial x^i} \\
&= -\boldsymbol{\nabla} \phi
\end{aligned}
\tag{57.18}
$$

Therefore in the limit $|\mathbf{v}| << c$ we have our agreement with the Newtonian EOM:

$$
\gamma \frac{d(\gamma \mathbf{p})}{dt} = -\boldsymbol{\nabla} \phi \approx \frac{d\mathbf{p}}{dt}
\tag{57.19}
$$

58

UNDERSTANDING FOUR VELOCITY TRANSFORM FROM REST FRAME

58.1

[10] writes $v = R\gamma_0 R^\dagger$, as a proper velocity expressed in terms of a rest frame velocity and a Lorentz boost. This was not clear to me, and would probably be a lot more obvious to me if I had fully read chapter 5, but in my defense it is a hard read without first getting more familiarity with basic relativity.

Let us just expand this out to see how this works. First thing to note is that there is an omitted factor of c, and I will add that back in here, since I am not comfortable enough without it explicitly for now.

With:

$$\begin{aligned}
\mathbf{v}/c &= \tanh(\alpha)\,\hat{\mathbf{v}} \\
R &= \exp(\alpha\hat{\mathbf{v}}/2)
\end{aligned} \tag{58.1}$$

We want to expansion this Lorentz boost exponential (see details section) and apply it to the rest frame basis vector. Writing $C = \cosh(\alpha/2)$, and $S = \sinh(\alpha/2)$, we have:

$$\begin{aligned}
v &= R(c\gamma_0)R^\dagger \\
&= c(C + \hat{\mathbf{v}}S)\gamma_0(C - \hat{\mathbf{v}}S) \\
&= c(C\gamma_0 + S\hat{\mathbf{v}}\gamma_0)(C - \hat{\mathbf{v}}S) \\
&= c\left(C^2\gamma_0 + SC\hat{\mathbf{v}}\gamma_0 - CS\gamma_0\hat{\mathbf{v}} - S^2\hat{\mathbf{v}}\gamma_0\hat{\mathbf{v}}\right)
\end{aligned} \tag{58.2}$$

Now, here things can start to get confusing since $\hat{\mathbf{v}}$ is a spatial quantity with vector-like spacetime basis bivectors $\sigma_i = \gamma_i\gamma_0$. Factoring out the γ_0 term, utilizing the fact that γ_0 and σ_i anticommute (see below).

$$
\begin{aligned}
v &= c\left(C^2 + S^2 + 2SC\hat{\mathbf{v}}\right)\gamma_0 \\
&= c\left(\cosh(\alpha) + \hat{\mathbf{v}}\sinh(\alpha)\right)\gamma_0 \\
&= c\cosh(\alpha)\left(1 + \hat{\mathbf{v}}\tanh(\alpha)\right)\gamma_0 \\
&= c\cosh(\alpha)\left(1 + \mathbf{v}/c\right)\gamma_0 \\
&= c\gamma\left(1 + \mathbf{v}/c\right)\gamma_0 \\
&= \gamma\left(c\gamma_0 + \sum v^i\gamma_i\right) \\
&= \frac{dt}{d\tau}\left(c\gamma_0 + \sum v^i\gamma_i\right) \\
&= \frac{dt}{d\tau}\frac{d}{dt}\left(ct\gamma_0 + \sum x^i\gamma_i\right) \\
&= \frac{dt}{d\tau}\frac{d}{dt}\sum x^\mu\gamma_\mu \\
&= \frac{d}{d\tau}\sum x^\mu\gamma_\mu \\
&= \frac{dx}{d\tau}
\end{aligned}
\tag{58.3}
$$

So, we get the end result that demonstrates that a Lorentz boost applied to the rest event vector $x = x^0\gamma_0 = ct\gamma_0$ directly produces the four velocity for the motion from the new viewpoint. This makes some intuitive sense, but I do not feel this is necessarily obvious without demonstration.

This also explains how the text is able to use the wedge and dot product ratios with the γ_0 basis vector to produce the relative spatial velocity. If one introduces a rest frame proper velocity of $w = \frac{d}{dt}\left(ct\gamma_0\right) = c\gamma_0$, then one has:

$$
\begin{aligned}
v \cdot w &= \left(\sum \frac{dx^\mu}{d\tau}\gamma_\mu\right) \cdot \left(c\gamma_0\right) \\
&= c^2\gamma
\end{aligned}
\tag{58.4}
$$

$$
\begin{aligned}
v \wedge w &= \left(\sum \frac{dx^\mu}{d\tau}\gamma_\mu\right) \wedge (c\gamma_0) \\
&= \left(\sum \frac{dx^i}{d\tau}\gamma_i\right) \wedge (c\gamma_0) \\
&= c \sum \frac{dx^i}{d\tau}\sigma_i \\
&= c\frac{dt}{d\tau} \sum \frac{dx^i}{dt}\sigma_i \\
&= c\gamma \sum \frac{dx^i}{dt}\sigma_i
\end{aligned}
\tag{58.5}
$$

Combining these one has the spatial observer dependent relative velocity:

$$
\frac{v \wedge w}{v \cdot w} = \frac{1}{c} \sum \frac{dx^i}{dt}\sigma_i = \frac{\mathbf{v}}{c}
\tag{58.6}
$$

58.1.1 *Invariance of relative velocity?*

What is not clear to me is whether this can be used to determine the relative velocity between two particles in the general case, when one of them is not a rest frame velocity (time progression only at a fixed point in space.) The text seems to imply this is the case, so perhaps it is obvious to them only and not me;)

This can be verified relatively easily for the extreme case, where one boosts both the w, and v velocities to measure v in its rest frame.

Expressed mathematically this is:

$$
\begin{aligned}
w &= c\gamma_0 \\
v &= RwR^\dagger \\
v' &= R^\dagger v R = R^\dagger R c\gamma_0 R^\dagger R = c\gamma_0 \\
w' &= R^\dagger w R
\end{aligned}
\tag{58.7}
$$

Now, this last expression for w' can be expanded brute force as was done initially to calculate v (and I in fact did that initially without thinking). The end result matches what should have been the intuitive expectation, with the velocity components all negated in a conjugate like fashion:

$$
w' = \gamma\left(c\gamma_0 - \sum v^i\gamma_i\right)
$$

With this result we have:

$$v' \cdot w' = c\gamma_0 \cdot \gamma \left(c\gamma_0 - \sum v^i \gamma_i \right) = \gamma c^2$$

$$\begin{aligned} v' \wedge w' &= c\gamma_0 \wedge \gamma \left(c\gamma_0 - \sum v^i \gamma_i \right) \\ &= -c\gamma \sum v^i \gamma_0 \gamma_i \\ &= c\gamma \sum v^i \sigma_i \end{aligned} \tag{58.8}$$

Dividing the two we have the following relative velocity between the two proper velocities:

$$\frac{v' \wedge w'}{v' \cdot w'} = \frac{1}{c} \sum v^i \sigma_i = \mathbf{v}/c.$$

Lo and behold, this is the same as when the first event worldline was in its rest frame, so we have the same relative velocity regardless of which of the two are observed at rest. The remaining obvious question is how to show that this is a general condition, assuming that it is.

58.1.2 *General invariance?*

Intuitively, I would guess that this is fact the case because when only two particles are considered, the result should be the same independent of which of the two is considered at rest.

Mathematically, I would express this statement by saying that if one has a Lorentz boost that takes $v' = TvT^\dagger$ to its rest frame, then application of this to both proper velocities leaves both the wedge and dot product parts of this ratio unchanged:

$$\begin{aligned} v \cdot w &= \left(T^\dagger v' T \right) \cdot \left(T^\dagger w' T \right) \\ &= \left\langle \left(T^\dagger v' T \right) \left(T^\dagger w' T \right) \right\rangle \\ &= \left\langle T^\dagger v' w' T \right\rangle \\ &= \left\langle T^\dagger v' \cdot w' T \right\rangle + \overset{= 0}{\boxed{\left\langle T^\dagger v' \wedge w' T \right\rangle}} \\ &= (v' \cdot w') \left\langle T^\dagger T \right\rangle \\ &= v' \cdot w' \end{aligned} \tag{58.9}$$

$$
\begin{aligned}
v \wedge w &= \left(T^{\dagger} v' T\right) \wedge \left(T^{\dagger} w' T\right) \\
&= \left\langle \left(T^{\dagger} v' T\right) \left(T^{\dagger} w' T\right) \right\rangle_2 \\
&= \left\langle T^{\dagger} v' w' T \right\rangle_2 \\
&\qquad\qquad\qquad = 0 \\
&= \boxed{\left\langle T^{\dagger} v' \cdot w' T \right\rangle_2} + \left\langle T^{\dagger} v' \wedge w' T \right\rangle_2 \\
&= T^{\dagger} \left(v' \wedge w'\right) T
\end{aligned}
\tag{58.10}
$$

FIXME: can not those last T factors be removed somehow?

58.2 APPENDIX. OMITTED DETAILS FROM ABOVE

58.2.1 *exponential of a vector*

Understanding the vector exponential is a prerequisite above. This is defined and interpreted by series expansion as with matrix exponentials. Expanding in series the exponential of a vector $\mathbf{x} = x\hat{\mathbf{x}}$, we have:

$$
\begin{aligned}
\exp\left(\mathbf{x}\right) &= \sum \frac{\mathbf{x}^{2k}}{(2k)\,!} + \sum \frac{\mathbf{x}^{2k+1}}{(2k+1)\,!} \\
&= \sum \frac{x^{2k}}{(2k)\,!} + \hat{\mathbf{x}} \sum \frac{x^{2k+1}}{(2k+1)\,!} \\
&= \cosh\left(x\right) + \hat{\mathbf{x}} \sinh\left(x\right)
\end{aligned}
\tag{58.11}
$$

Notationally this can also be written:

$$
\exp\left(\mathbf{x}\right) = \cosh\left(\mathbf{x}\right) + \sinh\left(\mathbf{x}\right)
$$

But doing so will not really help.

58.2.2 $\mathbf{v}$ *anticommutes with* γ_0

$$
\begin{aligned}
\mathbf{v}\gamma_0 &= \sum v^i \sigma_i \gamma_0 \\
&= \sum v^i \gamma_i \gamma_0 \gamma_0 \\
&= -\sum v^i \gamma_0 \gamma_i \gamma_0 \\
&= -\gamma_0 \sum v^i \gamma_i \gamma_0 \\
&= -\gamma_0 \sum v^i \sigma_0 \\
&= -\gamma_0 \mathbf{v}
\end{aligned}
\tag{58.12}
$$

59

FOUR VECTOR DOT PRODUCT INVARIANCE AND LORENTZ ROTORS

59.1

Prof. Ramamurti Shankar's In the relativity lectures of [40] Prof. Shankar indicates that the four vector dot product is a Lorentz invariant. This makes some logical sense, but lets demonstrate it explicitly.

Start with a Lorentz transform matrix between coordinates for two four vectors (omitting the components perpendicular to the motion) :

$$\begin{bmatrix} x^1 \\ x^0 \end{bmatrix}' = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} y^1 \\ y^0 \end{bmatrix}' = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} y^1 \\ y^0 \end{bmatrix}$$

Now write out the dot product between the two vectors given the perceived length and time measurements for the same events in the moving frame:

$$\begin{aligned} X' \cdot Y' &= \gamma^2 \left((-\beta x^1 + x^0)(-\beta y^1 + y^0) - (x^1 - \beta x^0)(y^1 - \beta y^0) \right) \\ &= \gamma^2 \left((\beta^2 x^1 y^1 + x^0 y^0) + x^0 y^1 (-\beta + \beta) + x^1 y^0 (-\beta + \beta) - (x^1 y^1 + \beta^2 x^0 y^0) \right) \\ &= \gamma^2 \left(x^0 y^0 (1 - \beta^2) - (1 - \beta^2) x^1 y^1 \right) \\ &= x^0 y^0 - x^1 y^1 \\ &= X \cdot Y \end{aligned} \tag{59.1}$$

This completes the proof of dot product Lorentz invariance. An automatic consequence of this is invariance of the Minkowski length.

59.1.1 *Invariance shown with hyperbolic trig functions*

Dot product or length invariance can also be shown with the hyperbolic representation of the Lorentz transformation:

$$\begin{bmatrix} x^1 \\ x^0 \end{bmatrix}' = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \begin{bmatrix} x^1 \\ x^0 \end{bmatrix} \tag{59.2}$$

Writing $S = \sinh(\alpha)$, and $C = \cosh(\alpha)$ for short, this gives:

$$\begin{aligned} X' \cdot Y' &= \left((-S x^1 + C x^0)(-S y^1 + C y^0) - (C x^1 - S x^0)(C y^1 - S y^0)\right) \\ &= \left((S^2 x^1 y^1 + C^2 x^0 y^0) + x^0 y^1 (-S C + S C) + x^1 y^0 (-S C + S C) - (C^2 x^1 y^1 + S^2 x^0 y^0)\right) \\ &= \left(x^0 y^0 (C^2 - S^2) - (C^2 - S^2) x^1 y^1\right) \\ &= x^0 y^0 - x^1 y^1 \\ &= X \cdot Y \end{aligned} \tag{59.3}$$

This is not really any less work.

59.2 GEOMETRIC PRODUCT FORMULATION OF LORENTZ TRANSFORM

We can show the above invariance almost trivially when we write the Lorentz boost in exponential form. However we first have to show how to do so.

Writing the spacetime bivector $\gamma_{10} = \gamma_1 \wedge \gamma_0$ for short, lets calculate the exponential of this spacetime bivector, as scaled with a rapidity angle α :

$$\exp(\gamma_{10}\alpha) = \sum \frac{(\gamma_{10}\alpha)^k}{k!} \tag{59.4}$$

Now, the spacetime bivector has a unit square:

$$\gamma_{10}{}^2 = \gamma_{1010} = -\gamma_{1001} = -\gamma_{11} = 1$$

so, we can split the sum of eq. (59.4) into even and odd parts, and pull out the common bivector factor:

$$\exp(\gamma_{10}\alpha) = \sum \frac{\alpha^{2k}}{(2k)!} + \gamma_{10} \sum \frac{\alpha^{2k+1}}{(2k+1)!} = \cosh(\alpha) + \gamma_{10}\sinh(\alpha) \tag{59.5}$$

59.2.1 *Spatial rotation*

So, this quite a similar form as bivector exponential with a Euclidean metric. For such a space the bivector had a negative square, just like the complex unit imaginary, which allowed for the normal trigonometric split of the exponential:

$$\exp(\mathbf{e}_{12}\theta) = \sum(-1)^k \frac{\theta^{2k}}{(2k)!} + \mathbf{e}_{12} \sum(-1)^k \frac{\theta^{2k+1}}{(2k+1)!} = \cos(\theta) + \mathbf{e}_{12}\sin(\theta) \tag{59.6}$$

Now, with the Minkowski metric having a negative square for purely spatial components, how does a purely spacial bivector behave when squared? Let us try it with

$$\gamma_{12}{}^2 = \gamma_{1212} = -\gamma_{1221} = \gamma_{11} = -1$$

This also has a square that behaves like the unit imaginary, so we can do spacial rotations with rotors like we can with Euclidean space. However, we have to invert the sign of the angle when using a Minkowski metric. Take a specific example of a 90 degree rotation in the x-y plane, expressed in complex form:

$$\begin{aligned} R_{\pi/2}(\gamma_1) &= \gamma_1 \exp(\gamma_{12}\pi/2) \\ &= \gamma_1(0 + \gamma_{12}) \\ &= -\gamma_2 \end{aligned} \tag{59.7}$$

In general our Rotor equation with a Minkowski $(+,-,-,-)$ metric will be thus be:

$$R_\theta(x) = \exp(i\theta/2)x\exp(-i\theta/2) \tag{59.8}$$

Here i is a spatial bivector (a bivector with negative square), such as $\gamma_1 \wedge \gamma_2$, and the rotation sense is with increasing angle from γ_1 towards γ_2.

59.2.2 *Validity of the double sided spatial rotor formula*

To demonstrate the validity of eq. (59.8) one has to observe how the unit vectors γ_μ behave with respect to commutation, and how that behavior results in either commutation or conjugate commutation with the exponential rotor. Without any loss of generality one can restrict attention to a specific example, such as bivector γ_{12}. By inspection, γ_0, and γ_3 both commute since an even number of exchanges in position is required for either:

$$\begin{aligned}\gamma_0\gamma_{12} &= \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \\ &= \gamma_1 \wedge \gamma_2 \wedge \gamma_0 \\ &= \gamma_{12}\gamma_0\end{aligned} \tag{59.9}$$

For this reason, application of the double sided rotation does not change any such (perpendicular) vector that commutes with the rotor:

$$\begin{aligned}R_\theta(x_\perp) &= \exp(i\theta/2)x_\perp \exp(-i\theta/2) \\ &= x_\perp \exp(i\theta/2)\exp(-i\theta/2) \\ &= x_\perp\end{aligned} \tag{59.10}$$

Now for the basis vectors that lie in the plane of the spatial rotation we have anticommutation:

$$\begin{aligned}\gamma_1\gamma_{12} &= -\gamma_1\gamma_{21} \\ &= -\gamma_{121} \\ &= -\gamma_{12}\gamma_1\end{aligned} \tag{59.11}$$

$$\begin{aligned}\gamma_2\gamma_{12} &= \gamma_{21}\gamma_2 \\ &= -\gamma_{12}\gamma_2\end{aligned} \tag{59.12}$$

Given an understanding of how the unit vectors either commute or anticommute with the bivector for the plane of rotation, one can now see how these behave when multiplied by a rotor expressed exponentially:

$$\gamma_\mu \exp(i\theta) = \gamma_\mu \left(\cos(\theta) + i\sin(\theta)\right) = \begin{cases} \left(\cos(\theta) + i\sin(\theta)\right)\gamma_\mu & \text{if } \gamma_\mu \cdot i = 0 \\ \left(\cos(\theta) - i\sin(\theta)\right)\gamma_\mu & \text{if } \gamma_\mu \cdot i \neq 0 \end{cases} \tag{59.13}$$

The condition $\gamma_\mu \cdot i = 0$ corresponds to a spacelike vector perpendicular to the plane of rotation, or a timelike vector, or any combination of the two, whereas $\gamma_\mu \cdot i \neq 0$ is true for any spacelike vector that lies completely in the plane of rotation.

Putting this information all together, we now complete the verification that the double sided rotor formula leaves the perpendicular spacelike or the timelike components untouched. For

for purely spacelike vectors in the plane of rotation we recover the single sided complex form rotation as illustrated by the following x-y plane rotation:

$$\begin{aligned} R_\theta(x_\parallel) &= \exp(\gamma_{12}\theta/2) x_\parallel \exp(-\gamma_{12}\theta/2) \\ &= x_\parallel \exp(-\gamma_{12}\theta/2) \exp(-\gamma_{12}\theta/2) \\ &= x_\parallel \exp(-\gamma_{12}\theta) \end{aligned} \tag{59.14}$$

59.2.3 *Back to time space rotation*

Now, like we can express a spatial rotation in exponential form, we can do the same for the hyperbolic "rotation" matrix of eq. (59.2). Direct expansion [1] of the product is the easiest way to see that this is the case:

$$\left(\gamma_1 x^1 + \gamma_0 x^0\right) \exp(\gamma_{10}\alpha) = \left(\gamma_1 x^1 + \gamma_0 x^0\right) \left(\cosh(\alpha) + \gamma_{10}\sinh(\alpha)\right) \tag{59.15}$$

$$\begin{aligned} &\left(\gamma_1 x^1 + \gamma_0 x^0\right) \exp(\gamma_{10}\alpha) \\ &\quad = \gamma_1 \left(x^1 \cosh(\alpha) - x^0 \sinh(\alpha)\right) + \gamma_0 \left(x^0 \cosh(\alpha) - x^1 \sinh(\alpha)\right) \end{aligned} \tag{59.16}$$

As with the spatial rotation, full characterization of this exponential rotation operator, in both single and double sided form requires that one looks at how the various unit vectors commute with the unit bivector. Without loss of generality one can restrict attention to a specific case, as done with the γ_{10} above.

As in the spatial case, γ_2, and γ_3 both commute with $\gamma_{10} = \gamma_1 \wedge \gamma_0$. Example:

$$\gamma_2\gamma_{10} = \gamma_2 \wedge \gamma_1 \wedge \gamma_0 = \gamma_1 \wedge \gamma_0 \wedge \gamma_2 = \gamma_{10}\gamma_2$$

Now, consider each of the basis vectors in the spacetime plane.

$$\gamma_0\gamma_{10} = \gamma_{010} = \gamma_{01}\gamma_0 = -\gamma_{10}\gamma_0$$

1 The paper "Generalized relativistic velocity addition with spacetime algebra", http://arxiv.org/pdf/physics/0511247.pdf derives the bivector form of this Lorentz boost directly in an interesting fashion. Simple relativistic arguments are used that are quite similar to those of Einstein in his "Relativity, the special and general theory" appendix. This paper is written in a form that requires you to work out many of the details yourself (likely for brevity). However, once that extra work is done, I found the first half of that paper quite readable.

$$\gamma_1\gamma_{10} = \gamma_{110} = -\gamma_{101} = -\gamma_{10}\gamma_1$$

Both of the basis vectors in the spacetime plane anticommute with the bivector that describes the plane, and as a result we have a conjugate change in the exponential comparing left and right multiplication as with a spatial rotor. Summarizing for the general case by introducing a spacetime rapidity plane described by a bivector
$\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}\alpha$, we have:

$$\begin{aligned}\gamma_\mu \exp(\boldsymbol{\alpha}) &= \gamma_\mu \left(\cosh(\alpha) + \hat{\boldsymbol{\alpha}} \sinh(\alpha)\right) \\ &= \begin{cases} \left(\cosh(\alpha) + \hat{\boldsymbol{\alpha}} \sinh(\alpha)\right) \gamma_\mu & \text{if } \gamma_\mu \cdot \hat{\boldsymbol{\alpha}} = 0 \\ \left(\cosh(\alpha) - \hat{\boldsymbol{\alpha}} \sinh(\alpha)\right) \gamma_\mu & \text{if } \gamma_\mu \cdot \hat{\boldsymbol{\alpha}} \neq 0 \end{cases}\end{aligned} \tag{59.17}$$

Observe the similarity between eq. (59.13), and eq. (59.17) for spatial and spacetime rotors. Regardless of whether the plane is spacelike, or a spacetime plane we have the same rule:

$$\gamma_\mu \exp(\mathbf{B}) = \begin{cases} \exp(\mathbf{B})\gamma_\mu & \text{if } \gamma_\mu \cdot \hat{\mathbf{B}} = 0 \\ \exp(-\mathbf{B})\gamma_\mu & \text{if } \gamma_\mu \cdot \hat{\mathbf{B}} \neq 0 \end{cases} \tag{59.18}$$

Here, if $\mathbf{B}$ is a spacelike bivector ($\mathbf{B}^2 < 0$) we get trigonometric functions generated by the exponentials, and if it represents the spacetime plane $\mathbf{B}^2 > 0$ we get the hyperbolic functions. As with the spatial rotor formulation, we have the same result for the general signature bivector, and can write the generalized spacetime or spatial rotation as:

$$R_{\mathbf{B}}(x) = \exp(-\mathbf{B}/2) x \exp(\mathbf{B}/2) \tag{59.19}$$

Some care is required assigning meaning to the bivector angle $\mathbf{B}$. We have seen that this is an negatively oriented spatial rotation in the $\hat{\mathbf{B}}$ plane when spacelike. How about for the spacetime case? Lets go back and rewrite eq. (59.16) in terms of vector relations, with $\mathbf{v} = v\hat{\mathbf{v}}$

$$\begin{aligned}&\left(x^1\hat{\mathbf{v}} + x^0\gamma_0\right)\left(\frac{1}{\sqrt{1-|(\mathbf{v}/c)|^2}} + \frac{(\mathbf{v}/c)\gamma_0}{\sqrt{1-|(\mathbf{v}/c)|^2}}\right) \\ &\quad = \hat{\mathbf{v}}\gamma\left(x^1 - x^0 v/c\right) + \gamma_0\gamma\left(x^0 - x^1 v/c\right)\end{aligned} \tag{59.20}$$

This allows for the following identification:

$$\cosh(\alpha) + \hat{\mathbf{v}}\gamma_0 \sinh(\alpha) = \exp(\hat{\mathbf{v}}\gamma_0\alpha) = \frac{1 + (\mathbf{v}/c)\gamma_0}{\sqrt{1 - |\mathbf{v}/c|^2}}$$

which gives us the rapidity bivector (**B** above) in terms of the values we are familiar with:

$$\hat{\mathbf{v}}\gamma_0\alpha = \log\left(\frac{1 + (\mathbf{v}/c)\gamma_0}{\sqrt{1 - |\mathbf{v}/c|^2}}\right)$$

Or,

$$\mathbf{B} = \hat{\mathbf{v}}\gamma_0\alpha = \tanh^{-1}(v/c)\hat{\mathbf{v}}\gamma_0$$

Now since $|v/c| < 1$, the hyperbolic inverse tangent here can be expanded in (the slowly convergent) power series:

$$\tanh^{-1}(x) = \sum_{k=0} \frac{x^{2k+1}}{2k+1}$$

Observe that this has only odd powers, and $((\mathbf{v}/c)\gamma_0)^{2k+1} = \hat{\mathbf{v}}\gamma_0(v/c)^{2k+1}$. This allows for the notational nicety of working with the spacetime bivector directly instead of only its magnitude:

$$\mathbf{B} = \tanh^{-1}((\mathbf{v}/c)\gamma_0) \tag{59.21}$$

59.2.4 *FIXME*

Revisit the equivalence of the two identities above. How can one get from the log expression to the hyperbolic inverse tangent directly?

59.2.5 *Apply to dot product invariance*

With composition of rotation and boost rotors we can form a generalized Lorentz transformation. For example application of a rotation with rotor R, to a boost with spacetime rotor L_0, we get a combined more general transformation:

$$L(x) = R(L_0 x {L_0}^\dagger) R^\dagger$$

In both cases, the rotor and its reverse when multiplied are identity:

$$1 = RR^\dagger = LL^\dagger$$

It is not hard to see one can also compose an arbitrary set of rotations and boosts in the same fashion. The new rotor will also satisfy $LL^\dagger = 1$.

Application of such a rotor to a four vector we have:

$$X' = LXL^\dagger$$

$$Y' = LYL^\dagger$$

$$\begin{aligned}
X' \cdot Y' &= (LXL^\dagger) \cdot (LYL^\dagger) \\
&= \left\langle LXL^\dagger LYL^\dagger \right\rangle \\
&= \left\langle LXYL^\dagger \right\rangle \\
&= \left\langle L(X \cdot Y)L^\dagger \right\rangle + \left\langle L(X \wedge Y)L^\dagger \right\rangle \\
&= (X \cdot Y)\left\langle LL^\dagger \right\rangle \\
&= X \cdot Y
\end{aligned} \tag{59.22}$$

It is also clear that the four bivector $X \wedge Y$ will also be Lorentz invariant. This also implies that the geometric product of two four vectors XY will also be Lorentz invariant.

UPDATE (Aug 14): I do not recall my reasons for thinking that the bivector invariance was clear initially. It does not seem so clear now after the fact so I should have written it down.

60

LORENTZ TRANSFORMATION OF SPACETIME GRADIENT

60.1 MOTIVATION

We have observed that the wave equation is Lorentz invariant, and conversely that invariance of the form of the wave equation under linear transformation for light can be used to calculate the Lorentz transformation. Specifically, this means that we require the equations of light (wave equation) retain its form after a change of variables that includes a (possibly scaled) translation. The wave equation should have no mixed partial terms, and retain the form:

$$\left(\boldsymbol{\nabla}^2 - \partial_{ct}^2\right) F = \left(\boldsymbol{\nabla}'^2 - \partial_{ct'}^2\right) F = 0$$

Having expressed the spacetime gradient with a (STA) Minkowski basis, and knowing that the Maxwell equation written using the spacetime gradient is Lorentz invariant:

$$\nabla F = J,$$

we therefore expect that the square root of the wave equation (Laplacian) operator is also Lorentz invariant. Here this idea is explored, and we look at how the spacetime gradient behaves under Lorentz transformation.

60.1.1 *Lets do it*

Our spacetime gradient is

$$\nabla = \sum \gamma^\mu \frac{\partial}{\partial x^\mu}$$

Under Lorentz transformation we can transform the $x^1 = x$, and $x^0 = ct$ coordinates:

$$\begin{bmatrix} x' \\ ct' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} x \\ ct \end{bmatrix}$$

Set $c = 1$ for convenience, and use this to transform the partials:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \\ &= \gamma\left(\frac{\partial}{\partial x'} - \beta\frac{\partial}{\partial t'}\right)\end{aligned} \tag{60.1}$$

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \\ &= \gamma\left(-\beta\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)\end{aligned} \tag{60.2}$$

Inserting this into our expression for the gradient we have

$$\begin{aligned}\nabla &= \gamma^0\frac{\partial}{\partial t} + \gamma^1\frac{\partial}{\partial x} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z} \\ &= \gamma^0\gamma\left(-\beta\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right) + \gamma^1\gamma\left(\frac{\partial}{\partial x'} - \beta\frac{\partial}{\partial t'}\right) + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}.\end{aligned} \tag{60.3}$$

Grouping by the primed partials this is:

$$\nabla = \gamma\left(\gamma^0 - \beta\gamma^1\right)\frac{\partial}{\partial t'} + \gamma\left(\gamma^1 - \beta\gamma^0\right)\frac{\partial}{\partial x'} + \gamma^2\frac{\partial}{\partial y} + \gamma^3\frac{\partial}{\partial z}. \tag{60.4}$$

Lo and behold, the basis vectors with respect to the new coordinates appear to themselves transform as a Lorentz pair. Specifically:

$$\begin{bmatrix}\gamma^{1'} \\ \gamma^{0'}\end{bmatrix} = \gamma\begin{bmatrix}1 & -\beta \\ -\beta & 1\end{bmatrix}\begin{bmatrix}\gamma^1 \\ \gamma^0\end{bmatrix}$$

Now this is a bit curious looking since these new basis vectors are a funny mix of the original time and space basis vectors. Observe however that these linear combinations of the basis vectors $\gamma^{0'}$, and $\gamma^{1'}$ do behave just as adequately as timelike and spacelike basis vectors:

$$\begin{aligned}\gamma^{0'}\gamma^{0'} &= \gamma^2\left(-\beta\gamma^1 + \gamma^0\right)\left(-\beta\gamma^1 + \gamma^0\right) \\ &= \gamma^2\left(-\beta^2 + 1 - \beta\gamma^0\gamma^1 - \beta\gamma^1\gamma^0\right) \\ &= \gamma^2(-\beta^2 + 1 + \overbrace{\boxed{\beta\gamma^1\gamma^0 - \beta\gamma^1\gamma^0}}^{=0}) \\ &= 1\end{aligned} \tag{60.5}$$

and for the transformed "spacelike" vector, it squares like a spacelike vector:

$$
\begin{aligned}
\gamma^{1\prime}\gamma^{1\prime} &= \gamma^2 \left(\gamma^1 - \beta\gamma^0\right)\left(\gamma^1 - \beta\gamma^0\right) \\
&= \gamma^2 \left(-1 + \beta^2 - \beta\gamma^0\gamma^1 - \beta\gamma^1\gamma^0\right) \\
&= \gamma^2 (-1 + \beta^2 + \underbrace{\boxed{\beta\gamma^1\gamma^0 - \beta\gamma^1\gamma^0}}_{=0}) \\
&= -1
\end{aligned}
\tag{60.6}
$$

The conclusion is that like the wave equation, its square root, the spacetime gradient is also Lorentz invariant, and to achieve this invariance we transform both the coordinates and the basis vectors (there was no need to transform the basis vectors for the wave equation since it is a scalar equation).

In fact, this gives a very interesting way to view the Lorentz transform. It is not just notational that we can think of the spacetime gradient as one of the square roots of the wave equation. Like the vector square root of a scalar there are infinitely many such roots, all differing by an angle or rotation in the vector space:

$$
(R\mathbf{n}R^\dagger)^2 = 1
$$

Requiring the mixed signature (Minkowski) metric for the space requires only that we need a slightly different meaning for any of the possible rotations applied to the vector.

60.1.2 *transform the spacetime bivector*

I am not sure of the significance of the following yet, but it is interesting to note that the spacetime bivector for the transformed coordinate pair is also invariant:

$$
\begin{aligned}
\gamma^{1\prime}\gamma^{0\prime} &= \gamma^2 \left(\gamma^1 - \beta\gamma^0\right)\left(-\beta\gamma^1 + \gamma^0\right) \\
&= \gamma^2 \left(\beta - \beta + \beta^2\gamma^0\gamma^1 + \gamma^1\gamma^0\right) \\
&= \gamma^2 \left(1 - \beta^2\right)\gamma^1\gamma^0 \\
&= \gamma^1\gamma^0
\end{aligned}
\tag{60.7}
$$

We can probably use this to figure out how to transform bivector quantities like the electromagnetic field F.

61 GRAVITOELECTROMAGNETISM

61.1 SOME ROUGH NOTES ON READING OF GRAVITOELECTROMAGNETISM REVIEW

I found the GEM equations interesting, and explored the surface of them slightly. Here are some notes, mostly as a reference for myself ... looking at the GEM equations mostly generates questions, especially since I do not have the GR background to understand where the potentials (ie: what is that stress energy tensor $T_{\mu\nu}$) nor the specifics of where the metric tensor (perturbation of the Minkowski metric) came from.

61.2 DEFINITIONS

The article [32] outlines the GEM equations, which in short are

Scalar and potential fields

$$\Phi \approx \frac{GM}{r}, \quad \mathbf{A} \approx \frac{G}{c}\frac{\mathbf{J}\times\mathbf{x}}{r^3} \tag{61.1}$$

Gauge condition

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\mathbf{A}\right) = 0. \tag{61.2}$$

GEM fields

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B}\right), \quad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{61.3}$$

and finally the Maxwell-like equations are

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B}\right) \\
\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\mathbf{B}\right) &= 0 \\
\boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi G\rho \\
\boldsymbol{\nabla}\times\left(\frac{1}{2}\mathbf{B}\right) &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi G}{c}\mathbf{J}
\end{aligned} \tag{61.4}$$

61.3 STA FORM

As with Maxwell's equations a Clifford algebra representation should be possible to put this into a more symmetric form. Combining the spatial div and grads, following conventions from [10] we have

$$
\begin{aligned}
\boldsymbol{\nabla}\mathbf{E} &= 4\pi G\rho + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}I\mathbf{B}\right) \\
\boldsymbol{\nabla}\left(\frac{1}{2}I\mathbf{B}\right) &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi G}{c}\mathbf{J}
\end{aligned}
\tag{61.5}
$$

Or

$$
\left(\boldsymbol{\nabla} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\mathbf{E} + \frac{1}{2}I\mathbf{B}\right) = \frac{4\pi G}{c}\left(c\rho + \mathbf{J}\right) \tag{61.6}
$$

Left multiplication with γ_0, using a time positive metric signature ($(\gamma_0)^2 = 1$),

$$
\left(\boldsymbol{\nabla} - \frac{1}{c}\frac{\partial}{\partial t}\right)\gamma_0\left(-\mathbf{E} + \frac{1}{2}I\mathbf{B}\right) = \frac{4\pi G}{c}\left(c\rho\gamma_0 + J^i\gamma_i\right) \tag{61.7}
$$

But $\left(\boldsymbol{\nabla} - \frac{1}{c}\frac{\partial}{\partial t}\right)\gamma_0 = \gamma_i\partial_i - \gamma_0\partial_0 = -\gamma^\mu\partial_\mu = -\nabla$. Introduction of a four vector mass density $J = c\rho\gamma_0 + J^i\gamma_i = J^\mu\gamma_\mu$, and a bivector field $F = \mathbf{E} - \frac{1}{2}I\mathbf{B}$ this is

$$
\nabla F = -\frac{4\pi G}{c}J \tag{61.8}
$$

The gauge condition suggests a four potential $V = \Phi\gamma_0 + \mathbf{A}\gamma_0 = V^\mu\gamma_\mu$, where $V^0 = \Phi$, and $V^i = A^i/2$. This merges the space and time parts of the gauge condition

$$
\nabla\cdot V = \gamma^\mu\partial_\mu\cdot\gamma_\nu V^\nu = \partial_\mu V^\mu = \frac{1}{c}\frac{\partial\Phi}{\partial t} + \frac{1}{2}\partial_i A^i. \tag{61.9}
$$

It is reasonable to assume that $F = \nabla\wedge V$ as in electromagnetism. Let us see if this is the case

$$
\begin{aligned}
\mathbf{E} - I\mathbf{B}/2 &= -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B}\right) - I\boldsymbol{\nabla}\times\mathbf{A}/2 \\
&= -\gamma_i\partial_i\gamma_0 V^0 - \frac{1}{2}\partial_0 A^i\gamma_i\gamma_0 + \boldsymbol{\nabla}\wedge\mathbf{A}/2 \\
&= \gamma^i\partial_i\gamma_0 V^0 + \gamma^0\partial_0\gamma_i A^i/2 - \gamma_i\partial_i\wedge\gamma_j V^j \\
&= \gamma^i\partial_i\gamma_0 V^0 + \gamma^0\partial_0\gamma_i V^i + \gamma^i\partial_i\wedge\gamma_j V^j \\
&= \gamma^\mu\partial_\mu\wedge\gamma_\nu V^\nu \\
&= \nabla\wedge V
\end{aligned}
\tag{61.10}
$$

Okay, so in terms of potential we have the form as Maxwell's equation

$$\nabla(\nabla \wedge V) = -\frac{4\pi G}{c} J. \tag{61.11}$$

With the gauge condition $\nabla \cdot V = 0$, this produces the wave equation

$$\nabla^2 V = -\frac{4\pi G}{c} J. \tag{61.12}$$

In terms of the author's original equation 1.2 it appears that roughly $V^\mu = \bar{h}_{0\mu}$, and $J^\mu \propto T_{0\mu}$.

This is logically how he is able to go from that equation to the Maxwell form since both have the same four-vector wave equation form (when $T_{ij} \approx 0$). To give the potentials specific values in terms of mass and current distribution appears to be where the retarded integrals are used.

The author expresses $T^{\mu\nu}$ in terms of ρ, and mass current j, but the field equations are in terms of $T_{\mu\nu}$. What metric tensor is used to translate from upper to lower indices in this case. ie: is it $g_{\mu\nu}$, or $\eta_{\mu\nu}$?

61.4 LAGRANGIANS

61.4.1 *Field Lagrangian*

Since the electrodynamic equation and corresponding field Lagrangian is

$$\begin{aligned} \nabla(\nabla \wedge A) &= \frac{J}{\epsilon_0 c} \\ \mathcal{L} &= -\frac{\epsilon_0 c}{2}(\nabla \wedge A)^2 + A \cdot J \end{aligned} \tag{61.13}$$

Then, from eq. (61.11), the GEM field Lagrangian in covariant form is

$$\mathcal{L} = \frac{c}{8\pi G}(\nabla \wedge V)^2 + V \cdot J \tag{61.14}$$

Writing $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$, the scalar part of this Lagrangian is:

$$\mathcal{L} = -\frac{c}{16\pi G} F^{\mu\nu} F_{\mu\nu} + V^\sigma J_\sigma \tag{61.15}$$

Is this expression hiding in the Einstein field equations?

What is the Lagrangian for Newtonian gravity, and how do they compare?

61.4.2 *Interaction Lagrangian*

The metric (equation 1.4) in the article is given to be

$$
\begin{aligned}
ds^2 &= -c^2\left(1 - 2\frac{\Phi}{c^2}\right)dt^2 + \frac{4}{c}\left(\mathbf{A}\cdot d\mathbf{x}\right)dt + \left(1 + 2\frac{\Phi}{c^2}\right)\delta_{ij}dx^i dx^j \\
\implies \left|ds^2\right| &= c^2(d\tau)^2 = (dx^0)^2 - \sum_i (dx^i)^2 - 2\frac{V_0}{c^2}(dx^0)^2 - \frac{8}{c^2}V_i dx^i dx^0 - 2\frac{V_0}{c^2}\delta_{ij}dx^i dx^j
\end{aligned}
\tag{61.16}
$$

With $v = \gamma_\mu dx^\mu/d\tau$, the Lagrangian for interaction is

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}m\left|\frac{ds}{d\tau}\right|^2 \\
&= \frac{1}{2}mc^2 \\
&= \frac{1}{2}mv^2 - 2\frac{mV_0}{c^2}\sum_\mu (\dot{x}^\mu)^2 - \frac{8m}{c^2}V_i\dot{x}^0\dot{x}^i
\end{aligned}
\tag{61.17}
$$

$$
\mathcal{L} = \frac{1}{2}mv^2 - 2m\left(V_0\sum_\mu (\dot{x}^\mu/c)^2 + 4V_i(\dot{x}^0/c)(\dot{x}^i/c)\right)
\tag{61.18}
$$

Now, unlike the Lorentz force Lagrangian

$$
\mathcal{L} = \frac{1}{2}mv^2 + qA\cdot v/c,
\tag{61.19}
$$

the Lagrangian of eq. (61.18) is quadratic in powers of $\dot{x}^\mu$. There are remarks in the article saying that the non-covariant Lagrangian used to arrive at the Lorentz force equivalent was a first order approximation. Evaluation of this interaction Lagrangian does not produce anything like the $\dot{p}_\mu = \kappa F_{\mu\nu}\dot{x}^\nu$ that we see in electrodynamics.

The calculation is not interesting but the end result for reference is

$$
\begin{aligned}
\dot{p} = &\frac{4m}{c^2}\left((v\cdot\nabla V_0)\gamma^\mu v^\mu + 2(v\cdot\nabla V_i)(v^i\gamma^0 + v^0\gamma^i)\right) \\
&+ \frac{4m}{c^2}\left(V_0\gamma^\mu a^\mu + 2V_i(a^i\gamma^0 + a^0\gamma^i)\right) \\
&- \frac{2m}{c^2}\left(\sum_\mu (v^\mu)^2\nabla V_0 + 4v^0 v^i\nabla V_i\right)
\end{aligned}
\tag{61.20}
$$

This can be simplified somewhat, but no matter what it will be quadratic in the velocity coordinates.

The article also says that the line element is approximate. Has some of what is required for a more symmetric covariant interaction proper force been discarded?

61.5 CONCLUSION

The ideas here are interesting. At a high level, roughly, as I see it, the equation

$$\nabla^2 h_{0\mu} = T_{0\mu} \tag{61.21}$$

has exactly the same form as Maxwell's equations in covariant form, so you can define an antisymmetric field tensor equation in the same way, treating these elements of h, and the corresponding elements of T as a four vector potential and mass current.

That said, I do not have the GR background to know understand the introduction. For example, how to actually arrive at 1.2 or how to calculated your metric tensor in equation 1.4. I would have expected 1.4 to have a more symmetric form like the covariant Lorentz force Lagrangian ($v^2 + kA.v$), since you can get a Lorentz force like equation out of it. Because of the quadratic velocity terms, no matter how one varies that metric with respect to s as a parameter, one cannot get anything at all close to the electrodynamics Lorentz force equation $m\ddot{x}^\mu = qF_\mu \nu \dot{x}_\nu$, so the correspondence between electromagnetism and GR breaks down once one considers the interaction.

62 RELATIVISTIC DOPPLER FORMULA

62.1 TRANSFORM OF ANGULAR VELOCITY FOUR VECTOR

It was possible to derive the Lorentz boost matrix by requiring that the wave equation operator

$$\nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 \tag{62.1}$$

retain its form under linear transformation (56). Applying spatial Fourier transforms (115), one finds that solutions to the wave equation

$$\nabla^2 \psi(t, \mathbf{x}) = 0 \tag{62.2}$$

Have the form

$$\psi(t, \mathbf{x}) = \int A(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} d^3 k \tag{62.3}$$

Provided that $\omega = \pm c|\mathbf{k}|$. Wave equation solutions can therefore be thought of as continuously weighted superpositions of constrained fundamental solutions

$$\begin{aligned}\psi &= e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \\ c^2\mathbf{k}^2 &= \omega^2\end{aligned} \tag{62.4}$$

The constraint on frequency and wave number has the look of a Lorentz square

$$\omega^2 - c^2\mathbf{k}^2 = 0 \tag{62.5}$$

Which suggests that in additional to the spacetime vector

$$X = (ct, \mathbf{x}) = x^\mu \gamma_\mu \tag{62.6}$$

evident in the wave equation fundamental solution, we also have a frequency-wavenumber four vector

$$K = (\omega/c, \mathbf{k}) = k^\mu \gamma_\mu \tag{62.7}$$

The pair of four vectors above allow the fundamental solutions to be put explicitly into covariant form

$$K \cdot X = \omega t - \mathbf{k} \cdot \mathbf{x} = k_\mu x^\mu \tag{62.8}$$

$$\psi = e^{-iK \cdot X} \tag{62.9}$$

Let us also examine the transformation properties of this fundamental solution, and see as a side effect that K has transforms appropriately as a four vector.

$$\begin{aligned}
0 &= \nabla^2 \psi(t, \mathbf{x}) \\
&= \nabla'^2 \psi(t', \mathbf{x}') \\
&= \nabla'^2 e^{i(\mathbf{x}' \cdot \mathbf{k}' - \omega' t')} \\
&= -\left(\frac{\omega'^2}{c^2} - \mathbf{k}'^2 \right) e^{i(\mathbf{x}' \cdot \mathbf{k}' - \omega' t')}
\end{aligned} \tag{62.10}$$

We therefore have the same form of frequency wave number constraint in the transformed frame (if we require that the wave function for light is unchanged under transformation)

$$\omega'^2 = c^2 \mathbf{k}'^2 \tag{62.11}$$

Writing this as

$$0 = \omega^2 - c^2 \mathbf{k}^2 = \omega'^2 - c^2 \mathbf{k}'^2 \tag{62.12}$$

singles out the Lorentz invariant nature of the $(\omega, \mathbf{k})$ pairing, and we conclude that this pairing does indeed transform as a four vector.

62.2 APPLICATION OF ONE DIMENSIONAL BOOST

Having attempted to justify the four vector nature of the wave number vector K, now move on to application of a boost along the x-axis to this vector.

$$\begin{aligned}\begin{bmatrix}\omega' \\ ck'\end{bmatrix} &= \gamma \begin{bmatrix}1 & -\beta \\ -\beta & 1\end{bmatrix}\begin{bmatrix}\omega \\ ck\end{bmatrix} \\ &= \begin{bmatrix}\omega - vk \\ ck - \beta\omega\end{bmatrix}\end{aligned} \tag{62.13}$$

We can take ratios of the frequencies if we make use of the dependency between ω and k. Namely, $\omega = \pm ck$. We then have

$$\begin{aligned}\frac{\omega'}{\omega} &= \gamma(1 \mp \beta) \\ &= \frac{1 \mp \beta}{\sqrt{1-\beta^2}} \\ &= \frac{1 \mp \beta}{\sqrt{1-\beta}\sqrt{1+\beta}}\end{aligned} \tag{62.14}$$

For the positive angular frequency this is

$$\frac{\omega'}{\omega} = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \tag{62.15}$$

and for the negative frequency the reciprocal.

Deriving this with a Lorentz boost is much simpler than the time dilation argument in wikipedia doppler article [45]. EDIT: Later found exactly the above boost argument in the wiki k-vector article [43].

What is missing here is putting this in a physical context properly with source and reciever frequencies spelled out. That would make this more than just math.

63 POINCARE TRANSFORMATIONS

63.1 motivation

In [35] a Poincare transformation is used to develop the symmetric stress energy tensor directly, in contrast to the non-symmetric canonical stress energy tensor that results from spacetime translation.

Attempt to decode one part of this article, the use of a Poincare transformation.

63.2 incremental transformation in ga form

Equation (11) in the article, is labeled an infinitesimal Poincare transformation

$$x'^{\mu} = x'^{\mu} + \epsilon^{\mu}{}_{\nu} x^{\nu} + \epsilon^{\mu} \tag{63.1}$$

It is stated that an antisymmetrization condition $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. This is somewhat confusing since the infinitesimal transformation is given by a mixed upper and lower index tensor. Due to the antisymmetry perhaps this all a coordinate statement of the following vector to vector linear transformation

$$x' = x + \epsilon + A \cdot x \tag{63.2}$$

This transformation is less restricted than a plain old spacetime transformation, as it also contains a projective term, where x is projected onto the spacetime (or spatial) plane A (a bivector), plus a rotation in that plane.

Writing as usual

$$x = \gamma_{\mu} x^{\mu} \tag{63.3}$$

So that components are recovered by taking dot products, as in

$$x^{\mu} = x \cdot \gamma^{\mu} \tag{63.4}$$

For the bivector term, write

$$A = c \wedge d = c^\alpha d^\beta (\gamma_\alpha \wedge \gamma_\beta) \tag{63.5}$$

For

$$\begin{aligned}(A \cdot x) \cdot \gamma^\mu &= c^\alpha d^\beta x_\sigma ((\gamma_\alpha \wedge \gamma_\beta) \cdot \gamma^\sigma) \cdot \gamma^\mu \\ &= c^\alpha d^\beta x_\sigma ({\delta_\alpha}^\mu {\delta_\beta}^\sigma - {\delta_\beta}^\mu {\delta_\alpha}^\sigma) \\ &= (c^\mu d^\sigma - c^\sigma d^\mu) x_\sigma\end{aligned} \tag{63.6}$$

This allows for an identification $\epsilon^{\mu\sigma} = c^\mu d^\sigma - c^\sigma d^\mu$ which is antisymmetric as required. With that identification we can write eq. (63.1) via the equivalent vector relation eq. (63.2) if we write

$${\epsilon^\mu}_\sigma x^\sigma = (c^\mu d_\sigma - c_\sigma d^\mu) x^\sigma \tag{63.7}$$

Where ${\epsilon^\mu}_\sigma$ is defined implicitly in terms of components of the bivector $A = c \wedge d$.

Is this what a Poincare transformation is? The Poincare Transformation article suggests not. This article suggests that the Poincare transformation is a spacetime translation plus a Lorentz transformation (composition of boosts and rotations). That Lorentz transformation will not be antisymmetric however, so how can these be reconciled? The key is probably the fact that this was an infinitesimal Poincare transformation so lets consider a Taylor expansion of the Lorentz boost or rotation rotor, considering instead a transformation of the following form

$$\begin{aligned}x' &= x + \epsilon + R x \tilde{R} \\ R\tilde{R} &= 1\end{aligned} \tag{63.8}$$

In particular, let us look at the Lorentz transformation in terms of the exponential form

$$R = e^{I\theta/2} \tag{63.9}$$

Here θ is either the angle of rotation (when the bivector is a unit spatial plane such as $I = \gamma_k \wedge \gamma_m$), or a rapidity angle (when the bivector is a unit spacetime plane such as $I = \gamma_k \wedge \gamma_0$).

Ignoring the translation in eq. (63.8) for now, to calculate the first order term in Taylor series we need

$$\begin{aligned}\frac{dx'}{d\theta} &= \frac{dR}{d\theta} x \tilde{R} + R x \frac{d\tilde{R}}{d\theta} \\ &= \frac{dR}{d\theta} \tilde{R} R x \tilde{R} + R x \tilde{R} R \frac{d\tilde{R}}{d\theta} \\ &= \frac{1}{2}(\Omega x' + x' \tilde{\Omega})\end{aligned} \tag{63.10}$$

where

$$\frac{1}{2}\Omega = \frac{dR}{d\theta}\tilde{R} \tag{63.11}$$

Now, what is the grade of the product Ω? We have both $dR/d\theta$ and R in $\{\bigwedge^0 \oplus \bigwedge^2\}$ so the product can only have even grades $\Omega \in \{\bigwedge^0 \oplus \bigwedge^2 \oplus \bigwedge^4\}$, but the unitary constraint on R restricts this

Since $R\tilde{R} = 1$ the derivative of this is zero

$$\frac{dR}{d\theta}\tilde{R} + R\frac{d\tilde{R}}{d\theta} = 0 \tag{63.12}$$

Or

$$\frac{dR}{d\theta}\tilde{R} = -\left(\frac{dR}{d\theta}\tilde{R}\right)^{\tilde{}} \tag{63.13}$$

Antisymmetry rules out grade zero and four terms, leaving only the possibility of grade 2. That leaves

$$\frac{dx'}{d\theta} = \frac{1}{2}(\Omega x' - x'\Omega) = \Omega \cdot x' \tag{63.14}$$

And the first order Taylor expansion around $\theta = 0$ is

$$\begin{aligned} x'(d\theta) &\approx x'(\theta = 0) + (\Omega d\theta) \cdot x' \\ &= x + (\Omega d\theta) \cdot x' \end{aligned} \tag{63.15}$$

This has close to the postulated form in eq. (63.2), but differs in one notable way. The dot product with the antisymmetric form $A = \frac{1}{2}\frac{dR}{d\theta}\tilde{R}d\theta$ is a dot product with x' and not x! One can however invert the identity writing x in terms of x' (to first order)

$$x = x' - (\Omega d\theta) \cdot x' \tag{63.16}$$

Replaying this argument in fast forward for the inverse transformation should give us a relation for x' in terms of x and the incremental Lorentz transform

$$\begin{aligned} x' &= Rx\tilde{R} \\ &\Longrightarrow \\ x &= \tilde{R}x'R \end{aligned} \tag{63.17}$$

$$
\begin{aligned}
\frac{dx}{d\theta} &= \frac{d\tilde{R}}{d\theta} R\tilde{R}x'R + \tilde{R}x'R\tilde{R}\frac{dR}{d\theta} \\
&= \left(2\frac{d\tilde{R}}{d\theta}R\right)\cdot x
\end{aligned}
\tag{63.18}
$$

So we have our incremental transformation given by

$$
x' = x - \left(2\frac{d\tilde{R}}{d\theta}Rd\theta\right)\cdot x
\tag{63.19}
$$

63.3 CONSIDER A SPECIFIC INFINITESIMAL SPATIAL ROTATION

The signs and primes involved in arriving at eq. (63.19) were a bit confusing. To firm things up a bit considering a specific example is called for.

For a rotation in the x, y plane, we have

$$
\begin{aligned}
R &= e^{\gamma_1\gamma_2\theta/2} \\
x' &= Rx\tilde{R}
\end{aligned}
\tag{63.20}
$$

Here also it is easy to get the signs wrong, and it is worth pointing out the sign convention picked here for the Dirac basis is ${\gamma_0}^2 = -{\gamma_k}^2 = 1$. To verify that R does the desired job, we have

$$
\begin{aligned}
R\gamma_1\tilde{R} &= \gamma_1\tilde{R}^2 \\
&= \gamma_1 e^{\gamma_2\gamma_1\theta} \\
&= \gamma_1(\cos\theta + \gamma_2\gamma_1\sin\theta) \\
&= \gamma_1(\cos\theta - \gamma_1\gamma_2\sin\theta) \\
&= \gamma_1\cos\theta + \gamma_2\sin\theta
\end{aligned}
\tag{63.21}
$$

and

$$
\begin{aligned}
R\gamma_2\tilde{R} &= \gamma_2\tilde{R}^2 \\
&= \gamma_2 e^{\gamma_2\gamma_1\theta} \\
&= \gamma_2(\cos\theta + \gamma_2\gamma_1\sin\theta) \\
&= \gamma_2\cos\theta - \gamma_1\sin\theta
\end{aligned}
\tag{63.22}
$$

For γ_3 or γ_0, the quaternion R commutes, so we have

$$
\begin{aligned}
R\gamma_3\tilde{R} &= R\tilde{R}\gamma_3 = \gamma_3 \\
R\gamma_0\tilde{R} &= R\tilde{R}\gamma_0 = \gamma_0
\end{aligned}
\tag{63.23}
$$

(leaving the perpendicular basis directions unchanged).
Summarizing the action on the basis vectors in matrix form this is

$$
\begin{bmatrix}\gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3\end{bmatrix} \rightarrow
\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}
\begin{bmatrix}\gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3\end{bmatrix}
\tag{63.24}
$$

Observe that the basis vectors transform with the transposed matrix to the coordinates, and we have

$$
\gamma_0 x^0 + \gamma_1 x^1 + \gamma_2 x^2 + \gamma_3 x^3 \rightarrow \gamma_0 x^0 + x^1(\gamma_1 \cos\theta + \gamma_2 \sin\theta) + x^2(\gamma_2 \cos\theta - \gamma_1 \sin\theta) + \gamma_3 x^3
\tag{63.25}
$$

Dotting $x'^\mu = x' \cdot \gamma^\mu$ we have

$$
\begin{aligned}
x^0 &\rightarrow x^0 \\
x^1 &\rightarrow x^1 \cos\theta - x^2 \sin\theta \\
x^2 &\rightarrow x^1 \sin\theta + x^2 \cos\theta \\
x^3 &\rightarrow x^3
\end{aligned}
\tag{63.26}
$$

In matrix form this is the expected and familiar rotation matrix in coordinate form

$$
\begin{bmatrix}x^0 \\ x^1 \\ x^2 \\ x^3\end{bmatrix} \rightarrow
\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}
\begin{bmatrix}x^0 \\ x^1 \\ x^2 \\ x^3\end{bmatrix}
\tag{63.27}
$$

Moving on to the initial verification we have

$$
\begin{aligned}
2\frac{d\tilde{R}}{d\theta} &= 2\frac{d}{d\theta} e^{\gamma_2\gamma_1\theta/2} \\
&= \gamma_1\gamma_2 e^{\gamma_2\gamma_1\theta/2}
\end{aligned}
\tag{63.28}
$$

So we have

$$
\begin{aligned}
2\frac{d\tilde{R}}{d\theta}R &= \gamma_2\gamma_1 e^{\gamma_2\gamma_1\theta/2} e^{\gamma_1\gamma_2\theta/2} \\
&= \gamma_2\gamma_1
\end{aligned}
\tag{63.29}
$$

The antisymmetric form $\epsilon_{\mu\nu}$ in this case therefore appears to be nothing more than the unit bivector for the plane of rotation! We should now be able to verify the incremental transformation result from eq. (63.19), which is in this specific case now calculated to be

$$
x' = x + d\theta(\gamma_1\gamma_2) \cdot x \tag{63.30}
$$

As a final check let us look at the action of rotation part of the transformation eq. (63.30) on the coordinates x^μ. Only the x^1 and x^2 coordinates need be considered since there is no projection of γ_0 or γ_3 components onto the plane $\gamma_1\gamma_2$.

$$
\begin{aligned}
d\theta(\gamma_1\gamma_2) \cdot (x^1\gamma_1 + x^2\gamma_2) &= d\theta\left\langle \gamma_1\gamma_2(x^1\gamma_1 + x^2\gamma_2)\right\rangle_1 \\
&= d\theta(\gamma_2 x^1 - \gamma_1 x^2)
\end{aligned}
\tag{63.31}
$$

Now compare to the incremental transformation on the coordinates in matrix form. That is

$$
\begin{aligned}
\delta R &= d\theta \frac{d}{d\theta} \left.\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right|_{\theta=0} \\
&= d\theta \left.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right|_{\theta=0} \\
&= d\theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{63.32}
$$

So acting on the coordinate vector

$$\begin{aligned}
\delta R &= d\theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \\
&= d\theta \begin{bmatrix} 0 \\ -x^2 \\ x^1 \\ 0 \end{bmatrix}
\end{aligned} \tag{63.33}$$

This is exactly what we got above with the bivector dot product. Good.

63.4 CONSIDER A SPECIFIC INFINITESIMAL BOOST

For a boost along the x axis we have

$$\begin{aligned}
R &= e^{\gamma_0\gamma_1\alpha/2} \\
x' &= Rx\tilde{R}
\end{aligned} \tag{63.34}$$

Verifying, we have

$$\begin{aligned}
x^0\gamma_0 &\rightarrow x^0(\cosh\alpha + \gamma_0\gamma_1 \sinh\alpha)\gamma_0 \\
&= x^0(\gamma_0 \cosh\alpha - \gamma_1 \sinh\alpha)
\end{aligned} \tag{63.35}$$

$$\begin{aligned}
x^1\gamma_1 &\rightarrow x^1(\cosh\alpha + \gamma_0\gamma_1 \sinh\alpha)\gamma_1 \\
&= x^1(\gamma_1 \cosh\alpha - \gamma_0 \sinh\alpha)
\end{aligned} \tag{63.36}$$

Dot products recover the familiar boost matrix

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}' = \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{63.37}$$

Now, how about the incremental transformation given by eq. (63.19). A quick calculation shows that we have

$$x' = x + d\alpha(\gamma_0\gamma_1) \cdot x \tag{63.38}$$

Just like the eq. (63.30) case for a rotation in the xy plane, the antisymmetric form is again the unit bivector of the rotation plane (this time the unit bivector in the spacetime plane of the boost.)

This completes the examination of two specific incremental Lorentz transformations. It is clear that the result will be the same for an arbitrarily oriented bivector, and the original guess eq. (63.2) of a geometric equivalent of tensor relation eq. (63.1) was correct, provided that A is a unit bivector scaled by the magnitude of the incremental transformation.

The specific case not treated however are those transformations where the orientation of the bivector is allowed to change. Parameterizing that by angle is not such an obvious procedure.

63.5 IN TENSOR FORM

For an arbitrary bivector $A = a \wedge b$, we can calculate $\epsilon^{\sigma}{}_{\alpha}$. That is

$$\begin{aligned}
\epsilon^{\sigma\alpha} x_\alpha &= d\theta \frac{((a^\mu\gamma_\mu \wedge b^\nu\gamma_\nu) \cdot (x_\alpha\gamma^\alpha)) \cdot \gamma^\sigma}{\left|((a^\mu\gamma_\mu) \wedge (b^\nu\gamma_\nu)) \cdot ((a_\alpha\gamma^\alpha) \wedge (b_\beta\gamma^\beta))\right|^{1/2}} \\
&= \frac{a^\sigma b^\alpha - a^\alpha b^\sigma}{\left|a^\mu b^\nu (a_\nu b_\mu - a_\mu b_\nu)\right|^{1/2}} x_\alpha
\end{aligned} \tag{63.39}$$

So we have

$$\epsilon^{\sigma}{}_{\alpha} = d\theta \frac{a^\sigma b_\alpha - a_\alpha b^\sigma}{\left|a^\mu b^\nu (a_\nu b_\mu - a_\mu b_\nu)\right|^{1/2}} \tag{63.40}$$

The denominator can be subsumed into $d\theta$, so the important factor is just the numerator, which encodes an incremental boost or rotational in some arbitrary spacetime or spatial plane (respectively). The associated antisymmetry can be viewed as a consequence of the bivector nature of the rotor derivative rotor product.

Part VII

ELECTRODYNAMICS

64

MAXWELL'S EQUATIONS EXPRESSED WITH GEOMETRIC ALGEBRA

64.1 ON DIFFERENT WAYS OF EXPRESSING MAXWELL'S EQUATIONS

One of the most striking applications of the geometric product is the ability to formulate the eight Maxwell's equations in a coherent fashion as a single equation.

This is not a new idea, and this has been done historically using formulations based on quaternions (1910. dig up citation). A formulation in terms of antisymmetric second rank tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ (See: wiki:Formulation of Maxwell's equations in special relativity) reduces the eight equations to two, but also introduces complexity and obfuscates the connection to the physically measurable quantities.

A formulation in terms of differential forms (See: wiki:Maxwell's equations) is also possible. This does not have the complexity of the tensor formulation, but requires the electromagnetic field to be expressed as a differential form. This is arguably strange given a traditional vector calculus education. One also does not have to integrate a field in any fashion, so what meaning should be given to a electrodynamic field as a differential form?

64.1.1 *Introduction of complex vector electromagnetic field*

To explore the ideas, the starting point is the traditional set of Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{64.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{64.2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{64.3}$$

$$c^2 \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \frac{\mathbf{J}}{\epsilon_0} \tag{64.4}$$

It is customary in relativistic treatments of electrodynamics to introduce a four vector (x, y, z, ict). Using this as a hint, one can write the time partials in terms of ict and regrouping slightly

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{64.5}$$

$$\nabla \cdot (ic\mathbf{B}) = 0 \tag{64.6}$$

$$\nabla \times \mathbf{E} + \frac{\partial(ic\mathbf{B})}{\partial(ict)} = 0 \tag{64.7}$$

$$\nabla \times (ic\mathbf{B}) + \frac{\partial \mathbf{E}}{\partial(ict)} = i\frac{\mathbf{J}}{\epsilon_0 c} \tag{64.8}$$

There is no use of geometric or wedge products here, but the opposing signs in the two sets of curl and time partial equations is removed. The pairs of equations can be added together without loss of information since the original equations can be recovered by taking real and imaginary parts.

$$\nabla \cdot (\mathbf{E} + ic\mathbf{B}) = \frac{\rho}{\epsilon_0} \tag{64.9}$$

$$\nabla \times (\mathbf{E} + ic\mathbf{B}) + \frac{\partial(\mathbf{E} + ic\mathbf{B})}{\partial(ict)} = i\frac{\mathbf{J}}{\epsilon_0 c} \tag{64.10}$$

It is thus natural to define a combined electrodynamic field as a complex vector, expressing the natural orthogonality of the electric and magnetic fields

$$\mathbf{F} = \mathbf{E} + ic\mathbf{B} \tag{64.11}$$

The electric and magnetic fields can be recovered from this composite field by taking real and imaginary parts respectively, and we can now write write Maxwell's equations in terms of this single electrodynamic field

$$\nabla \cdot \mathbf{F} = \frac{\rho}{\epsilon_0} \tag{64.12}$$

$$\nabla \times \mathbf{F} + \frac{\partial \mathbf{F}}{\partial(ict)} = i\frac{\mathbf{J}}{\epsilon_0 c} \tag{64.13}$$

64.1.2 *Converting the curls in the pair of Maxwell's equations for the electrodynamic field to wedge and geometric products*

The above manipulations didn't make any assumptions about the structure of the "imaginary" denoted i above. What was implied was a requirement that $i^2 = -1$, and that i commutes with vectors. Both of these conditions are met by the use of the pseudoscalar for 3D Euclidean space $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. This is usually denoted I and we'll now switch notations for clarity. XX With multiplication of the second by a I factor to convert to a wedge product representation the remaining pair of equations can be written

$$\nabla \cdot \mathbf{F} = \frac{\rho}{\epsilon_0} \tag{64.14}$$

$$I\nabla \times \mathbf{F} + \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} = -\frac{\mathbf{J}}{\epsilon_0 c} \tag{64.15}$$

This last, in terms of the geometric product is,

$$\nabla \wedge \mathbf{F} + \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} = -\frac{\mathbf{J}}{\epsilon_0 c} \tag{64.16}$$

These equations can be added without loss

$$\nabla \cdot \mathbf{F} + \nabla \wedge \mathbf{F} + \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} = \frac{\rho}{\epsilon_0} - \frac{\mathbf{J}}{\epsilon_0 c} \tag{64.17}$$

Leading to the end result

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)\mathbf{F} = \frac{1}{\epsilon_0}\left(\rho - \frac{\mathbf{J}}{c}\right) \tag{64.18}$$

Here we have all of Maxwell's equations as a single differential equation. This gives a hint why it is hard to separately solve these equations for the electric or magnetic field components (the partials of which are scattered across the original eight different equations.) Logically the electric and magnetic field components have to be kept together.

Solution of this equation will require some new tools. Minimally, some relearning of existing vector calculus tools is required.

64.1.3 *Components of the geometric product Maxwell equation*

Explicit expansion of this equation, again using $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, will yield a scalar, vector, bivector, and pseudoscalar components, and is an interesting exercise to verify the simpler field equation really describes the same thing.

FIXME: the following is busted. Both $\boldsymbol{\nabla} \cdot (I\mathbf{B})$ and $\boldsymbol{\nabla} \wedge (I\mathbf{B})$ are malformed.

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right)\mathbf{F} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + I\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E} + \nabla \cdot I\mathbf{B} + \nabla \wedge I\mathbf{B} \tag{64.19}$$

The imaginary part of the field can be multiplied out as bivector components explicitly

$$\begin{aligned} I\mathbf{B} &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(\mathbf{e}_1 B_1 + \mathbf{e}_2 B_2 + \mathbf{e}_3 B_3) \\ &= \mathbf{e}_2\mathbf{e}_3 B_1 + \mathbf{e}_3\mathbf{e}_1 B_2 + \mathbf{e}_1\mathbf{e}_2 B_3 \end{aligned} \tag{64.20}$$

which allows for direct calculation of the following

$$\nabla \wedge I\mathbf{B} = I\nabla \cdot \mathbf{B} \tag{64.21}$$

$$\nabla \cdot I\mathbf{B} = -\nabla \times \mathbf{B} \tag{64.22}$$

That, plus writing the electric field curl term in terms of the cross product

$$\nabla \wedge \mathbf{E} = I\nabla \times \mathbf{E} \tag{64.23}$$

This allows for grouping of real and imaginary scalar and real and imaginary vector (bivector) components

$$(\nabla \cdot \mathbf{E}) + I(\nabla \cdot \mathbf{B}) + \left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B}\right) + I\left(\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right) \tag{64.24}$$

$$= \frac{\rho}{\epsilon_0} + I(0) + \left(-\frac{\mathbf{J}}{\epsilon_0 c}\right) + I\mathbf{0} \tag{64.25}$$

Comparing each of the left and right side components recovers the original set of four (or eight depending on your point of view) Maxwell's equations.

64.2 FUTURE: COMPARISON TO GRAVITATION?

The high school electrostatics equation, where ρ is either a continuous distribution or a spatial delta function for point masses:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{(\mathbf{r}-\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^2}dV' \tag{64.26}$$

As a field equation this is written:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \tag{64.27}$$

but this is both not relativistically correct nor does is include the propagation effects for "electrostatics" interactions which occur at the speed of light.

We need the other three components of the Maxwell's equation eq. (64.18), to get the propagation and relativistic corrections.

Compare this to newton's gravitational field equation:

$$\mathbf{G}(\mathbf{r}) = -G \int \rho(\mathbf{r}') \frac{(\mathbf{r} - \mathbf{r}')}{(\mathbf{r} - \mathbf{r}')^2} dV' \tag{64.28}$$

which can be written as a field equation as:

$$\nabla \cdot \mathbf{G}(\mathbf{r}) = 4\pi G \rho(\mathbf{r}). \tag{64.29}$$

If one assumes that electrodynamics and gravitation have the same form then is the corrected form of the gravitational field equation with respect to relativity and propagation at the speed of light as follows:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla\right) \mathbf{G}(\mathbf{r}) = 4\pi G \rho(\mathbf{r}) \tag{64.30}$$

Is this correct in any sense? Perhaps it matches the special relativity results but not the general relativity ones?

65 BACK TO MAXWELL'S EQUATIONS

65.1

Having observed and demonstrated that the Lorentz transformation is a natural consequence of requiring the electromagnetic wave equation retains the form of the wave equation under change of space and time variables that includes a velocity change in one spacial direction.

Lets step back and look at Maxwell's equations in more detail. In particular looking at how we get from integral to differential to GA form. Some of this is similar to the approach in GAFP, but that text is intended for more mathematically sophisticated readers.

We start with the equations in SI units:

$$
\begin{aligned}
\int_{S\,\text{(closed boundary of V)}} \mathbf{E}\cdot\hat{\mathbf{n}}dA &= \frac{1}{\epsilon_0}\int_V \rho dV \\
\int_{S\,\text{(any closed surface)}} \mathbf{B}\cdot\hat{\mathbf{n}}dA &= 0 \\
\int_{C\text{(boundary of S)}} \mathbf{E}\cdot d\mathbf{x} &= -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot\hat{\mathbf{n}}dA \\
\int_{C\text{(boundary of S)}} \mathbf{B}\cdot d\mathbf{x} &= \mu_0\left(I + \epsilon_0\int_S \frac{\partial \mathbf{E}}{\partial t}\cdot\hat{\mathbf{n}}dA\right)
\end{aligned}
\tag{65.1}
$$

As the surfaces and corresponding loops or volumes are made infinitely small, these equations (FIXME: demonstrate), can be written in differential form:

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\
\boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)
\end{aligned}
\tag{65.2}
$$

These are respectively, Gauss's Law for E, Gauss's Law for B, Faraday's Law, and the Ampere/Maxwell's Law.

This differential form can be manipulated to derive the wave equation for free space, or the wave equation with charge and current forcing terms in other space.

65.1.1 *Regrouping terms for dimensional consistency*

Derivation of the wave equation can be done nicely using geometric algebra, but first is it helpful to put these equations in a more dimensionally pleasant form. Lets relate the dimensions of the electric and magnetic fields and the constants μ_0, ϵ_0.

From Faraday's equation we can relate the dimensions of **B**, and **E**:

$$\frac{[\mathbf{E}]}{[d]} = \frac{[\mathbf{B}]}{[t]} \tag{65.3}$$

We therefore see that **B**, and **E** are related dimensionally by a velocity factor.

Looking at the dimensions of the displacement current density in the Ampere/Maxwell equation we see:

$$\frac{[\mathbf{B}]}{[d]} = [\mu_0\epsilon_0]\frac{[\mathbf{E}]}{[t]} \tag{65.4}$$

From the two of these the dimensions of the $\mu_0\epsilon_0$ product can be seen to be:

$$[\mu_0\epsilon_0] = \frac{[t]^2}{[d]^2} \tag{65.5}$$

So, we see that we have a velocity factor relating **E**, and **B**, and we also see that we have a squared velocity coefficient in Ampere/Maxwell's law. Let us factor this out explicitly so that **E** and **B** take dimensionally consistent form:

$$\begin{aligned}
\tau &= \frac{t}{\sqrt{\mu_0\epsilon_0}} \\
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \cdot \frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial}{\partial\tau}\frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}} \\
\boldsymbol{\nabla} \times \frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}} &= \sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{J} + \frac{\partial\mathbf{E}}{\partial\tau}
\end{aligned} \tag{65.6}$$

65.1.2 *Refactoring the equations with the geometric product*

Now that things are dimensionally consistent, we are ready to group these equations using the geometric product

$$\mathbf{A}\mathbf{B} = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\wedge\mathbf{B} = \mathbf{A}\cdot\mathbf{B} + i\mathbf{A}\times\mathbf{B} \tag{65.7}$$

where $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the spatial pseudoscalar. By grouping the divergence and curl terms for each of $\mathbf{B}$, and $\mathbf{E}$ we can write vector gradient equations for each of the Electric and Magnetic fields:

$$\boldsymbol{\nabla}\mathbf{E} = \frac{\rho}{\epsilon_0} - i\frac{\partial}{\partial\tau}\frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}} \tag{65.8}$$

$$\boldsymbol{\nabla}\frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}} = i\sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{J} + i\frac{\partial\mathbf{E}}{\partial\tau} \tag{65.9}$$

Multiplication of eq. (65.9) with i, and adding to eq. (65.8), we have Maxwell's equations consolidated into:

$$\boldsymbol{\nabla}\left(\mathbf{E} + i\frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}}\right) = \left(\frac{\rho}{\epsilon_0} - \sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{J}\right) - \frac{\partial}{\partial\tau}\left(\mathbf{E} + \frac{i\mathbf{B}}{\sqrt{\mu_0\epsilon_0}}\right) \tag{65.10}$$

We see that we have a natural combined Electrodynamic field:

$$\mathbf{F} = \epsilon_0\left(\mathbf{E} + i\frac{\mathbf{B}}{\sqrt{\mu_0\epsilon_0}}\right) = \epsilon_0\left(\mathbf{E} + ic\mathbf{B}\right) \tag{65.11}$$

Note that here the ϵ_0 factor has been included as a convenience to remove it from the charge and current density terms later. We have also looked ahead slightly and written:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \tag{65.12}$$

The dimensional analysis above showed that this had dimensions of velocity. This velocity is in fact the speed of light, and we will see this more exactly when looking at the wave equation for electrodynamics. Until that this can be viewed as a nothing more than a convenient shorthand.

We use this to write (Maxwell's) eq. (65.10) as:

$$\left(\boldsymbol{\nabla} + \frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{F} = \rho - \frac{\mathbf{J}}{c}. \tag{65.13}$$

These are still four equations, and the originals can be recovered by taking scalar, vector, bivector and trivector parts. However, in this consolidated form, we are able to see the structure more easily.

65.1.3 *Grouping by charge and current density*

Before moving on to the wave equation, lets put equations eq. (65.8) and eq. (65.9) in a slightly more symmetric form, grouping by charge and current density respectively:

$$\boldsymbol{\nabla}\mathbf{E} + \frac{\partial ic\mathbf{B}}{\partial ct} = \frac{\rho}{\epsilon_0} \tag{65.14}$$

$$\boldsymbol{\nabla} ic\mathbf{B} + \frac{\partial \mathbf{E}}{\partial ct} = -\frac{\mathbf{J}}{\epsilon_0 c} \tag{65.15}$$

Here we see how spatial electric field variation and magnetic field time variation are related to charge density. We also see the opposite pairing, where spatial magnetic field variation and electric field variation with time are related to current density.

TODO: examine Lorentz transformations of the coordinates here.

Perhaps the most interesting feature here is how the spacetime gradient ends up split across the **E** and **B** fields, but it may not be worth revisiting this. Let us move on.

65.1.4 *Wave equation for light*

To arrive at the wave equation, we take apply the gradient twice to calculate the Laplacian. First vector gradient is:

$$\boldsymbol{\nabla}\mathbf{F} = -\frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} + \left(\rho - \frac{\mathbf{J}}{c}\right). \tag{65.16}$$

Second application gives:

$$\boldsymbol{\nabla}^2\mathbf{F} = -\frac{1}{c}\boldsymbol{\nabla}\frac{\partial \mathbf{F}}{\partial t} + \boldsymbol{\nabla}\left(\rho - \frac{\mathbf{J}}{c}\right).$$

Assuming continuity sufficient for mixed partial equality, we can swap the order of spatial and time derivatives, and substitute eq. (65.16) back in.

$$\boldsymbol{\nabla}^2\mathbf{F} = -\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} + \left(\rho - \frac{\mathbf{J}}{c}\right)\right) + \boldsymbol{\nabla}\left(\rho - \frac{\mathbf{J}}{c}\right) \tag{65.17}$$

Or,

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_{tt}\right)\mathbf{F} = \left(\boldsymbol{\nabla} - \frac{1}{c}\partial_t\right)\left(\rho - \frac{\mathbf{J}}{c}\right) \tag{65.18}$$

Now there are a number of things that can be read out of this equation. The first is that in a charge and current free region the electromagnetic field is described by an unforced wave equation:

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_{tt}\right)\mathbf{F} = 0 \tag{65.19}$$

This confirms the meaning that was assigned to c. It is the speed that an electrodynamic wave propagates in a charge and current free region of space.

65.1.5 *Charge and current density conservation*

Now, lets look at the right hand side of eq. (65.18) a bit closer:

$$(\boldsymbol{\nabla} - \partial_{ct})\left(\rho - \frac{\mathbf{J}}{c}\right) = -\frac{1}{c}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}\right) + \boldsymbol{\nabla}\rho - \frac{1}{c}\boldsymbol{\nabla}\wedge\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t} \tag{65.20}$$

Compare this to the left hand side of eq. (65.18) which has only vector and bivector parts. This implies that the scalar components of the right hand side are zero. Specifically:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$$

This is a statement of charge conservation, and is more easily interpreted in integral form:

$$-\int_{S\,\text{(closed boundary of V)}} \mathbf{J}\cdot\hat{\mathbf{n}}\,dA = \frac{\partial}{\partial t}\int_V \rho\, dV = \frac{\partial Q_{enc}}{\partial t} \tag{65.21}$$

FIXME: think about signs fully here.

The flux of the current density vector through a closed surface equals the time rate of change of the charge enclosed by that volume (ie: the current). This could perhaps be viewed as the definition of the current density itself. This fact would probably be more obvious if I did the math myself to demonstrate exactly how to take Maxwell's equations in integral form and convert those to their differential form. In lieu of having done that proof myself I can at least determine this as a side effect of a bit of math.

65.1.6 *Electric and Magnetic field dependence on charge and current density*

Removing the explicit scalar terms from eq. (65.18) we have:

$$\left(\boldsymbol{\nabla}^2 - \partial_{ct,ct}\right)\mathbf{F} = \frac{1}{c}\left(\boldsymbol{\nabla} c\rho + \frac{\partial \mathbf{J}}{\partial ct}\right) - \frac{1}{c}\boldsymbol{\nabla} \wedge \mathbf{J}$$

This shows explicitly how the charge and current forced wave equations for the electric and magnetic fields is split:

$$\left(\boldsymbol{\nabla}^2 - \partial_{ct,ct}\right)\mathbf{E} = \frac{1}{c}\left(\boldsymbol{\nabla} c\rho + \frac{\partial \mathbf{J}}{\partial ct}\right)$$

$$\left(\boldsymbol{\nabla}^2 - \partial_{ct,ct}\right)\mathbf{B} = -\frac{1}{c^2}\boldsymbol{\nabla} \times \mathbf{J}$$

65.1.7 *Spacetime basis*

Now, if we look back to Maxwell's equation in the form of eq. (65.13), we have a spacetime "gradient" with vector and scalar parts, an electrodynamic field with vector and trivector parts, and a charge and current density term with scalar and vector parts.

It is still rather confused, but it all works out, and one can recover the original four vector equations by taking scalar, vector, bivector, and trivector parts.

We want however to put this into a natural orderly fashion, and can do so if we use a normal bivector basis for all the spatial basis vectors, and factor out a basis vector from that for each of the scalar (timelike) factors.

Since bivectors over a Euclidean space have negative square, and this is not what we want for our Euclidean basis, and will have to pick a bivector basis with a mixed metric. We will see that this defines a Minkowski metric space. Amazingly, by the simple desire that we want to express Maxwell's equations be written in the most orderly fashion, we arrive at the mixed signature spacetime metric that is the basis of special relativity.

Now, perhaps the reasons why to try to factor the spatial basis into a bivector basis are not obvious. It is worth noting that we have suggestions of conjugate operations above. Examples of this are the charge and current terms with alternate signs, and the alternation in sign in the wave equation itself. Also worth pointing out is the natural appearance of a complex factor i in Maxwell's equation coupled with the time term (that idea is explored more in ../maxwell/-maxwell.pdf). This coupling was observed long ago and Minkowski's original paper refactors Maxwell's equation using it. Now we have also seen that complex numbers are isomorphic with a scalar plus vector representation. Quaternions, which were originally "designed" to fit naturally in Maxwell's equation and express the inherent structure are exactly this, a scalar and bivector sum. There is a lot of history that leads up to this idea, and the ideas here are not too surprising with some reading of the past attempts to put structure to these equations.

On to the math...

Having chosen to find a bivector representation for our spatial basis vectors we write:

$$\mathbf{e}_i = \gamma_i \wedge \gamma_0 = \gamma_i \gamma_0 = \gamma^0 \wedge \gamma^i = \gamma^0 \gamma^i$$

For our Euclidean space we want

$$(\mathbf{e}_i)^2 = \gamma_i \gamma_0 \gamma_i \gamma_0 = -(\gamma_i)^2 (\gamma_0)^2 = 1$$

This implies the mixed signature:

$$(\gamma_i)^2 = -(\gamma_0)^2 = \pm 1$$

We are free to pick either γ_0 or γ_i to have a negative square, but following GAFP we use:

$$\begin{aligned}
(\gamma_0)^2 &= 1 \\
(\gamma_i)^2 &= -1 \\
\gamma^0 &= \gamma_0 \\
\gamma^i &= -\gamma_i
\end{aligned} \tag{65.22}$$

Now, lets translate the other scalar, vector, bivector, and trivector representations to use this alternate basis, and see what we get. Start with the spacial pseudoscalar that is part of our magnetic field:

$$
\begin{aligned}
i &= \mathbf{e}_{123} \\
&= \gamma_{102030} \\
&= -\gamma_{012030} \\
&= \gamma_{012300} \\
&= \gamma_{0123}
\end{aligned}
\tag{65.23}
$$

We see that the three dimensional pseudoscalar represented with this four dimensional basis is in fact also a pseudoscalar for that space. Lets now use this to expand the trivector part of our electromagnetic field in this new basis:

$$
i\mathbf{B} = \sum i\mathbf{e}_i B^i \quad = \sum \gamma_{0123i0} B^i = \gamma_{32} B^1 + \gamma_{13} B^2 + \gamma_{21} B^3 \tag{65.24}
$$

So we see that our electromagnetic field has a bivector only representation with this mixed signature basis:

$$
\mathbf{F} = \mathbf{E} + ic\mathbf{B} = \gamma_{10} E^1 + \gamma_{20} E^2 + \gamma_{30} E^3 + \gamma_{32} c B^1 + \gamma_{13} c B^2 + \gamma_{21} c B^3 \tag{65.25}
$$

Each of the possible bivector basis vectors is associated with a component of the combined electromagnetic field. I had the signs wrong initially for the **B** components, but I think it is right now (and signature independent in fact). ? If I did get it wrong the idea is the same ... F is naturally viewed as a pure bivector, which fits well with the fact that the tensor formulation is two completely antisymmetric rank two tensors.

Now, lets look at the spacetime gradient terms, first writing the spacial gradient in index form:

$$
\begin{aligned}
\boldsymbol{\nabla} &= \sum e^i \frac{\partial}{\partial x^i} \\
&= \sum e_i \frac{\partial}{\partial x^i} \\
&= \sum \gamma_i \gamma_0 \frac{\partial}{\partial x^i} \\
&= \gamma_0 \sum \gamma^i \frac{\partial}{\partial x^i}.
\end{aligned}
\tag{65.26}
$$

This allows the spacetime gradient to be written in vector form replacing the vector plus scalar formulation:

$$
\begin{aligned}
\boldsymbol{\nabla} + \partial_{ct} &= \gamma_0 \sum \gamma^i \frac{\partial}{\partial x^i} + \partial_{ct} \\
&= \gamma_0 \left(\sum \gamma^i \frac{\partial}{\partial x^i} + \gamma^0 \partial_{ct} \right) \\
&= \gamma_0 \sum \gamma^\mu \frac{\partial}{\partial x^i}
\end{aligned}
\tag{65.27}
$$

Observe that after writing $x^0 = ct$ we can factor out the γ_0, and write the spacetime gradient in pure vector form, using this mixed signature basis.

Now, let us do the same thing for the charge and current density terms, writing $\mathbf{J} = e_i J^i$:

$$
\begin{aligned}
\rho - \frac{\mathbf{J}}{c} &= \frac{1}{c} \left(c\rho - \sum \mathbf{e}_i J^i \right) \\
&= \frac{1}{c} \left(c\rho - \sum \gamma_i \gamma_0 J^i \right) \\
&= \frac{1}{c} \left(c\rho + \gamma_0 \sum \gamma_i J^i \right) \\
&= \gamma_0 \frac{1}{c} \left(\gamma_0 c\rho + \sum \gamma_i J^i \right)
\end{aligned}
\tag{65.28}
$$

Thus after writing $J^0 = c\rho$, we have:

$$
\rho - \frac{\mathbf{J}}{c} = \gamma_0 \frac{1}{c} \sum \gamma_\mu J^\mu
$$

Putting these together and canceling out the leading γ_0 terms we have the final result:

$$
\sum \gamma^\mu \frac{\partial}{\partial x^i} \mathbf{F} = \frac{1}{c} \sum \gamma_\mu J^\mu . \tag{65.29}
$$

Or with a four-gradient $\nabla = \sum \gamma^\mu \frac{\partial}{\partial x^i}$, and four current $J = \sum \gamma_\mu J^\mu$, we have Maxwell's equation in their most compact and powerful form:

$$
\nabla \mathbf{F} = \frac{J}{c} . \tag{65.30}
$$

65.1.8 *Examining the GA form Maxwell equation in more detail*

From eq. (65.30), the wave equation becomes quite simple to derive. Lets look at this again from this point of view. Applying the gradient we have:

$$\nabla^2 \mathbf{F} = \frac{\nabla J}{c}. \tag{65.31}$$

$$\nabla^2 = \nabla \cdot \nabla = \sum (\gamma^\mu)^2 \partial_{x^\mu, x^\mu} = -\boldsymbol{\nabla}^2 + \frac{1}{c^2}\partial_{tt}. \tag{65.32}$$

Thus for a charge and current free region, we still have the wave equation.

Now, lets look at the right hand side, and verify that it meets the expectations:

$$\frac{1}{c}\nabla J = \frac{1}{c}\left(\nabla \cdot J + \nabla \wedge J\right) \tag{65.33}$$

First thing to observe is that the left hand side is a pure spacetime bivector, which implies that the scalar part of eq. (65.33) is zero as we previously observed. Lets verify that this is still the charge conservation condition:

$$\begin{aligned}
0 &= \nabla \cdot J \\
&= (\sum \gamma^\mu \partial_\mu) \cdot \sum \gamma_\nu J^\nu \\
&= \sum \gamma^\mu \cdot \gamma_\nu \partial_\mu J^\nu \\
&= \sum \delta^\mu_\nu \partial_\mu J^\nu \\
&= \sum \partial_\mu J^\mu \\
&= \partial_{ct}(c\rho) + \sum \partial_i J^i
\end{aligned} \tag{65.34}$$

This is our previous result:

$$\frac{\partial \rho}{\partial_t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0 \tag{65.35}$$

This allows a slight simplification of the current forced wave equation for an electrodynamic field, by taking just the bivector parts:

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_{tt}\right)\mathbf{F} = -\nabla \wedge \frac{J}{c} \tag{65.36}$$

Now we know how to solve the left hand side of this equation in its homogeneous form, but the four space curl term on the right is new.

This is really a set of six equations, subject to coupled boundary value conditions. Written this out in components, one for each $F \cdot (\gamma^\nu \wedge \gamma^\mu)$ term and the corresponding terms of the right hand side one ends up with:

$$-\nabla^2 \mathbf{E} = \boldsymbol{\nabla}\rho/\epsilon_0 + \mu_0 \partial_t \mathbf{J}$$

$$-\nabla^2 \mathbf{B} = -\mu_0 \boldsymbol{\nabla} \times \mathbf{J}$$

I have not bothered transcribing my notes for how to get this. One way (messy) was starting with eq. (65.36) and dotting with $\gamma^{\nu\mu}$ to calculate the tensor $F^{\mu\nu}$ (components of which are E and B components). Doing the same for the spacetime curl term the end result is:

$$(\nabla \wedge J) \cdot (\gamma^{\nu\mu}) = \partial_\mu J^\nu (\gamma^\mu)^2 - \partial_\nu J^\mu (\gamma^\nu)^2$$

For a spacetime split of indices one gets the $\boldsymbol{\nabla}\rho$, and $\partial_t \mathbf{J}$ term, and for a space-space pair of indices one gets the spacial curl in the $\mathbf{B}$ equation.

An easier starting point for this is actually using equations eq. (65.14) and eq. (65.15) since they are already split into $\mathbf{E}$, and $\mathbf{B}$ fields.

65.1.9 *Minkowski metric*

Having observed that a mixed signature bivector basis with a space time mix of underlying basis vectors is what we want to express Maxwell's equation in its most simple form, now lets step back and look at that in a bit more detail. In particular lets examine the dot product of a four vector with such a basis. Our current density four vector is one such vector:

$$J^2 = J \cdot J = \sum (J^\mu)^2 (\gamma_\mu)^2 = (c\rho)^2 - \mathbf{J}^2 \tag{65.37}$$

The coordinate vector that is forms the partials of our four gradient is another such vector:

$$x = (ct, x^1, x^2, x^3) = \sum \gamma_\mu x^\mu$$

Again, the length applied to this vector is:

$$x^2 = x \cdot x = (ct)^2 - \mathbf{x}^2 \tag{65.38}$$

As a result of nothing more than a desire to put Maxwell's equations into structured form, we have the special relativity metric of Minkowski and Einstein.

66

MACROSCOPIC MAXWELL'S EQUATION

66.1 MOTIVATION

In [22] the macroscopic Maxwell's equations are given as

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= 4\pi\rho \\
\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \\
\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0
\end{aligned}
\tag{66.1}
$$

The $\mathbf{H}$ and $\mathbf{D}$ fields are then defined in terms of dipole, and quadrupole fields

$$
\begin{aligned}
D_\alpha &= E_\alpha + 4\pi \left(P_\alpha - \sum_\beta \frac{\partial Q'_{\alpha\beta}}{\partial x_\beta} + \cdots \right) \\
H_\alpha &= B_\alpha - 4\pi \left(M_\alpha + \cdots \right)
\end{aligned}
\tag{66.2}
$$

Can this be put into the Geometric Algebra formulation that works so nicely for microscopic Maxwell's equations, and if so what will it look like?

66.2 CONSOLIDATION ATTEMPT

Let us try this, writing

$$
\begin{aligned}
\mathbf{P} &= \sigma^\alpha \left(P_\alpha - \sum_\beta \frac{\partial Q'_{\alpha\beta}}{\partial x_\beta} + \cdots \right) \\
\mathbf{M} &= \sigma^\alpha \left(M_\alpha + \cdots \right)
\end{aligned}
\tag{66.3}
$$

We can then express the $\mathbf{E}$, $\mathbf{B}$ in terms of the derived fields

$$
\begin{aligned}
\mathbf{E} &= \mathbf{D} - 4\pi\mathbf{P} \\
\mathbf{B} &= \mathbf{H} + 4\pi\mathbf{M}
\end{aligned}
\tag{66.4}
$$

and in turn can write the macroscopic Maxwell equations eq. (66.1) in terms of just the derived fields, the material properties, and the charges and currents

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= 4\pi\rho \\
\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \\
\boldsymbol{\nabla} \times \mathbf{D} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} &= 4\pi\boldsymbol{\nabla} \times \mathbf{P} + \frac{4\pi}{c}\frac{\partial \mathbf{M}}{\partial t} \\
\boldsymbol{\nabla} \cdot \mathbf{H} &= -4\pi\boldsymbol{\nabla} \cdot \mathbf{M}
\end{aligned}
\tag{66.5}
$$

Now, using $\mathbf{a} \times \mathbf{b} = -i(\mathbf{a} \wedge \mathbf{b})$, we have

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{D} &= 4\pi\rho \\
i\boldsymbol{\nabla} \wedge \mathbf{H} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= -\frac{4\pi}{c}\mathbf{J} \\
\boldsymbol{\nabla} \wedge \mathbf{D} + \frac{1}{c}\frac{\partial i\mathbf{H}}{\partial t} &= 4\pi i\boldsymbol{\nabla} \times \mathbf{P} + \frac{4\pi}{c}\frac{\partial i\mathbf{M}}{\partial t} \\
i\boldsymbol{\nabla} \cdot \mathbf{H} &= -4\pi i\boldsymbol{\nabla} \cdot \mathbf{M}
\end{aligned}
\tag{66.6}
$$

Summing these in pairs with $\boldsymbol{\nabla}\mathbf{a} = \boldsymbol{\nabla} \cdot \mathbf{a} + \boldsymbol{\nabla} \wedge \mathbf{a}$, and writing $\partial/\partial(ct) = \partial_0$ we have

$$
\begin{aligned}
\boldsymbol{\nabla}\mathbf{D} + \partial_0 i\mathbf{H} &= 4\pi\rho + 4\pi\boldsymbol{\nabla} \wedge \mathbf{P} + 4\pi\partial_0 i\mathbf{M} \\
i\boldsymbol{\nabla}\mathbf{H} + \partial_0\mathbf{D} &= -\frac{4\pi}{c}\mathbf{J} - 4\pi i\boldsymbol{\nabla} \cdot \mathbf{M}
\end{aligned}
\tag{66.7}
$$

Note that while had $i\boldsymbol{\nabla} \cdot \mathbf{a} \neq \boldsymbol{\nabla} \cdot (i\mathbf{a})$, and $i\boldsymbol{\nabla} \wedge \mathbf{a} \neq \boldsymbol{\nabla} \wedge (i\mathbf{a})$ (instead $i\boldsymbol{\nabla} \cdot \mathbf{a} = \boldsymbol{\nabla} \wedge (i\mathbf{a})$, and $i\boldsymbol{\nabla} \wedge \mathbf{a} = \boldsymbol{\nabla} \cdot (i\mathbf{a})$), but now that these are summed we can take advantage of the fact that the pseudoscalar i commutes with all vectors (such as $\boldsymbol{\nabla}$). So, summing once again we have

$$
(\partial_0 + \boldsymbol{\nabla})(\mathbf{D} + i\mathbf{H}) = \frac{4\pi}{c}\left(c\rho - \mathbf{J}\right) + 4\pi\left(\boldsymbol{\nabla} \wedge \mathbf{P} + \partial_0 i\mathbf{M} - \boldsymbol{\nabla} \wedge (i\mathbf{M})\right)
\tag{66.8}
$$

Finally, premultiplication by γ_0, where $\mathbf{J} = \sigma_k J^k = \gamma_k\gamma_0 J^k$, and $\boldsymbol{\nabla} = \sum_k \gamma_k\gamma_0\partial_k$ we have

$$
\gamma^\mu\partial_\mu(\mathbf{D} + i\mathbf{H}) = \frac{4\pi}{c}\left(c\rho\gamma_0 + J^k\gamma_k\right) + 4\pi\gamma_0\left(\boldsymbol{\nabla} \wedge \mathbf{P} + \partial_0 i\mathbf{M} - \boldsymbol{\nabla} \wedge (i\mathbf{M})\right)
\tag{66.9}
$$

With

$$
\begin{aligned}
J^0 &= c\rho \\
J &= \gamma_\mu J^\mu \\
\nabla &= \gamma^\mu\partial_\mu \\
F &= \mathbf{D} + i\mathbf{H}
\end{aligned}
\tag{66.10}
$$

For the remaining terms we have $\boldsymbol{\nabla} \wedge \mathbf{P}, i\mathbf{M} \in \text{span}\{\gamma_a \gamma_b\}$, and $\gamma_0 \boldsymbol{\nabla} \wedge (iM) \in \text{span}\, \gamma_1 \gamma_2 \gamma_3$, so between the three of these we have a (Dirac) trivector, so it would be reasonable to write

$$T = \gamma_0 \left(\boldsymbol{\nabla} \wedge \mathbf{P} + \partial_0 i\mathbf{M} - \boldsymbol{\nabla} \wedge (i\mathbf{M}) \right) \in \text{span}\{\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\sigma\} \tag{66.11}$$

Putting things back together we have

$$\nabla F = \frac{4\pi}{c} J + 4\pi T \tag{66.12}$$

This has a nice symmetry, almost nicer than the original microscopic version of Maxwell's equation since we now have matched grades (vector plus trivector in the Dirac vector space) on both sides of the equation.

66.2.1 *Continuity equation*

Also observe that interestingly we still have the same continuity equation as in the microscopic case. Application of another spacetime gradient and then selecting scalar grades we have

$$\begin{aligned}
\langle \nabla \nabla F \rangle &= 4\pi \left\langle \nabla \left(\frac{J}{c} + T \right) \right\rangle \\
\nabla^2 \langle F \rangle &= \\
&= \frac{4\pi}{c} \langle J \rangle \\
&= \frac{4\pi}{c} \partial_\mu J^\mu
\end{aligned} \tag{66.13}$$

Since F is a Dirac bivector it has no scalar part, so this whole thing is zero by the grade selection on the LHS. So, from the RHS we have

$$\begin{aligned}
0 &= \partial_\mu J^\mu \\
&= \frac{1}{c} \frac{\partial c\rho}{\partial t} + \partial_k J^k \\
&= \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}
\end{aligned} \tag{66.14}$$

Despite the new trivector term in the equation due to the matter properties!

67

EXPRESSING WAVE EQUATION EXPONENTIAL SOLUTIONS USING FOUR VECTORS

67.1 MECHANICAL WAVE EQUATION SOLUTIONS

For the unforced wave equation in 3D one wants solutions to

$$\left(\frac{1}{\mathbf{v}^2}\partial_{tt} - \sum_{j=1}^{3}\partial_{jj}\right)\phi = 0 \tag{67.1}$$

For the single spatial variable case one can verify that $\phi = f(\mathbf{x} \pm |\mathbf{v}|t)$ is a solution for any function f. In particular $\phi = \exp(i(\pm|\mathbf{v}|t + x))$ is a solution. Similarly $\phi = \exp(i(\pm|\mathbf{v}|t + \hat{\mathbf{k}} \cdot \mathbf{x}))$ is a solution in the 3D case.

Can the relativistic four vector notation be used to put this in a more symmetric form with respect to time and position? For the four vector

$$x = x^\mu \gamma_\mu \tag{67.2}$$

Lets try the following as a possible solution to eq. (67.1)

$$\phi = \exp(ik \cdot x) \tag{67.3}$$

verifying that this can be a solution, and determining the constraints required on the four vector k.

Observe that

$$x \cdot k = x^\mu k_\mu \tag{67.4}$$

so

$$\begin{aligned}\phi_\mu &= ik_\mu \\ \phi_{\mu\mu} &= (ik_\mu)^2\phi = -(k_\mu)^2\phi\end{aligned} \tag{67.5}$$

Since $\partial_t = c\partial_0$, we have $\phi_t t = c^2\phi_{00}$, and

$$\left(\frac{1}{\mathbf{v}^2}\partial_{tt} - \sum_{j=1}^{3}\partial_{jj}\right)\phi = \left(-\frac{1}{\mathbf{v}^2}c^2{k_0}^2 - \sum_{j=1}^{3} -(k_j)^2\right)\phi \tag{67.6}$$

For equality with zero, and $\boldsymbol{\beta} = \mathbf{v}/c$, we require

$$\boldsymbol{\beta}^2 = \frac{(k_0)^2}{\sum_j (k_j)^2} \tag{67.7}$$

Now want the components of $k = k_\mu \gamma^\mu$ in terms of k directly. First

$$k_0 = k \cdot \gamma_0 \tag{67.8}$$

The spacetime relative vector for k is

$$\begin{aligned}
\mathbf{k} &= k \wedge \gamma_0 = \sum k_\mu \gamma^\mu \wedge \gamma_0 = (\gamma_1)^2 \sum_j k_j \sigma_j \\
\mathbf{k}^2 &= (\pm 1)^2 \sum_j (k_j)^2
\end{aligned} \tag{67.9}$$

So the constraint on the four vector parameter k is

$$\begin{aligned}
\boldsymbol{\beta}^2 &= \frac{(k_0)^2}{\sum_j (k_j)^2} \\
&= \frac{(k \cdot \gamma_0)^2}{(k \wedge \gamma_0)^2}
\end{aligned} \tag{67.10}$$

It is interesting to compare this to the relative spacetime bivector for x

$$\begin{aligned}
v &= \frac{dx}{d\tau} = c\frac{dt}{d\tau}\gamma_0 + \frac{dx^i}{d\tau}\gamma_i \\
v \cdot \gamma^0 &= \frac{dx}{d\tau} \cdot \gamma^0 = c\frac{dt}{d\tau} \\
v \wedge \gamma_0 &= \frac{dx}{d\tau} \wedge \gamma_0 \\
&= \frac{dx^i}{d\tau}\sigma_i \\
&= \frac{dx^i}{dt}\frac{dt}{d\tau}\sigma_i
\end{aligned} \tag{67.11}$$

$$
\begin{aligned}
\mathbf{v}/c &= \frac{d(x^i \sigma_i)}{dt} \\
&= \frac{v \wedge \gamma_0}{v \cdot \gamma^0}
\end{aligned}
\tag{67.12}
$$

So, for $\phi = \exp(ik \cdot x)$ to be a solution to the wave equation for a wave traveling with velocity $|\mathbf{v}|$, the constraint on k in terms of proper velocity v is

$$
\left|\frac{k \wedge \gamma_0}{k \cdot \gamma^0}\right|^{-1} = \left|\frac{v \wedge \gamma_0}{v \cdot \gamma^0}\right|
\tag{67.13}
$$

So we see the relative spacetime vector of k has an inverse relationship with the relative spacetime velocity vector $v = dx/d\tau$.

68

GAUSSIAN SURFACE INVARIANCE FOR RADIAL FIELD

68.1 FLUX INDEPENDENCE OF SURFACE

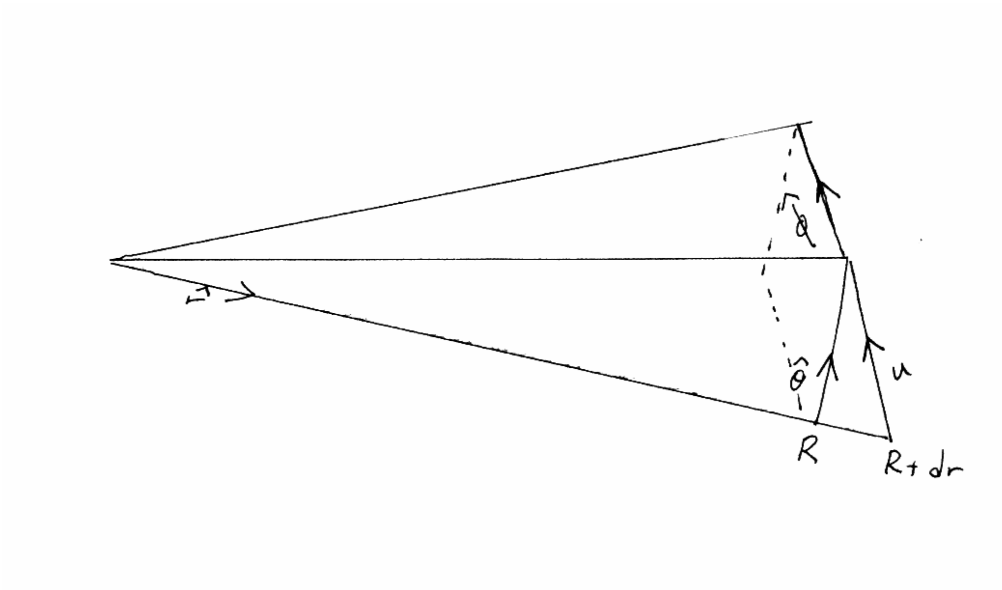

Figure 68.1: Flux through tilted spherical surface element

In [37], section 1.10 is a demonstration that the flux through any closed surface is the same as that through a sphere.

A similar demonstration of the same is possible using a spherical polar basis $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ with an element of surface area that is tilted slightly as illustrated in fig. 68.1.

The tangential surface on the sphere at radius r will have bivector

$$d\mathbf{A}_r = r^2 d\theta d\phi \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\phi}} \tag{68.1}$$

where $d\theta$, and $d\phi$ are the subtended angles (should have put them in the figure).

Now, as in the figure we want to compute the bivector for the tilted surface at radius R. The vector $\mathbf{u}$ in the figure is required. This is $\hat{\mathbf{r}}R + Rd\theta\hat{\boldsymbol{\theta}} - \hat{\mathbf{r}}(R + dr)$, so the bivector for that area element is

$$\left(R\hat{\mathbf{r}} + Rd\theta\hat{\boldsymbol{\theta}} - (R + dr)\hat{\mathbf{r}}\right) \wedge Rd\theta\hat{\boldsymbol{\phi}} = \left(Rd\theta\hat{\boldsymbol{\theta}} - dr\hat{\mathbf{r}}\right) \wedge Rd\phi\hat{\boldsymbol{\phi}} \tag{68.2}$$

For

$$d\mathbf{A}_R = R^2 d\theta d\phi\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} - Rdrd\phi\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \tag{68.3}$$

Now normal area elements can be calculated by multiplication with a $\mathbb{R}^3$ pseudoscalar such as $I = \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}$.

$$\begin{aligned} \hat{\mathbf{n}}_r|d\mathbf{A}_r| &= r^2 d\theta d\phi\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} \\ &= -r^2 d\theta d\phi\hat{\mathbf{r}} \end{aligned} \tag{68.4}$$

And

$$\begin{aligned} \hat{\mathbf{n}}_R|d\mathbf{A}_R| &= \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}}\left(R^2 d\theta d\phi\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\phi}} - Rdrd\phi\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\right) \\ &= -R^2 d\theta d\phi\hat{\mathbf{r}} - Rdrd\phi\hat{\boldsymbol{\theta}} \end{aligned} \tag{68.5}$$

Calculating $\mathbf{E} \cdot \hat{\mathbf{n}}dA$ for the spherical surface element at radius r we have

$$\begin{aligned} \mathbf{E}(r) \cdot \hat{\mathbf{n}}_r|d\mathbf{A}_r| &= \frac{1}{4\pi\epsilon_0 r^2} q\hat{\mathbf{r}} \cdot (-r^2 d\theta d\phi\hat{\mathbf{r}}) \\ &= \frac{-d\theta d\phi q}{4\pi\epsilon_0} \end{aligned} \tag{68.6}$$

and for the tilted surface at R

$$\begin{aligned} \mathbf{E}(R) \cdot \hat{\mathbf{n}}_R|d\mathbf{A}_R| &= \frac{q}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}} \cdot \left(-R^2 d\theta d\phi\hat{\mathbf{r}} - Rdrd\phi\hat{\boldsymbol{\theta}}\right) \\ &= \frac{-d\theta d\phi q}{4\pi\epsilon_0} \end{aligned} \tag{68.7}$$

The $\hat{\boldsymbol{\theta}}$ component of the surface normal has no contribution to the flux since it is perpendicular to the outwards ($\hat{\mathbf{r}}$ facing) field. Here the particular normal to the surface happened to be inwards facing due to choice of the pseudoscalar, but because the normals chosen in each case had the same orientation this does not make a difference to the equivalence result.

68.1.1 *Suggests dual form of Gauss's law can be natural*

The fact that the bivector area elements work well to describe the surface can also be used to write Gauss's law in an alternate form. Let $\hat{\mathbf{n}} dA = -I d\mathbf{A}$

$$
\begin{aligned}
\mathbf{E} \cdot \hat{\mathbf{n}} dA &= -\mathbf{E} \cdot (I d\mathbf{A}) \\
&= \frac{-1}{2}(\mathbf{E} I d\mathbf{A} + I d\mathbf{A}\mathbf{E}) \\
&= \frac{-I}{2}(\mathbf{E} d\mathbf{A} + d\mathbf{A}\mathbf{E}) \\
&= -I(\mathbf{E} \wedge d\mathbf{A})
\end{aligned}
\tag{68.8}
$$

So for

$$
\int \mathbf{E} \cdot \hat{\mathbf{n}} dA = \int \frac{\rho}{\epsilon_0} dV \tag{68.9}
$$

with $d\mathbf{V} = I dV$, we have Gauss's law in dual form:

$$
\int \mathbf{E} \wedge d\mathbf{A} = \int \frac{\rho}{\epsilon_0} d\mathbf{V} \tag{68.10}
$$

Writing Gauss's law in this form it becomes almost obvious that we can deform the surface without changing the flux, since all the non-tangential surface elements will have an $\hat{\mathbf{r}}$ factor and thus produce a zero once wedged with the radial field.

69

ELECTRODYNAMIC WAVE EQUATION SOLUTIONS

69.1 MOTIVATION

In 67 four vector solutions to the mechanical wave equations were explored. What was obviously missing from that was consideration of the special case for $\mathbf{v}^2 = c^2$.

Here solutions to the electrodynamic wave equation will be examined. Consideration of such solutions in more detail will is expected to be helpful as background for the more complex study of quantum (matter) wave equations.

69.2 ELECTROMAGNETIC WAVE EQUATION SOLUTIONS

For electrodynamics our equation to solve is

$$\nabla F = J/\epsilon_0 c \tag{69.1}$$

For the unforced (vacuum) solutions, with $F = \nabla \wedge A$, and the Coulomb gauge $\nabla \cdot A = 0$ this reduces to

$$\begin{aligned} 0 &= \left((\gamma^\mu)^2 \partial_{\mu\mu}\right) A \\ &= \left(\frac{1}{c^2}\partial_{tt} - \partial_{jj}\right) A \end{aligned} \tag{69.2}$$

These equations have the same form as the mechanical wave equation where the wave velocity $\mathbf{v}^2 = c^2$ is the speed of light

$$\left(\frac{1}{\mathbf{v}^2}\partial_{tt} - \sum_{j=1}^{3} \partial_{jj}\right)\psi = 0 \tag{69.3}$$

69.2.1 *Separation of variables solution of potential equations*

Let us solve this using separation of variables, and write $A^\nu = XYZT = \Pi_\mu X^\mu$

From this we have

$$\sum_{\mu} (\gamma^{\mu})^2 \frac{(X^{\mu})''}{X^{\mu}} = 0 \tag{69.4}$$

and can proceed with the normal procedure of assuming that a solution can be found by separately equating each term to a constant. Writing those constants explicitly as $(m_{\mu})^2$, which we allow to be potentially complex we have (no sum)

$$X^{\mu} = \exp\left(\pm \sqrt{(\gamma^{\mu})^2} m_{\mu} x^{\mu}\right) \tag{69.5}$$

Now, let $k_{\mu} = \pm\sqrt{(\gamma^{\mu})^2} m_{\mu}$, folding any sign variation and complex factors into these constants. Our complete solution is thus

$$\Pi_{\mu} X^{\mu} = \exp\left(\sum k_{\mu} x^{\mu}\right) \tag{69.6}$$

However, for this to be a solution, the wave equation imposes the constraint

$$\sum_{\mu} (\gamma^{\mu})^2 (k_{\mu})^2 = 0 \tag{69.7}$$

Or

$$(k_0)^2 - \sum_{j} (k_j)^2 = 0 \tag{69.8}$$

Summarizing each potential term has a solution expressible in terms of null "wave-number" vectors K_{ν}

$$\begin{aligned} A_{\nu} &= \exp\left(K_{\nu} \cdot x\right) \\ |K_{\nu}| &= 0 \end{aligned} \tag{69.9}$$

69.2.2 *Faraday bivector and tensor from the potential solutions*

From the components of the potentials eq. (69.9) we can compute the curl for the complete field. That is

$$\begin{aligned} F &= \nabla \wedge A \\ A &= \gamma^{\nu} \exp\left(K_{\nu} \cdot x\right) \end{aligned} \tag{69.10}$$

This is

$$
\begin{aligned}
F &= (\gamma^\mu \wedge \gamma^\nu)\, \partial_\mu \exp\left(K_\nu \cdot x\right) \\
&= (\gamma^\mu \wedge \gamma^\nu)\, \partial_\mu \exp\left(\gamma^\alpha K_{\nu\alpha} \cdot \gamma_\sigma x^\sigma\right) \\
&= (\gamma^\mu \wedge \gamma^\nu)\, \partial_\mu \exp\left(K_{\nu\sigma} x^\sigma\right) \\
&= (\gamma^\mu \wedge \gamma^\nu)\, K_{\nu\mu} \exp\left(K_{\nu\sigma} x^\sigma\right) \\
&= (\gamma^\mu \wedge \gamma^\nu)\, K_{\nu\mu} \exp\left(K_\nu \cdot x\right) \\
&= (\gamma^\mu \wedge \gamma^\nu)\, \frac{1}{2} \left(K_{\nu\mu} \exp\left(K_\nu \cdot x\right) - K_{\mu\nu} \exp\left(K_\mu \cdot x\right)\right)
\end{aligned}
\tag{69.11}
$$

Writing our field in explicit tensor form

$$
F = F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu \tag{69.12}
$$

our vacuum solution is therefore

$$
F_{\mu\nu} = \frac{1}{2} \left(K_{\nu\mu} \exp\left(K_\nu \cdot x\right) - K_{\mu\nu} \exp\left(K_\mu \cdot x\right)\right) \tag{69.13}
$$

but subject to the null wave number and Lorentz gauge constraints

$$
\begin{aligned}
\left|K_\mu\right| &= 0 \\
\nabla \cdot \left(\gamma^\mu \exp\left(K_\mu \cdot x\right)\right) &= 0
\end{aligned}
\tag{69.14}
$$

69.2.3 *Examine the Lorentz gauge constraint*

That Lorentz gauge constraint on the potential is a curious looking beastie. Let us expand that out in full to examine it closer

$$
\begin{aligned}
\nabla \cdot \left(\gamma^\mu \exp\left(K_\mu \cdot x\right)\right) &= \gamma^\alpha \partial_\alpha \cdot \left(\gamma^\mu \exp\left(K_\mu \cdot x\right)\right) \\
&= \sum_\mu (\gamma^\mu)^2 \partial_\mu \exp\left(K_\mu \cdot x\right) \\
&= \sum_\mu (\gamma^\mu)^2 \partial_\mu \exp\left(\sum \gamma^\nu K_{\mu\nu} \cdot \gamma_\alpha x^\alpha\right) \\
&= \sum_\mu (\gamma^\mu)^2 \partial_\mu \exp\left(\sum K_{\mu\alpha} x^\alpha\right) \\
&= \sum_\mu (\gamma^\mu)^2 K_{\mu\mu} \exp\left(K_\mu \cdot x\right)
\end{aligned}
\tag{69.15}
$$

If this must be zero for any x it must also be zero for $x = 0$, so the Lorentz gauge imposes an additional restriction on the wave number four vectors K_μ

$$\sum_\mu (\gamma^\mu)^2 K_{\mu\mu} = 0 \tag{69.16}$$

Expanding in time and spatial coordinates this is

$$K_{00} - \sum_j K_{jj} = 0 \tag{69.17}$$

One obvious way to satisfy this is to require that the tensor $K_{\mu\nu}$ be diagonal, but since we also have the null vector requirement on each of the K_μ four vectors it is not clear that this is an acceptable choice.

69.2.4 *Summarizing so far*

We have found that our field solution has the form

$$F_{\mu\nu} = \frac{1}{2}\left(K_{\nu\mu} \exp\left(K_\nu \cdot x\right) - K_{\mu\nu} \exp\left(K_\mu \cdot x\right)\right) \tag{69.18}$$

Where the vectors K_μ have coordinates

$$K_\mu = \gamma^\nu K_{\mu\nu} \tag{69.19}$$

This last allows us to write the field tensor completely in tensor formalism

$$F_{\mu\nu} = \frac{1}{2}\left(K_{\nu\mu} \exp\left(K_{\nu\sigma} x^\sigma\right) - K_{\mu\nu} \exp\left(K_{\mu\sigma} x^\sigma\right)\right) \tag{69.20}$$

Note that we also require the constraints

$$\begin{aligned} 0 &= \sum_\mu (\gamma^\mu)^2 K_{\mu\mu} \\ 0 &= \sum_\mu (\gamma^\mu)^2 (K_{\nu\mu})^2 \end{aligned} \tag{69.21}$$

Alternately, calling out the explicit space time split of the constraint, we can remove the explicit γ^μ factors

$$0 = K_{00} - \sum_j K_{jj} = (K_{00})^2 - \sum_j (K_{jj})^2 \tag{69.22}$$

69.3 LOOKING FOR MORE GENERAL SOLUTIONS

69.3.1 *Using mechanical wave solutions as a guide*

In the mechanical wave equation, we had exponential solutions of the form

$$f(\mathbf{x}, t) = \exp\left(\mathbf{k} \cdot \mathbf{x} + \omega t\right) \tag{69.23}$$

which were solutions to eq. (69.3) provided that

$$\frac{1}{\mathbf{v}^2}\omega^2 - \mathbf{k}^2 = 0. \tag{69.24}$$

This meant that

$$\omega = \pm|\mathbf{v}||\mathbf{k}| \tag{69.25}$$

and our function takes the (hyperbolic) form, or (sinusoidal) form respectively

$$\begin{aligned} f(\mathbf{x}, t) &= \exp\left(|\mathbf{k}|\left(\hat{\mathbf{k}} \cdot \mathbf{x} \pm |\mathbf{v}|t\right)\right) \\ f(\mathbf{x}, t) &= \exp\left(i|\mathbf{k}|\left(\hat{\mathbf{k}} \cdot \mathbf{x} \pm |\mathbf{v}|t\right)\right) \end{aligned} \tag{69.26}$$

Fourier series superposition of the latter solutions can be used to express any spatially periodic function, while Fourier transforms can be used to express the non-periodic cases.

These superpositions, subject to boundary value conditions, allow for writing solutions to the wave equation in the form

$$f(\mathbf{x}, t) = g\left(\hat{\mathbf{k}} \cdot \mathbf{x} \pm |\mathbf{v}|t\right) \tag{69.27}$$

Showing this logically follows from the original separation of variables approach has not been done. However, despite this, it is simple enough to confirm that, this more general function does satisfy the unforced wave equation eq. (69.3).

TODO: as followup here would like to go through the exercise of showing that the solution of eq. (69.27) follows from a Fourier transform superposition. Intuition says this is possible, and I have said so without backing up the statement.

69.3.2 *Back to the electrodynamic case*

Using the above generalization argument as a guide we should be able to do something similar for the electrodynamic wave solution.

We want to solve for F the following gradient equation for the field in free space

$$\nabla F = 0 \tag{69.28}$$

Let us suppose that the following is a solution and find the required constraints

$$F = \gamma^\mu \wedge \gamma^\nu \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right) \tag{69.29}$$

We have two different grade equations built into Maxwell's equation eq. (69.28), one of which is the vector equation, and the other trivector. Those are respectively

$$\begin{aligned} \nabla \cdot F &= 0 \\ \nabla \wedge F &= 0 \end{aligned} \tag{69.30}$$

69.3.2.1 *zero wedge*

For the grade three term we have we can substitute eq. (69.29) and see what comes out

$$\nabla \wedge F = \left(\gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu\right) \partial_\alpha \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right) \tag{69.31}$$

For the partial we will want the following

$$\begin{aligned} \partial_\mu (x \cdot K_\beta) &= \partial_\mu (x^\nu \gamma_\nu \cdot K_{\beta\sigma} \gamma^\sigma) \\ &= \partial_\mu (x^\sigma K_{\beta\sigma} \\ &= K_{\beta\mu} \end{aligned} \tag{69.32}$$

and application of this with the chain rule we have

$$\begin{aligned} \nabla \wedge F &= \left(\gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu\right) \left(K_{\mu\nu} K_{\mu\alpha} f'(x \cdot K_\mu) - K_{\nu\mu} K_{\nu\alpha} f'(x \cdot K_\nu)\right) \\ &= 2 \left(\gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu\right) K_{\mu\nu} K_{\mu\alpha} f'(x \cdot K_\mu) \end{aligned} \tag{69.33}$$

So, finally for this to be zero uniformly for all f, we require

$$K_{\mu\nu} K_{\mu\alpha} = 0 \tag{69.34}$$

69.3.2.2 *zero divergence*

Now for the divergence term, corresponding to the current four vector condition $J = 0$, we have

$$
\begin{aligned}
&\nabla \cdot F \\
&= \gamma^\alpha \cdot (\gamma^\mu \wedge \gamma^\nu)\, \partial_\alpha \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right) \\
&= (\gamma_\alpha)^2 \left(\gamma^\nu {\delta_\alpha}^\mu - \gamma^\mu {\delta_\alpha}^\nu\right) \partial_\alpha \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right) \\
&= \left((\gamma_\mu)^2 \gamma^\nu \partial_\mu - (\gamma_\nu)^2 \gamma^\mu \partial_\nu\right) \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right) \\
&= (\gamma_\mu)^2 \gamma^\nu \partial_\mu \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right) - (\gamma_\mu)^2 \gamma^\nu \partial_\mu \left(K_{\nu\mu} f(x \cdot K_\nu) - K_{\mu\nu} f(x \cdot K_\mu)\right) \\
&= 2(\gamma_\mu)^2 \gamma^\nu \partial_\mu \left(K_{\mu\nu} f(x \cdot K_\mu) - K_{\nu\mu} f(x \cdot K_\nu)\right)
\end{aligned}
\tag{69.35}
$$

Application of the chain rule, and $\partial_\mu(x \cdot K_\beta) = K_{\beta\mu}$, gives us

$$
\nabla \cdot F = 2(\gamma_\mu)^2 \gamma^\nu \left(K_{\mu\nu} K_{\mu\mu} f'(x \cdot K_\mu) - K_{\nu\mu} K_{\nu\mu} f'(x \cdot K_\nu)\right) \tag{69.36}
$$

For $\mu = \nu$ this is zero, which is expected since that should follow from the wedge product itself, but for the $\mu \neq \nu$ case it is not clear cut.

Damn. On paper I missed some terms and it all canceled out nicely giving only a condition on $K_{\mu\nu}$ from the wedge term. The only conclusion possible is that we require $x \cdot K_\nu = x \cdot K_\mu$ for this form of solution, and therefore need to restrict the test solution to a fixed spacetime direction.

69.4 TAKE II. A BOGUS ATTEMPT AT A LESS GENERAL PLANE WAVE LIKE SOLUTION

Let us try instead

$$
F = \gamma^\mu \wedge \gamma^\nu A_{\mu\nu} f(x \cdot k) \tag{69.37}
$$

and see if we can find conditions on the vector k, and the tensor A that make this a solution to the unforced Maxwell equation eq. (69.28).

69.4.1 *curl term*

Taking the curl is straightforward

$$
\begin{aligned}
\nabla \wedge F &= \gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu \partial_\alpha A_{\mu\nu} f(x \cdot k) \\
&= \gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu A_{\mu\nu} \partial_\alpha f(x^\sigma k_\sigma) \\
&= \gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu A_{\mu\nu} k_\alpha f'(x \cdot k) \\
&= \frac{1}{2} \gamma^\alpha \wedge \gamma^\mu \wedge \gamma^\nu (A_{\mu\nu} - A_{\nu\mu}) k_\alpha f'(x \cdot k)
\end{aligned}
\tag{69.38}
$$

Curiously, the only condition that this yields is that we have

$$
A_{\mu\nu} - A_{\nu\mu} = 0 \tag{69.39}
$$

which is a symmetry requirement for the tensor

$$
A_{\mu\nu} = A_{\nu\mu} \tag{69.40}
$$

69.4.2 *divergence term*

Now for the divergence

$$
\begin{aligned}
\nabla \cdot F &= \gamma_\alpha \cdot (\gamma^\mu \wedge \gamma^\nu) \partial^\alpha A_{\mu\nu} f(x_\sigma k^\sigma) \\
&= \left({\delta_\alpha}^\mu \gamma^\nu - {\delta_\alpha}^\nu \gamma^\mu \right) k^\alpha A_{\mu\nu} f'(x \cdot k) \\
&= \gamma^\nu k^\mu A_{\mu\nu} f'(x \cdot k) - \gamma^\mu k^\nu A_{\mu\nu} f'(x \cdot k) \\
&= \gamma^\nu k^\mu (A_{\mu\nu} - A_{\nu\mu}) f'(x \cdot k)
\end{aligned}
\tag{69.41}
$$

So, again, as in the divergence part of Maxwell's equation for the vacuum ($\nabla F = 0$), we require, and it is sufficient that

$$
A_{\mu\nu} - A_{\nu\mu} = 0, \tag{69.42}
$$

for eq. (69.37) to be a solution. This is somewhat surprising since I would not have expected a symmetric tensor to fall out of the analysis.

Actually, this is more than surprising and amounts to a requirement that the field solution is zero. Going back to the proposed solution we have

$$
\begin{aligned}
F &= \gamma^\mu \wedge \gamma^\nu A_{\mu\nu} f(x \cdot k) \\
&= \gamma^\mu \wedge \gamma^\nu \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) f(x \cdot k)
\end{aligned}
\tag{69.43}
$$

So, any symmetric components of the tensor A automatically cancel out.

69.5 SUMMARY

A few dead ends have been chased and I am left with the original attempt summarized by eq. (69.18), eq. (69.19), and eq. (69.21).

It appears that the TODO noted above to attempt the Fourier transform treatment will likely be required to put these exponentials into a more general form. I had also intended to try to cover phase and group velocities for myself here but took too much time chasing the dead ends. Will have to leave that to another day.

70

MAGNETIC FIELD BETWEEN TWO PARALLEL WIRES

70.1 STUDENT'S GUIDE TO MAXWELL'S' EQUATIONS. PROBLEM 4.1

The problem is:

Two parallel wires carry currents I1 and 2I1 in opposite directions. Use Ampere is law to find the magnetic field at a point midway between the wires.

Do this instead (visualizing the cross section through the wires) for N wires located at points P_k, with currents I_k.

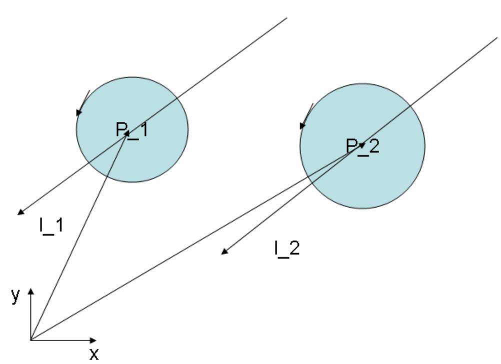

Figure 70.1: Currents through parallel wires

This is illustrated for two wires in fig. 70.1.

70.1.1

Consider first just the magnetic field for one wire, temporarily putting the origin at the point of the current.

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$$

At a point $\mathbf{r}$ from the local origin the tangent vector is obtained by rotation of the unit vector:

$$\hat{\mathbf{y}} \exp\left(\hat{\mathbf{x}}\hat{\mathbf{y}} \log\left(\frac{\mathbf{r}}{||\mathbf{r}||}\right)\right) = \hat{\mathbf{y}}\left(\frac{\mathbf{r}}{||\mathbf{r}||}\right)^{\hat{\mathbf{x}}\hat{\mathbf{y}}}$$

Thus the magnetic field at the point $\mathbf{r}$ due to this particular current is:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I \hat{\mathbf{y}}}{2\pi ||\mathbf{r}||}\left(\frac{\mathbf{r}}{||\mathbf{r}||}\right)^{\hat{\mathbf{x}}\hat{\mathbf{y}}}$$

Considering additional currents with the wire centers at points P_k, and measurement of the field at point $\mathbf{R}$ we have for each of those:

$$\mathbf{r} = \mathbf{R} - \mathbf{P}$$

Thus the total field at point $\mathbf{R}$ is:

$$\mathbf{B}(\mathbf{R}) = \frac{\mu_0 \hat{\mathbf{y}}}{2\pi} \sum_k \frac{I_k}{||\mathbf{R} - \mathbf{P}_k||}\left(\frac{\mathbf{R} - \mathbf{P}_k}{||\mathbf{R} - \mathbf{P}_k||}\right)^{\hat{\mathbf{x}}\hat{\mathbf{y}}} \tag{70.1}$$

70.1.2 *Original problem*

For the problem as stated, put the origin between the two points with those two points on the x-axis.

$$\begin{aligned} \mathbf{P}_1 &= -\hat{\mathbf{x}} d/2 \\ \mathbf{P}_2 &= \hat{\mathbf{x}} d/2 \end{aligned} \tag{70.2}$$

Here $\mathbf{R} = 0$, so $\mathbf{r}_1 = \mathbf{R} - \mathbf{P}_1 = \hat{\mathbf{x}} d/2$ and $\mathbf{r}_2 = -\hat{\mathbf{x}} d/2$. With $\hat{\mathbf{x}}\hat{\mathbf{y}} = i$, this is:

$$\begin{aligned} \mathbf{B}(0) &= \frac{\mu_0 \hat{\mathbf{y}}}{\pi d}\left(I_1 (-\hat{\mathbf{x}})^i + I_2 \hat{\mathbf{x}}^i\right) \\ &= \frac{\mu_0 \hat{\mathbf{y}}}{\pi d}\left(-I - 2I\right) \\ &= \frac{-3 I \mu_0 \hat{\mathbf{y}}}{\pi d} \end{aligned} \tag{70.3}$$

Here unit vectors exponentials were evaluated with the equivalent complex number manipulations:

$$
\begin{aligned}
(-1)^i &= x \\
i \log(-1) &= \log x \\
i\pi &= \log x \\
\exp(i\pi) &= \log x \\
x &= -1
\end{aligned}
\tag{70.4}
$$

$$
\begin{aligned}
(1)^i &= x \\
i \log(1) &= \log x \\
0 &= \log x \\
x &= 1
\end{aligned}
\tag{70.5}
$$

71

FIELD DUE TO LINE CHARGE IN ARC

71.1 MOTIVATION

Problem 1.5 from [37], is to calculate the field at the center of a half circular arc of line charge. Do this calculation and setup for the calculation at other points.

71.2 JUST THE STATED PROBLEM

To solve for the field at just the center point in the plane of the arc, given line charge density λ, and arc radius R one has, and pseudoscalar for the plane $i = \mathbf{e}_1\mathbf{e}_2$ one has

$$
\begin{aligned}
dq &= \lambda R d\theta \\
d\mathbf{E} &= \frac{1}{4\pi\epsilon_0 R^2} dq(-\mathbf{e}_1 e^{i\theta})
\end{aligned}
\tag{71.1}
$$

Straight integration gives the result in short order

$$
\begin{aligned}
\mathbf{E} &= \frac{-\lambda \mathbf{e}_1}{4\pi\epsilon_0 R} \int_0^\pi e^{i\theta} d\theta \\
&= \frac{\lambda \mathbf{e}_2}{4\pi\epsilon_0 R} \left. e^{i\theta} \right|_0^\pi \\
&= \frac{-\lambda \mathbf{e}_2}{2\pi\epsilon_0 R}
\end{aligned}
\tag{71.2}
$$

So, if the total charge is $Q = \pi R \lambda$, the field is then

$$
\mathbf{E} = \frac{-Q\mathbf{e}_2}{2\pi^2\epsilon_0 R^2}
\tag{71.3}
$$

So, at the center point the semicircular arc of charge behaves as if it is a point charge of magnitude $2Q/\pi$ at the point $R\mathbf{e}_2$

$$
\mathbf{E} = \frac{-Q\mathbf{e}_2}{4\pi\epsilon_0 R^2} \frac{2}{\pi}
\tag{71.4}
$$

71.3 FIELD AT OTHER POINTS

Now, how about at points outside of the plane of the charge?

Suppose our point of measurement is expressed in cylindrical polar coordinates

$$P = \rho \mathbf{e}_1 e^{i\alpha} + z\mathbf{e}_3 \tag{71.5}$$

So that the vector from the element of charge at θ is

$$\mathbf{u} = P - R\mathbf{e}_1 e^{i\theta} = \mathbf{e}_1(\rho e^{i\alpha} - Re^{i\theta}) + z\mathbf{e}_3 \tag{71.6}$$

Relative to θ, writing $\theta = \alpha + \beta$ this is

$$\mathbf{u} = \mathbf{e}_1 e^{i\alpha}(\rho - Re^{i\beta}) + z\mathbf{e}_3 \tag{71.7}$$

The squared magnitude of this vector is

$$\begin{aligned} \mathbf{u}^2 &= \left|\rho - Re^{i\beta}\right|^2 + z^2 \\ &= z^2 + \rho^2 + R^2 - 2\rho R \cos\beta \end{aligned} \tag{71.8}$$

The field is thus

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\lambda R \int_{\beta=\theta_1-\alpha}^{\beta=\theta_2-\alpha} \left(z^2 + \rho^2 + R^2 - 2\rho R\cos\beta\right)^{-3/2} \left(\mathbf{e}_1 e^{i\alpha}(\rho - Re^{i\beta}) + z\mathbf{e}_3\right) d\beta \tag{71.9}$$

This integral has two variations

$$\begin{aligned} &\int \left(a^2 - b^2\cos\beta\right)^{-3/2} d\beta \\ &\int \left(a^2 - b^2\cos\beta\right)^{-3/2} e^{i\beta} d\beta \end{aligned} \tag{71.10}$$

or

$$\begin{aligned} I_1 &= \int \left(a^2 - b^2\cos\beta\right)^{-3/2} d\beta \\ I_2 &= \int \left(a^2 - b^2\cos\beta\right)^{-3/2} \cos\beta d\beta \\ I_3 &= \int \left(a^2 - b^2\cos\beta\right)^{-3/2} \sin\beta d\beta \end{aligned} \tag{71.11}$$

Of these when only the last is obviously integrable (at least for $b \neq 0$)

$$\begin{aligned} I_3 &= \int \left(a^2 - b^2\cos\beta\right)^{-3/2} \sin\beta d\beta \\ &= -2\left(a^2 - b^2\cos\beta\right)^{-1/2} \end{aligned} \tag{71.12}$$

Having solved for the imaginary component can the Cauchy Riemann equations be used to supply the real part? How about I_1 ?

71.3.1 *On the z-axis*

Not knowing how to solve the integral of eq. (71.9) (elliptic?), the easy case of $\rho = 0$ (up the z-axis) can at least be obtained

$$
\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\lambda R\left(z^2 + R^2\right)^{-3/2}\int_{\theta_1}^{\theta_2}\left(-\mathbf{e}_1 R e^{i\theta} + z\mathbf{e}_3\right)d\theta \\
&= \frac{1}{4\pi\epsilon_0}\lambda R\left(z^2 + R^2\right)^{-3/2}\left(\mathbf{e}_2 R(e^{i\theta_2} - e^{i\theta_1}) + z\mathbf{e}_3\Delta\theta\right) \\
&= \frac{1}{4\pi\epsilon_0}\lambda R\left(z^2 + R^2\right)^{-3/2}\left(\mathbf{e}_2 R e^{i(\theta_1+\theta_2)/2}\left(e^{i(\theta_2-\theta_1)/2} - e^{-i(\theta_2-\theta_1)/2}\right) + z\mathbf{e}_3\Delta\theta\right) \\
&= \frac{1}{4\pi\epsilon_0}\lambda R\left(z^2 + R^2\right)^{-3/2}\left(-2\mathbf{e}_1 R e^{i(\theta_1+\theta_2)/2}\sin(\Delta\theta/2) + z\mathbf{e}_3\Delta\theta\right) \\
&= \frac{1}{4\pi\epsilon_0\Delta\theta}Q\left(z^2 + R^2\right)^{-3/2}\left(-2\mathbf{e}_1 R e^{i(\theta_1+\theta_2)/2}\sin(\Delta\theta/2) + z\mathbf{e}_3\Delta\theta\right)
\end{aligned}
\tag{71.13}
$$

Eliminating the explicit imaginary, and writing $\bar{\theta} = (\theta_1 + \theta_2)/2$, we have in vector form the field on any position up and down the z-axis

$$
\mathbf{E} = \frac{1}{4\pi\epsilon_0\Delta\theta}Q\left(z^2 + R^2\right)^{-3/2}\left(-2R\left(\mathbf{e}_1\cos\bar{\theta} + \mathbf{e}_2\sin\bar{\theta}\right)\sin(\Delta\theta/2) + z\mathbf{e}_3\Delta\theta\right)
\tag{71.14}
$$

For $z = 0$, $\theta_1 = 0$, and $\theta_2 = \pi$, this matches with eq. (71.4) as expected, but expressing this as an equivalent to a point charge is no longer possible at any point off the plane of the charge.

72

CHARGE LINE ELEMENT

72.1 MOTIVATION

In [37] the electric field for an infinite length charged line element is derived in two ways. First using summation directly, then with Gauss's law. Associated with the first was the statement that the field must be radial by symmetry. This was not obvious to me when initially taking my E&M course, so I thought it was worth revisiting.

72.2 CALCULATION OF ELECTRIC FIELD FOR NON-INFINITE LENGTH LINE ELEMENT

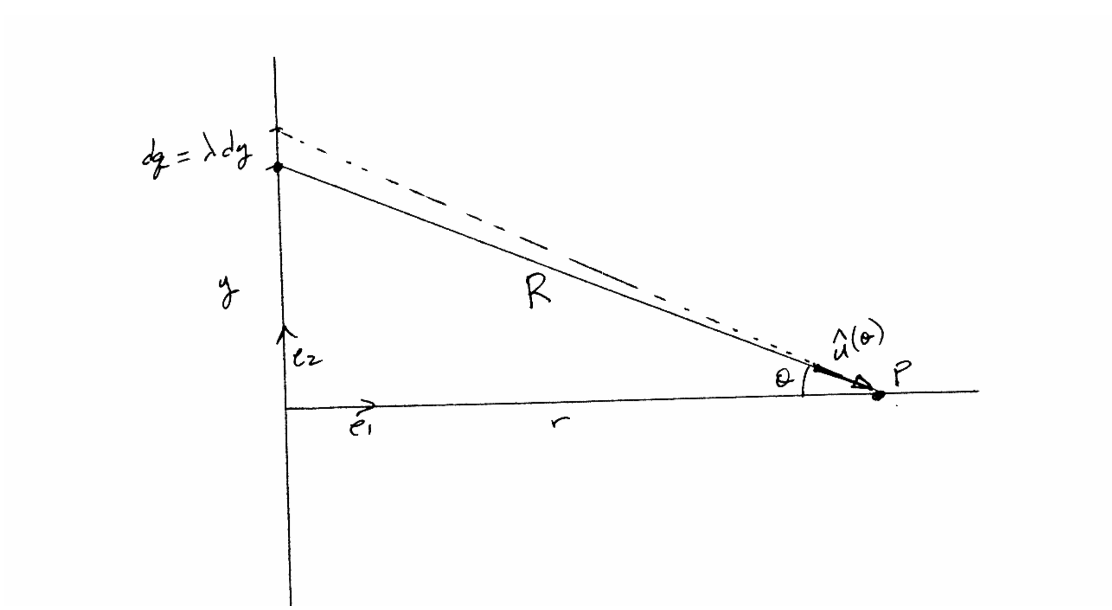

Figure 72.1: Charge on wire

This calculation will be done with a thickness neglected wire running up and down along the y axis as illustrated in fig. 72.1, where the field is being measured at $P = r\mathbf{e}_1$, and the field contributions due to all charge elements $dq = \lambda dy$ are to be summed.

We want to sum each of the field contributions along the line, so with

$$\begin{aligned}
d\mathbf{E} &= \frac{dq\hat{\mathbf{u}}(\theta)}{4\pi\epsilon_0 R^2} \\
r/R &= \cos\theta \\
dy &= rd(\tan\theta) = r\sec^2\theta \\
\hat{\mathbf{u}}(\theta) &= \mathbf{e}_1 e^{i\theta} \\
i &= \mathbf{e}_1\mathbf{e}_2
\end{aligned} \tag{72.1}$$

Putting things together we have

$$\begin{aligned}
d\mathbf{E} &= \frac{\lambda r\sec^2\theta\mathbf{e}_1 e^{i\theta}d\theta}{4\pi\epsilon_0 r^2\sec^2\theta} \\
&= \frac{\lambda\mathbf{e}_1 e^{i\theta}d\theta}{4\pi\epsilon_0 r} \\
&= -\frac{\lambda\mathbf{e}_1 id(e^{i\theta})}{4\pi\epsilon_0 r}
\end{aligned} \tag{72.2}$$

Thus the total field is

$$\begin{aligned}
\mathbf{E} &= \int d\mathbf{E} \\
&= -\frac{\lambda\mathbf{e}_2}{4\pi\epsilon_0 r}\int d(e^{i\theta})
\end{aligned} \tag{72.3}$$

We see that the integration, which has the value

$$\mathbf{E} = -\frac{\lambda}{4\pi\epsilon_0 r}\mathbf{e}_2 e^{i\delta\theta} \tag{72.4}$$

The integration range for the infinite wire is $\theta \in [3\pi/2, \pi/2]$ so the field for the infinite wire is

$$\begin{aligned}
\mathbf{E} &= -\frac{\lambda}{4\pi\epsilon_0 r}\mathbf{e}_2 \left. e^{i\theta}\right|_{\theta=3\pi/2}^{\theta=\pi/2} \\
&= -\frac{\lambda}{4\pi\epsilon_0 r}\mathbf{e}_2(e^{i\pi/2} - e^{3i\pi/2}) \\
&= -\frac{\lambda}{4\pi\epsilon_0 r}\mathbf{e}_2(\mathbf{e}_1\mathbf{e}_2 - (-\mathbf{e}_1\mathbf{e}_2)) \\
&= \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e}_1
\end{aligned} \tag{72.5}$$

Invoking symmetry was done in order to work with coordinates, but working with the vector quantities directly avoids this requirement and gives the general result for any subset of angles.

For a finite length wire all that is required is an angle parametrization of that wire's length

$$[\theta_1, \theta_2] = [\tan^{-1}(y_1/r), \tan^{-1}(y_2/r)] \tag{72.6}$$

For such a range the exponential difference for the integral is

$$\begin{aligned} e^{i\theta}\Big|_{\theta_1}^{\theta_2} &= e^{i\theta_2} - e^{i\theta_1} \\ &= e^{i(\theta_1+\theta_2)/2}\left(e^{i(\theta_2-\theta_1)/2} - e^{i(\theta_2-\theta_1)/2}\right) \\ &= 2ie^{i(\theta_1+\theta_2)/2}\sin((\theta_2-\theta_1)/2) \end{aligned} \tag{72.7}$$

thus the associated field is

$$\begin{aligned} \mathbf{E} &= -\frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e}_2 ie^{i(\theta_1+\theta_2)/2}\sin((\theta_2-\theta_1)/2) \\ &= \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e}_1 e^{i(\theta_1+\theta_2)/2}\sin((\theta_2-\theta_1)/2) \end{aligned} \tag{72.8}$$

73

BIOT SAVART DERIVATION

73.1 MOTIVATION

Looked at my Biot-Savart derivation in 96. There I was playing with doing this without first dropping down to the familiar vector relations, and end up with an expression of the Biot Savart law in terms of the complete Faraday bivector. This is an excessive approach, albeit interesting (to me). Let us try this again in terms of just the magnetic field.

73.2 DO IT

73.2.1 *Setup. Ampere-Maxwell equation for steady state*

The starting point can still be Maxwell's equation

$$\nabla F = J/\epsilon_0 c \tag{73.1}$$

and the approach taken will be the more usual consideration of a loop of steady-state (no-time variation) current.

In the steady state we have

$$\nabla = \gamma^0 \frac{1}{c}\partial_t + \gamma^k \partial_k = \gamma^k \partial_k \tag{73.2}$$

and in particular

$$\begin{aligned}
\gamma_0 \nabla F &= \gamma_0 \gamma^k \partial_k F \\
&= \gamma_k \gamma_0 \partial_k F \\
&= \sigma_k \partial_k F \\
&= \boldsymbol{\nabla}(\mathbf{E} + Ic\mathbf{B})
\end{aligned} \tag{73.3}$$

and for the RHS,

$$\begin{aligned}\gamma_0 J/\epsilon_0 c &= \gamma_0(c\rho\gamma_0 + J^k\gamma_k)/\epsilon_0 c \\ &= (c\rho - J^k\sigma_k)/\epsilon_0 c \\ &= (c\rho - \mathbf{j})/\epsilon_0 c\end{aligned} \tag{73.4}$$

So we have

$$\boldsymbol{\nabla}(\mathbf{E} + Ic\mathbf{B}) = \frac{1}{\epsilon_0}\rho - \frac{\mathbf{j}}{\epsilon_0 c} \tag{73.5}$$

Selection of the (spatial) vector grades gives

$$Ic(\boldsymbol{\nabla} \wedge \mathbf{B}) = -\frac{\mathbf{j}}{\epsilon_0 c} \tag{73.6}$$

or with $\mathbf{a} \wedge \mathbf{b} = I(\mathbf{a} \times \mathbf{b})$, and $\epsilon_0\mu_0 c^2 = 1$, this is the familiar Ampere-Maxwell equation when $\partial\mathbf{E}/\partial t = 0$.

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} \tag{73.7}$$

73.2.2 *Three vector potential solution*

With $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (the trivector part of eq. (73.5)), we can write

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{73.8}$$

For some vector potential $\mathbf{A}$. In particular, we have in eq. (73.7),

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{B} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) \\ &= -I(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \times \mathbf{A})) \\ &= -\frac{I}{2}(\boldsymbol{\nabla}(\boldsymbol{\nabla} \times \mathbf{A}) - (\boldsymbol{\nabla} \times \mathbf{A})\boldsymbol{\nabla}) \\ &= \frac{I^2}{2}(\boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{A}) - (\boldsymbol{\nabla} \wedge \mathbf{A})\boldsymbol{\nabla}) \\ &= -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A})\end{aligned} \tag{73.9}$$

Therefore the three vector potential equation for the magnetic field is

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A} = \mu_0 \mathbf{j} \tag{73.10}$$

73.2.3 *Gauge freedom*

We have the freedom to set $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, in eq. (73.10). To see this suppose that the vector potential is expressed in terms of some other potential $\mathbf{A}'$ that does have zero divergence ($\boldsymbol{\nabla} \cdot \mathbf{A}' = 0$) plus a (spatial) gradient

$$\mathbf{A} = \mathbf{A}' + \boldsymbol{\nabla}\phi \tag{73.11}$$

Provided such a construction is possible, then we have

$$\begin{aligned} \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2\mathbf{A} &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\mathbf{A}' + \boldsymbol{\nabla}\phi)) - \boldsymbol{\nabla}^2(\mathbf{A}' + \boldsymbol{\nabla}\phi) \\ &= -\boldsymbol{\nabla}^2\mathbf{A}' \end{aligned} \tag{73.12}$$

and can instead solve the simpler equivalent problem

$$\boldsymbol{\nabla}^2\mathbf{A}' = -\mu_0\mathbf{j} \tag{73.13}$$

Addition of the gradient $\boldsymbol{\nabla}\phi$ to A' will not change the magnetic field $\mathbf{B}$ since $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0$.

FIXME: what was not shown here is that it is possible to express any vector potential $\mathbf{A}$ in terms of a divergence free potential and a gradient. How would one show this?

73.2.4 *Solution to the vector Poisson equation*

The solution (dropping primes) to the Poisson eq. (73.13) is

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}}{r} dV \tag{73.14}$$

(See [39] for example.)

The magnetic field follows by taking the spatial curl

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \\ &= \frac{\mu_0}{4\pi}\boldsymbol{\nabla} \times \int \frac{\mathbf{j}'}{|\mathbf{r} - \mathbf{r}'|} dV' \end{aligned} \tag{73.15}$$

Pulling the curl into the integral and writing the gradient in terms of radial components

$$\boldsymbol{\nabla} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial}{\partial|\mathbf{r} - \mathbf{r}'|} \tag{73.16}$$

we have

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \times \mathbf{j}' \frac{\partial}{\partial |\mathbf{r} - \mathbf{r}'|} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' \\
&= -\frac{\mu_0}{4\pi} \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \times \mathbf{j}' dV'
\end{aligned}
\tag{73.17}
$$

Finally with $\mathbf{j}' dV' = I \hat{\mathbf{j}}' dl'$, we have

$$
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dl' \hat{\mathbf{j}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}
\tag{73.18}
$$

74

VECTOR FORMS OF MAXWELL'S EQUATIONS AS PROJECTION AND REJECTION OPERATIONS

74.1 VECTOR FORM OF MAXWELL'S EQUATIONS

We saw how to extract the tensor formulation of Maxwell's equations from $\nabla F = J$. A little bit of play shows how to pick off the divergence equations we are used to as well.

The end result is that we can pick off two of the eight coordinate equations with specific product operations.

It is helpful in the following to write ∇F in index notation

$$\nabla F = \frac{\partial E^i}{\partial x^\mu} {\gamma^\mu}_{i0} - \epsilon_{ijk} c \frac{\partial B^i}{\partial x^\mu} {\gamma^\mu}_{jk} \tag{74.1}$$

In particular, look at the span of the vector, or trivector multiplicands of the partials of the electric and magnetic field coordinates

$${\gamma^\mu}_{i0} \in \text{span}\{\gamma_\mu, \gamma_{0ij}\} \tag{74.2}$$

$${\gamma^\mu}_{jk} \in \text{span}\{\gamma_{ij\mu}, \gamma_i\} \tag{74.3}$$

74.1.1 *Gauss's law for electrostatics*

For extract Gauss's law for electric fields that operation is to take the scalar parts of the product with γ^0.

Dotting with γ^0 will pick off the ρ term from J

$$\frac{J}{\epsilon_0 c} \cdot \gamma^0 = \rho/\epsilon_0,$$

We see that dotting with γ_0 will leave bivector parts contributed by the trivectors in the span of eq. (74.2). Similarly the magnetic partials will contribute bivectors and scalars with this product. Therefore to get an equation with strictly scalar parts equal to ρ/ϵ_0 we need to compute

$$\begin{aligned}
\left\langle \left(\nabla F - J/\epsilon_0 c\right)\gamma^0 \right\rangle &= \left\langle \nabla \mathbf{E} \gamma^0 \right\rangle - \rho/\epsilon_0 \\
&= \left\langle \nabla E^k {\gamma_{k0}}^0 \right\rangle - \rho/\epsilon_0 \\
&= \left\langle \gamma^j \partial_j E^k \gamma_k \right\rangle - \rho/\epsilon_0 \\
&= {\delta^j}_k \partial_j E^k - \rho/\epsilon_0 \\
&= \partial_k E^k - \rho/\epsilon_0
\end{aligned} \tag{74.4}$$

This is Gauss's law for electrostatics:

$$\left\langle \left(\nabla F - J/\epsilon_0 c\right)\gamma^0 \right\rangle = \boldsymbol{\nabla} \cdot \mathbf{E} - \rho/\epsilon_0 = 0 \tag{74.5}$$

74.1.2 *Gauss's law for magnetostatics*

Here we are interested in just the trivector terms that are equal to zero that we saw before in $\nabla \wedge \nabla \wedge A = 0$.

The divergence like equation of these four can be obtained by dotting with $\gamma_{123} = \gamma^0 I$. From the span enumerated in eq. (74.3), we see that only the **B** field contributes such a trivector. An addition scalar part selection is used to eliminate the bivector that J contributes.

$$\begin{aligned}
\left\langle \left(\nabla F - J/\epsilon_0 c\right) \cdot \left(\gamma^0 I\right) \right\rangle &= (\nabla I c \mathbf{B}) \cdot \left(\gamma^0 I\right) \\
&= \left\langle \nabla I c \mathbf{B} \gamma^0 I \right\rangle \\
&= \left\langle I \nabla I c \mathbf{B} \gamma^0 \right\rangle \\
&= -c \left\langle I^2 \nabla \mathbf{B} \gamma^0 \right\rangle \\
&= c \left\langle \nabla \mathbf{B} \gamma^0 \right\rangle \\
&= c \left\langle \gamma^\mu \partial_\mu B^k \gamma_k \right\rangle \\
&= c {\delta^\mu}_k \partial_\mu B^k \\
&= c \partial_k B^k \\
&= 0
\end{aligned} \tag{74.6}$$

This is just the divergence, and therefore yields Gauss's law for magnetostatics:

$$\left(\nabla F - J/\epsilon_0 c\right) \cdot \left(\gamma^0 I/c\right) = \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{74.7}$$

74.1.3 *Faraday's Law*

We have three more trivector equal zero terms to extract from our field equation.

Taking dot products for those remaining three trivectors we have

$$(\nabla F - J/\epsilon_0 c) \cdot (\gamma^j I) \tag{74.8}$$

This will leave a contribution from J, so to exclude that we want to calculate

$$\left\langle (\nabla F - J/\epsilon_0 c) \cdot (\gamma^j I) \right\rangle \tag{74.9}$$

The electric field contribution gives us

$$\partial_\mu E^k \left\langle \gamma^\mu \gamma_{k0} \gamma^j{}_{0123} \right\rangle = -\partial_\mu E^k (\gamma_0)^2 \left\langle \gamma^\mu \gamma_k \gamma^j{}_{123} \right\rangle \tag{74.10}$$

the terms $\mu = 0$ will not produce a scalar, so this leaves

$$\begin{aligned}
-\partial_i E^k (\gamma_0)^2 \left\langle \gamma^i \gamma_k \gamma^j{}_{123} \right\rangle &= -\partial_i E^k (\gamma_0)^2 (\gamma_k)^2 \epsilon_{jki} \\
&= \partial_i E^k \epsilon_{jki} \\
&= -\partial_i E^k \epsilon_{jik}
\end{aligned} \tag{74.11}$$

Now, for the magnetic field contribution we have

$$\begin{aligned}
c\partial_\mu B^k \left\langle \gamma^\mu I \gamma_{k0} \gamma^j I \right\rangle &= -c\partial_\mu B^k \left\langle I \gamma^\mu \gamma_{k0} \gamma^j I \right\rangle \\
&= -c\partial_\mu B^k \left\langle I^2 \gamma^\mu \gamma_{k0} \gamma^j \right\rangle \\
&= c\partial_\mu B^k \left\langle \gamma^\mu \gamma_{k0} \gamma^j \right\rangle
\end{aligned} \tag{74.12}$$

For a scalar part we need $\mu = 0$ leaving

$$\begin{aligned}
c\partial_0 B^k \left\langle \gamma^0 \gamma_{k0} \gamma^j \right\rangle &= -\partial_t B^k \left\langle \gamma_k \gamma^j \right\rangle \\
&= -\partial_t B^k \delta_k{}^j \\
&= -\partial_t B^j
\end{aligned} \tag{74.13}$$

Combining the results and summing as a vector we have:

$$\begin{aligned}
\sum \sigma_j \left\langle (\nabla F - J/\epsilon_0 c) \cdot (\gamma^j I) \right\rangle &= -\partial_i E^k \epsilon_{jik} \sigma_j - \partial_t B^j \sigma_j \\
&= -\partial_j E^k \epsilon_{ijk} \sigma_i - \partial_t B^i \sigma_i \\
&= -\boldsymbol{\nabla} \times \mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} \\
&= 0
\end{aligned} \tag{74.14}$$

Moving one term to the opposite side of the equation yields the familiar vector form for Faraday's law

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{74.15}$$

74.1.4 *Ampere Maxwell law*

For the last law, we want the current density, so to extract the Ampere Maxwell law we must have to wedge with γ^0. Such a wedge will eliminate all the trivectors from the span of eq. (74.2), but can contribute pseudoscalar components from the trivectors in eq. (74.3). Therefore the desired calculation is

$$\begin{aligned}
\left\langle (\nabla F - J/\epsilon_0 c) \wedge \gamma^0 \right\rangle_2 &= \left\langle ((\gamma^\mu{}_{j0}) \wedge \gamma^0 \partial_\mu E^j + (\nabla I c B) \wedge \gamma^0 \right\rangle_2 - (\gamma_0)^2 \mathbf{J}/\epsilon_0 c \\
&= \left\langle -((\gamma^0{}_{0j}) \wedge \gamma^0 \partial_0 E^j + (\nabla I c B) \wedge \gamma^0 \right\rangle_2 - (\gamma_0)^2 \mathbf{J}/\epsilon_0 c \\
&= -\gamma_j{}^0 \frac{1}{c} \partial_t E^j + \left\langle (\nabla I c B) \wedge \gamma^0 \right\rangle_2 - (\gamma_0)^2 \mathbf{J}/\epsilon_0 c \\
&= -\frac{(\gamma_0)^2}{c} \frac{\partial \mathbf{E}}{\partial t} + c \langle \nabla I B \rangle_1 \wedge \gamma^0 - (\gamma_0)^2 \mathbf{J}/\epsilon_0 c
\end{aligned} \tag{74.16}$$

Let us take just that middle term

$$\begin{aligned}
\langle \nabla I B \rangle_1 \wedge \gamma^0 &= -\left\langle I \gamma^\mu \partial_\mu B^k \gamma_{k0} \right\rangle_1 \wedge \gamma^0 \\
&= -\partial_\mu B^k \langle \gamma_{0123} \gamma^\mu \gamma_{k0} \rangle_1 \wedge \gamma^0 \\
&= \partial_\mu B^k \left(\langle \gamma_{0123} \gamma^\mu \gamma_0 \rangle_2 \cdot \gamma_k \right) \wedge \gamma^0
\end{aligned} \tag{74.17}$$

Here $\mu \neq 0$ since that leaves just a pseudoscalar in the grade two selection.

$$\begin{aligned}
\langle \nabla I B \rangle_1 \wedge \gamma^0 &= \partial_j B^k \left(\left\langle \gamma_{0123} \gamma^j \gamma_0 \right\rangle_2 \cdot \gamma_k \right) \wedge \gamma^0 \\
&= (\gamma_0)^2 \partial_j B^k \left(\left\langle \gamma_{123} \gamma^j \right\rangle_2 \cdot \gamma_k \right) \wedge \gamma^0 \\
&= (\gamma_0)^2 \partial_j B^k \left(\left\langle \epsilon^{hkj} \gamma_{hkj} \gamma^j \right\rangle_2 \cdot \gamma_k \right) \wedge \gamma^0 \\
&= \partial_j B^k \epsilon^{hkj} (\gamma_0)^2 (\gamma_k)^2 \gamma_h{}^0 \\
&= -(\gamma_0)^2 \partial_j B^k \epsilon^{hkj} \sigma_h \\
&= (\gamma_0)^2 \boldsymbol{\nabla} \times \mathbf{B}
\end{aligned} \tag{74.18}$$

Putting things back together and factoring out the common metric dependent $(\gamma_0)^2$ term we have

$$\begin{aligned}
-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + c\boldsymbol{\nabla}\times\mathbf{B} - \mathbf{J}/\epsilon_0 c = 0 \\
\implies \\
-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B} - \mathbf{J}/\epsilon_0 c^2 = 0
\end{aligned} \tag{74.19}$$

With $\frac{1}{c^2} = \mu_0\epsilon_0$ this is the Ampere Maxwell law

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \tag{74.20}$$

which we can put in the projection form of eq. (74.5) and eq. (74.7) as:

$$\left\langle (\nabla F - J/\epsilon_0 c)\wedge(\gamma_0/c)\right\rangle_2 = \boldsymbol{\nabla}\times\mathbf{B} - \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) = 0 \tag{74.21}$$

74.2 SUMMARY OF TRADITIONAL MAXWELL'S EQUATIONS AS PROJECTIVE OPERATIONS ON MAXWELL EQUATION

$$\begin{aligned}
\left\langle (\nabla F - J/\epsilon_0 c)\,\gamma^0\right\rangle &= \boldsymbol{\nabla}\cdot\mathbf{E} - \rho/\epsilon_0 = 0 \\
\left\langle (\nabla F - J/\epsilon_0 c)\cdot\left(\gamma^0 I/c\right)\right\rangle &= \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \\
\sum \sigma_j \left\langle (\nabla F - J/\epsilon_0 c)\cdot(\gamma^j I)\right\rangle &= -\boldsymbol{\nabla}\times\mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0 \\
\left\langle (\nabla F - J/\epsilon_0 c)\wedge(\gamma_0/c)\right\rangle_2 &= \boldsymbol{\nabla}\times\mathbf{B} - \mu_0\left(\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) = 0
\end{aligned} \tag{74.22}$$

Faraday's law requiring a sum suggests that this can likely be written instead using a rejective operation. Will leave that as a possible future followup.

75

APPLICATION OF STOKES INTEGRALS TO MAXWELL'S EQUATION

75.1 PUTTING MAXWELL'S EQUATION IN CURL FORM

These notes contain an application of the bivector Stokes equations detailed in **??** . Background of interest can also be found in [8], which contained the core statement of the multivector form of Stokes equation and Biot-Savart like application of it. Also informative as background is the following excellent [7]. introduction to the STA form of Maxwell's equation.

Stokes equation applied to a bivector takes the following form

$$\iiint (\nabla \wedge F) \cdot d^3\mathbf{x} = \oiint F \cdot d^2\mathbf{x}, \tag{75.1}$$

where we will write F as the electromagnetic field bivector, and apply it to Maxwell's equation

$$\nabla F = J/\epsilon_0 c. \tag{75.2}$$

Taking vector and trivector parts we have two equations

$$\nabla \cdot F = J/\epsilon_0 c, \tag{75.3}$$

and

$$\nabla \wedge F = 0. \tag{75.4}$$

75.1.1 *Trivector equation part*

The second of these, eq. (75.4), we can apply Stokes to directly:

$$\iiint (\nabla \wedge F) \cdot d^3\mathbf{x} = \oiint F \cdot d^2\mathbf{x} = 0. \tag{75.5}$$

This area integral is a flux like quantity. Suppose we call this the field flux, then this says says the flux of the combined electromagnetic field through any surface is zero independent of the charge or current densities. Note that here $d^3\mathbf{x}$ can be a regular spatial volume trivector element, but one can also pick a spacetime (area times time) "volume" to integrate over, in which case $d^2\mathbf{x}$ are the oriented "surfaces" of such a spacetime volume.

This does not seem like a result that I am familiar with based on the traditional vector forms of Maxwell's equation. Perhaps it is recognizable in terms of $\mathbf{E}$ and $\mathbf{B}$ explicitly:

$$\oiint \mathbf{E} \cdot d^2\mathbf{x} = -c \oiint \mathbf{B} \cdot (d^2\mathbf{x} I) \tag{75.6}$$

On the surface this does not look like a familiar identity. It is in fact Gauss's law for magnetostatics, which will be shown later.

Note also the subtle difference from traditional vector treatments where $\mathbf{E}$ and $\mathbf{B}$ were spatial vectors. Here they are written as spacetime bivectors, $\mathbf{E} = E^i \sigma_i = E^i \gamma_i \wedge \gamma_0$, $\mathbf{B} = B^i \sigma_i = B^i \gamma_i \wedge \gamma_0$.

75.1.2 *Vector part*

Moving on to the charge and current dependent vector terms of Maxwell's equation, we want express eq. (75.3) as a spacetime curl so that we can apply stokes to it.

We can do this by temporarily writing our field in terms of a potential as well its dual bivector.

$$F = \nabla \wedge A = ID \tag{75.7}$$

$$\begin{aligned}
\nabla F &= \nabla(\nabla \wedge A) \\
&= \nabla \cdot (\nabla \wedge A) + \nabla \wedge (\nabla \wedge A) \\
&= \nabla \cdot (ID) \\
&= \langle \nabla I D \rangle_1 \\
&= -\left\langle I(\overbrace{\boxed{\nabla \cdot D}}^{\text{1 vector}} + \underbrace{\boxed{\nabla \wedge D}}_{\text{3 vector}})\right\rangle_1 \\
&= -I(\nabla \wedge D)
\end{aligned} \tag{75.8}$$

or

$$I \nabla F = \nabla \wedge D. \tag{75.9}$$

Applying stokes we have

$$
\begin{aligned}
\int (\nabla \wedge D) \cdot d^3\mathbf{x} &= \oiint D \cdot d^2\mathbf{x} \\
\int (I\nabla F) \cdot d^3\mathbf{x} &= \oiint (-IF) \cdot d^2\mathbf{x} \\
&= \oiint \left\langle -F d^2\mathbf{x} I \right\rangle \\
&= -\oiint F \cdot (d^2\mathbf{x} I) \\
\frac{1}{\epsilon_0 c} \int (IJ) \cdot d^3\mathbf{x} &= \\
\frac{1}{\epsilon_0 c} \int \left\langle I J d^3\mathbf{x} \right\rangle &= \\
\frac{1}{\epsilon_0 c} \int \left\langle J d^3\mathbf{x} I \right\rangle &= \\
\frac{1}{\epsilon_0 c} \int J \cdot (d^3\mathbf{x} I) &=
\end{aligned}
\tag{75.10}
$$

Or

$$
\oiint F \cdot (d^2\mathbf{x} I) = \int \frac{J}{\epsilon_0 c} \cdot (d^3\mathbf{x} I) \tag{75.11}
$$

This is the integral form of the vector part of Maxwell's equation eq. (75.2). This does not look terribly familiar, but we are not used to seeing Maxwell's equations in a non-disassembled form. Hiding in there should be a subset of the traditional eight Maxwell's equations in integral form. It will be possible to extract these by considering variations of current and charge density and different volume and surface integration regions.

75.2 EXTRACTING THE VECTOR INTEGRAL FORMS OF MAXWELL'S EQUATIONS

One can extract the integral forms of Maxwell's equations from eq. (75.2), by first extracting the differential vector equations, and then using the spatial divergence and stokes equations. However, having formulated Stokes equation in its bivector form we can go directly to those equations by appropriate selection of spatial or spacetime volumes. Of course we also now have new tools to work with the field in its entirety, but lets use this as an exercise to verify that all the previous computation that led to Stokes equation gives us the expected results. In particular this should be a good way to verify that sign mistakes or other similar small errors (which would not be too hard) have not been made.

75.2.1 *Zero current density. Gauss's law for Electrostatics*

With $J = c\rho\gamma_0$, the integral form of Maxwell's equation becomes

$$\begin{aligned}\oiint F \cdot (d^2\mathbf{x}I) &= \int \frac{\rho}{\epsilon_0}\left\langle \gamma_0 d^3\mathbf{x}I \right\rangle \\ &= \int \frac{\rho}{\epsilon_0}\left\langle \gamma_{0123}\gamma_0 d^3\mathbf{x} \right\rangle \\ &= -\frac{1}{\epsilon_0}{\gamma_0}^2 \int \frac{\rho}{\epsilon_0}\left\langle \gamma_{123} d^3\mathbf{x} \right\rangle\end{aligned} \tag{75.12}$$

From this we see that, in the absence of currents the LHS integral must be zero unless the volume is purely spatial. Denoting the boundary of a spacetime volume as ∂Act, this is

$$\oiint_{\partial Act} F \cdot (d^2\mathbf{x}I) = 0. \tag{75.13}$$

For a purely spatial volume the dual surfaces $d^2\mathbf{x}I$ always includes a spacetime bivector, therefore the magnetic field contributes nothing

$$\oiint_{\partial V} IcB \cdot (d^2\mathbf{x}I) = -c \oiint_{\partial V} B \cdot d^2\mathbf{x} = 0$$

Although this looks similar to the integral equivalent of $\boldsymbol{\nabla} \cdot B = 0$, we should look elsewhere for that since that is true for the non-zero current density case too.

That leaves

$$\oiint E \cdot (d^2\mathbf{x}I) = -\frac{1}{\epsilon_0}{\gamma_0}^2 \int_V \rho\left\langle \gamma_{123} d^3\mathbf{x} \right\rangle \tag{75.14}$$

Letting $d^3\mathbf{x} = dx^1 dx^2 dx^3 \gamma_{123}$. Within the charge integral becomes

$$-\frac{1}{\epsilon_0}{\gamma_0}^2 \int_V \rho\left\langle \gamma_{123} d^3\mathbf{x} \right\rangle = \frac{1}{\epsilon_0} \overbrace{\boxed{{\gamma_0}^2{\gamma_1}^2}}^{=-1} \underbrace{\boxed{{\gamma_2}^2{\gamma_3}^2}}_{=(\pm 1)^2} \int_V \rho dx^1 dx^2 dx^3 \quad = -\frac{1}{\epsilon_0} \int_V \rho dx^1 dx^2 dx^3 \tag{75.15}$$

To put this in correspondence with the forms we are used to consider the surfaces separately. For the dual to the front surface (see: **??**) we have

$$
\begin{aligned}
d^2\mathbf{x}I &= dx^1 dx^2 \gamma_{12} I \\
&= dx^1 dx^2 \gamma_{120123} \\
&= dx^1 dx^2 \gamma_{112023} \\
&= -dx^1 dx^2 \gamma_{112203} \\
&= -(\pm 1)^2 dx^1 dx^2 \gamma_{03} \\
&= dx^1 dx^2 \sigma_3
\end{aligned}
\tag{75.16}
$$

For the left surface

$$
\begin{aligned}
d^2\mathbf{x}I &= dx^3 dx^2 \gamma_{32} I \\
&= dx^3 dx^2 \gamma_{320123} \\
&= dx^3 dx^2 \gamma_{332012} \\
&= dx^3 dx^2 \gamma_{332201} \\
&= dx^3 dx^2 (\pm 1)^2 \gamma_{01} \\
&= -dx^3 dx^2 \sigma_1
\end{aligned}
\tag{75.17}
$$

and for the top

$$
\begin{aligned}
d^2\mathbf{x}I &= dx^1 dx^3 \gamma_{13} I \\
&= dx^1 dx^3 \gamma_{130123} \\
&= dx^1 dx^3 \gamma_{113023} \\
&= dx^1 dx^3 \gamma_{113302} \\
&= -dx^1 dx^3 \sigma_2
\end{aligned}
\tag{75.18}
$$

Assembling results, writing $(x^1, x^2, x^3) = (x, y, z)$ we have

$$
\begin{aligned}
\frac{1}{\epsilon_0} \int_V \rho dx dy dz = &\iint (E_x(x, y, z_1) - E_x(x, y, z_0)) dx dy \\
&+ \iint (E_y(x_1, y, z) - E_y(x_0, y, z)) dy dz \\
&+ \iint (E_z(x, y_1, z) - E_z(x, y_0, z)) dx dz
\end{aligned}
\tag{75.19}
$$

This is Gauss's law for electrostatics in integral form

$$
\iint \mathbf{E} \cdot \hat{\mathbf{n}} dA = \iiint \frac{\rho}{\epsilon_0} dV
\tag{75.20}
$$

Although this extraction method is easy to understand, it is apparent that having only a pictorial way of enumerating the oriented bivector area elements is not efficient for high level computation. Revisiting the stokes derivation with a more algebraic enumeration of the surfaces should be done!

75.2.2 *Gauss's law for magneto-statics*

Return now to eq. (75.6), which resulted from considering the trivector part of Maxwell's equation

$$\oiint \mathbf{E} \cdot d^2\mathbf{x} = -c \oiint \mathbf{B} \cdot (d^2\mathbf{x} I). \tag{75.21}$$

To start some observations can be made.

Only the spacetime surfaces of the volume contribute to the LHS integral since $\sigma_i \cdot (\gamma_j \wedge \gamma_k) = 0$.

For the RHS, only the purely spatial surfaces contribute to that $\mathbf{B}$ integral, since the dual surface $d^2\mathbf{x}I$ must have a spacetime component for that dot product to be non-zero. We have also just enumerated these dual surface area elements $d^2\mathbf{x}I$ for a purely spatial surface, therefore with a E, B substitution we must have

$$\begin{aligned}
0 = &\iint (B_x(x, y, z_1) - B_x(x, y, z_0))dxdy \\
&+ \iint (B_y(x_1, y, z) - B_y(x_0, y, z))dydz \\
&+ \iint (B_z(x, y_1, z) - B_z(x, y_0, z))dxdz
\end{aligned} \tag{75.22}$$

or, more compactly

$$\iint \mathbf{B} \cdot \hat{\mathbf{n}} dA = 0 \tag{75.23}$$

For any current or charge distribution. We have therefore obtained two of the eight Maxwell's equations.

75.2.3 *Zero charge. Current density in single direction*

Next to consider is $J = j^i \gamma_i$. For simplicity, consider current in only one direction, taking $J = j^1 \gamma_1$. The exercise will be to compute the integrals of eq. (75.11).

$$
\begin{aligned}
\oiint F \cdot (I d^2\mathbf{x}) &= \int \frac{J}{\epsilon_0 c} \cdot (I d^3\mathbf{x}) \\
&= \int \frac{j^1}{\epsilon_0 c} \gamma_1 \cdot (I d^3\mathbf{x})
\end{aligned}
\tag{75.24}
$$

Unlike the calculations for the Gauss's law equations above, this one will be done using the area orientation methods from **??** since algebraically enumerating the surfaces should make life easier. The two Gauss's law results above were done without this, which was not too bad for a purely spatial volume, but with spacetime volumes this is probably confusing in addition to being harder.

Starting with the volume element, one can observe that the current density will not contribute to the boundary integral unless $d^3\mathbf{x}$ has no γ_1 component, thus for a rectangular prism integration spacetime volume let $d^3\mathbf{x} = \gamma_{023} dx^0 dx^2 dx^3$

$$
\begin{aligned}
\gamma_1 \cdot (I d^3\mathbf{x}) &= \gamma_1 \cdot \gamma_{0123023} dx^0 dx^2 dx^3 \\
&= \gamma_1 \cdot \gamma_{0012233} dx^0 dx^2 dx^3 \\
&= \gamma_1 \cdot \gamma_{111} dx^0 dx^2 dx^3 \\
&= -\gamma_1 \cdot \gamma^1 dx^0 dx^2 dx^3 \\
&= -dx^0 dx^2 dx^3
\end{aligned}
\tag{75.25}
$$

Now for all the surfaces we want to calculate $I d^2\mathbf{x}$ for each of the surfaces. For each of $\mu \in \{0, 2, 3\}$, calculation of $I(d^2\mathbf{x})_\mu$ is required where

$$
\begin{aligned}
(d^2\mathbf{x})_\mu &= d^3\mathbf{x} \cdot \mathbf{r}^\mu \\
\mathbf{r} &= x^i \gamma_i \\
\mathbf{r}_\mu &= \frac{\partial \mathbf{r}}{\partial x^\mu} \\
&= \gamma_\mu \\
\mathbf{r}^\mu &= \gamma^\mu
\end{aligned}
\tag{75.26}
$$

Calculating the surfaces

$$
\begin{aligned}
I(d^2\mathbf{x})_\mu \frac{dx^\mu}{dx^0 dx^2 dx^3} &= \langle \gamma_{0123}(\gamma_{023} \cdot \gamma^\mu) \rangle_2 \\
&= \frac{1}{2} \langle \gamma_{0123}(\gamma_{023}\gamma^\mu + \gamma^\mu \gamma_{023}) \rangle_2 \\
&= \frac{1}{2} \langle \gamma_{0123}(\gamma_{023}\gamma^\mu + \gamma_{023}\gamma^\mu) \rangle_2 \\
&= \langle \gamma_{0012233}\gamma^\mu \rangle_2 \\
&= -\langle \gamma_{133}\gamma^\mu \rangle_2 \\
&= -\left\langle \gamma^1 \gamma^\mu \right\rangle_2 \\
&= \gamma^\mu \wedge \gamma^1
\end{aligned}
\tag{75.27}
$$

Putting things back together we have

$$
-\int j^1 dx^0 dx^2 dx^3 = \int \sum_{\mu=0,2,3} F \cdot \left(\gamma^\mu \wedge \gamma^1\right)\Big|_{\partial x^\mu} \frac{dx^0 dx^2 dx^3}{dx^\mu}
\tag{75.28}
$$

Now, for $\mu = 0$ we pick up the electric field component of the field

$$
\begin{aligned}
F \cdot \gamma^{01} &= \left(E^i \gamma_{i0} - \epsilon_{ijk} c B^k \gamma_{ij}\right) \cdot \gamma^{01} \\
&= E^i,
\end{aligned}
\tag{75.29}
$$

and for $\mu = 2, 3$ we pick up magnetic field components

$$
\begin{aligned}
F \cdot \gamma^{\mu 1} &= \left(E^i \gamma_{i0} - \epsilon_{ijk} c B^k \gamma_{ij}\right) \cdot \gamma^{\mu 1} \\
&= -\epsilon_{1\mu k} c B^k \gamma_{1\mu} \cdot \gamma^{\mu 1}.
\end{aligned}
\tag{75.30}
$$

For $\mu = 2$ this is $-cB^3$, and for $\mu = 3$, $-\epsilon_{132} c B^2 = cB^2$, so we have

$$
\begin{aligned}
0 &= \int \frac{j^1}{c\epsilon_0} dx^0 dx^2 dx^3 + \int E^1 dx^2 dx^3\Big|_{\partial x^0} + c \int B^2 dx^0 dx^2\Big|_{\partial x^3} - c \int B^3 dx^0 dx^3\Big|_{\partial x^2} \\
&= \int \frac{j^1}{c\epsilon_0} dx^0 dx^2 dx^3 + \int \frac{\partial E^1}{\partial x^0} dx^0 dx^2 dx^3 + c \int \frac{\partial B^2}{\partial x^3} dx^3 dx^2 dx^0 - c \int \frac{\partial B^3}{\partial x^2} dx^2 dx^0 dx^3 \\
&= \int dx^0 \int dx^2 dx^3 \left(\frac{j^1}{c\epsilon_0} + \frac{1}{c}\frac{\partial E^1}{\partial t} + c\frac{\partial B^2}{\partial x^3} - c\frac{\partial B^3}{\partial x^2} \right)
\end{aligned}
\tag{75.31}
$$

If this is zero for all time intervals, then the inner integral is also zero. Utilizing $c^2\mu_0\epsilon_0 = 1$ this is

$$0 = \int dx^2 dx^3 \left(\mu_0 \left(j^1 + \epsilon_0 \frac{\partial E^1}{\partial t}\right) + \left(\frac{\partial B^2}{\partial x^3} - \frac{\partial B^3}{\partial x^2}\right)\right). \tag{75.32}$$

Writing $d\mathbf{A} = \sigma_1 dx^2 dx^3$, $\mathbf{j} = j^1\sigma_1$, $\mathbf{E} = E^1\sigma_1$, and $\mathbf{B} = B^i\sigma_i$ we can pick off the differential form of the Maxwell-Ampere equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right), \tag{75.33}$$

as well as the integral form

$$\int (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{A} = \mu_0 \left(\int \mathbf{j} \cdot d\mathbf{A} + \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{A}\right) \tag{75.34}$$

Both of these forms come straight from the application of the generalized Stokes equation integrating an appropriate spacetime volume.

Now it is normal to have the spatial curl of **B** written as a closed loop integral. Stokes can be employed again to get exactly that form. This really just undoes the fact that the partials to used as a convenience enumerate exactly those loop boundaries (although they were originally oriented area boundaries).

$$\begin{aligned} \int \frac{\partial B^2}{\partial x^3} dx^3 &= B^2(t, x, y, z_1) - B^2(t, x, y, z_0) \\ \int \frac{\partial B^3}{\partial x^2} dx^2 &= B^3(t, x, y_1, z) - B^3(t, x, y_0, z) \end{aligned} \tag{75.35}$$

Also observe that this whole treatment was done with $J = j^1\gamma_1$ only. It is not hard to see that doing the same with j^i and summing over σ_i will produce the same result. Of course more care is required to handle the more abstract symbolic indices since a nice hard-coded number is easier. On the other hand the usual dodge, employing freedom to orient the coordinate system along the γ_1 direction makes the more general algebraic approach a less interesting exercise.

75.2.4 *Faraday's law*

We have five of the eight Maxwell's equations. Gauss's law for electrostatics from the vector part of eq. (75.3), integrating over a spatial volume, and the Maxwell-Ampere equation from the same, integrating over a spacetime volume. Gauss's law for magneto-statics from the trivector part of eq. (75.3), integrating over a spatial volume. This suggests that our remaining three (one three-vector) equation will come from integrating the trivector parts over a spacetime volume.

Stokes' gives us

$$\int_V (\nabla \wedge F) \cdot d^3\mathbf{x} = \int_{\partial V} F \cdot (d^2\mathbf{x})$$

Picking a spacetime volume element, and corresponding area elements

$$\begin{aligned} d^3\mathbf{x} &= \gamma_{0ij} dx^0 dx^i dx^j \\ (d^2\mathbf{x})_\mu &= (\gamma_{0ij} \cdot \gamma^\mu) \frac{dx^0 dx^i dx^j}{dx^\mu} \end{aligned} \tag{75.36}$$

Our area integral (expanding boundaries as one more integral of partials) is

$$\int \sum_{\mu=0,i,j} dx^0 dx^i dx^j \left(\frac{\partial F}{\partial x^\mu} \cdot (\gamma_{0ij} \cdot \gamma^\mu) \right).$$

For the dot products of the area elements we have

$$\begin{cases} \gamma_{ij} & \text{if } \mu = 0 \\ \gamma_{0i} = -\sigma_i & \text{if } \mu = j \\ -\gamma_{0j} = \sigma_j & \text{if } \mu = i \end{cases}$$

Our field derivatives in coordinates are

$$\frac{\partial F}{\partial x^\mu} = \frac{\partial E^m}{\partial x^\mu} \sigma_m - \epsilon_{klm} c \frac{\partial B^m}{\partial x^\mu} \gamma_{kl}$$

Observe that $\mu \neq 0$ selects only the electric field components, and $\mu = 0$ only the magnetic field components are selected. Specifically

$$\frac{\partial F}{\partial x^\mu} = \begin{cases} -\epsilon_{jim} c \frac{\partial B^m}{\partial x^0} (\gamma_i)^2 (\gamma_j)^2 & = \epsilon_{ijk} \frac{\partial B^k}{\partial t} & \text{if } \mu = 0 \\ \frac{\partial E^m}{\partial x^j} \sigma_m \cdot (-\sigma_i) & = -\frac{\partial E^i}{\partial x^j} & \text{if } \mu = j \\ \frac{\partial E^m}{\partial x^i} \sigma_m \cdot (\sigma_j) & = \frac{\partial E^j}{\partial x^i} & \text{if } \mu = i \end{cases}$$

Reassembling the integral we have

$$\begin{aligned} 0 &= \int dx^0 dx^i dx^j \left(\frac{\partial E^j}{\partial x^i} - \frac{\partial E^i}{\partial x^j} + \epsilon_{ijk} \frac{\partial B^k}{\partial t} \right) \\ &= \int dx^0 \epsilon_{ijk} \int dx^i dx^j \sigma_k \left(\sigma_k \epsilon_{ijk} \left(\frac{\partial E^j}{\partial x^i} - \frac{\partial E^i}{\partial x^j} \right) + \sigma_k \frac{\partial B^k}{\partial t} \right) \end{aligned} \tag{75.37}$$

Summing over k, we can pick out the differential form of Faraday's law

$$0 = \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} \tag{75.38}$$

as well as the integral form

$$\begin{aligned} 0 &= \sum_k \int dx^i dx^j \sigma_k \left(\sigma_k \epsilon_{ijk} \left(\frac{\partial E^j}{\partial x^i} - \frac{\partial E^i}{\partial x^j} \right) + \sigma_k \frac{\partial B^k}{\partial t} \right) \\ &= \sum_k \epsilon_{ijk} \int dx^j \left. E^j \right|_{\partial x^i} - \sum_k \epsilon_{ijk} \int dx^i \left. E^i \right|_{\partial x^j} + \int \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} d\mathbf{A} \end{aligned} \tag{75.39}$$

which is

$$0 = \oint \mathbf{E} \cdot d\mathbf{r} + \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}. \tag{75.40}$$

75.3 CONCLUSION

In the treatment of these notes, the traditional integral form of Maxwell's equations are obtained directly from the STA Maxwell's equation using the bivector Stokes equation, and various space-time integration volumes.

75.3.1 *Summary of results*

We started with the bivector form of Stokes law

$$\iiint (\nabla \wedge F) \cdot d^3\mathbf{x} = \oiint F \cdot d^2\mathbf{x}, \tag{75.41}$$

and the multivector Maxwell equation

$$\nabla F = J/\epsilon_0 c. \tag{75.42}$$

The trivector parts of this can be integrated directly. This integral is always zero for all spacetime or spatial surfaces

$$\int (\nabla \wedge F) \cdot d^3\mathbf{x} = 0 \tag{75.43}$$

Duality relations were used to put the vector parts of eq. (75.42) into a form that Stokes can be applied to. This gives us

$$\oiint F \cdot (d^2\mathbf{x}I) = \int \frac{J}{\epsilon_0 c} \cdot (d^3\mathbf{x}I). \tag{75.44}$$

Integration of the trivector parts

$$\iiint (\nabla \wedge F) \cdot d^3\mathbf{x} = \oiint F \cdot d^2\mathbf{x} = 0, \tag{75.45}$$

produces a combined electric and magnetic field form of a Faraday's law and Gauss' magnetostatics law that does not look terribly familiar

$$\oiint \mathbf{E} \cdot d^2\mathbf{x} = -c \oiint \mathbf{B} \cdot (d^2\mathbf{x}I), \tag{75.46}$$

but integration of this using a spatial volume produces the familiar Gauss's magneto-static law

$$\begin{aligned} \iint \mathbf{B} \cdot d\mathbf{A} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0. \end{aligned} \tag{75.47}$$

Integration and summation of the same trivector parts in eq. (75.46) over each of the possible three spacetime volumes gives us Faraday's law in its familiar forms

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} &= 0 \\ \oint \mathbf{E} \cdot d\mathbf{r} + \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} &= 0. \end{aligned} \tag{75.48}$$

Now, the vector parts of Maxwell's multivector equation integrated over a spatial volume produces Gauss's law for electrostatics

$$\begin{aligned} \iint \mathbf{E} \cdot d\mathbf{A} &= \int \frac{\rho}{\epsilon_0} dV \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}. \end{aligned} \tag{75.49}$$

Finally, integration of the same with summation over all spacetime volumes gives us the famous Maxwell-Ampere equation

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\
\oint \mathbf{B} \cdot d\mathbf{r} &= \mu_0 \left(\int \mathbf{j} \cdot d\mathbf{A} + \epsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{A} \right).
\end{aligned}
\tag{75.50}
$$

In the process of arriving at these results it appears that some of the use of Stokes equation was actually superfluous. One of the first things that was done once the area elements were established was to undo the boundary integral writing things once more in terms of the partials over those boundaries. Doing all this with just the volume integrals would possibly have been simpler. That said, as an exercise to validate the generalized Stokes equation formulation it worked well!

Conceptually the idea that integration of Maxwell's equation over various volumes produces all the traditional vector differential and integral forms that we are used to is quite nice. It seems less arbitrary than trying to figure out the exactly what specific projection like operations, as done in 74, will produce the various traditional vector differential equations. Of course those can be used once found to develop the integral relations, but here we get them all in one shot.

75.3.2 *Getting a glimpse of how the pieces fit together?*

I think I am starting to see a bit of the big picture for electrodynamics. In 65, an earlier treatment of Maxwell's equations in a GA context, I used dimensional analysis to group electric and magnetic fields in a logical way, and employs the spatial pseudoscalar to combine divergence and curl terms. This I thought was a good motivation for the STA form of the equation, using ideas familiar from school. Similar treatments can be found elsewhere such as in [10] but understanding that takes a lot more work.

Once the STA form is taken as more fundamental, one can take that and show the types of spacetime projection operations, as in 74, and produce the various traditional vector differential forms of Maxwell's equations. Alternatively, as in 76, we can extract the traditional tensor form of the equations.

From an even higher level point of view we can relate the STA Maxwell's equations to the least action principles, as done in [25], to find the Lorentz force law in STA form using the Euler-Lagrange equations, and finally in [24] where the STA form of Maxwell's equation is obtained directly from a complex valued field Lagrangian.

Goldstein [16] has an interesting treatment of a combined Lagrangian for both the Lorentz force law and the field equations (using spatial delta functions). Minimization of the action for that Lagrangian with respect to the potential produces the field equations, and with respect to coordinates produces the Lorentz force law. Have to work through that in a covariant form to see how this relates to my previous treatments.

75.3.3 *Followup*

It would be interesting to see if any of the problems in a Maxwell's equation text like [15] would be any easier with a combined field as is possible in the STA formulation (ie: the ones based on just current or charge distributions).

There is also some interesting looking treatments of complex number residue like integrals for the field equation in references such as [20]. I re-encountered that paper after writing up these notes. I had seen it before but those parts that cover (tersely) the same material as above did not make much sense until I had independently worked it all out in detail myself. Perhaps I am dense, but I find that many academic papers are ironically not very good at all for learning from!

I believe these residue/green's function ideas both relate to the Biot-Savart law, as mentioned in [20], [10], and [8]. All of those are either too terse or have details missing that indicate I need to study the ideas in more depth to understand.

76 TENSOR RELATIONS FROM BIVECTOR FIELD EQUATION

76.1 MOTIVATION

This contains a somewhat unstructured collection of notes translating between tensor and bivector forms of Maxwell's equation(s).

76.2 ELECTRODYNAMIC TENSOR

John Denker's paper [7] writes:

$$F = (\mathbf{E} + ic\mathbf{B})\gamma_0 \tag{76.1}$$

with

$$\begin{aligned} \mathbf{E} &= E^i \gamma_i \\ \mathbf{B} &= B^i \gamma_i \end{aligned} \tag{76.2}$$

Since he uses the positive end of the metric for spatial indices this works fine. Contrast to [10] who write:

$$F = \mathbf{E} + ic\mathbf{B} \tag{76.3}$$

with the following implied spatial bivector representation:

$$\begin{aligned} \mathbf{E} &= E^i \sigma_i = E^i \gamma_{i0} \\ \mathbf{B} &= B^i \sigma_i = B^i \gamma_{i0}. \end{aligned} \tag{76.4}$$

That implied representation was not obvious to me, but I eventually figured out what they meant. They also use $c = 1$, so I have added it back in here for clarity.

The end result in both cases is a pure bivector representation for the complete field:

$$F = E^j \gamma_{j0} + icB^j \gamma_{j0}$$

Let us look at the B^j basis bivectors a bit more closely:

$$i\gamma_{j0} = \gamma_{0123j0} = -\gamma_{01230j} = +\gamma_{00123j} = (\gamma_0)^2\gamma_{123j}$$

Where,

$$\gamma_{123j} = \begin{cases} (\gamma_j)^2\gamma_{23} & \text{if } j = 1 \\ (\gamma_j)^2\gamma_{31} & \text{if } j = 2 \\ (\gamma_j)^2\gamma_{12} & \text{if } j = 3 \end{cases}$$

Combining these results we have a $(\gamma_0)^2(\gamma_j)^2 = -1$ coefficient that is metric invariant, and can write:

$$i\sigma_j = i\gamma_{j0} = \begin{cases} \gamma_{32} & \text{if } j = 1 \\ \gamma_{13} & \text{if } j = 2 \\ \gamma_{21} & \text{if } j = 3 \end{cases}$$

Or, more compactly:

$$i\sigma_a = i\gamma_{a0} = -\epsilon_{abc}\gamma_{bc}$$

Putting things back together, our bivector field in index notation is:

$$F = E^i\gamma_{i0} - \epsilon_{ijk}cB^i\gamma_{jk} \tag{76.5}$$

76.2.1 *Tensor components*

Now, given a grade two multivector such as our field, how can we in general compute the components of that field given any arbitrary basis. This can be done using the reciprocal bivector frame:

$$F = \sum a_{\mu\nu}(e_\mu \wedge e_\nu)$$

To calculate the coordinates $a_{\mu\nu}$ we can dot with $e^\nu \wedge e^\mu$:

$$\begin{aligned} F \cdot (e^\nu \wedge e^\mu) &= \sum a_{\alpha\beta}(e_\alpha \wedge e_\beta) \cdot (e^\nu \wedge e^\mu) \\ &= (a_{\mu\nu}(e_\mu \wedge e_\nu) + a_{\nu\mu}(e_\nu \wedge e_\mu)) \cdot (e^\nu \wedge e^\mu) \\ &= a_{\mu\nu} - a_{\nu\mu} \\ &= 2a_{\mu\nu} \end{aligned} \tag{76.6}$$

Therefore

$$F = \frac{1}{2}\sum(F \cdot (e^\nu \wedge e^\mu))(e_\mu \wedge e_\nu) = \sum_{\mu<\nu}(F \cdot (e^\nu \wedge e^\mu))(e_\mu \wedge e_\nu)$$

Or, with $F^{\mu\nu} = F \cdot (e^\nu \wedge e^\mu)$ and summation convention:

$$F = \frac{1}{2}F^{\mu\nu}(e_\mu \wedge e_\nu) \tag{76.7}$$

It is not hard to see that the representation with respect to the reciprocal frame, with $F_{\mu\nu} = F \cdot (e_\nu \wedge e_\mu)$ must be:

$$F = \frac{1}{2}F_{\mu\nu}(e^\mu \wedge e^\nu) \tag{76.8}$$

Writing $F^{\mu\nu}$ or $F_{\mu\nu}$ leaves a lot unspecified. You will get a different tensor for each choice of basis. Using this form amounts to the equivalent of using the matrix of a linear transformation with respect to a specified basis.

76.2.2 *Electromagnetic tensor components*

Next, let us calculate these $F_{\mu\nu}$, and $F^{\mu\nu}$ values and relate them to our electric and magnetic fields so we can work in or translate to and from all of the traditional vector, the tensor, and the Clifford/geometric languages.

$$F^{\mu\nu} = \left(E^i\gamma_{i0} - \epsilon_{ijk}cB^i\gamma_{jk}\right) \cdot \gamma^{\nu\mu}$$

By inspection our electric field components we have:

$$F^{i0} = E^i,$$

and for the magnetic field:

$$F^{ij} = -\epsilon_{kij}cB^k = -\epsilon_{ijk}cB^k.$$

Putting in sample numbers this is:

$$\begin{aligned}
F^{32} &= -\epsilon_{321}cB^1 = cB^1 \\
F^{13} &= -\epsilon_{132}cB^2 = cB^2 \\
F^{21} &= -\epsilon_{213}cB^3 = cB^3
\end{aligned} \tag{76.9}$$

This can be summarized in matrix form:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \tag{76.10}$$

Observe that no specific reference to a metric was required to evaluate these components.

76.2.3 *reciprocal tensor (name?)*

The reciprocal frame representation of eq. (76.5) is

$$\begin{aligned} F &= E^i \gamma_{i0} - \epsilon_{ijk} c B^i \gamma_{jk} \\ &= -E^i \gamma^{i0} - \epsilon_{ijk} c B^i \gamma^{jk} \end{aligned} \tag{76.11}$$

Calculation of the reciprocal representation of the field tensor $F_{\mu\nu} = F \cdot \gamma_{\nu\mu}$ is now possible, and by inspection

$$\begin{aligned} F_{i0} &= -E^i = -F^{i0} \\ F_{ij} &= -\epsilon_{ijk} c B^k = F^{ij} \end{aligned} \tag{76.12}$$

So, all the electric field components in the tensor have inverted sign:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -cB^3 & cB^2 \\ -E^2 & cB^3 & 0 & -cB^1 \\ -E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}$$

This is metric independent with this bivector based definition of $F_{\mu\nu}$, and $F^{\mu\nu}$. Surprising, since I thought I had read otherwise.

76.2.4 *Lagrangian density*

[10] write the Lagrangian density in terms of $\left\langle F^2 \right\rangle$, whereas Denker writes it in terms of $\left\langle F\tilde{F} \right\rangle$. Is their alternate choice in metric responsible for this difference.

Reversing the field since it is a bivector, just inverts the sign:

$$\begin{aligned} F &= E^i \gamma_{i0} - \epsilon_{ijk} c B^i \gamma_{jk} \\ \tilde{F} &= E^i \gamma_{0i} - \epsilon_{ijk} c B^i \gamma_{kj} = -F \end{aligned} \tag{76.13}$$

So the choice of $\left\langle F^2 \right\rangle$ vs. $\left\langle F\tilde{F} \right\rangle$ is just a sign choice, and does not have anything to do with the metric.

Let us evaluate one of these:

$$\begin{aligned} F^2 &= (E^i \gamma_{i0} - \epsilon_{ijk} c B^i \gamma_{jk})(E^u \gamma_{u0} - \epsilon_{uvw} c B^u \gamma_{vw}) \\ &= E^i E^u \gamma_{i0} \gamma_{u0} - \epsilon_{uvw} E^i c B^u \gamma_{vw} \gamma_{i0} - \epsilon_{ijk} E^u c B^i \gamma_{jk} \gamma_{u0} + \epsilon_{ijk} \epsilon_{uvw} c^2 B^i B^u \gamma_{vw} \gamma_{jk} \end{aligned} \tag{76.14}$$

That first term is:

$$\begin{aligned} E^i E^u \gamma_{i0} \gamma_{u0} &= \mathbf{E}^2 + \sum_{i \neq j} E^i E^j (\sigma_i \sigma_j + \sigma_j \sigma_i) \\ &= \mathbf{E}^2 + \sum_{i \neq j} 2 E^i E^j \sigma_i \cdot \sigma_j \\ &= \mathbf{E}^2 \end{aligned} \tag{76.15}$$

Hmm. This is messy. Let us try with $F = \mathbf{E} + ic\mathbf{B}$ directly (with the Doran/Lasenby convention: $\mathbf{E} = E^k \sigma_k$) :

$$\begin{aligned} F^2 &= (\mathbf{E} + ic\mathbf{B})(\mathbf{E} + ic\mathbf{B}) \\ &= \mathbf{E}^2 + c^2 (i\mathbf{B})(i\mathbf{B}) + c(i\mathbf{B}\mathbf{E} + \mathbf{E}i\mathbf{B}) \\ &= \mathbf{E}^2 + c^2 (\mathbf{B}i)(i\mathbf{B}) + ic(\mathbf{B}\mathbf{E} + \mathbf{E}\mathbf{B}) \\ &= \mathbf{E}^2 - c^2 \mathbf{B}^2 + 2ic(\mathbf{B} \cdot \mathbf{E}) \end{aligned} \tag{76.16}$$

76.2.4.1 *Compared to tensor form*

Now lets compare to the tensor form, where the Lagrangian density is written in terms of the product of upper and lower index tensors:

$$
\begin{aligned}
F_{\mu\nu}F^{\mu\nu} &= F_{i0}F^{i0} + F_{0i}F^{0i} + \sum_{i<j} F_{ij}F^{ij} + \sum_{j<i} F_{ij}F^{ij} \\
&= 2F_{i0}F^{i0} + 2\sum_{i<j} F_{ij}F^{ij} \\
&= 2(-E^i)(E^i) + 2\sum_{i<j}(F^{ij})^2 \\
&= -2\mathbf{E}^2 + 2\sum_{i<j}(-\epsilon_{ijk}cB^k)^2 \\
&= -2(\mathbf{E}^2 - c^2\mathbf{B}^2)
\end{aligned}
\tag{76.17}
$$

Summarizing with a comparison of the bivector and tensor forms we have:

$$
\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = c^2\mathbf{B}^2 - \mathbf{E}^2 = -\left\langle F^2 \right\rangle = \left\langle F\tilde{F} \right\rangle
\tag{76.18}
$$

But to put this in context we need to figure out how to apply this in the Lagrangian. That appears to require a potential formulation of the field equations, so that is the next step.

76.2.4.2 *Potential and relation to electromagnetic tensor*

Since the field is a bivector is it reasonable to assume that it may be possible to express as the curl of a vector

$$
F = \nabla \wedge A.
$$

Inserting this into the field equation we have:

$$
\begin{aligned}
\nabla(\nabla \wedge A) &= \nabla \cdot (\nabla \wedge A) + \overbrace{\boxed{\nabla \wedge \nabla}}^{=0} \wedge A \\
&= \nabla^2 A - \nabla(\nabla \cdot A) \\
&= \frac{1}{\epsilon_0 c} J
\end{aligned}
\tag{76.19}
$$

With application of the gauge condition $\nabla \cdot A = 0$, one is left with the four scalar equations:

$$\nabla^2 A = \frac{1}{\epsilon_0 c} J \tag{76.20}$$

This can also be seen more directly since the gauge condition implies:

$$\nabla \wedge A = \nabla \wedge A + \nabla \cdot A = \nabla A$$

from which eq. (76.20) follows directly. Observe that although the field equation was not metric dependent, the equivalent potential equation is since it has a squared Laplacian.

76.2.4.3 *Index raising or lowering*

Any raising or lowering of indices, whether it be in the partials or the basis vectors corresponds to a multiplication by a $(\gamma_\alpha)^2 = \pm 1$ value, so doing this twice cancels out $(\pm 1)^2 = 1$.

Vector coordinates in the reciprocal basis is translated by such a squared factor when we are using an orthonormal basis:

$$\begin{aligned} x &= \sum \gamma^\mu (\gamma_\mu \cdot x) \\ &= \sum \gamma^\mu x_\mu \\ &= \sum \gamma^\mu (\gamma^\mu \gamma_\mu) x_\mu \\ &= \sum (\gamma^\mu)^2 \gamma_\mu x_\mu \end{aligned} \tag{76.21}$$

therefore

$$x^\mu = x \cdot \gamma^\mu = (\gamma^\mu)^2 x_\mu$$

Similarly our partial derivatives can be raised or lowered since they are just derivatives in terms of one of the choices of coordinates

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial (\gamma_\mu)^2 x_\mu} = (\gamma_\mu)^2 \frac{\partial}{\partial x_\mu} = (\gamma_\mu)^2 \partial^\mu$$

when written as a gradient, we have two pairs of $(\gamma_\mu)^2$ factors that cancel if we switch both indices:

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = (\gamma_\mu)^2(\gamma_\mu)^2\gamma_\mu \frac{\partial}{\partial x_\mu} = (\pm 1)^2\gamma_\mu \frac{\partial}{\partial x_\mu}$$

Or in short with the notation above

$$\nabla = \gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu$$

76.2.4.4 *Back to tensor in terms of potential*

Utilizing matched raising and lowering of indices, our field can be written in any of the following ways

$$\begin{aligned}
\nabla \wedge A &= \gamma_\mu \wedge \gamma_\nu \partial^\mu A^\nu = \sum_{\mu<\nu} \gamma_\mu \wedge \gamma_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\
&= \gamma^\mu \wedge \gamma^\nu \partial_\mu A_\nu = \sum_{\mu<\nu} \gamma^\mu \wedge \gamma^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\
&= \gamma_\mu \wedge \gamma^\nu \partial^\mu A_\nu = \sum_{\mu<\nu} \gamma_\mu \wedge \gamma^\nu \left(\partial^\mu A_\nu - \partial_\nu A^\mu\right) \\
&= \gamma^\mu \wedge \gamma_\nu \partial_\mu A^\nu = \sum_{\mu<\nu} \gamma^\mu \wedge \gamma_\nu \left(\partial_\mu A^\nu - \partial^\nu A_\mu\right)
\end{aligned} \tag{76.22}$$

These implicitly define the tensor in terms of potential, so we can write: Calculating the tensor in terms of the bivector we have:

$$\begin{aligned}
F^{\mu\nu} &= F \cdot (\gamma^\nu \wedge \gamma^\mu) = \partial^\mu A^\nu - \partial^\nu A^\mu \\
F_{\mu\nu} &= F \cdot (\gamma_\nu \wedge \gamma_\mu) = \partial_\mu A_\nu - \partial_\nu A_\mu \\
{F^\mu}_\nu &= F \cdot (\gamma^\nu \wedge \gamma_\mu) = \partial^\mu A_\nu - \partial_\nu A^\mu \\
{F_\mu}^\nu &= F \cdot (\gamma_\nu \wedge \gamma^\mu) = \partial_\mu A^\nu - \partial^\nu A_\mu
\end{aligned} \tag{76.23}$$

These potential based equations of eq. (76.23), are consistent with the definition of the field tensor in terms of potential in the wikipedia Covariant electromagnetism article. That article's definition of the field tensor is also consistent with the field tensor in matrix form of eq. (76.10).

However, the wikipedia Electromagnetic Tensor uses different conventions (at the time of this writing), but both claim a $- + ++$ metric, so I think one is wrong. I had naturally favor the covariant article since it agrees with my results.

76.2.5 *Field equations in tensor form*

$$\begin{aligned} J/c\epsilon_0 &= \nabla(\nabla \wedge A) \\ &= \nabla \cdot (\nabla \wedge A) + \nabla \wedge \nabla \wedge A \end{aligned} \tag{76.24}$$

This produces two equations

$$\nabla \cdot (\nabla \wedge A) = J/c\epsilon_0$$

$$\nabla \wedge \nabla \wedge A = 0$$

76.2.5.1 *Vector equation part*

Expanding the first in coordinates we have

$$\begin{aligned} J/c\epsilon_0 &= \gamma^\alpha \partial_\alpha \cdot (\gamma^\mu \wedge \gamma_\nu \partial_\mu A^\nu) \\ &= (\gamma^\alpha \cdot \gamma_{\mu\nu}) \partial_\alpha \partial^\mu A^\nu \\ &= (\delta^\alpha_\mu \gamma_\nu - \delta^\alpha_\nu \gamma_\mu) \partial_\alpha \partial^\mu A^\nu \\ &= (\gamma_\nu \partial_\mu - \gamma_\mu \partial_\nu) \partial^\mu A^\nu \\ &= \gamma_\nu \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \gamma_\nu \partial_\mu F^{\mu\nu} \end{aligned} \tag{76.25}$$

Dotting the LHS with γ^α we have

$$\begin{aligned} \gamma^\alpha \cdot J/c\epsilon_0 &= \gamma^\alpha \cdot \gamma_\beta J^\beta / c\epsilon_0 \\ &= \delta^\alpha_\beta J^\beta / c\epsilon_0 \\ &= J^\alpha / c\epsilon_0 \end{aligned} \tag{76.26}$$

and for the RHS

$$\gamma^\alpha \cdot \gamma_\nu \partial_\mu F^{\mu\nu} = \partial_\mu F^{\mu\alpha} \tag{76.27}$$

Or,

$$\partial_\mu F^{\mu\alpha} = J^\alpha / c\epsilon_0 \tag{76.28}$$

This is exactly (with index switch) the tensor equation in wikipedia Covariant electromagnetism article. It however, differs from the wikipedia Electromagnetic Tensor article.

76.2.5.2 *Trivector part*

Now, the trivector part of this equation does not seem like it is worth much consideration

$$\nabla \wedge \nabla \wedge A = 0 \tag{76.29}$$

But this is four of the eight traditional Maxwell's equations when written out in terms of coordinates. Let us write this out in tensor form and see how this follows.

$$\begin{aligned}
\nabla \wedge \nabla \wedge A &= (\gamma^\alpha \partial_\alpha) \wedge (\gamma^\beta \partial_\beta) \wedge (\gamma^\sigma A_\sigma) \\
&= (\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\sigma) \partial_\alpha \partial_\beta A_\sigma \\
&= (\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\sigma) \partial_\alpha \partial_\beta A_\sigma + (\gamma^\alpha \wedge \gamma^\sigma \wedge \gamma^\beta) \partial_\alpha \partial_\sigma A_\beta \\
&= (\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\sigma) \partial_\alpha (\partial_\beta A_\sigma - \partial_\sigma A_\beta) \\
&= (\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\sigma) \partial_\alpha F_{\beta\sigma}
\end{aligned} \tag{76.30}$$

For each of the four trivectors that span the trivector space the coefficients of those trivectors must all therefore equal zero. The duality set

$$\{i\gamma^\mu\}$$

can be used to enumerate these four equations, so to separate these from the wedge products we have to perform the dot products. Here i can be any pseudoscalar associated with the four vector space, and it will be convenient to use an "index-up" pseudoscalar $i = \gamma^{0123}$. This will still anticommute with any of the γ^μ vectors.

$$\begin{aligned}
(\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\sigma) \cdot (i\gamma^\mu) &= \left\langle (\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\sigma)(i\gamma^\mu) \right\rangle \\
&= -\left\langle \gamma^\alpha \gamma^\beta \gamma^\sigma \gamma^{\mu 0123} \right\rangle \\
&= -\left\langle \gamma^{\alpha\beta\sigma\mu 0123} \right\rangle \\
&= \epsilon^{\alpha\beta\sigma\mu}
\end{aligned} \tag{76.31}$$

The last line follows with the observation that the scalar part will be zero unless α, β, σ, and μ are all unique. When they are $0, 1, 2, 3$ for example then we have $i^2 = -1$, and any odd permutation will change the sign.

Application of this to our curl of curl expression we have

$$(\nabla \wedge \nabla \wedge A) \cdot (i\gamma^\mu) = \epsilon^{\alpha\beta\sigma\mu} \partial_\alpha F_{\beta\sigma}$$

Which is exactly the remaining four equations of Maxwell's equation in standard tensor form

$$\epsilon^{\alpha\beta\sigma\mu}\partial_\alpha F_{\beta\sigma} = 0 \tag{76.32}$$

One of these will be Gauss's law $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, and the other three can be summed in vector form for Faraday's law $\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$.

76.2.6 *Lagrangian density in terms of potential*

We have seen that we can write the core of the Lagrangian density in two forms:

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = -\left\langle F^2 \right\rangle = c^2\mathbf{B}^2 - \mathbf{E}^2$$

where summarizing the associated relations we have:

$$F = \mathbf{E} + ic\mathbf{B} = \frac{1}{2}F^{\mu\nu}\gamma_{\mu\nu} = \nabla \wedge A = E^i\gamma_{i0} - \epsilon_{ijk}cB^i\gamma_{jk}$$

$$\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
F^{i0} &= E^i = -F_{i0} \\
F^{ij} &= -\epsilon_{ijk}cB^k = F_{ij}
\end{aligned} \tag{76.33}$$

Now, if we want the density in terms of potential, by inspection we can form this from the tensor as:

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

We should also be able to calculate this directly from the bivector square. Lets verify this:

$$
\begin{aligned}
\left\langle F^2 \right\rangle &= \langle (\nabla \wedge A)(\nabla \wedge A) \rangle \\
&= \left\langle (\gamma^\mu \wedge \gamma_\nu \partial_\mu A^\nu)(\gamma^\alpha \wedge \gamma_\beta \partial_\alpha A^\beta) \right\rangle \\
&= (\gamma^\mu \wedge \gamma^\nu \partial_\mu A_\nu) \cdot (\gamma_\alpha \wedge \gamma_\beta \partial^\alpha A^\beta) \\
&= (((\gamma^\mu \wedge \gamma^\nu) \cdot \gamma_\alpha) \cdot \gamma_\beta) \partial_\mu A_\nu \partial^\alpha A^\beta \\
&= \left(\delta^\mu_\beta \delta^\nu_\alpha - \delta^\mu_\alpha \delta^\nu_\beta \right) \partial_\mu A_\nu \partial^\alpha A^\beta \\
&= \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\mu A_\nu \partial^\mu A^\nu \\
&= \partial_\mu A_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) \\
&= \frac{1}{2} \left(\partial_\mu A_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) + \partial_\nu A_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) \right) \\
&= \frac{1}{2} \left(\partial_\nu A_\mu - \partial_\mu A_\nu \right) \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) \\
&= -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}
\end{aligned}
\tag{76.34}
$$

as expected.

The factor of $1/2$ appearance is a $x = (1/2)(x + x)$ operation, plus a switch of dummy indices in one half of the sum.

With the density expanded completely in terms of potentials things are in a form for an attempt to evaluate the Lagrangian equations or do the variational exercise (as in Feynman [12] with the electrostatic case) and see that this recovers the field equations (covered in a subsequent set of notes in both fashions).

77

FOUR VECTOR POTENTIAL

77.1

Goldstein's classical mechanics, and many other texts, will introduce the four potential starting with Maxwell's equation in scalar, vector, bivector, trivector expanded form:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)
\end{aligned}
\tag{77.1}
$$

ie: E can not be a gradient, since it has a curl, but B can be the curl of something since it has zero divergence, so we have $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Faraday's law above gives:

$$
\begin{aligned}
0 &= \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \boldsymbol{\nabla} \times \mathbf{A}}{\partial t} \\
&= \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right)
\end{aligned}
\tag{77.2}
$$

Because this curl is zero, one can write it as a gradient, say $-\boldsymbol{\nabla}\phi$.

The end result are the equations:

$$
\mathbf{E} = -\left(\boldsymbol{\nabla}\phi + \partial_t \mathbf{A} \right) \tag{77.3}
$$

$$
\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{77.4}
$$

Looking at what Goldstein does with this (which I re-derived above to put in the SI form I am used to), my immediate question is how would the combined bivector field look when expressed using an STA basis, and then once that is resolved, how would his Lagrangian for a charged point particle look in explicit four vector form?

Intuition says that this is all going to work out to be a spacetime gradient of a four vector, but I am not sure how the Lorentz gauge freedom will turn out. Here is an exploration of this.

77.1.1

Forming as usual

$$\mathbf{F} = \mathbf{E} + ic\mathbf{B} \tag{77.5}$$

We can combine the equations eq. (77.3) and eq. (77.4) into bivector form

$$\mathbf{F} = -\left(\boldsymbol{\nabla}\phi + \partial_t \mathbf{A}\right) + c\boldsymbol{\nabla} \wedge \mathbf{A} \tag{77.6}$$

77.1.2 *Dimensions*

Let us do a dimensional check before continuing:

Equation (77.5) gives:

$$[\mathbf{E}] = \frac{[m][d]}{[q][t]^2}$$

That and eq. (77.6) gives

$$[\phi] = \frac{[m][d]^2}{[q][t]^2}$$

And the two $\mathbf{A}$ terms of eq. (77.6) both give:

$$[\mathbf{A}] = \frac{[m][d]}{[q][t]}.$$

Therefore if we create a four vector out of ϕ, and $\mathbf{A}$ in SI units we will need that factor c with $\mathbf{A}$ with velocity dimensions to fix things up.

77.1.3 *Expansion in terms of components. STA split*

$$\begin{aligned}
\mathbf{F} &= -\left(\boldsymbol{\nabla}\phi + \partial_t \mathbf{A}\right) + c\boldsymbol{\nabla} \wedge \mathbf{A} \\
&= -\sum \gamma_i\gamma_0 \partial_{x^i}\phi - \sum \gamma_i\gamma_0 \partial_t A^i + c\left(\sum \sigma_i \partial_{x^i}\right) \wedge \left(\sum \sigma_j A^j\right) \\
&= \sum \gamma^i \partial_{x^i}(\gamma_0\phi) + \sum \gamma_0 \partial_{ct} c\gamma_i A^i - \left(\sum \gamma_i \partial_{x^i}\right) \wedge \left(\sum \gamma_j c A^j\right) \\
&= \sum \gamma^i \wedge \gamma_0 \partial_{x^i}\phi + \sum \gamma^0 \wedge \gamma_i \partial_{x^0} c A^i + \sum \gamma^i \wedge \gamma_j \partial_{x^i} c A^j \\
&= \left(\sum \gamma^i \partial_{x^i}\right) \wedge \left(\gamma_0\phi + \gamma_i c A^i\right) \\
&= \nabla \wedge \left(\gamma_0\phi + \sum \gamma_i c A^i\right)
\end{aligned} \tag{77.7}$$

Once the electric and magnetic fields are treated as one entity, the separate equations of eq. (77.3) and eq. (77.4) become nothing more than a statement that the bivector field $\mathbf{F}$ is the spacetime curl of a four vector potential $A = \gamma_0\phi + \sum \gamma_i c A^i$.

This original choice of components A^i, defined such that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is a bit unfortunate in SI units. Setting $\mathcal{A}^i = cA^i$, and $\mathcal{A}^0 = \phi$, one then has the more symmetric form.

$$A = \sum \gamma_\mu \mathcal{A}^\mu.$$

Of course the same thing could be achieved with $c = 1$;)

Anyways, substitution of this back into Maxwell's equation gives:

$$\nabla(\nabla \wedge A) = \nabla \cdot (\nabla \wedge A) + \overbrace{\boxed{\nabla \wedge \nabla \wedge A}}^{=0} = J$$

One can see an immediate simplification possible if one requires:

$$\nabla \cdot A = 0.$$

Then we are left with a forced wave equation to solve for the four potential:

$$\nabla^2 A = -\left(\sum \partial_{x^i x^i} - \frac{1}{c^2}\partial_{tt}\right) A = J.$$

Now, without all this mess of algebra, I could have gone straight to this end result (and had done so previously). I just wanted to see where I would get applying the STA basis to the classical vector+scalar four vector ideas.

77.1.4 *Lorentz gauge*

Looking at $\nabla \cdot A = 0$, I was guessing that this was what I recalled being called the Lorentz gauge, but in a slightly different form.

If one expands this you get:

$$\begin{aligned}
0 &= \nabla \cdot A \\
&= \sum \gamma^\mu \partial_\mu \cdot \left(\gamma_0 \phi + c \sum \gamma_j A^j\right) \\
&= \partial_{ct}\phi + c \sum \partial_{x^i} A^i \\
&= \partial_{ct}\phi + c \boldsymbol{\nabla} \cdot \mathbf{A}
\end{aligned} \tag{77.8}$$

Or,

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\partial_t \phi \tag{77.9}$$

Checked my Feynman book. Yes, this is the Lorentz Gauge.

Another note. Again the SI units make things ugly. With the above modification of components that hide this, where one sets $A = \sum \gamma_i \mathcal{A}^i$, this gauge equation also takes a simpler form:

$$0 = \nabla \cdot A = \left(\sum \gamma^\mu \partial_{x^\mu}\right) \cdot \left(\sum \gamma_\nu \mathcal{A}^\nu\right) = \sum \partial_{x^\mu} \mathcal{A}^\mu.$$

77.2 APPENDIX

77.2.1 *wedge of spacetime bivector basis elements*

For $i \neq j$:

$$\begin{aligned}
\sigma_i \wedge \sigma_j &= \frac{1}{2}\left(\sigma_i \sigma_j - \sigma_j \sigma_i\right) \\
&= \frac{1}{2}\left(\gamma_{i0j0} - \gamma_{j0i0}\right) \\
&= \frac{1}{2}\left(-\gamma_{ij} + \gamma_{ji}\right) \\
&= \gamma_{ji}
\end{aligned} \tag{77.10}$$

78 METRIC SIGNATURE DEPENDENCIES

78.1 MOTIVATION

Doran/Lasenby use a $+,-,-,-$ signature, and I had gotten used to that. On first seeing the alternate signature used by John Denker's excellent GA explanatory paper , I found myself disoriented. How many of the identities that I was used to were metric dependent? Here are some notes that explore some of the metric dependencies of STA, in particular observing which identities are metric dependent and which are not.

In the end this exploration turned into a big meandering examination and comparison of the bivector and tensor forms of Maxwell's equation. That part has been split into a different writeup.

78.2 THE GUTS

78.2.1 *Spatial basis*

Our spatial (bivector) basis:

$$\sigma_i = \gamma_i \wedge \gamma_0 = \gamma_{i0},$$

that behaves like Euclidean vectors (positive square) still behave as desired, regardless of the signature:

$$\begin{aligned}
\sigma_i \cdot \sigma_j &= \left\langle \gamma_{i0j0} \right\rangle \\
&= -\left\langle \gamma_{ij} \right\rangle (\gamma_0)^2 \\
&= -\delta_{ij} (\gamma_i)^2 (\gamma_0)^2
\end{aligned} \tag{78.1}$$

Regardless of the signature the pair of products $(\gamma_i)^2(\gamma_0)^2 = -1$, so our spatial bivectors are metric invariant.

78.2.2 *How about commutation?*

Commutation with

$$i\gamma_\mu = \gamma_{0123\mu} = \gamma_{\mu 0123}$$

μ has to "pass" three indices regardless of metric, so anticommutes for any μ.

$$\sigma_k \gamma_\mu = \gamma_{k0\mu}$$

If $k = \mu$, or $0 = \mu$, then we get a sign inversion, and otherwise commute (pass two indices). This is also metric invariant.

78.2.3 *Spatial and time component selection*

With a positive time metric (Doran/Lasenby) selection of the x^0 component of a vector x requires a dot product:

$$x = x^0\gamma_0 + x^i\gamma_i$$

$$x \cdot \gamma_0 = x^0(\gamma_0)^2$$

Obviously this is a metric dependent operation. To generalize it appropriately, we need to dot with γ^0 instead:

$$x \cdot \gamma^0 = x^0$$

Now, what do we get when wedging with this upper index quantity instead.

$$\begin{aligned}
x \wedge \gamma^0 &= \left(x^0\gamma_0 + x^i\gamma_i\right) \wedge \gamma^0 \\
&= x^i\gamma_i \wedge \gamma^0 \\
&= x^i\gamma_{i0}(\gamma^0)^2 \\
&= x^i\sigma_i(\gamma^0)^2 \\
&= \mathbf{x}\left(\gamma^0\right)^2
\end{aligned} \tag{78.2}$$

Not quite the usual expression we are used to, but it still behaves as a Euclidean vector (positive square), regardless of the metric:

$$(x \wedge \gamma^0)^2 = (\pm\mathbf{x})^2 = \mathbf{x}^2$$

This suggests that we should define our spatial projection vector as $x \wedge \gamma^0$ instead of $x \wedge \gamma_0$ as done in Doran/Lasenby (where a positive time metric is used).

78.2.3.1 *Velocity*

Variation of a event path with some parameter we have:

$$\begin{aligned}\frac{dx}{d\lambda} &= \frac{dx^\mu}{d\lambda}\gamma_\mu = c\frac{dt}{d\lambda}\gamma_0 + \frac{dx^i}{d\lambda}\gamma_i \\ &= \frac{dt}{d\lambda}\left(c\gamma_0 + \frac{dx^i}{dt}\gamma_i\right)\end{aligned} \tag{78.3}$$

The square of this is:

$$\begin{aligned}\frac{1}{c^2}\left(\frac{dx}{d\lambda}\right)^2 &= \left(\frac{dt}{d\lambda}\right)^2(\gamma_0)^2\left(1 + \frac{1}{c^2}\left(\frac{dx^i}{dt}\right)^2(\gamma_i)^2(\gamma_0)^2\right) \\ &= \left(\frac{dt}{d\lambda}\right)^2(\gamma_0)^2\left(1 - (\mathbf{v}/c)^2\right) \\ \frac{(\gamma_0)^2}{c^2}\left(\frac{dx}{d\lambda}\right)^2 &= \left(\frac{dt}{d\lambda}\right)^2\left(1 - (\mathbf{v}/c)^2\right)\end{aligned} \tag{78.4}$$

We define the proper time τ as that particular parametrization $c\tau = \lambda$ such that the LHS equals 1. This is implicitly defined via the integral

$$\tau = \int \sqrt{1 - (\mathbf{v}/c)^2}dt = \int \sqrt{1 - \left(\frac{1}{c}\frac{dx^i}{d\alpha}\right)^2}d\alpha$$

Regardless of this parametrization $\alpha = \alpha(t)$, this velocity scaled 4D arc length is the same. This is a bit of a digression from the ideas of metric dependence investigation. There is however a metric dependence in the first steps arriving at this result.

with proper velocity defined in terms of proper time $v = dx/d\tau$, we also have:

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - (\mathbf{v}/c)^2}} \tag{78.5}$$

$$v = \gamma \left(c\gamma_0 + \frac{dx^i}{dt}\gamma_i \right) \tag{78.6}$$

Therefore we can select this quantity γ, and our spatial velocity components, from our proper velocity:

$$c\gamma = v \cdot \gamma^0$$

In eq. (78.5) we did not define $\mathbf{v}$, only implicitly requiring that its square was $\sum (dx^i/dt)^2$, as we require for correspondence with Euclidean meaning. This can be made more exact by taking wedge products to weed out the time component:

$$v \wedge \gamma^0 = \gamma \frac{dx^i}{dt} \gamma_i \wedge \gamma^0$$

With a definition of $\mathbf{v} = \frac{dx^i}{dt}\gamma_i \wedge \gamma^0$ (which has the desired positive square), we therefore have:

$$\begin{aligned} \mathbf{v} &= \frac{v \wedge \gamma^0}{\gamma} \\ &= \frac{v \wedge \gamma^0}{v/c \cdot \gamma^0} \end{aligned} \tag{78.7}$$

Or,

$$\mathbf{v}/c = \frac{v/c \wedge \gamma^0}{v/c \cdot \gamma^0} \tag{78.8}$$

All the lead up to this allows for expression of the spatial component of the proper velocity in a metric independent fashion.

78.2.4 *Reciprocal Vectors*

By reciprocal frame we mean the set of vectors $\{u^\alpha\}$ associated with a basis for some linear subspace $\{u_\alpha\}$ such that:

$$u_\alpha \cdot u^\beta = \delta_\alpha^\beta$$

In the special case of orthonormal vectors $u_\alpha \cdot u_\beta = \pm\delta_{\alpha\beta}$ the reciprocal frame vectors are just the inverses (literally reciprocals), which can be verified by taking dot products:

$$\begin{aligned}
\frac{1}{u_\alpha} \cdot u_\alpha &= \left\langle \frac{1}{u_\alpha} u_\alpha \right\rangle \\
&= \left\langle \frac{1}{u_\alpha} \frac{u_\alpha}{u_\alpha} u_\alpha \right\rangle \\
&= \left\langle \frac{(u_\alpha)^2}{(u_\alpha)^2} \right\rangle \\
&= 1
\end{aligned} \tag{78.9}$$

Written out explicitly for our positive "orthonormal" time metric:

$$\begin{aligned}
(\gamma_0)^2 &= 1 \\
(\gamma_i)^2 &= -1,
\end{aligned} \tag{78.10}$$

we have the reciprocal vectors:

$$\begin{aligned}
\gamma_0 &= \gamma^0 \\
\gamma_i &= -\gamma^i
\end{aligned} \tag{78.11}$$

Note that this last statement is consistent with $(\gamma_i)^2 = -1$, since $(\gamma_i)^2 = \gamma_i(-\gamma^i) = -\delta_i^i = -1$ Contrast this with a positive spatial metric:

$$\begin{aligned}
(\gamma_0)^2 &= -1 \\
(\gamma_i)^2 &= 1,
\end{aligned} \tag{78.12}$$

with reciprocal vectors:

$$\begin{aligned}
\gamma_0 &= -\gamma^0 \\
\gamma_i &= \gamma^i
\end{aligned} \tag{78.13}$$

where we have the opposite.

78.2.5 *Reciprocal Bivectors*

Now, let us examine the bivector reciprocals. Given our orthonormal vector basis, let us invert the bivector and verify that is what we want:

$$
\begin{aligned}
\frac{1}{\gamma_{\mu\nu}} &= \frac{1}{\gamma_{\mu\nu}} \frac{\gamma_{\nu\mu}}{\gamma_{\nu\mu}} \\
&= \frac{1}{\gamma_{\mu\nu}} \frac{1}{\gamma_{\nu\mu}} \gamma_{\nu\mu} \\
&= \frac{1}{\gamma_{\mu\nu\nu\mu}} \gamma_{\nu\mu} \\
&= \frac{1}{(\gamma_\mu)^2 (\gamma_\nu)^2} \gamma_{\nu\mu}
\end{aligned}
\tag{78.14}
$$

Multiplication with our vector we will get 1 if this has the required reciprocal relationship:

$$
\begin{aligned}
\frac{1}{\gamma_{\mu\nu}} \gamma_{\mu\nu} &= \frac{1}{(\gamma_\mu)^2 (\gamma_\nu)^2} \gamma_{\nu\mu} \gamma_{\mu\nu} \\
&= \frac{(\gamma_\mu)^2 (\gamma_\nu)^2}{(\gamma_\mu)^2 (\gamma_\nu)^2} \\
&= 1
\end{aligned}
\tag{78.15}
$$

Observe that unlike our basis vectors the bivector reciprocals are metric independent. Let us verify this explicitly:

$$
\begin{aligned}
\frac{1}{\gamma_{i0}} &= \frac{1}{(\gamma_i)^2 (\gamma_0)^2} \gamma_{0i} \\
\frac{1}{\gamma_{ij}} &= \frac{1}{(\gamma_i)^2 (\gamma_j)^2} \gamma_{ji} \\
\frac{1}{\gamma_{0i}} &= \frac{1}{(\gamma_0)^2 (\gamma_i)^2} \gamma_{i0}
\end{aligned}
\tag{78.16}
$$

With a spacetime mix of indices we have a -1 denominator for either metric. With a spatial only mix (B components) we have 1 in the denominator $1^2 = (-1)^2$ for either metric.

Now, perhaps counter to intuition the reciprocal $\frac{1}{\gamma_{\mu\nu}}$ of $\gamma_{\mu\nu}$ is not $\gamma^{\mu\nu}$, but instead $\gamma^{\nu\mu}$. Here the shorthand can be deceptive and it is worth verifying this statement explicitly:

$$
\begin{aligned}
\gamma_{\mu\nu} \cdot \gamma^{\alpha\beta} &= (\gamma_\mu \wedge \gamma_\nu) \cdot (\gamma^\alpha \wedge \gamma^\beta) \\
&= ((\gamma_\mu \wedge \gamma_\nu) \cdot \gamma^\alpha) \cdot \gamma^\beta) \\
&= (\gamma_\mu (\gamma_\nu \cdot \gamma^\alpha) - \gamma_\nu (\gamma_\mu \cdot \gamma^\alpha)) \cdot \gamma^\beta) \\
&= (\gamma_\mu {\delta_\nu}^\alpha - \gamma_\nu {\delta_\mu}^\alpha) \cdot \gamma^\beta
\end{aligned}
\tag{78.17}
$$

Or,

$$\gamma_{\mu\nu} \cdot \gamma^{\alpha\beta} = {\delta_\mu}^\beta {\delta_\nu}^\alpha - {\delta_\nu}^\beta {\delta_\mu}^\alpha \tag{78.18}$$

In particular for matched pairs of indices we have:

$$\gamma_{\mu\nu} \cdot \gamma^{\nu\mu} = {\delta_\mu}^\mu {\delta_\nu}^\nu - {\delta_\nu}^\mu {\delta_\mu}^\nu = 1$$

78.2.6 *Pseudoscalar expressed with reciprocal frame vectors*

With a positive time metric

$$\gamma_{0123} = -\gamma^{0123}$$

(three inversions for each of the spatial quantities). This is metric invariant too since it will match the single negation for the same operation using a positive spatial metric.

78.2.7 *Spatial bivector basis commutation with pseudoscalar*

I have been used to writing:

$$\sigma_j = \gamma_{j0}$$

as a spatial basis, and having this equivalent to the four-pseudoscalar, but this only works with a time positive metric:

$$i_3 = \sigma_{123} = \gamma_{102030} = \gamma_{0123}(\gamma_0)^2$$

With the spatial positive spacetime metric we therefore have:

$$i_3 = \sigma_{123} = \gamma_{102030} = -i_4$$

instead of $i_3 = i_4$ as is the case with a time positive spacetime metric. We see that the metric choice can also be interpreted as a choice of handedness.

That choice allowed Doran/Lasenby to initially write the field as a vector plus trivector where i is the spatial pseudoscalar:

$$F = \mathbf{E} + ic\mathbf{B}, \tag{78.19}$$

and then later switch the interpretation of i to the four space pseudoscalar. The freedom to do so is metric dependent freedom, but eq. (78.19) works regardless of metric when i is uniformly interpreted as the spacetime pseudoscalar.

Regardless of the metric the spacetime pseudoscalar commutes with $\sigma_j = \gamma_{j0}$, since it anti-commutes twice to cross:

$$\sigma_j i = \gamma_{j00123} = \gamma_{00123j} = \gamma_{0123j0} = i\sigma_j$$

78.2.8 *Gradient and Laplacian*

As seen by the Lagrangian based derivation of the (spacetime or spatial) gradient, the form is metric independent and valid even for non-orthonormal frames:

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu}$$

78.2.8.1 *Vector derivative*

A cute aside, as pointed out in John Denker's paper, for orthonormal frames, this can also be written as:

$$\nabla = \frac{1}{\gamma_\mu} \frac{\partial}{\partial x^\mu} \tag{78.20}$$

as a mnemonic for remembering where the signs go, since in that form the upper and lower indices are nicely matched in summation convention fashion.

Now, γ_μ is a constant when we are not working in curvilinear coordinates, and for constants we are used to the freedom to pull them into our derivatives as in:

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial (ct)}$$

Supposing that one had an orthogonal vector decomposition:

$$\mathbf{x} = \sum \gamma_i x^i = \sum \mathbf{x}_i$$

then, we can abuse notation and do the same thing with our unit vectors, rewriting the gradient eq. (78.20) as:

$$\nabla = \frac{\partial}{\partial(\gamma_\mu x^\mu)} = \sum \frac{\partial}{\partial \mathbf{x}_i} \tag{78.21}$$

Is there anything to this that is not just abuse of notation? I think so, and I am guessing the notational freedom to do this is closely related to what Hestenes calls geometric calculus.

Expanding out the gradient in the form of eq. (78.21) as a limit statement this becomes, rather loosely:

$$\nabla = \sum_i \lim_{d\mathbf{x}_i \rightarrow 0} \frac{1}{d\mathbf{x}_i} \left(f(\mathbf{x} + d\mathbf{x}_i) - f(\mathbf{x}) \right)$$

If nothing else this justifies the notation for the polar form gradient of a function that is only radially dependent, where the quantity:

$$\boldsymbol{\nabla} = \hat{\mathbf{r}} \frac{\partial}{\partial r} = \frac{1}{\hat{\mathbf{r}}} \frac{\partial}{\partial r}$$

is sometimes written:

$$\boldsymbol{\nabla} = \frac{\partial}{\partial \mathbf{r}}$$

Tong does this for example in his online dynamics paper, although there it appears to be not much more than a fancy shorthand for gradient.

78.2.9 *Four-Laplacian*

Now, although our gradient is metric invariant, its square the four-Laplacian is not. There we have:

$$\begin{aligned}
\nabla^2 &= \sum (\gamma^\mu)^2 \frac{\partial^2}{\partial^2 x^\mu} \\
&= (\gamma^0)^2 \left(\frac{\partial^2}{\partial^2 x^0} + (\gamma^0)^2 (\gamma^i)^2 \frac{\partial^2}{\partial^2 x^i} \right) \\
&= (\gamma^0)^2 \left(\frac{\partial^2}{\partial^2 x^0} - \frac{\partial^2}{\partial^2 x^i} \right)
\end{aligned} \tag{78.22}$$

This makes the metric dependency explicit so that we have:

$$\begin{aligned}
\nabla^2 &= \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} - \frac{\partial^2}{\partial^2 x^i} \quad \text{if } (\gamma^0)^2 = 1 \\
\nabla^2 &= \frac{\partial^2}{\partial^2 x^i} - \frac{1}{c^2} \frac{\partial^2}{\partial^2 t} \quad \text{if } (\gamma^0)^2 = -1
\end{aligned}$$

79 WAVE EQUATION FORM OF MAXWELL'S EQUATIONS

79.1 MOTIVATION

In [22], on plane waves, he writes "we find easily..." to show that the wave equation for each of the components of $\mathbf{E}$, and $\mathbf{B}$ in the absence of current and charge satisfy the wave equation. Do this calculation.

79.2 VACUUM CASE

Avoiding the non-vacuum medium temporarily, Maxwell's vacuum equations (in SI units) are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \tag{79.1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{79.2}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{79.3}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{79.4}$$

The last two curl equations can be decoupled by once more calculating the curl. Illustrating by example

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \mathbf{B} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{79.5}$$

Digging out vector identities and utilizing the zero divergence we have

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 \mathbf{E} = -\boldsymbol{\nabla}^2 \mathbf{E} \tag{79.6}$$

Putting eq. (79.5), and eq. (79.6) together provides a wave equation for the electric field vector

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{E} = 0 \tag{79.7}$$

Operating with curl on the remaining Maxwell equation similarly produces a wave equation for the magnetic field vector

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \boldsymbol{\nabla}^2 \mathbf{B} = 0 \tag{79.8}$$

This is really six wave equations, one for each of the field coordinates.

79.3 WITH GEOMETRIC ALGEBRA

Arriving at eq. (79.7), and eq. (79.8) is much easier using the GA formalism of [10].

Pre or post multiplication of the gradient with the observer frame time basis unit vector γ_0 has a conjugate like action

$$\begin{aligned}\nabla\gamma_0 &= \gamma^0\gamma_0\partial_0 + \gamma^k\gamma_0\partial_k \\ &= \partial_0 - \boldsymbol{\nabla}\end{aligned} \tag{79.9}$$

(where as usual our spatial basis is $\sigma_k = \gamma_k\gamma_0$).

Similarly

$$\gamma_0\nabla = \partial_0 + \boldsymbol{\nabla} \tag{79.10}$$

For the vacuum Maxwell's equation is just

$$\nabla F = \nabla(\mathbf{E} + Ic\mathbf{B}) = 0 \tag{79.11}$$

With nothing more than an algebraic operation we have

$$\begin{aligned}0 &= \nabla\gamma_0\gamma_0\nabla F \\ &= (\partial_0 - \boldsymbol{\nabla})(\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + Ic\mathbf{B}) \\ &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)(\mathbf{E} + Ic\mathbf{B})\end{aligned} \tag{79.12}$$

This equality is true independently for each of the components of **E** and **B**, so we have as before

These wave equations are still subject to the constraints of the original Maxwell equations.

$$\begin{aligned} 0 &= \gamma_0 \nabla F \\ &= (\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + Ic\mathbf{B}) \\ &= \boldsymbol{\nabla} \cdot \mathbf{E} + (\partial_0 \mathbf{E} - c\boldsymbol{\nabla} \times \mathbf{B}) + I(c\partial_0 \mathbf{B} + \boldsymbol{\nabla} \times \mathbf{E}) + Ic\boldsymbol{\nabla} \cdot \mathbf{B} \end{aligned} \tag{79.13}$$

79.4 TENSOR APPROACH?

In both the traditional vector and the GA form one can derive the wave equation relations of eq. (79.7), eq. (79.8). One can obviously summarize these in tensor form as

$$\partial_\mu \partial^\mu F^{\alpha\beta} = 0 \tag{79.14}$$

working backwards from the vector or GA result. In this notation, the coupling constraint would be that the field variables $F^{\alpha\beta}$ are subject to the Maxwell divergence equation (name?)

$$\partial_\mu F^{\mu\nu} = 0 \tag{79.15}$$

and also the dual tensor relation

$$\epsilon^{\sigma\mu\alpha\beta} \partial_\mu F_{\alpha\beta} = 0 \tag{79.16}$$

I cannot seem to figure out how to derive eq. (79.14) starting from these tensor relations?

This probably has something to do with the fact that we require both the divergence and the dual relations eq. (79.15), eq. (79.16) expressed together to do this.

79.5 ELECTROMAGNETIC WAVES IN MEDIA

Jackson lists the Macroscopic Maxwell equations in (6.70) as

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \cdot \mathbf{D} &= 4\pi\rho \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \end{aligned} \tag{79.17}$$

(for this note this means unfortunately a switch from SI to CGS midstream)

For linear material ($\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$) that is devoid of unbound charge and current ($\rho = 0$, and $\mathbf{J} = 0$), we can assemble these into his (7.1) equations

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{B} - \frac{\epsilon\mu}{c}\frac{\partial \mathbf{E}}{\partial t} &= 0
\end{aligned}
\tag{79.18}
$$

In this macroscopic form, it is not obvious how to assemble the equations into a nice tidy GA form. A compromise is

$$
\begin{aligned}
\boldsymbol{\nabla}\mathbf{E} + \partial_0(I\mathbf{B}) &= 0 \\
\boldsymbol{\nabla}(I\mathbf{B}) + \epsilon\mu\partial_0\mathbf{E} &= 0
\end{aligned}
\tag{79.19}
$$

Although not as pretty, we can at least derive the wave equations from these. For example for $\mathbf{E}$, we apply one additional spatial gradient

$$
\begin{aligned}
0 &= \boldsymbol{\nabla}^2\mathbf{E} + \partial_0(\boldsymbol{\nabla} I\mathbf{B}) \\
&= \boldsymbol{\nabla}^2\mathbf{E} + \partial_0(-\epsilon\mu\partial_0\mathbf{E})
\end{aligned}
\tag{79.20}
$$

For $\mathbf{B}$ we get the same, and have two wave equations

$$
\begin{aligned}
\frac{\mu\epsilon}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{E} &= 0 \\
\frac{\mu\epsilon}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{B} &= 0
\end{aligned}
\tag{79.21}
$$

The wave velocity is thus not c, but instead the reduced speed of $c/\sqrt{\mu\epsilon}$.

The fact that it is possible to assemble wave equations of this form means that there must also be a simpler form than eq. (79.19). The reduced velocity is the clue, and that can be used to refactor the constants

$$
\begin{aligned}
\boldsymbol{\nabla}\mathbf{E} + \sqrt{\mu\epsilon}\partial_0\left(\frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) &= 0 \\
\boldsymbol{\nabla}\left(\frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) + \sqrt{\mu\epsilon}\partial_0\mathbf{E} &= 0
\end{aligned}
\tag{79.22}
$$

These can now be added

$$\left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0\right)\left(\mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) = 0 \tag{79.23}$$

This allows for the one liner derivation of eq. (79.21) by premultiplying by the conjugate operator $-\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0$

$$\begin{aligned}
0 &= \left(-\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0\right)\left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0\right)\left(\mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) \\
&= \left(-\boldsymbol{\nabla}^2 + \frac{\mu\epsilon}{c^2}\partial_{tt}\right)\left(\mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right)
\end{aligned} \tag{79.24}$$

Using the same hint, and doing some rearrangement, we can write Jackson's equations (6.70) as

$$\left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0\right)\left(\mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) = \frac{4\pi}{\epsilon}\left(\rho - \frac{\sqrt{\mu\epsilon}}{c}\mathbf{J}\right) \tag{79.25}$$

80

SPACE TIME ALGEBRA SOLUTIONS OF THE MAXWELL EQUATION FOR DISCRETE FREQUENCIES

80.1 MOTIVATION

How to obtain solutions to Maxwell's equations in vacuum is well known. The aim here is to explore the same problem starting with the Geometric Algebra (GA) formalism [10] of the Maxwell equation.

$$
\begin{aligned}
\nabla F &= J/\epsilon_0 c \\
F &= \nabla \wedge A = \mathbf{E} + ic\mathbf{B}
\end{aligned}
\tag{80.1}
$$

A Fourier transformation attack on the equation should be possible, so let us see what falls out doing so.

80.1.1 *Fourier problem*

Picking an observer bias for the gradient by premultiplying with γ_0 the vacuum equation for light can therefore also be written as

$$
\begin{aligned}
0 &= \gamma_0 \nabla F \\
&= \gamma_0 (\gamma^0 \partial_0 + \gamma^k \partial_k) F \\
&= (\partial_0 - \gamma^k \gamma_0 \partial_k) F \\
&= (\partial_0 + \sigma^k \partial_k) F \\
&= \left(\frac{1}{c} \partial_t + \boldsymbol{\nabla} \right) F
\end{aligned}
\tag{80.2}
$$

A Fourier transformation of this equation produces

$$
0 = \frac{1}{c} \frac{\partial F}{\partial t}(\mathbf{k}, t) + \frac{1}{(\sqrt{2\pi})^3} \int \sigma^m \partial_m F(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3 x
\tag{80.3}
$$

and with a single integration by parts one has

$$
\begin{aligned}
0 &= \frac{1}{c}\frac{\partial F}{\partial t}(\mathbf{k}, t) - \frac{1}{(\sqrt{2\pi})^3}\int \sigma^m F(\mathbf{x}, t)(-ik_m)e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \\
&= \frac{1}{c}\frac{\partial F}{\partial t}(\mathbf{k}, t) + \frac{1}{(\sqrt{2\pi})^3}\int \mathbf{k}F(\mathbf{x}, t)ie^{-i\mathbf{k}\cdot\mathbf{x}} d^3x \\
&= \frac{1}{c}\frac{\partial F}{\partial t}(\mathbf{k}, t) + i\mathbf{k}\hat{F}(\mathbf{k}, t)
\end{aligned}
\tag{80.4}
$$

The flexibility to employ the pseudoscalar as the imaginary $i = \gamma_0\gamma_1\gamma_2\gamma_3$ has been employed above, so it should be noted that pseudoscalar commutation with Dirac bivectors was implied above, but also that we do not have the flexibility to commute $\mathbf{k}$ with F.

Having done this, the problem to solve is now Maxwell's vacuum equation in the frequency domain

$$
\frac{\partial F}{\partial t}(\mathbf{k}, t) = -ic\mathbf{k}\hat{F}(\mathbf{k}, t) \tag{80.5}
$$

Introducing an angular frequency (spatial) bivector, and its vector dual

$$
\begin{aligned}
\Omega &= -ic\mathbf{k} \\
\omega &= c\mathbf{k}
\end{aligned}
\tag{80.6}
$$

This becomes

$$
\hat{F}' = \Omega F \tag{80.7}
$$

With solution

$$
\hat{F} = e^{\Omega t}\hat{F}(\mathbf{k}, 0) \tag{80.8}
$$

Differentiation with respect to time verifies that the ordering of the terms is correct and this does in fact solve eq. (80.7). This is something we have to be careful of due to the possibility of non-commuting variables.

Back substitution into the inverse transform now supplies the time evolution of the field given the initial time specification

$$
\begin{aligned}
F(\mathbf{x}, t) &= \frac{1}{(\sqrt{2\pi})^3}\int e^{\Omega t}\hat{F}(\mathbf{k}, 0)e^{i\mathbf{k}\cdot\mathbf{x}} d^3k \\
&= \frac{1}{(2\pi)^3}\int e^{\Omega t}\left(\int F(\mathbf{x}', 0)e^{-i\mathbf{k}\cdot\mathbf{x}'} d^3x'\right)e^{i\mathbf{k}\cdot\mathbf{x}} d^3k
\end{aligned}
\tag{80.9}
$$

Observe that Pseudoscalar exponentials commute with the field because i commutes with spatial vectors and itself

$$\begin{aligned}
Fe^{i\theta} &= (\mathbf{E} + ic\mathbf{B})(C + iS) \\
&= C(\mathbf{E} + ic\mathbf{B}) + S(\mathbf{E} + ic\mathbf{B})i \\
&= C(\mathbf{E} + ic\mathbf{B}) + Si(\mathbf{E} + ic\mathbf{B}) \\
&= e^{i\theta}F
\end{aligned} \tag{80.10}$$

This allows the specifics of the initial time conditions to be suppressed

$$F(\mathbf{x}, t) = \int d^3k e^{\Omega t} e^{i\mathbf{k}\cdot\mathbf{x}} \int \frac{1}{(2\pi)^3} F(\mathbf{x}', 0) e^{-i\mathbf{k}\cdot\mathbf{x}'} d^3x' \tag{80.11}$$

The interior integral has the job of a weighting function over plane wave solutions, and this can be made explicit writing

$$\begin{aligned}
D(\mathbf{k}) &= \frac{1}{(2\pi)^3} \int F(\mathbf{x}', 0) e^{-i\mathbf{k}\cdot\mathbf{x}'} d^3x' \\
F(\mathbf{x}, t) &= \int e^{\Omega t} e^{i\mathbf{k}\cdot\mathbf{x}} D(\mathbf{k}) d^3k
\end{aligned} \tag{80.12}$$

Many assumptions have been made here, not the least of which was a requirement for the Fourier transform of a bivector valued function to be meaningful, and have an inverse. It is therefore reasonable to verify that this weighted plane wave result is in fact a solution to the original Maxwell vacuum equation. Differentiation verifies that things are okay so far

$$\begin{aligned}
\gamma_0 \nabla F(\mathbf{x}, t) &= \left(\frac{1}{c}\partial_t + \boldsymbol{\nabla}\right) \int e^{\Omega t} e^{i\mathbf{k}\cdot\mathbf{x}} D(\mathbf{k}) d^3k \\
&= \int \left(\frac{1}{c}\Omega e^{\Omega t} + \sigma^m e^{\Omega t} ik_m\right) e^{i\mathbf{k}\cdot\mathbf{x}} D(\mathbf{k}) d^3k \\
&= \int \left(\frac{1}{c}(-i\mathbf{k}c) + i\mathbf{k}\right) e^{\Omega t} e^{i\mathbf{k}\cdot\mathbf{x}} D(\mathbf{k}) d^3k \\
&= 0 \qquad \square
\end{aligned} \tag{80.13}$$

80.1.2 *Discretizing and grade restrictions*

The fact that it the integral has zero gradient does not mean that it is a bivector, so there must also be at least also be restrictions on the grades of $D(\mathbf{k})$.

To simplify discussion, let us discretize the integral writing

$$D(\mathbf{k}') = D_{\mathbf{k}}\delta^3(\mathbf{k} - \mathbf{k}') \tag{80.14}$$

So we have

$$\begin{aligned} F(\mathbf{x}, t) &= \int e^{\Omega t} e^{i\mathbf{k}'\cdot\mathbf{x}} D(\mathbf{k}') d^3k' \\ &= \int e^{\Omega t} e^{i\mathbf{k}'\cdot\mathbf{x}} D_{\mathbf{k}}\delta^3(\mathbf{k} - \mathbf{k}') d^3k' \end{aligned} \tag{80.15}$$

This produces something planewave-ish

$$F(\mathbf{x}, t) = e^{\Omega t} e^{i\mathbf{k}\cdot\mathbf{x}} D_{\mathbf{k}} \tag{80.16}$$

Observe that at $t = 0$ we have

$$\begin{aligned} F(\mathbf{x}, 0) &= e^{i\mathbf{k}\cdot\mathbf{x}} D_{\mathbf{k}} \\ &= (\cos(\mathbf{k} \cdot \mathbf{x}) + i \sin(\mathbf{k} \cdot \mathbf{x}))D_{\mathbf{k}} \end{aligned} \tag{80.17}$$

There is therefore a requirement for $D_{\mathbf{k}}$ to be either a spatial vector or its dual, a spatial bivector. For example taking D_k to be a spatial vector we can then identify the electric and magnetic components of the field

$$\begin{aligned} \mathbf{E}(\mathbf{x}, 0) &= \cos(\mathbf{k} \cdot \mathbf{x})D_{\mathbf{k}} \\ c\mathbf{B}(\mathbf{x}, 0) &= \sin(\mathbf{k} \cdot \mathbf{x})D_{\mathbf{k}} \end{aligned} \tag{80.18}$$

and if D_k is taken to be a spatial bivector, this pair of identifications would be inverted.

Considering eq. (80.16) at $\mathbf{x} = 0$, we have

$$\begin{aligned} F(0, t) &= e^{\Omega t} D_{\mathbf{k}} \\ &= (\cos(|\Omega|t) + \hat{\Omega} \sin(|\Omega|t))D_{\mathbf{k}} \\ &= (\cos(|\Omega|t) - i\hat{\mathbf{k}} \sin(|\Omega|t))D_{\mathbf{k}} \end{aligned} \tag{80.19}$$

If $D_{\mathbf{k}}$ is first assumed to be a spatial vector, then F would have a pseudoscalar component if $D_{\mathbf{k}}$ has any component parallel to $\hat{\mathbf{k}}$.

$$D_{\mathbf{k}} \in \{\sigma^m\} \implies D_{\mathbf{k}} \cdot \hat{\mathbf{k}} = 0 \tag{80.20}$$

$$D_{\mathbf{k}} \in \{\sigma^a \wedge \sigma^b\} \implies D_{\mathbf{k}} \cdot (i\hat{\mathbf{k}}) = 0 \tag{80.21}$$

Since we can convert between the spatial vector and bivector cases using a duality transformation, there may not appear to be any loss of generality imposing a spatial vector restriction on $D_{\mathbf{k}}$, at least in this current free case. However, an attempt to do so leads to trouble. In particular, this leads to collinear electric and magnetic fields, and thus the odd seeming condition where the field energy density is non-zero but the field momentum density (Poynting vector $\mathbf{P} \propto \mathbf{E} \times \mathbf{B}$) is zero. In retrospect being forced down the path of including both grades is not unreasonable, especially since this gives $D_{\mathbf{k}}$ precisely the form of the field itself $F = \mathbf{E} + ic\mathbf{B}$.

80.2 ELECTRIC AND MAGNETIC FIELD SPLIT

With the basic form of the Maxwell vacuum solution determined, we are now ready to start extracting information from the solution and making comparisons with the more familiar vector form. To start doing the phasor form of the fundamental solution can be expanded explicitly in terms of two arbitrary spatial parametrization vectors $\mathbf{E}_{\mathbf{k}}$ and $\mathbf{B}_{\mathbf{k}}$.

$$F = e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_{\mathbf{k}} + ic\mathbf{B}_{\mathbf{k}}) \tag{80.22}$$

Whether these parametrization vectors have any relation to electric and magnetic fields respectively will have to be determined, but making that assumption for now to label these uniquely does not seem unreasonable.

From eq. (80.22) we can compute the electric and magnetic fields by the conjugate relations eq. (80.49). Our conjugate is

$$\begin{aligned} F^\dagger &= (\mathbf{E}_{\mathbf{k}} - ic\mathbf{B}_{\mathbf{k}}) e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t} \\ &= e^{-i\omega t} e^{-i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_{\mathbf{k}} - ic\mathbf{B}_{\mathbf{k}}) \end{aligned} \tag{80.23}$$

Thus for the electric field

$$\begin{aligned} F + F^\dagger &= e^{-i\omega t} \left(e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_{\mathbf{k}} + ic\mathbf{B}_{\mathbf{k}}) + e^{-i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_{\mathbf{k}} - ic\mathbf{B}_{\mathbf{k}}) \right) \\ &= e^{-i\omega t} \left(2\cos(\mathbf{k}\cdot\mathbf{x}) \mathbf{E}_{\mathbf{k}} + ic(2i)\sin(\mathbf{k}\cdot\mathbf{x}) \mathbf{B}_{\mathbf{k}} \right) \\ &= 2\cos(\omega t) \left(\cos(\mathbf{k}\cdot\mathbf{x}) \mathbf{E}_{\mathbf{k}} - c\sin(\mathbf{k}\cdot\mathbf{x}) \mathbf{B}_{\mathbf{k}} \right) \\ &+ 2\sin(\omega t) \hat{\mathbf{k}} \times \left(\cos(\mathbf{k}\cdot\mathbf{x}) \mathbf{E}_{\mathbf{k}} - c\sin(\mathbf{k}\cdot\mathbf{x}) \mathbf{B}_{\mathbf{k}} \right) \end{aligned} \tag{80.24}$$

So for the electric field $\mathbf{E} = \frac{1}{2}(F + F^\dagger)$ we have

$$\mathbf{E} = \left(\cos(\omega t) + \sin(\omega t)\hat{\mathbf{k}}\times\right)\left(\cos(\mathbf{k}\cdot\mathbf{x})\mathbf{E}_\mathbf{k} - c\sin(\mathbf{k}\cdot\mathbf{x})\mathbf{B}_\mathbf{k}\right) \tag{80.25}$$

Similarly for the magnetic field we have

$$\begin{aligned} F - F^\dagger &= e^{-i\omega t}\left(e^{i\mathbf{k}\cdot\mathbf{x}}(\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) - e^{-i\mathbf{k}\cdot\mathbf{x}}(\mathbf{E}_\mathbf{k} - ic\mathbf{B}_\mathbf{k})\right) \\ &= e^{-i\omega t}\left(2i\sin(\mathbf{k}\cdot\mathbf{x})\mathbf{E}_\mathbf{k} + 2ic\cos(\mathbf{k}\cdot\mathbf{x})\mathbf{B}_\mathbf{k}\right) \end{aligned} \tag{80.26}$$

This gives $c\mathbf{B} = \frac{1}{2i}(F - F^\dagger)$ we have

$$c\mathbf{B} = \left(\cos(\omega t) + \sin(\omega t)\hat{\mathbf{k}}\times\right)\left(\sin(\mathbf{k}\cdot\mathbf{x})\mathbf{E}_\mathbf{k} + c\cos(\mathbf{k}\cdot\mathbf{x})\mathbf{B}_\mathbf{k}\right) \tag{80.27}$$

Observe that the action of the time dependent phasor has been expressed, somewhat abusively and sneakily, in a scalar plus cross product operator form. The end result, when applied to a vector perpendicular to $\hat{\mathbf{k}}$, is still a vector

$$e^{-i\omega t}\mathbf{a} = \left(\cos(\omega t) + \sin(\omega t)\hat{\mathbf{k}}\times\right)\mathbf{a} \tag{80.28}$$

Also observe that the Hermitian conjugate split of the total field bivector F produces vectors $\mathbf{E}$ and $\mathbf{B}$, not phasors. There is no further need to take real or imaginary parts nor treat the phasor eq. (80.22) as an artificial mathematical construct used for convenience only.

With $\mathbf{E}\cdot\hat{\mathbf{k}} = \mathbf{B}\cdot\hat{\mathbf{k}} = 0$, we have here what Jackson ([22], ch7), calls a transverse wave.

80.2.1 *Polar Form*

Suppose an explicit polar form is introduced for the plane vectors $\mathbf{E}_\mathbf{k}$, and $\mathbf{B}_\mathbf{k}$. Let

$$\begin{aligned} \mathbf{E}_\mathbf{k} &= E\hat{\mathbf{E}}_k \\ \mathbf{B}_\mathbf{k} &= B\hat{\mathbf{E}}_k e^{i\hat{\mathbf{k}}\theta} \end{aligned} \tag{80.29}$$

Then for the field we have

$$F = e^{-i\omega t}e^{i\mathbf{k}\cdot\mathbf{x}}(E + icBe^{-i\hat{\mathbf{k}}\theta})\hat{\mathbf{E}}_k \tag{80.30}$$

For the conjugate

$$\begin{aligned} F^\dagger &= \hat{\mathbf{E}}_k(E - icBe^{i\hat{\mathbf{k}}\theta})e^{-i\mathbf{k}\cdot\mathbf{x}}e^{i\omega t} \\ &= e^{-i\omega t}e^{-i\mathbf{k}\cdot\mathbf{x}}(E - icBe^{-i\hat{\mathbf{k}}\theta})\hat{\mathbf{E}}_k \end{aligned} \tag{80.31}$$

So, in the polar form we have for the electric, and magnetic fields

$$\begin{aligned}\mathbf{E} &= e^{-i\omega t}(E\cos(\mathbf{k}\cdot\mathbf{x}) - cB\sin(\mathbf{k}\cdot\mathbf{x})e^{-i\hat{\mathbf{k}}\theta})\hat{\mathbf{E}}_k \\ c\mathbf{B} &= e^{-i\omega t}(E\sin(\mathbf{k}\cdot\mathbf{x}) + cB\cos(\mathbf{k}\cdot\mathbf{x})e^{-i\hat{\mathbf{k}}\theta})\hat{\mathbf{E}}_k\end{aligned} \tag{80.32}$$

Observe when θ is an integer multiple of π, $\mathbf{E}$ and $\mathbf{B}$ are colinear, having the zero Poynting vector mentioned previously. Now, for arbitrary θ it does not appear that there is any inherent perpendicularity between the electric and magnetic fields. It is common to read of light being the propagation of perpendicular fields, both perpendicular to the propagation direction. We have perpendicularity to the propagation direction by virtue of requiring that the field be a (Dirac) bivector, but it does not look like the solution requires any inherent perpendicularity for the field components. It appears that a normal triplet of field vectors and propagation directions must actually be a special case. Intuition says that this freedom to pick different magnitude or angle between $\mathbf{E_k}$ and $\mathbf{B_k}$ in the plane perpendicular to the transmission direction may correspond to different mixes of linear, circular, and elliptic polarization, but this has to be confirmed.

Working towards confirming (or disproving) this intuition, lets find the constraints on the fields that lead to normal electric and magnetic fields. This should follow by taking dot products

$$\begin{aligned}\mathbf{E}\cdot\mathbf{B}c &= \left\langle e^{-i\omega t}(E\cos(\mathbf{k}\cdot\mathbf{x}) - cB\sin(\mathbf{k}\cdot\mathbf{x})e^{-i\hat{\mathbf{k}}\theta})\hat{\mathbf{E}}_k\hat{\mathbf{E}}_k e^{i\omega t}(E\sin(\mathbf{k}\cdot\mathbf{x}) + cB\cos(\mathbf{k}\cdot\mathbf{x})e^{i\hat{\mathbf{k}}\theta})\right\rangle \\ &= \left\langle (E\cos(\mathbf{k}\cdot\mathbf{x}) - cB\sin(\mathbf{k}\cdot\mathbf{x})e^{-i\hat{\mathbf{k}}\theta})(E\sin(\mathbf{k}\cdot\mathbf{x}) + cB\cos(\mathbf{k}\cdot\mathbf{x})e^{i\hat{\mathbf{k}}\theta})\right\rangle \\ &= (E^2 - c^2B^2)\cos(\mathbf{k}\cdot\mathbf{x})\sin(\mathbf{k}\cdot\mathbf{x}) + cEB\left\langle\cos^2(\mathbf{k}\cdot\mathbf{x})e^{i\hat{\mathbf{k}}\theta} - \sin^2(\mathbf{k}\cdot\mathbf{x})e^{-i\hat{\mathbf{k}}\theta}\right\rangle \\ &= (E^2 - c^2B^2)\cos(\mathbf{k}\cdot\mathbf{x})\sin(\mathbf{k}\cdot\mathbf{x}) + cEB\cos(\theta)(\cos^2(\mathbf{k}\cdot\mathbf{x}) - \sin^2(\mathbf{k}\cdot\mathbf{x})) \\ &= (E^2 - c^2B^2)\cos(\mathbf{k}\cdot\mathbf{x})\sin(\mathbf{k}\cdot\mathbf{x}) + cEB\cos(\theta)(\cos^2(\mathbf{k}\cdot\mathbf{x}) - \sin^2(\mathbf{k}\cdot\mathbf{x})) \\ &= \frac{1}{2}(E^2 - c^2B^2)\sin(2\mathbf{k}\cdot\mathbf{x}) + cEB\cos(\theta)\cos(2\mathbf{k}\cdot\mathbf{x})\end{aligned} \tag{80.33}$$

The only way this can be zero for any $\mathbf{x}$ is if the left and right terms are separately zero, which means

$$\begin{aligned}|\mathbf{E}_k| &= c|\mathbf{B}_k| \\ \theta &= \frac{\pi}{2} + n\pi\end{aligned} \tag{80.34}$$

This simplifies the phasor considerably, leaving

$$\begin{aligned}E + icBe^{-i\hat{\mathbf{k}}\theta} &= E(1 + i(\mp i\hat{\mathbf{k}})) \\ &= E(1 \pm \hat{\mathbf{k}})\end{aligned} \tag{80.35}$$

So the field is just

$$F = e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (1 \pm \hat{\mathbf{k}}) \mathbf{E_k} \tag{80.36}$$

Using this, and some regrouping, a calculation of the field components yields

$$\begin{aligned} \mathbf{E} &= e^{i\hat{\mathbf{k}}(\pm\mathbf{k}\cdot\mathbf{x}-\omega t)} \mathbf{E_k} \\ c\mathbf{B} &= \pm e^{i\hat{\mathbf{k}}(\pm\mathbf{k}\cdot\mathbf{x}-\omega t)} i\mathbf{k}\mathbf{E_k} \end{aligned} \tag{80.37}$$

Observe that $i\mathbf{k}$ rotates any vector in the plane perpendicular to $\hat{\mathbf{k}}$ by 90 degrees, so we have here $c\mathbf{B} = \pm\hat{\mathbf{k}} \times \mathbf{E}$. This is consistent with the transverse wave restriction (7.11) of Jackson [22], where he says, the "curl equations provide a further restriction, namely", and

$$\mathcal{B} = \sqrt{\mu\epsilon}\mathbf{n} \times \mathcal{E} \tag{80.38}$$

He works in explicit complex phasor form and CGS units. He also allows $\mathbf{n}$ to be complex. With real $\mathbf{k}$, and no $\mathbf{E} \cdot \mathbf{B} = 0$ constraint, it appears that we cannot have such a simple coupling between the field components? Is it possible that allowing $\mathbf{k}$ to be complex allows this cross product coupling constraint on the fields without the explicit 90 degree phase difference between the electric and magnetic fields?

80.3 ENERGY AND MOMENTUM FOR THE PHASOR

To calculate the field energy density we can work with the two fields of equations eq. (80.32), or work with the phasor eq. (80.22) directly. From the phasor and the energy-momentum four vector eq. (80.52) we have for the energy density

$$
\begin{aligned}
U &= T(\gamma_0) \cdot \gamma_0 \\
&= \frac{-\epsilon_0}{2} \langle F\gamma_0 F\gamma_0 \rangle \\
&= \frac{-\epsilon_0}{2} \left\langle e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) \gamma_0 e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) \gamma_0 \right\rangle \\
&= \frac{-\epsilon_0}{2} \left\langle e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) (\gamma_0)^2 e^{-i\omega t} e^{-i\mathbf{k}\cdot\mathbf{x}} (-\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) \right\rangle \\
&= \frac{-\epsilon_0}{2} \left\langle e^{-i\omega t} (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) e^{-i\omega t} (-\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) \right\rangle \\
&= \frac{\epsilon_0}{2} \langle (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k})(\mathbf{E}_\mathbf{k} - ic\mathbf{B}_\mathbf{k}) \rangle \\
&= \frac{\epsilon_0}{2} \left((\mathbf{E}_k)^2 + c^2 (\mathbf{B}_\mathbf{k})^2 \right) + c\epsilon_0 \langle i\mathbf{E}_\mathbf{k} \wedge \mathbf{B}_\mathbf{k} \rangle \\
&= \frac{\epsilon_0}{2} \left((\mathbf{E}_k)^2 + c^2 (\mathbf{B}_\mathbf{k})^2 \right) + c\epsilon_0 \langle \mathbf{B}_\mathbf{k} \times \mathbf{E}_\mathbf{k} \rangle
\end{aligned}
\tag{80.39}
$$

Quite anticlimactically we have for the energy the sum of the energies associated with the parametrization constants, lending some justification for the initial choice to label these as electric and magnetic fields

$$
U = \frac{\epsilon_0}{2} \left((\mathbf{E}_k)^2 + c^2 (\mathbf{B}_\mathbf{k})^2 \right) \tag{80.40}
$$

For the momentum, we want the difference of $FF^\dagger$, and $F^\dagger F$

$$
\begin{aligned}
FF^\dagger &= e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k})(\mathbf{E}_\mathbf{k} - ic\mathbf{B}_\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t} \\
&= (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k})(\mathbf{E}_\mathbf{k} - ic\mathbf{B}_\mathbf{k}) \\
&= (\mathbf{E}_\mathbf{k})^2 + c^2 (\mathbf{B}_\mathbf{k})^2 - 2c\mathbf{B}_\mathbf{k} \times \mathbf{E}_\mathbf{k}
\end{aligned}
\tag{80.41}
$$

$$
\begin{aligned}
FF^\dagger &= (\mathbf{E}_\mathbf{k} - ic\mathbf{B}_\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t} e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}} (\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) \\
&= (\mathbf{E}_\mathbf{k} - ic\mathbf{B}_\mathbf{k})(\mathbf{E}_\mathbf{k} + ic\mathbf{B}_\mathbf{k}) \\
&= (\mathbf{E}_\mathbf{k})^2 + c^2 (\mathbf{B}_\mathbf{k})^2 + 2c\mathbf{B}_\mathbf{k} \times \mathbf{E}_\mathbf{k}
\end{aligned}
\tag{80.42}
$$

So we have for the momentum, also anticlimactically

$$
\mathbf{P} = \frac{1}{c} T(\gamma_0) \wedge \gamma_0 = \epsilon_0 \mathbf{E}_\mathbf{k} \times \mathbf{B}_\mathbf{k} \tag{80.43}
$$

80.4 FOLLOWUP

Well, that is enough for one day. Understanding how to express circular and elliptic polarization is one of the logical next steps. I seem to recall from Susskind's QM lectures that these can be considered superpositions of linearly polarized waves, so examining a sum of two co-directionally propagating fields would seem to be in order. Also there ought to be a more natural way to express the perpendicularity requirement for the field and the propagation direction. The fact that the field components and propagation direction when all multiplied is proportional to the spatial pseudoscalar can probably be utilized to tidy this up and also produce a form that allows for simpler summation of fields in different propagation directions. It also seems reasonable to consider a planar Fourier decomposition of the field components, perhaps framing the superposition of multiple fields in that context.

Reconsilation of the Jackson's (7.11) restriction for perpendicularity of the fields noted above has not been done. If such a restriction is required with an explicit dot and cross product split of Maxwell's equation, it would make sense to also have this required of a GA based solution. Is this just a conquense of the differences between his explicit phasor representation, and this geometric approach where the phasor has an explicit representation in terms of the transverse plane?

80.5 APPENDIX. BACKGROUND DETAILS

80.5.1 *Conjugate split*

The Hermitian conjugate is defined as

$$A^\dagger = \gamma_0 \tilde{A} \gamma_0 \tag{80.44}$$

The conjugate action on a multivector product is straightforward to calculate

$$\begin{aligned}
(AB)^\dagger &= \gamma_0 \tilde{(AB)} \gamma_0 \\
&= \gamma_0 \tilde{B} \tilde{A} \gamma_0 \\
&= \gamma_0 \tilde{B} {\gamma_0}^2 \tilde{A} \gamma_0 \\
&= B^\dagger A^\dagger
\end{aligned} \tag{80.45}$$

For a spatial vector Hermitian conjugation leaves the vector unaltered

$$\begin{aligned}
\mathbf{a} &= \gamma_0 \tilde{(\gamma_k \gamma_0)} a^k \gamma_0 \\
&= \gamma_0 (\gamma_0 \gamma_k) a^k \gamma_0 \\
&= \gamma_k a^k \gamma_0 \\
&= \mathbf{a}
\end{aligned} \tag{80.46}$$

But the pseudoscalar is negated

$$\begin{aligned}
i^\dagger &= \gamma_0 \tilde{i} \gamma_0 \\
&= \gamma_0 i \gamma_0 \\
&= -\gamma_0 \gamma_0 i \\
&= -i
\end{aligned} \tag{80.47}$$

This allows for a split by conjugation of the field into its electric and magnetic field components.

$$\begin{aligned}
F^\dagger &= -\gamma_0 (\mathbf{E} + ic\mathbf{B}) \gamma_0 \\
&= -\gamma_0^2 (-\mathbf{E} + ic\mathbf{B}) \\
&= \mathbf{E} - ic\mathbf{B}
\end{aligned} \tag{80.48}$$

So we have

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2}(F + F^\dagger) \\
c\mathbf{B} &= \frac{1}{2i}(F - F^\dagger)
\end{aligned} \tag{80.49}$$

80.5.2 *Field Energy Momentum density four vector*

In the GA formalism the energy momentum tensor is

$$T(a) = \frac{\epsilon_0}{2} F a \tilde{F} \tag{80.50}$$

It is not necessarily obvious this bivector-vector-bivector product construction is even a vector quantity. Expansion of $T(\gamma_0)$ in terms of the electric and magnetic fields demonstrates this vectorial nature.

$$\begin{aligned}
F\gamma_0\tilde{F} &= -(\mathbf{E} + ic\mathbf{B})\gamma_0(\mathbf{E} + ic\mathbf{B}) \\
&= -\gamma_0(-\mathbf{E} + ic\mathbf{B})(\mathbf{E} + ic\mathbf{B}) \\
&= -\gamma_0(-\mathbf{E}^2 - c^2\mathbf{B}^2 + ic(\mathbf{B}\mathbf{E} - \mathbf{E}\mathbf{B})) \\
&= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2) - 2\gamma_0 ic(\mathbf{B} \wedge \mathbf{E})) \\
&= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2) + 2\gamma_0 c(\mathbf{B} \times \mathbf{E}) \\
&= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2) + 2\gamma_0 c\gamma_k\gamma_0(\mathbf{B} \times \mathbf{E})^k \\
&= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2) + 2\gamma_k(\mathbf{E} \times (c\mathbf{B}))^k
\end{aligned} \tag{80.51}$$

Therefore, $T(\gamma_0)$, the energy momentum tensor biased towards a particular observer frame γ_0 is

$$T(\gamma_0) = \gamma_0\frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) + \gamma_k\epsilon_0(\mathbf{E} \times (c\mathbf{B}))^k \tag{80.52}$$

Recognizable here in the components $T(\gamma_0)$ are the field energy density and momentum density. In particular the energy density can be obtained by dotting with γ_0, whereas the (spatial vector) momentum by wedging with γ_0.

These are

$$\begin{aligned}
U \equiv T(\gamma_0) \cdot \gamma_0 &= \frac{1}{2}\left(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right) \\
c\mathbf{P} \equiv T(\gamma_0) \wedge \gamma_0 &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}
\end{aligned} \tag{80.53}$$

In terms of the combined field these are

$$\begin{aligned}
U &= \frac{-\epsilon_0}{2}(F\gamma_0F\gamma_0 + \gamma_0F\gamma_0F) \\
c\mathbf{P} &= \frac{-\epsilon_0}{2}(F\gamma_0F\gamma_0 - \gamma_0F\gamma_0F)
\end{aligned} \tag{80.54}$$

Summarizing with the Hermitian conjugate

$$\begin{aligned}
U &= \frac{\epsilon_0}{2}(FF^\dagger + F^\dagger F) \\
c\mathbf{P} &= \frac{\epsilon_0}{2}(FF^\dagger - F^\dagger F)
\end{aligned} \tag{80.55}$$

80.5.2.1 *Divergence*

Calculation of the divergence produces the components of the Lorentz force densities

$$
\begin{aligned}
\nabla \cdot T(a) &= \frac{\epsilon_0}{2} \langle \nabla (F a F) \rangle \\
&= \frac{\epsilon_0}{2} \langle (\nabla F) a F + (F \nabla) F a \rangle
\end{aligned}
\tag{80.56}
$$

Here the gradient is used implicitly in bidirectional form, where the direction is implied by context. From Maxwell's equation we have

$$
\begin{aligned}
J/\epsilon_0 c &= (\nabla F)\tilde{} \\
&= (\tilde{F} \tilde{\nabla}) \\
&= -(F \nabla)
\end{aligned}
\tag{80.57}
$$

and continuing the expansion

$$
\begin{aligned}
\nabla \cdot T(a) &= \frac{1}{2c} \langle J a F - J F a \rangle \\
&= \frac{1}{2c} \langle F J a - J F a \rangle \\
&= \frac{1}{2c} \langle (F J - J F) a \rangle
\end{aligned}
\tag{80.58}
$$

Wrapping up, the divergence and the adjoint of the energy momentum tensor are

$$
\begin{aligned}
\nabla \cdot T(a) &= \frac{1}{c} (F \cdot J) \cdot a \\
\overline{T}(\nabla) &= F \cdot J/c
\end{aligned}
\tag{80.59}
$$

When integrated over a volume, the quantities $F \cdot J/c$ are the components of the RHS of the Lorentz force equation $\dot{p} = qF \cdot v/c$.

81 TRANSVERSE ELECTRIC AND MAGNETIC FIELDS

81.1 MOTIVATION

In Eli's Transverse Electric and Magnetic Fields in a Conducting Waveguide blog entry he works through the algebra calculating the transverse components, the perpendicular to the propagation direction components.

This should be possible using Geometric Algebra too, and trying this made for a good exercise.

81.2 SETUP

The starting point can be the same, the source free Maxwell's equations. Writing $\partial_0 = (1/c)\partial/\partial t$, we have

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\partial_0 \mathbf{B} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu\epsilon\partial_0 \mathbf{E}
\end{aligned} \tag{81.1}$$

Multiplication of the last two equations by the spatial pseudoscalar I, and using $I\mathbf{a} \times \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$, the curl equations can be written in their dual bivector form

$$\begin{aligned}
\boldsymbol{\nabla} \wedge \mathbf{E} &= -\partial_0 I\mathbf{B} \\
\boldsymbol{\nabla} \wedge \mathbf{B} &= \mu\epsilon\partial_0 I\mathbf{E}
\end{aligned} \tag{81.2}$$

Now adding the dot and curl equations using $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$ eliminates the cross products

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{E} &= -\partial_0 I\mathbf{B} \\
\boldsymbol{\nabla}\mathbf{B} &= \mu\epsilon\partial_0 I\mathbf{E}
\end{aligned} \tag{81.3}$$

These can be further merged without any loss, into the GA first order equation for Maxwell's equation in cgs units

$$\left(\boldsymbol{\nabla} + \frac{\sqrt{\mu\epsilon}}{c}\partial_t\right)\left(\mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) = 0. \tag{81.4}$$

We are really after solutions to the total multivector field $F = \mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}$. For this problem where separate electric and magnetic field components are desired, working from eq. (81.3) is perhaps what we want?

Following Eli and Jackson, write $\boldsymbol{\nabla} = \boldsymbol{\nabla}_t + \hat{\mathbf{z}}\partial_z$, and

$$\begin{aligned}
\mathbf{E}(x, y, z, t) &= \mathbf{E}(x, y)e^{\pm ikz - i\omega t} \\
\mathbf{B}(x, y, z, t) &= \mathbf{B}(x, y)e^{\pm ikz - i\omega t}
\end{aligned} \tag{81.5}$$

Evaluating the z and t partials we have

$$\begin{aligned}
(\boldsymbol{\nabla}_t \pm ik\hat{\mathbf{z}})\mathbf{E}(x, y) &= \frac{i\omega}{c} I\mathbf{B}(x, y) \\
(\boldsymbol{\nabla}_t \pm ik\hat{\mathbf{z}})\mathbf{B}(x, y) &= -\mu\epsilon\frac{i\omega}{c} I\mathbf{E}(x, y)
\end{aligned} \tag{81.6}$$

For the remainder of these notes, the explicit (x, y) dependence will be assumed for $\mathbf{E}$ and $\mathbf{B}$.

An obvious thing to try with these equations is just substitute one into the other. If that is done we get the pair of second order harmonic equations

$$\boldsymbol{\nabla}_t{}^2 \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \left(k^2 - \mu\epsilon\frac{\omega^2}{c^2}\right)\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \tag{81.7}$$

One could consider the problem solved here. Separately equating both sides of this equation to zero, we have the $k^2 = \mu\epsilon\omega^2/c^2$ constraint on the wave number and angular velocity, and the second order Laplacian on the left hand side is solved by the real or imaginary parts of any analytic function. Especially when one considers that we are after a multivector field that of intrinsic complex nature.

However, that is not really what we want as a solution. Doing the same on the unified Maxwell equation eq. (81.4), we have

$$\left(\boldsymbol{\nabla}_t \pm ik\hat{\mathbf{z}} - \sqrt{\mu\epsilon}\frac{i\omega}{c}\right)\left(\mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}\right) = 0 \tag{81.8}$$

Selecting scalar, vector, bivector and trivector grades of this equation produces the following respective relations between the various components

$$
\begin{aligned}
0 &= \langle \cdots \rangle = \boldsymbol{\nabla}_t \cdot \mathbf{E} \pm ik\hat{\mathbf{z}} \cdot \mathbf{E} \\
0 &= \langle \cdots \rangle_1 = I\boldsymbol{\nabla}_t \wedge \mathbf{B}/\sqrt{\mu\epsilon} \pm iIk\hat{\mathbf{z}} \wedge \mathbf{B}/\sqrt{\mu\epsilon} - i\sqrt{\mu\epsilon}\frac{\omega}{c}\mathbf{E} \\
0 &= \langle \cdots \rangle_2 = \boldsymbol{\nabla}_t \wedge \mathbf{E} \pm ik\hat{\mathbf{z}} \wedge \mathbf{E} - i\frac{\omega}{c}I\mathbf{B} \\
0 &= \langle \cdots \rangle_3 = I\boldsymbol{\nabla}_t \cdot \mathbf{B}/\sqrt{\mu\epsilon} \pm iIk\hat{\mathbf{z}} \cdot \mathbf{B}/\sqrt{\mu\epsilon}
\end{aligned}
\tag{81.9}
$$

From the scalar and pseudoscalar grades we have the propagation components in terms of the transverse ones

$$
\begin{aligned}
E_z &= \frac{\pm i}{k}\boldsymbol{\nabla}_t \cdot \mathbf{E}_t \\
B_z &= \frac{\pm i}{k}\boldsymbol{\nabla}_t \cdot \mathbf{B}_t
\end{aligned}
\tag{81.10}
$$

But this is the opposite of the relations that we are after. On the other hand from the vector and bivector grades we have

$$
\begin{aligned}
i\frac{\omega}{c}\mathbf{E} &= -\frac{1}{\mu\epsilon}\left(\boldsymbol{\nabla}_t \times \mathbf{B}_z \pm ik\hat{\mathbf{z}} \times \mathbf{B}_t\right) \\
i\frac{\omega}{c}\mathbf{B} &= \boldsymbol{\nabla}_t \times \mathbf{E}_z \pm ik\hat{\mathbf{z}} \times \mathbf{E}_t
\end{aligned}
\tag{81.11}
$$

81.3 A CLUE FROM THE FINAL RESULT

From eq. (81.11) and a lot of messy algebra we should be able to get the transverse equations. Is there a slicker way? The end result that Eli obtained suggests a path. That result was

$$
\mathbf{E}_t = \frac{i}{\mu\epsilon\frac{\omega^2}{c^2} - k^2}\left(\pm k\boldsymbol{\nabla}_t E_z - \frac{\omega}{c}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z\right)
\tag{81.12}
$$

The numerator looks like it can be factored, and after a bit of playing around a suitable factorization can be obtained:

$$\begin{aligned}\left\langle\left(\pm k + \frac{\omega}{c}\hat{\mathbf{z}}\right)\boldsymbol{\nabla}_t \hat{\mathbf{z}}\left(\mathbf{E}_z + I\mathbf{B}_z\right)\right\rangle_1 &= \left\langle\left(\pm k + \frac{\omega}{c}\hat{\mathbf{z}}\right)\boldsymbol{\nabla}_t\left(E_z + IB_z\right)\right\rangle_1 \\ &= \pm k\boldsymbol{\nabla}E_z + \frac{\omega}{c}\langle I\hat{\mathbf{z}}\boldsymbol{\nabla}_t B_z\rangle_1 \\ &= \pm k\boldsymbol{\nabla}E_z + \frac{\omega}{c} I\hat{\mathbf{z}} \wedge \boldsymbol{\nabla}_t B_z \\ &= \pm k\boldsymbol{\nabla}E_z - \frac{\omega}{c}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z\end{aligned} \tag{81.13}$$

Observe that the propagation components of the field $\mathbf{E}_z + I\mathbf{E}_z$ can be written in terms of the symmetric product

$$\begin{aligned}\frac{1}{2}\left(\hat{\mathbf{z}}(\mathbf{E} + I\mathbf{B}) + (\mathbf{E} + I\mathbf{B})\hat{\mathbf{z}}\right) &= \frac{1}{2}\left(\hat{\mathbf{z}}\mathbf{E} + \mathbf{E}\hat{\mathbf{z}}\right) + \frac{I}{2}\left(\hat{\mathbf{z}}\mathbf{B} + \mathbf{B}\hat{\mathbf{z}} + I\right) \\ &= \hat{\mathbf{z}} \cdot \mathbf{E} + I\hat{\mathbf{z}} \cdot \mathbf{B}\end{aligned} \tag{81.14}$$

Now the total field in CGS units was actually $F = \mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}$, not $F = \mathbf{E} + I\mathbf{B}$, so the factorization above is not exactly what we want. It does however, provide the required clue. We probably get the result we want by forming the symmetric product (a hybrid dot product selecting both the vector and bivector terms).

81.4 SYMMETRIC PRODUCT OF THE FIELD WITH THE DIRECTION VECTOR

Rearranging Maxwell's equation eq. (81.8) in terms of the transverse gradient and the total field F we have

$$\boldsymbol{\nabla}_t F = \left(\mp ik\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{i\omega}{c}\right)F \tag{81.15}$$

With this our symmetric product is

$$\begin{aligned}\boldsymbol{\nabla}_t(F\hat{\mathbf{z}} + \hat{\mathbf{z}}F) &= (\boldsymbol{\nabla}_t F)\hat{\mathbf{z}} - \hat{\mathbf{z}}(\boldsymbol{\nabla}_t F) \\ &= \left(\mp ik\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{i\omega}{c}\right)F\hat{\mathbf{z}} - \hat{\mathbf{z}}\left(\mp ik\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{i\omega}{c}\right)F \\ &= i\left(\mp k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)(F\hat{\mathbf{z}} - \hat{\mathbf{z}}F)\end{aligned} \tag{81.16}$$

The antisymmetric product on the right hand side should contain the desired transverse field components. To verify multiply it out

$$
\begin{aligned}
\frac{1}{2}(F\hat{\mathbf{z}} - \hat{\mathbf{z}}F) &= \frac{1}{2}\left(\left(\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}\right)\hat{\mathbf{z}} - \hat{\mathbf{z}}\left(\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}\right)\right) \\
&= \mathbf{E} \wedge \hat{\mathbf{z}} + I\mathbf{B}/\sqrt{\mu\epsilon} \wedge \hat{\mathbf{z}} \\
&= (\mathbf{E}_t + I\mathbf{B}_t/\sqrt{\mu\epsilon})\hat{\mathbf{z}}
\end{aligned}
\tag{81.17}
$$

Now, with multiplication by the conjugate quantity $-i(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\omega/c)$, we can extract these transverse components.

$$
\left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)\left(\mp k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)(F\hat{\mathbf{z}} - \hat{\mathbf{z}}F) = \left(-k^2 + \mu\epsilon\frac{\omega^2}{c^2}\right)(F\hat{\mathbf{z}} - \hat{\mathbf{z}}F) \tag{81.18}
$$

Rearranging, we have the transverse components of the field

$$
(\mathbf{E}_t + I\mathbf{B}_t/\sqrt{\mu\epsilon})\hat{\mathbf{z}} = \frac{i}{k^2 - \mu\epsilon\frac{\omega^2}{c^2}}\left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)\boldsymbol{\nabla}_t\frac{1}{2}(F\hat{\mathbf{z}} + \hat{\mathbf{z}}F) \tag{81.19}
$$

With left multiplication by $\hat{\mathbf{z}}$, and writing $F = F_t + F_z$ we have

$$
F_t = \frac{i}{k^2 - \mu\epsilon\frac{\omega^2}{c^2}}\left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)\boldsymbol{\nabla}_t F_z \tag{81.20}
$$

While this is a complete solution, we can additionally extract the electric and magnetic fields to compare results with Eli's calculation. We take vector grades to do so with $\mathbf{E}_t = \langle F_t\rangle_1$, and $\mathbf{B}_t/\sqrt{\mu\epsilon} = \langle -IF_t\rangle_1$. For the transverse electric field

$$
\begin{aligned}
\left\langle\left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)\boldsymbol{\nabla}_t(\mathbf{E}_z + I\mathbf{B}_z/\sqrt{/\mu\epsilon)}\right\rangle_1 &= \pm k\hat{\mathbf{z}}(-\hat{\mathbf{z}})\boldsymbol{\nabla}_t E_z + \frac{\omega}{c}\overset{-I^2\hat{\mathbf{z}}\times\boldsymbol{\nabla}_t}{\boxed{\langle I\boldsymbol{\nabla}_t\hat{\mathbf{z}}\rangle_1}}B_z \\
&= \mp k\boldsymbol{\nabla}_t E_z + \frac{\omega}{c}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z
\end{aligned}
\tag{81.21}
$$

and for the transverse magnetic field

$$
\begin{aligned}
&\left\langle -I\left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)\boldsymbol{\nabla}_t(\mathbf{E}_z + I\mathbf{B}_z/\sqrt{\mu\epsilon})\right\rangle_1 \\
&= -I\sqrt{\mu\epsilon}\frac{\omega}{c}\boldsymbol{\nabla}_t\mathbf{E}_z + \left\langle\left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)\boldsymbol{\nabla}_t\mathbf{B}_z/\sqrt{\mu\epsilon}\right\rangle_1 \\
&= -\sqrt{\mu\epsilon}\frac{\omega}{c}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z \mp k\boldsymbol{\nabla}_t B_z/\sqrt{\mu\epsilon}
\end{aligned}
\tag{81.22}
$$

Thus the split of transverse field into the electric and magnetic components yields

$$\begin{aligned}\mathbf{E}_t &= \frac{i}{k^2 - \mu\epsilon\frac{\omega^2}{c^2}} \left(\mp k \boldsymbol{\nabla}_t E_z + \frac{\omega}{c} \hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z \right) \\ \mathbf{B}_t &= \frac{i}{k^2 - \mu\epsilon\frac{\omega^2}{c^2}} \left(-\mu\epsilon \frac{\omega}{c} \hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z \mp k \boldsymbol{\nabla}_t B_z \right)\end{aligned} \tag{81.23}$$

Compared to Eli's method using messy traditional vector algebra, this method also has a fair amount of messy tricky algebra, but of a different sort.

81.5 SUMMARY

There is potentially a lot of new ideas above (some for me even with previous exposure to the Geometric Algebra formalism). There was no real attempt to teach GA here, but for completeness the GA form of Maxwell's equation was developed from the traditional divergence and curl formulation of Maxwell's equations. That was mainly due to use of CGS units which differ since this makes Maxwell's equation take a different form from the usual (see [10]).

Here a less exploratory summary of the previous results above is assembled.

In these CGS units our field F, and Maxwell's equation (in absence of charge and current), take the form

$$\begin{aligned}F &= \mathbf{E} + \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}} \\ 0 &= \left(\boldsymbol{\nabla} + \frac{\sqrt{\mu\epsilon}}{c} \partial_t \right) F\end{aligned} \tag{81.24}$$

The electric and magnetic fields can be picked off by selecting the grade one (vector) components

$$\begin{aligned}\mathbf{E} &= \langle F \rangle_1 \\ \mathbf{B} &= \sqrt{\mu\epsilon} \langle -IF \rangle_1\end{aligned} \tag{81.25}$$

With an explicit sinusoidal and z-axis time dependence for the field

$$F(x, y, z, t) = F(x, y) e^{\pm ikz - i\omega t} \tag{81.26}$$

and a split of the gradient into transverse and z-axis components $\boldsymbol{\nabla} = \boldsymbol{\nabla}_t + \hat{\mathbf{z}}\partial_z$, Maxwell's equation takes the form

$$\left(\boldsymbol{\nabla}_t \pm ik\hat{\mathbf{z}} - \sqrt{\mu\epsilon}\frac{i\omega}{c}\right) F(x, y) = 0 \tag{81.27}$$

Writing for short $F = F(x, y)$, we can split the field into transverse and z-axis components with the commutator and anticommutator products respectively. For the z-axis components we have

$$F_z\hat{\mathbf{z}} \equiv E_z + IB_z = \frac{1}{2}(F\hat{\mathbf{z}} + \hat{\mathbf{z}}F) \tag{81.28}$$

The projections onto the z-axis and and transverse directions are respectively

$$\begin{aligned} F_z &= \mathbf{E}_z + I\mathbf{B}_z = \frac{1}{2}(F + \hat{\mathbf{z}}F\hat{\mathbf{z}}) \\ F_t &= \mathbf{E}_t + I\mathbf{B}_t = \frac{1}{2}(F - \hat{\mathbf{z}}F\hat{\mathbf{z}}) \end{aligned} \tag{81.29}$$

With an application of the transverse gradient to the z-axis field we easily found the relation between the two field components

$$\boldsymbol{\nabla}_t F_z = i\left(\pm k\hat{\mathbf{z}} - \sqrt{\mu\epsilon}\frac{\omega}{c}\right) F_t \tag{81.30}$$

A left division by the multivector factor gives the total transverse field

$$F_t = \frac{1}{i\left(\pm k\hat{\mathbf{z}} - \sqrt{\mu\epsilon}\frac{\omega}{c}\right)} \boldsymbol{\nabla}_t F_z \tag{81.31}$$

Multiplication of both the numerator and denominator by the conjugate normalizes this

$$F_t = \frac{i}{k^2 - \mu\epsilon\frac{\omega^2}{c^2}} \left(\pm k\hat{\mathbf{z}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right) \boldsymbol{\nabla}_t F_z \tag{81.32}$$

From this the transverse electric and magnetic fields may be picked off using the projective grade selection operations of eq. (81.25), and are

$$\begin{aligned} \mathbf{E}_t &= \frac{i}{\mu\epsilon\frac{\omega^2}{c^2} - k^2} \left(\pm k\boldsymbol{\nabla}_t E_z - \frac{\omega}{c}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z\right) \\ \mathbf{B}_t &= \frac{i}{\mu\epsilon\frac{\omega^2}{c^2} - k^2} \left(\mu\epsilon\frac{\omega}{c}\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z \pm k\boldsymbol{\nabla}_t B_z\right) \end{aligned} \tag{81.33}$$

82

COMPARING PHASOR AND GEOMETRIC TRANSVERSE SOLUTIONS TO THE MAXWELL EQUATION

82.1 MOTIVATION

In (80) a phasor like form of the transverse wave equation was found by considering Fourier solutions of the Maxwell equation. This will be called the "geometric phasor" since it is hard to refer and compare it without giving it a name. Curiously no perpendicularity condition for **E** and **B** seemed to be required for this geometric phasor. Why would that be the case? In Jackson's treatment, which employed the traditional dot and cross product form of Maxwell's equations, this followed by back substituting the assumed phasor solution back into the equations. This back substitution was not done in (80). If we attempt this we should find the same sort of additional mutual perpendicularity constraints on the fields.

Here we start with the equations from Jackson ([22], ch7), expressed in GA form. Using the same assumed phasor form we should get the same results using GA. Anything else indicates a misunderstanding or mistake, so as an intermediate step we should at least recover the Jackson result.

After using a more traditional phasor form (where one would have to take real parts) we revisit the geometric phasor found in (80). It will be found that the perpendicular constraints of the Jackson phasor solution lead to a representation where the geometric phasor is reduced to the Jackson form with a straight substitution of the imaginary i with the pseudoscalar $I = \sigma_1\sigma_2\sigma_3$. This representation however, like the more general geometric phasor requires no selection of real or imaginary parts to construct a "physical" solution.

82.2 WITH ASSUMED PHASOR FIELD

Maxwell's equations in absence of charge and current ((7.1) of Jackson) can be summarized by

$$0 = (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \tag{82.1}$$

The F above is a composite electric and magnetic field merged into a single multivector. In the spatial basic the electric field component $\mathbf{E}$ is a vector, and the magnetic component $I\mathbf{B}$ is a bivector (in the Dirac basis both are bivectors).

$$F = \mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon} \tag{82.2}$$

With an assumed phasor form

$$F = \mathcal{F} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = (\mathcal{E} + I\mathcal{B}/\sqrt{\mu\epsilon})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{82.3}$$

Although there are many geometric multivectors that square to -1, we do not assume here that the imaginary i has any specific geometric meaning, and in fact commutes with all multivectors. Because of this we have to take the real parts later when done.

Operating on F with Maxwell's equation we have

$$0 = (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F = i\left(\mathbf{k} - \sqrt{\mu\epsilon}\frac{\omega}{c}\right)F \tag{82.4}$$

Similarly, left multiplication of Maxwell's equation by the conjugate operator $\boldsymbol{\nabla} - \sqrt{\mu\epsilon}\partial_0$, we have the wave equation

$$0 = \left(\boldsymbol{\nabla}^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)F \tag{82.5}$$

and substitution of the assumed phasor solution gives us

$$0 = (\boldsymbol{\nabla}^2 - \mu\epsilon\partial_{00})F = -\left(\mathbf{k}^2 - \mu\epsilon\frac{\omega^2}{c^2}\right)F \tag{82.6}$$

This provides the relation between the magnitude of $\mathbf{k}$ and ω, namely

$$|\mathbf{k}| = \pm\sqrt{\mu\epsilon}\frac{\omega}{c} \tag{82.7}$$

Without any real loss of generality we can pick the positive root, so the result of the Maxwell equation operator on the phasor is

$$0 = (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F = i\sqrt{\mu\epsilon}\frac{\omega}{c}\left(\hat{\mathbf{k}} - 1\right)F \tag{82.8}$$

Rearranging we have the curious property that the field F can "swallow" a left multiplication by the propagation direction unit vector

$$\hat{\mathbf{k}}F = F \tag{82.9}$$

Selection of the scalar and pseudoscalar grades of this equation shows that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are both completely transverse to the propagation direction $\hat{\mathbf{k}}$. For the scalar grades we have

$$\begin{aligned} 0 &= \left\langle \hat{\mathbf{k}}F - F \right\rangle \\ &= \hat{\mathbf{k}} \cdot \mathbf{E} \end{aligned} \tag{82.10}$$

and for the pseudoscalar

$$\begin{aligned} 0 &= \left\langle \hat{\mathbf{k}}F - F \right\rangle_3 \\ &= I\hat{\mathbf{k}} \cdot \mathbf{B} \end{aligned} \tag{82.11}$$

From this we have $\hat{\mathbf{k}} \cdot \mathbf{B} = \hat{\mathbf{k}} \cdot \mathbf{B} = 0$. Because of this transverse property we see that the $\hat{\mathbf{k}}$ multiplication of F in eq. (82.9) serves to map electric field (vector) components into bivectors, and the magnetic bivector components into vectors. For the result to be the same means we must have an additional coupling between the field components. Writing out eq. (82.9) in terms of the field components we have

$$\begin{aligned} \mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon} &= \hat{\mathbf{k}}(\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}) \\ &= \hat{\mathbf{k}} \wedge \mathbf{E} + I(\hat{\mathbf{k}} \wedge \mathbf{B})/\sqrt{\mu\epsilon} \\ &= I\hat{\mathbf{k}} \times \mathbf{E} + I^2(\hat{\mathbf{k}} \times \mathbf{B})/\sqrt{\mu\epsilon} \end{aligned} \tag{82.12}$$

Equating left and right hand grades we have

$$\begin{aligned} \mathbf{E} &= -(\hat{\mathbf{k}} \times \mathbf{B})/\sqrt{\mu\epsilon} \\ \mathbf{B} &= \sqrt{\mu\epsilon}(\hat{\mathbf{k}} \times \mathbf{E}) \end{aligned} \tag{82.13}$$

Since $\mathbf{E}$ and $\mathbf{B}$ both have the same phase relationships we also have

$$\begin{aligned} \mathcal{E} &= -(\hat{\mathbf{k}} \times \mathcal{B})/\sqrt{\mu\epsilon} \\ \mathcal{B} &= \sqrt{\mu\epsilon}(\hat{\mathbf{k}} \times \mathcal{E}) \end{aligned} \tag{82.14}$$

With phasors as used in electrical engineering it is usual to allow the fields to have complex values. Assuming this is allowed here too, taking real parts of F, and separating by grade, we have for the electric and magnetic fields

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = \mathrm{Re}\begin{pmatrix}\mathcal{E}\\ \mathcal{B}\end{pmatrix}\cos(\mathbf{k}\cdot\mathbf{x}-\omega t) + \mathrm{Im}\begin{pmatrix}\mathcal{E}\\ \mathcal{B}\end{pmatrix}\sin(\mathbf{k}\cdot\mathbf{x}-\omega t) \tag{82.15}$$

We will find a slightly different separation into electric and magnetic fields with the geometric phasor.

82.3 GEOMETRIZED PHASOR

Translating from SI units to the CGS units of Jackson the geometric phasor representation of the field was found previously to be

$$F = e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}(\mathcal{E}+I\mathcal{B}/\sqrt{\mu\epsilon}) \tag{82.16}$$

As above the transverse requirement $\mathcal{E}\cdot\mathbf{k} = \mathcal{B}\cdot\mathbf{k} = 0$ was required. Application of Maxwell's equation operator should show if we require any additional constraints. That is

$$\begin{aligned}
0 &= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \\
&= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}(\mathcal{E}+I\mathcal{B}/\sqrt{\mu\epsilon}) \\
&= \sum \sigma_m e^{-I\hat{\mathbf{k}}\omega t}(Ik^m)e^{I\mathbf{k}\cdot\mathbf{x}}(\mathcal{E}+I\mathcal{B}/\sqrt{\mu\epsilon}) - I\hat{\mathbf{k}}\sqrt{\mu\epsilon}\frac{\omega}{c}e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}(\mathcal{E}+I\mathcal{B}/\sqrt{\mu\epsilon}) \\
&= I\left(\mathbf{k} - \hat{\mathbf{k}}\sqrt{\mu\epsilon}\frac{\omega}{c}\right)e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}(\mathcal{E}+I\mathcal{B}/\sqrt{\mu\epsilon})
\end{aligned} \tag{82.17}$$

This is zero for any combinations of $\mathcal{E}$ or $\mathcal{B}$ since $\mathbf{k} = \hat{\mathbf{k}}\sqrt{\mu\epsilon}\omega/c$. It therefore appears that this geometric phasor has a fundamentally different nature than the non-geometric version. We have two exponentials that commute, but due to the difference in grades of the arguments, it does not appear that there is any easy way to express this as an single argument exponential. Multiplying these out, and using the trig product to sum identities helps shed some light on the differences between the geometric phasor and the one using a generic imaginary. Starting off we have

$$\begin{aligned}
&e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}} \\
&= (\cos(\omega t) - I\hat{\mathbf{k}}\sin(\omega t))(\cos(\mathbf{k}\cdot\mathbf{x}) + I\sin(\mathbf{k}\cdot\mathbf{x})) \\
&= \cos(\omega t)\cos(\mathbf{k}\cdot\mathbf{x}) + \hat{\mathbf{k}}\sin(\omega t)\sin(\mathbf{k}\cdot\mathbf{x}) - I\hat{\mathbf{k}}\sin(\omega t)\cos(\mathbf{k}\cdot\mathbf{x}) + I\cos(\omega t)\sin(\mathbf{k}\cdot\mathbf{x})
\end{aligned} \tag{82.18}$$

In this first expansion we see that this product of exponentials has scalar, vector, bivector, and pseudoscalar grades, despite the fact that we have only vector and bivector terms in the end result. That will be seen to be due to the transverse nature of $\mathcal{F}$ that we multiply with. Before performing that final multiplication, writing $C_- = \cos(\omega t - \mathbf{k} \cdot \mathbf{x})$, $C_+ = \cos(\omega t + \mathbf{k} \cdot \mathbf{x})$, $S_- = \sin(\omega t - \mathbf{k} \cdot \mathbf{x})$, and $S_+ = \sin(\omega t + \mathbf{k} \cdot \mathbf{x})$, we have

$$e^{-I\hat{\mathbf{k}}\omega t} e^{I\mathbf{k}\cdot\mathbf{x}} = \frac{1}{2}\Big((C_- + C_+) + \hat{\mathbf{k}}(C_- - C_+) - I\hat{\mathbf{k}}(S_- + S_+) - I(S_- - S_+)\Big) \tag{82.19}$$

As an operator the left multiplication of $\hat{\mathbf{k}}$ on a transverse vector has the action

$$\begin{aligned} \hat{\mathbf{k}}(\cdot) &= \hat{\mathbf{k}} \wedge (\cdot) \\ &= I(\hat{\mathbf{k}} \times (\cdot)) \end{aligned} \tag{82.20}$$

This gives

$$e^{-I\hat{\mathbf{k}}\omega t} e^{I\mathbf{k}\cdot\mathbf{x}} = \frac{1}{2}\Big((C_- + C_+) + (C_- - C_+)I\hat{\mathbf{k}} \times +(S_- + S_+)\hat{\mathbf{k}} \times -I(S_- - S_+)\Big) \tag{82.21}$$

Now, lets apply this to the field with $\mathcal{F} = \mathcal{E} + I\mathcal{B}/\sqrt{\mu\epsilon}$. To avoid dragging around the $\sqrt{\mu\epsilon}$ factors, let us also temporarily work with units where $\mu\epsilon = 1$. We then have

$$\begin{aligned} 2e^{-I\hat{\mathbf{k}}\omega t} e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} &= (C_- + C_+)(\mathcal{E} + I\mathcal{B}) \\ &+ (C_- - C_+)(I(\hat{\mathbf{k}} \times \mathcal{E}) - \hat{\mathbf{k}} \times \mathcal{B}) \\ &+ (S_- + S_+)(\hat{\mathbf{k}} \times \mathcal{E} + I(\hat{\mathbf{k}} \times \mathcal{B})) \\ &+ (S_- - S_+)(-I\mathcal{E} + \mathcal{B}) \end{aligned} \tag{82.22}$$

Rearranging explicitly in terms of the electric and magnetic field components this is

$$\begin{aligned} 2e^{-I\hat{\mathbf{k}}\omega t} e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} &= (C_- + C_+)\mathcal{E} - (C_- - C_+)(\hat{\mathbf{k}} \times \mathcal{B}) + (S_- + S_+)(\hat{\mathbf{k}} \times \mathcal{E}) + (S_- - S_+)\mathcal{B} \\ &+ I\Big((C_- + C_+)\mathcal{B} + (C_- - C_+)(\hat{\mathbf{k}} \times \mathcal{E}) + (S_- + S_+)(\hat{\mathbf{k}} \times \mathcal{B}) - (S_- - S_+)\mathcal{E}\Big) \end{aligned} \tag{82.23}$$

Quite a mess! A first observation is that the application of the perpendicularity conditions eq. (82.14) we have a remarkable reduction in complexity. That is

$$\begin{aligned} 2e^{-I\hat{\mathbf{k}}\omega t} e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} &= (C_- + C_+)\mathcal{E} + (C_- - C_+)\mathcal{E} + (S_- + S_+)\mathcal{B} + (S_- - S_+)\mathcal{B} \\ &+ I\left((C_- + C_+)\mathcal{B} + (C_- - C_+)\mathcal{B} - (S_- + S_+)\mathcal{E} - (S_- - S_+)\mathcal{E}\right) \end{aligned} \tag{82.24}$$

This wipes out the receding wave terms leaving only the advanced wave terms, leaving

$$
\begin{aligned}
e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} &= C_-\mathcal{E} + S_-(\hat{\mathbf{k}}\times\mathcal{E}) + I\left(C_-\mathcal{B} + S_-\hat{\mathbf{k}}\times\mathcal{B}\right) \\
&= C_-(\mathcal{E} + I\mathcal{B}) + S_-(\mathcal{B} - I\mathcal{E}) \\
&= (C_- - IS_-)(\mathcal{E} + I\mathcal{B})
\end{aligned}
\tag{82.25}
$$

We see therefore for this special case of mutually perpendicular (equ-magnitude) field components, our geometric phasor has only the advanced wave term

$$
e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} = e^{-I(\omega t - \mathbf{k}\cdot\mathbf{x})}\mathcal{F}
\tag{82.26}
$$

If we pick this as the starting point for the assumed solution, it is clear that the same perpendicularity constraints will follow as in Jackson's treatment, or the GA version of it above. We have something that is slightly different though, for we have no requirement to take real parts of this simplified geometric phasor, since the result already contains just the vector and bivector terms of the electric and magnetic fields respectively.

A small aside, before continuing. Having made this observation that we can write the assumed phasor for this transverse field in the form of eq. (82.26) an easier way to demonstrate that the product of exponentials reduces only to the advanced wave term is now clear. Instead of using eq. (82.14) we could start back at eq. (82.19) and employ the absorption property $\hat{\mathbf{k}}\mathcal{F} = \mathcal{F}$. That gives

$$
\begin{aligned}
e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} &= \frac{1}{2}\left((C_- + C_+) + (C_- - C_+) - I(S_- + S_+) - I(S_- - S_+)\right)\mathcal{F} \\
&= \left(C_- - IS_-\right)\mathcal{F}
\end{aligned}
\tag{82.27}
$$

That is the same result, obtained in a slicker manner.

82.4 EXPLICIT SPLIT OF GEOMETRIC PHASOR INTO ADVANCED AND RECEDING PARTS

For a more general split of the geometric phasor into advanced and receding wave terms, will there be interdependence between the electric and magnetic field components? Going back to eq. (82.19), and rearranging, we have

$$
2e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}} = (C_- - IS_-) + \hat{\mathbf{k}}(C_- - IS_-) + (C_+ + IS_+) - \hat{\mathbf{k}}(C_+ + IS_+)
\tag{82.28}
$$

So we have

$$
e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}} = \frac{1}{2}(1 + \hat{\mathbf{k}})e^{-I(\omega t - \mathbf{k}\cdot\mathbf{x})} + \frac{1}{2}(1 - \hat{\mathbf{k}})e^{I(\omega t + \mathbf{k}\cdot\mathbf{x})}
\tag{82.29}
$$

As observed if we have $\hat{\mathbf{k}}\mathcal{F} = \mathcal{F}$, the result is only the advanced wave term

$$e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} = e^{-I(\omega t-\mathbf{k}\cdot\mathbf{x})}\mathcal{F} \tag{82.30}$$

Similarly, with absorption of $\hat{\mathbf{k}}$ with the opposing sign $\hat{\mathbf{k}}\mathcal{F} = -\mathcal{F}$, we have only the receding wave

$$e^{-I\hat{\mathbf{k}}\omega t}e^{I\mathbf{k}\cdot\mathbf{x}}\mathcal{F} = e^{I(\omega t+\mathbf{k}\cdot\mathbf{x})}\mathcal{F} \tag{82.31}$$

Either of the receding or advancing wave solutions should independently satisfy the Maxwell equation operator. Let us verify both of these, and verify that for either the $\pm$ cases the following is a solution and examine the constraints for that to be the case.

$$F = \frac{1}{2}(1 \pm \hat{\mathbf{k}})e^{\pm I(\omega t\pm\mathbf{k}\cdot\mathbf{x})}\mathcal{F} \tag{82.32}$$

Now we wish to apply the Maxwell equation operator $\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0$ to this assumed solution. That is

$$\begin{aligned}
0 &= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \\
&= \sigma_m\frac{1}{2}(1 \pm \hat{\mathbf{k}})(\pm I \pm k^m)e^{\pm I(\omega t\pm\mathbf{k}\cdot\mathbf{x})}\mathcal{F} + \frac{1}{2}(1 \pm \hat{\mathbf{k}})(\pm I\sqrt{\mu\epsilon}\omega/c)e^{\pm I(\omega t\pm\mathbf{k}\cdot\mathbf{x})}\mathcal{F} \\
&= \frac{\pm I}{2}\left(\pm\hat{\mathbf{k}} + \sqrt{\mu\epsilon}\frac{\omega}{c}\right)(1 \pm \hat{\mathbf{k}})e^{\pm I(\omega t\pm\mathbf{k}\cdot\mathbf{x})}\mathcal{F}
\end{aligned} \tag{82.33}$$

By left multiplication with the conjugate of the Maxwell operator $\nabla - \sqrt{\mu\epsilon}\partial_0$ we have the wave equation operator, and applying that, we have as before, a magnitude constraint on the wave number $\mathbf{k}$

$$\begin{aligned}
0 &= (\boldsymbol{\nabla} - \sqrt{\mu\epsilon}\partial_0)(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \\
&= (\boldsymbol{\nabla}^2 - \mu\epsilon\partial_{00})F \\
&= \frac{-1}{2}(1 \pm \hat{\mathbf{k}})\left(\mathbf{k}^2 - \mu\epsilon\frac{\omega^2}{c^2}\right)e^{\pm I(\omega t\pm\mathbf{k}\cdot\mathbf{x})}\mathcal{F}
\end{aligned} \tag{82.34}$$

So we have as before $|\mathbf{k}| = \sqrt{\mu\epsilon}\omega/c$. Substitution into the the first order operator result we have

$$\begin{aligned}
0 &= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \\
&= \frac{\pm I}{2}\sqrt{\mu\epsilon}\frac{\omega}{c}\left(\pm\hat{\mathbf{k}} + 1\right)(1 \pm \hat{\mathbf{k}})e^{\pm I(\omega t\pm\mathbf{k}\cdot\mathbf{x})}\mathcal{F}
\end{aligned} \tag{82.35}$$

Observe that the multivector $1 \pm \hat{\mathbf{k}}$, when squared is just a multiple of itself

$$(1 \pm \hat{\mathbf{k}})^2 = 1 + \hat{\mathbf{k}}^2 \pm 2\hat{\mathbf{k}} = 2(1 \pm \hat{\mathbf{k}}) \tag{82.36}$$

So we have

$$\begin{aligned} 0 &= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \\ &= \pm I \sqrt{\mu\epsilon}\frac{\omega}{c}(1 \pm \hat{\mathbf{k}})e^{\pm I(\omega t \pm \mathbf{k}\cdot\mathbf{x})}\mathcal{F} \end{aligned} \tag{82.37}$$

So we see that the constraint again on the individual assumed solutions is again that of absorption. Separately the advanced or receding parts of the geometric phasor as expressed in eq. (82.32) are solutions provided

$$\hat{\mathbf{k}}F = \mp F \tag{82.38}$$

The geometric phasor is seen to be curious superposition of both advancing and receding states. Independently we have something pretty much like the standard transverse phasor wave states. Is this superposition state physically meaningful. It is a solution to the Maxwell equation (without any constraints on $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$).

83

COVARIANT MAXWELL EQUATION IN MEDIA

83.1 MOTIVATION, SOME NOTATION, AND REVIEW

Adjusting to Jackson's of CGS [22] and Maxwell's equations in matter takes some work. A first pass at a GA form was assembled in (66), based on what was in the introduction chapter for media that includes **P**, and **M** properties. He later changes conventions, and also assumes linear media in most cases, so we want something different than what was previously derived.

The non-covariant form of Maxwell's equation in absence of current and charge has been convenient to use in some initial attempts to look at wave propagation. That was

$$\begin{aligned} F &= \mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon} \\ 0 &= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \end{aligned} \tag{83.1}$$

To examine the energy momentum tensor, it is desirable to express this in a fashion that has no such explicit spacetime dependence. This suggests a spacetime gradient definition that varies throughout the media.

$$\nabla \equiv \gamma^m \partial_m + \sqrt{\mu\epsilon}\gamma^0 \partial_0 \tag{83.2}$$

Observe that this spacetime gradient is adjusted by the speed of light in the media, and is not one that is naturally relativistic. Even though the differential form of Maxwell's equation is implicitly defined only in a neighborhood of the point it is evaluated at, we now have a reason to say this explicitly, because this non-isotropic condition is now hiding in the (perhaps poor) notation for the operator. Ignoring the obscuring nature of this operator, and working with it, we can can that Maxwell's equation in the neighborhood (where $\mu\epsilon$ is "fixed") is

$$\nabla F = 0 \tag{83.3}$$

We also want a variant of this that includes the charge and current terms.

83.2 LINEAR MEDIA

Lets pick Jackson's equation (6.70) as the starting point. A partial translation to GA form, with $\mathbf{D} = \epsilon\mathbf{E}$, and $\mathbf{B} = \mu\mathbf{H}$, and $\partial_0 = \partial/\partial ct$ is

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \cdot \epsilon\mathbf{E} &= 4\pi\rho \\
-I\boldsymbol{\nabla} \wedge \mathbf{E} + \partial_0\mathbf{B} &= 0 \\
-I\boldsymbol{\nabla} \wedge \mathbf{B}/\mu - \partial_0\epsilon\mathbf{E} &= \frac{4\pi}{c}\mathbf{J}
\end{aligned} \tag{83.4}$$

Scaling and adding we have

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{E} + \partial_0 I\mathbf{B} &= \frac{4\pi\rho}{\epsilon} \\
\boldsymbol{\nabla}\mathbf{B} - I\partial_0\mu\epsilon\mathbf{E} &= \frac{4\pi\mu I}{c}\mathbf{J}
\end{aligned} \tag{83.5}$$

Once last scaling prepares for addition of these last two equations

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{E} + \sqrt{\mu\epsilon}\partial_0 I\mathbf{B}/\sqrt{\mu\epsilon} &= \frac{4\pi\rho}{\epsilon} \\
\boldsymbol{\nabla} I\mathbf{B}/\sqrt{\mu\epsilon} + \partial_0\sqrt{\mu\epsilon}\mathbf{E} &= -\frac{4\pi\mu}{c\sqrt{\mu\epsilon}}\mathbf{J}
\end{aligned} \tag{83.6}$$

This gives us a non-covariant assembly of Maxwell's equations in linear media

$$(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F = \frac{4\pi}{c}\left(\frac{c\rho}{\epsilon} - \sqrt{\frac{\mu}{\epsilon}}\mathbf{J}\right) \tag{83.7}$$

Premultiplication by γ_0, and utilizing the definition of eq. (83.2) we have

$$\nabla F = \frac{4\pi}{c}\left(c\frac{\rho}{\epsilon}\gamma_0 + \sqrt{\frac{\mu}{\epsilon}}J^m\gamma_m\right) \tag{83.8}$$

We can then define

$$J \equiv \frac{c\rho}{\epsilon}\gamma_0 + \sqrt{\frac{\mu}{\epsilon}}J^m\gamma_m \tag{83.9}$$

and are left with an expression of Maxwell's equation that puts space and time on a similar footing. It is probably not really right to call this a covariant expression since it is not naturally relativistic.

$$\nabla F = \frac{4\pi}{c} J \tag{83.10}$$

83.3 ENERGY MOMENTUM TENSOR

My main goal was to find the GA form of the stress energy tensor in media. With the requirement for both an alternate spacetime gradient and the inclusion of the scaling factors for the media it is not obviously clear to me how to do translate from the vacuum expression in SI units to the CGS in media form. It makes sense to step back to see how the divergence conservation equation translates with both of these changes. In SI units our tensor (a four vector parametrized by another direction vector a) was

$$T(a) \equiv \frac{-1}{2\epsilon_0} F a F \tag{83.11}$$

Ignoring units temporarily, let us calculate the media-spacetime divergence of $-FaF/2$. That is

$$\begin{aligned}
-\frac{1}{2}\nabla \cdot (FaF) &= -\frac{1}{2}\langle \nabla (FaF) \rangle \\
&= -\frac{1}{2}\left\langle (F(\overrightarrow{\nabla} F) + (F\overleftarrow{\nabla})F) a \right\rangle \\
&= -\frac{4\pi}{c}\left\langle \frac{1}{2}(FJ - JF) a \right\rangle \\
&= -\frac{4\pi}{c}(F \cdot J) \cdot a
\end{aligned} \tag{83.12}$$

We want the $T^{\mu 0}$ components of the tensor $T(\gamma_0)$. Noting the anticommutation relation for the pseudoscalar $I\gamma_0 = -\gamma_0 I$, and the anticommutation behavior for spatial vectors such as $\mathbf{E}\gamma_0 = -\gamma_0$ we have

$$
\begin{aligned}
-\frac{1}{2}(\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon})\gamma_0(\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}) &= \frac{\gamma_0}{2}(\mathbf{E} - I\mathbf{B}/\sqrt{\mu\epsilon})(\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}) \\
&= \frac{\gamma_0}{2}\left((\mathbf{E}^2 + \mathbf{B}^2/\mu\epsilon) + I\frac{1}{\sqrt{\mu\epsilon}}(\mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E})\right) \\
&= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2/\mu\epsilon) + \gamma_0 I\frac{1}{\sqrt{\mu\epsilon}}(\mathbf{E}\wedge\mathbf{B}) \\
&= \frac{\gamma_0}{2}(\mathbf{E}^2 + \mathbf{B}^2/\mu\epsilon) - \gamma_0\frac{1}{\sqrt{\mu\epsilon}}(\mathbf{E}\times\mathbf{B}) \\
&= \frac{\gamma_0}{2}(\mathbf{E}^2 + \mathbf{B}^2/\mu\epsilon) - \gamma_0\frac{1}{\sqrt{\mu\epsilon}}\gamma_m\gamma_0(\mathbf{E}\times\mathbf{B})^m \\
&= \frac{\gamma_0}{2}(\mathbf{E}^2 + \mathbf{B}^2/\mu\epsilon) + \frac{1}{\sqrt{\mu\epsilon}}\gamma_m(\mathbf{E}\times\mathbf{B})^m
\end{aligned}
\tag{83.13}
$$

Calculating the divergence of this using the media spacetime gradient we have

$$
\begin{aligned}
\nabla\cdot\left(-\frac{1}{2}F\gamma_0 F\right) &= \frac{\sqrt{\mu\epsilon}}{c}\frac{\partial}{\partial t}\frac{1}{2}\left(\mathbf{E}^2 + \frac{1}{\mu\epsilon}\mathbf{B}^2\right) + \sum_m\frac{\partial}{\partial x^m}\left(\frac{1}{\sqrt{\mu\epsilon}}(\mathbf{E}\times\mathbf{B})^m\right) \\
&= \frac{\sqrt{\mu\epsilon}}{c}\frac{\partial}{\partial t}\frac{1}{2}\left(\mathbf{E}^2 + \frac{1}{\mu\epsilon}\mathbf{B}^2\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\sqrt{\mu\epsilon}}(\mathbf{E}\times\mathbf{B})^m\right)
\end{aligned}
\tag{83.14}
$$

Multiplying this by $(c/4\pi)\sqrt{\epsilon/\mu}$, we have

$$
\begin{aligned}
\nabla\cdot\left(-\frac{c}{8\pi}\sqrt{\frac{\epsilon}{\mu}}F\gamma_0 F\right) &= \frac{\partial}{\partial t}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) + \boldsymbol{\nabla}\cdot\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) \\
&= -\sqrt{\frac{\epsilon}{\mu}}(F\cdot J)\cdot\gamma_0
\end{aligned}
\tag{83.15}
$$

Now expand the RHS. We have

$$
\begin{aligned}
\sqrt{\frac{\epsilon}{\mu}}(F\cdot J)\cdot\gamma_0 &= \left((\mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon})\cdot\left(\frac{\rho}{\sqrt{\mu\epsilon}}\gamma_0 + J^m\gamma_m\right)\right)\cdot\gamma_0 \\
&= \left\langle E^q\gamma_q\gamma_0 J^m\gamma_m\gamma_0\right\rangle \\
&= \mathbf{E}\cdot\mathbf{J}
\end{aligned}
\tag{83.16}
$$

Assembling results the energy conservation relation, first in covariant form is

$$\nabla \cdot \left(-\frac{c}{8\pi} \sqrt{\frac{\epsilon}{\mu}} F a F \right) = -\sqrt{\frac{\epsilon}{\mu}} (F \cdot J) \cdot a \tag{83.17}$$

and the same with an explicit spacetime split in vector quantities is

$$\frac{\partial}{\partial t} \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) + \boldsymbol{\nabla} \cdot \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}) = -\mathbf{E} \cdot \mathbf{J} \tag{83.18}$$

The first of these two eq. (83.17) is what I was after for application to optics where the radiation field in media can be expressed directly in terms of F instead of **E** and **B**. The second sets the dimensions appropriately and provides some confidence in the result since we can compare to the well known Poynting results in these units.

84

ELECTROMAGNETIC GAUGE INVARIANCE

At the end of section 12.1 in Jackson [22] he states that it is obvious that the Lorentz force equations are gauge invariant.

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= e\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) \\ \frac{dE}{dt} &= e\mathbf{u} \cdot \mathbf{E}\end{aligned} \tag{84.1}$$

Since I did not remember what Gauge invariance was, it was not so obvious. But if I looking ahead to one of the problem 12.2 on this invariance we have a Gauge transformation defined in four vector form as

$$A^\alpha \rightarrow A^\alpha + \partial^\alpha \psi \tag{84.2}$$

In vector form with $A = \gamma_\alpha A^\alpha$, this gauge transformation can be written

$$A \rightarrow A + \nabla \psi \tag{84.3}$$

so this is really a statement that we add a spacetime gradient of something to the four vector potential. Given this, how does the field transform?

$$\begin{aligned}F &= \nabla \wedge A \\ &\rightarrow \nabla \wedge (A + \nabla \psi) \\ &= F + \nabla \wedge \nabla \psi\end{aligned} \tag{84.4}$$

But $\nabla \wedge \nabla \psi = 0$ (assuming partials are interchangeable) so the field is invariant regardless of whether we are talking about the Lorentz force

$$\nabla F = J/\epsilon_0 c \tag{84.5}$$

or the field equations themselves

$$\frac{dp}{d\tau} = eF \cdot v/c \tag{84.6}$$

So, once you know the definition of the gauge transformation in four vector form, yes this justifiably obvious, however, to anybody who is not familiar with Geometric Algebra, perhaps this is still not so obvious. How does this translate to the more common place tensor or space time vector notations? The tensor four vector translation is the easier of the two, and there we have

$$\begin{aligned}
F^{\alpha\beta} &= \partial^\alpha A^\beta - \partial^\beta A^\alpha \\
&\rightarrow \partial^\alpha (A^\beta + \partial^\beta \psi) - \partial^\beta (A^\alpha + \partial^\alpha \psi) \\
&= F^{\alpha\beta} + \partial^\alpha \partial^\beta \psi - \partial^\beta \partial^\alpha \psi
\end{aligned} \tag{84.7}$$

As required for $\nabla \wedge \nabla\psi = 0$ interchange of partials means the field components $F^{\alpha\beta}$ are unchanged by adding this gradient. Finally, in plain old spatial vector form, how is this gauge invariance expressed?

In components we have

$$\begin{aligned}
A^0 &\rightarrow A^0 + \partial^0 \psi = \phi + \frac{1}{c}\frac{\partial \psi}{\partial t} \\
A^k &\rightarrow A^k + \partial^k \psi = A^k - \frac{\partial \psi}{\partial x^k}
\end{aligned} \tag{84.8}$$

This last in vector form is $\mathbf{A} \rightarrow \mathbf{A} - \boldsymbol{\nabla}\psi$, where the sign inversion comes from $\partial^k = -\partial_k = -\partial/\partial x^k$, assuming a $+---$ metric.

We want to apply this to the electric and magnetic field components

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}
\end{aligned} \tag{84.9}$$

The electric field transforms as

$$\begin{aligned}
\mathbf{E} &\rightarrow -\boldsymbol{\nabla}\left(\phi + \frac{1}{c}\frac{\partial \psi}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{A} - \boldsymbol{\nabla}\psi\right) \\
&= \mathbf{E} - \frac{1}{c}\boldsymbol{\nabla}\frac{\partial \psi}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{\nabla}\psi
\end{aligned} \tag{84.10}$$

With partial interchange this is just $\mathbf{E}$. For the magnetic field we have

$$\begin{aligned}
\mathbf{B} &\rightarrow \boldsymbol{\nabla} \times \left(\mathbf{A} - \boldsymbol{\nabla}\psi\right) \\
&= \mathbf{B} - \boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi
\end{aligned} \tag{84.11}$$

Again since the partials interchange we have $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$, so this is just the magnetic field. Alright. Worked this in three different ways, so now I can say its obvious.

85

MULTIVECTOR COMMUTATORS AND LORENTZ BOOSTS

85.1 MOTIVATION

In some reading there I found that the electrodynamic field components transform in a reversed sense to that of vectors, where instead of the perpendicular to the boost direction remaining unaffected, those are the parts that are altered.

To explore this, look at the Lorentz boost action on a multivector, utilizing symmetric and antisymmetric products to split that vector into portions effected and unaffected by the boost. For the bivector (electrodynamic case) and the four vector case, examine how these map to dot and wedge (or cross) products.

The underlying motivator for this boost consideration is an attempt to see where equation (6.70) of [9] comes from. We get to this by the very end.

85.2 GUTS

85.2.1 *Structure of the bivector boost*

Recall that we can write our Lorentz boost in exponential form with

$$
\begin{aligned}
L &= e^{\alpha \boldsymbol{\sigma}/2} \\
X' &= L^\dagger X L,
\end{aligned}
\tag{85.1}
$$

where $\boldsymbol{\sigma}$ is a spatial vector. This works for our bivector field too, assuming the composite transformation is an outermorphism of the transformed four vectors. Applying the boost to both the gradient and the potential our transformed field is then

$$
\begin{aligned}
F' &= \nabla' \wedge A' \\
&= (L^\dagger \nabla L) \wedge (L^\dagger A L) \\
&= \frac{1}{2}\left((L^\dagger \overrightarrow{\nabla} L)(L^\dagger A L) - (L^\dagger A L)(L^\dagger \overleftarrow{\nabla} L) \right) \\
&= \frac{1}{2} L^\dagger \left(\overrightarrow{\nabla} A - A \overleftarrow{\nabla} \right) L \\
&= L^\dagger (\nabla \wedge A) L.
\end{aligned}
\tag{85.2}
$$

Note that arrows were used briefly to indicate that the partials of the gradient are still acting on A despite their vector components being to one side. We are left with the very simple transformation rule

$$F' = L^\dagger F L, \tag{85.3}$$

which has exactly the same structure as the four vector boost.

85.2.2 *Employing the commutator and anticommutator to find the parallel and perpendicular components*

If we apply the boost to a four vector, those components of the four vector that commute with the spatial direction $\boldsymbol{\sigma}$ are unaffected. As an example, which also serves to ensure we have the sign of the rapidity angle α correct, consider $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1$. We have

$$X' = e^{-\alpha\sigma/2}(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3)(\cosh\alpha/2 + \gamma_1\gamma_0\sinh\alpha/2) \tag{85.4}$$

We observe that the scalar and $\boldsymbol{\sigma}_1 = \gamma_1\gamma_0$ components of the exponential commute with γ_2 and γ_3 since there is no vector in common, but that $\boldsymbol{\sigma}_1$ anticommutes with γ_0 and γ_1. We can therefore write

$$\begin{aligned} X' &= x^2\gamma_2 + x^3\gamma_3 + (x^0\gamma_0 + x^1\gamma_1 +)(\cosh\alpha + \gamma_1\gamma_0\sinh\alpha) \\ &= x^2\gamma_2 + x^3\gamma_3 + \gamma_0(x^0\cosh\alpha - x^1\sinh\alpha) + \gamma_1(x^1\cosh\alpha - x^0\sinh\alpha) \end{aligned} \tag{85.5}$$

reproducing the familiar matrix result should we choose to write it out. How can we express the commutation property without resorting to components. We could write the four vector as a spatial and timelike component, as in

$$X = x^0\gamma_0 + \mathbf{x}\gamma_0, \tag{85.6}$$

and further separate that into components parallel and perpendicular to the spatial unit vector $\boldsymbol{\sigma}$ as

$$X = x^0\gamma_0 + (\mathbf{x}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\gamma_0 + (\mathbf{x}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma}\gamma_0. \tag{85.7}$$

However, it would be nicer to group the first two terms together, since they are ones that are affected by the transformation. It would also be nice to not have to resort to spatial dot and wedge products, since we get into trouble too easily if we try to mix dot and wedge products of four vector and spatial vector components.

What we can do is employ symmetric and antisymmetric products (the anticommutator and commutator respectively). Recall that we can write any multivector product this way, and in particular

$$M\boldsymbol{\sigma} = \frac{1}{2}(M\boldsymbol{\sigma} + \boldsymbol{\sigma}M) + \frac{1}{2}(M\boldsymbol{\sigma} - \boldsymbol{\sigma}M). \tag{85.8}$$

Left multiplying by the unit spatial vector $\boldsymbol{\sigma}$ we have

$$M = \frac{1}{2}(M + \boldsymbol{\sigma}M\boldsymbol{\sigma}) + \frac{1}{2}(M - \boldsymbol{\sigma}M\boldsymbol{\sigma}) = \frac{1}{2}\{M, \boldsymbol{\sigma}\}\boldsymbol{\sigma} + \frac{1}{2}\left[M, \boldsymbol{\sigma}\right]\boldsymbol{\sigma}. \tag{85.9}$$

When $M = \mathbf{a}$ is a spatial vector this is our familiar split into parallel and perpendicular components with the respective projection and rejection operators

$$\mathbf{a} = \frac{1}{2}\{\mathbf{a}, \boldsymbol{\sigma}\}\boldsymbol{\sigma} + \frac{1}{2}\left[\mathbf{a}, \boldsymbol{\sigma}\right]\boldsymbol{\sigma} = (\mathbf{a}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + (\mathbf{a}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma}. \tag{85.10}$$

However, the more general split employing symmetric and antisymmetric products in eq. (85.9), is something we can use for our four vector and bivector objects too.

Observe that we have the commutation and anti-commutation relationships

$$\begin{aligned}\left(\frac{1}{2}\{M, \boldsymbol{\sigma}\}\boldsymbol{\sigma}\right)\boldsymbol{\sigma} &= \boldsymbol{\sigma}\left(\frac{1}{2}\{M, \boldsymbol{\sigma}\}\boldsymbol{\sigma}\right) \\ \left(\frac{1}{2}\left[M, \boldsymbol{\sigma}\right]\boldsymbol{\sigma}\right)\boldsymbol{\sigma} &= -\boldsymbol{\sigma}\left(\frac{1}{2}\left[M, \boldsymbol{\sigma}\right]\boldsymbol{\sigma}\right).\end{aligned} \tag{85.11}$$

This split therefore serves to separate the multivector object in question nicely into the portions that are acted on by the Lorentz boost, or left unaffected.

85.2.3 *Application of the symmetric and antisymmetric split to the bivector field*

Let us apply eq. (85.9) to the spacetime event X again with an x-axis boost $\sigma = \sigma_1$. The anticommutator portion of X in this boost direction is

$$\begin{aligned}\frac{1}{2}\{X, \boldsymbol{\sigma}_1\}\boldsymbol{\sigma}_1 &= \frac{1}{2}\left(\left(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3\right) + \gamma_1\gamma_0\left(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3\right)\gamma_1\gamma_0\right) \\ &= x^2\gamma_2 + x^3\gamma_3,\end{aligned}$$

$$(85.12)$$

whereas the commutator portion gives us

$$\begin{aligned}\frac{1}{2}[X, \boldsymbol{\sigma}_1]\boldsymbol{\sigma}_1 &= \frac{1}{2}\left(\left(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3\right) - \gamma_1\gamma_0\left(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3\right)\gamma_1\gamma_0\right) \\ &= x^0\gamma_0 + x^1\gamma_1.\end{aligned} \tag{85.13}$$

We have seen that only these commutator portions are acted on by the boost. We have therefore found the desired logical grouping of the four vector X into portions that are left unchanged by the boost and those that are affected. That is

$$\begin{aligned}\frac{1}{2}[X, \boldsymbol{\sigma}]\boldsymbol{\sigma} &= x^0\gamma_0 + (\mathbf{x}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\gamma_0 \\ \frac{1}{2}\{X, \boldsymbol{\sigma}\}\boldsymbol{\sigma} &= (\mathbf{x}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma}\gamma_0\end{aligned} \tag{85.14}$$

Let us now return to the bivector field $F = \nabla \wedge A = \mathbf{E} + Ic\mathbf{B}$, and split that multivector into boostable and unboostable portions with the commutator and anticommutator respectively.

Observing that our pseudoscalar I commutes with all spatial vectors we have for the anticommutator parts that will not be affected by the boost

$$\frac{1}{2}\{\mathbf{E} + Ic\mathbf{B}, \boldsymbol{\sigma}\}\boldsymbol{\sigma} = (\mathbf{E}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + Ic(\mathbf{B}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}, \tag{85.15}$$

and for the components that will be boosted we have

$$\frac{1}{2}[\mathbf{E} + Ic\mathbf{B}, \boldsymbol{\sigma}]\boldsymbol{\sigma} = (\mathbf{E}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma} + Ic(\mathbf{B}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma}. \tag{85.16}$$

For the four vector case we saw that the components that lay "perpendicular" to the boost direction, were unaffected by the boost. For the field we see the opposite, and the components of the individual electric and magnetic fields that are parallel to the boost direction are unaffected.

Our boosted field is therefore

$$F' = (\mathbf{E}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + Ic(\mathbf{B}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \left((\mathbf{E}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma} + Ic(\mathbf{B}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma}\right)\left(\cosh\alpha + \boldsymbol{\sigma}\sinh\alpha\right) \tag{85.17}$$

Focusing on just the non-parallel terms we have

$$\begin{aligned}&\left((\mathbf{E}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma} + Ic(\mathbf{B}\wedge\boldsymbol{\sigma})\boldsymbol{\sigma}\right)\left(\cosh\alpha + \boldsymbol{\sigma}\sinh\alpha\right) \\ &= (\mathbf{E}_\perp + Ic\mathbf{B}_\perp)\cosh\alpha + (I\mathbf{E}\times\boldsymbol{\sigma} - c\mathbf{B}\times\boldsymbol{\sigma})\sinh\alpha \\ &= \mathbf{E}_\perp\cosh\alpha - c(\mathbf{B}\times\boldsymbol{\sigma})\sinh\alpha + I(c\mathbf{B}_\perp\cosh\alpha + (\mathbf{E}\times\boldsymbol{\sigma})\sinh\alpha) \\ &= \gamma\left(\mathbf{E}_\perp - c(\mathbf{B}\times\boldsymbol{\sigma})|\mathbf{v}|/c + I(c\mathbf{B}_\perp + (\mathbf{E}\times\boldsymbol{\sigma})|\mathbf{v}|/c)\right)\end{aligned} \tag{85.18}$$

A final regrouping gives us

$$F' = \mathbf{E}_{\parallel} + \gamma \left(\mathbf{E}_{\perp} - \mathbf{B} \times \mathbf{v}\right) + Ic \left(\mathbf{B}_{\parallel} + \gamma \left(\mathbf{B}_{\perp} + \mathbf{E} \times \mathbf{v}/c^2\right)\right) \tag{85.19}$$

In particular when we consider the proton, electron system as in equation (6.70) of [9] where it is stated that the electron will feel a magnetic field given by

$$\mathbf{B} = -\frac{\mathbf{v}}{c} \times \mathbf{E} \tag{85.20}$$

we can see where this comes from. If $F = \mathbf{E} + Ic(0)$ is the field acting on the electron, then application of a $\mathbf{v}$ boost to the electron perpendicular to the field (ie: radial motion), we get

$$F' = \gamma \mathbf{E} + Ic\gamma \mathbf{E} \times \mathbf{v}/c^2 = \gamma \mathbf{E} + -Ic\gamma \frac{\mathbf{v}}{c^2} \times \mathbf{E} \tag{85.21}$$

We also have an additional $1/c$ factor in our result, but that is a consequence of the choice of units where the dimensions of $\mathbf{E}$ match $c\mathbf{B}$, whereas in the text we have $\mathbf{E}$ and $\mathbf{B}$ in the same units. We also have an additional γ factor, so we must presume that $|\mathbf{v}| << c$ in this portion of the text. That is actually a requirement here, for if the electron was already in motion, we would have to boost a field that also included a magnetic component. A consequence of this is that the final interaction Hamiltonian of (6.75) is necessarily non-relativistic.

86

A CYLINDRICAL LIENARD-WIECHERT POTENTIAL CALCULATION USING MULTIVECTOR MATRIX PRODUCTS

86.1 MOTIVATION

A while ago I worked the problem of determining the equations of motion for a chain like object [26]. This was idealized as a set of N interconnected spherical pendulums. One of the aspects of that problem that I found fun was that it allowed me to use a new construct, factoring vectors into multivector matrix products, multiplied using the Geometric (Clifford) product. It seemed at the time that this made the problem tractable, whereas a traditional formulation was much less so. Later I realized that a very similar factorization was possible with matrices directly [27]. This was a bit disappointing since I was enamored by my new calculation tool, and realized that the problem could be tackled with much less learning cost if the same factorization technique was applied using plain old matrices.

I have now encountered a new use for this idea of factoring a vector into a product of multivector matrices. Namely, a calculation of the four vector Lienard-Wiechert potentials, given a general motion described in cylindrical coordinates. This I thought I had try since we had a similar problem on our exam (with the motion of the charged particle additionally constrained to a circle).

86.2 THE GOAL OF THE CALCULATION

Our problem is to calculate

$$\begin{aligned} A^0 &= \frac{q}{R^*} \\ \mathbf{A} &= \frac{q\mathbf{v}_c}{cR^*} \end{aligned} \tag{86.1}$$

where $\mathbf{x}_c(t)$ is the location of the charged particle, $\mathbf{r}$ is the point that the field is measured, and

$$
\begin{aligned}
R^* &= R - \frac{\mathbf{v}_c}{c} \cdot \mathbf{R} \\
R^2 &= \mathbf{R}^2 = c^2(t - t_r)^2 \\
\mathbf{R} &= \mathbf{r} - \mathbf{x}_c(t_r) \\
\mathbf{v}_c &= \frac{\partial \mathbf{x}_c}{\partial t_r}.
\end{aligned}
\tag{86.2}
$$

86.3 CALCULATING THE POTENTIALS FOR AN ARBITRARY CYLINDRICAL MOTION

Suppose that our charged particle has the trajectory

$$
\mathbf{x}_c(t) = h(t)\mathbf{e}_3 + a(t)\mathbf{e}_1 e^{i\theta(t)}
\tag{86.3}
$$

where $i = \mathbf{e}_1\mathbf{e}_2$, and we measure the field at the point

$$
\mathbf{r} = z\mathbf{e}_3 + \rho\mathbf{e}_1 e^{i\phi}
\tag{86.4}
$$

The vector separation between the two is

$$
\begin{aligned}
\mathbf{R} &= \mathbf{r} - \mathbf{x}_c \\
&= (z - h)\mathbf{e}_3 + \mathbf{e}_1(\rho e^{i\phi} - a e^{i\theta}) \\
&= \begin{bmatrix} \mathbf{e}_1 e^{i\phi} & -\mathbf{e}_1 e^{i\theta} & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z - h \end{bmatrix}
\end{aligned}
\tag{86.5}
$$

Transposition does not change this at all, so the (squared) length of this vector difference is

$$
\begin{aligned}
\mathbf{R}^2 &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 e^{i\phi} \\ -\mathbf{e}_1 e^{i\theta} \\ \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 e^{i\phi} & -\mathbf{e}_1 e^{i\theta} & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 e^{i\phi} \mathbf{e}_1 e^{i\phi} & -\mathbf{e}_1 e^{i\phi} \mathbf{e}_1 e^{i\theta} & \mathbf{e}_1 e^{i\phi} \mathbf{e}_3 \\ -\mathbf{e}_1 e^{i\theta} \mathbf{e}_1 e^{i\phi} & \mathbf{e}_1 e^{i\theta} \mathbf{e}_1 e^{i\theta} & -\mathbf{e}_1 e^{i\theta} \mathbf{e}_3 \\ \mathbf{e}_3 \mathbf{e}_1 e^{i\phi} & -\mathbf{e}_3 \mathbf{e}_1 e^{i\theta} & \mathbf{e}_3 \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 1 & -e^{i(\theta-\phi)} & \mathbf{e}_1 e^{i\phi} \mathbf{e}_3 \\ -e^{i(\phi-\theta)} & 1 & -\mathbf{e}_1 e^{i\theta} \mathbf{e}_3 \\ \mathbf{e}_3 \mathbf{e}_1 e^{i\phi} & -\mathbf{e}_3 \mathbf{e}_1 e^{i\theta} & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix}
\end{aligned}
\tag{86.6}
$$

86.3.1 *A motivation for a Hermitian like transposition operation*

There are a few things of note about this matrix. One of which is that it is not symmetric. This is a consequence of the non-commutative nature of the vector products. What we do have is a Hermitian transpose like symmetry. Observe that terms like the $(1, 2)$ and the $(2, 1)$ elements of the matrix are equal after all the vector products are reversed.

Using tilde to denote this reversion, we have

$$
\begin{aligned}
(e^{i(\theta-\phi)})\tilde{} &= \cos(\theta-\phi) + (\mathbf{e}_1\mathbf{e}_2)\tilde{} \sin(\theta-\phi) \\
&= \cos(\theta-\phi) + \mathbf{e}_2\mathbf{e}_1 \sin(\theta-\phi) \\
&= \cos(\theta-\phi) - \mathbf{e}_1\mathbf{e}_2 \sin(\theta-\phi) \\
&= e^{-i(\theta-\phi)}.
\end{aligned}
\tag{86.7}
$$

The fact that all the elements of this matrix, if non-scalar, have their reversed value in the transposed position, is sufficient to show that the end result is a scalar as expected. Consider a general quadratic form where the matrix has scalar and bivector grades as above, where there is reversion in all the transposed positions. That is

$$
b^{\mathrm{T}} A b
\tag{86.8}
$$

where $A = \left\|A_{ij}\right\|$, a $m \times m$ matrix where $A_{ij} = \tilde{A_{ji}}$ and contains scalar and bivector grades, and $b = \|b_i\|$, a $m \times 1$ column matrix of scalars. Then the product is

$$
\begin{aligned}
\sum_{ij} b_i A_{ij} b_j &= \sum_{i<j} b_i A_{ij} b_j + \sum_{j<i} b_i A_{ij} b_j + \sum_k b_k A_{kk} b_k \\
&= \sum_{i<j} b_i A_{ij} b_j + \sum_{i<j} b_j A_{ji} b_i + \sum_k b_k A_{kk} b_k \\
&= \sum_k b_k A_{kk} b_k + 2 \sum_{i<j} b_i (A_{ij} + A_{ji}) b_j \\
&= \sum_k b_k A_{kk} b_k + 2 \sum_{i<j} b_i (A_{ij} + \tilde{A_{ij}}) b_j
\end{aligned}
\tag{86.9}
$$

The quantity in braces $A_{ij} + \tilde{A_{ij}}$ is a scalar since any of the bivector grades in A_{ij} cancel out.
Consider a similar general product of a vector after the vector has been factored into a product of matrices of multivector elements

$$
\mathbf{x} = \begin{bmatrix} a_1 & a_2 & \dots & a_m \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
\tag{86.10}
$$

The (squared) length of the vector is

$$
\begin{aligned}
\mathbf{x}^2 &= (a_i b_i)(a_j b_j) \\
&= (a_i b_i)\tilde{} a_j b_j \\
&= \tilde{b}_i \tilde{a}_i a_j b_j \\
&= \tilde{b}_i (\tilde{a}_i a_j) b_j.
\end{aligned}
\tag{86.11}
$$

It is clear that we want a transposition operation that includes reversal of its elements, so with a general factorization of a vector into matrices of multivectors $\mathbf{x} = Ab$, its square will be $\mathbf{x} = \tilde{b}^{\mathrm{T}} \tilde{A}^{\mathrm{T}} A b$.

As with purely complex valued matrices, it is convenient to use the dagger notation, and define

$$
A^{\dagger} = \tilde{A}^{\mathrm{T}}
\tag{86.12}
$$

where $\tilde{A}$ contains the reversed elements of A. By extension, we can define dot and wedge products of vectors expressed as products of multivector matrices. Given $\mathbf{x} = Ab$, a row vector and column vector product, and $\mathbf{y} = Cd$, where each of the rows or columns has m elements, the dot and wedge products are

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= \left\langle d^\dagger C^\dagger A b \right\rangle \\ \mathbf{x} \wedge \mathbf{y} &= \left\langle d^\dagger C^\dagger A b \right\rangle_2 .\end{aligned} \tag{86.13}$$

In particular, if b and d are matrices of scalars we have

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= d^{\mathrm{T}} \left\langle C^\dagger A \right\rangle b = d^{\mathrm{T}} \frac{C^\dagger A + A^\dagger C}{2} b \\ \mathbf{x} \wedge \mathbf{y} &= d^{\mathrm{T}} \left\langle C^\dagger A \right\rangle_2 b = d^{\mathrm{T}} \frac{C^\dagger A - A^\dagger C}{2} b.\end{aligned} \tag{86.14}$$

The dot product is seen as a generator of symmetric matrices, and the wedge product a generator of purely antisymmetric matrices.

86.3.2 *Back to the problem*

Now, returning to the example above, where we want $\mathbf{R}^2$. We have seen that we can drop any bivector terms from the matrix, so that the squared length can be reduced as

$$\begin{aligned}\mathbf{R}^2 &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 1 & -e^{i(\theta-\phi)} & 0 \\ -e^{i(\phi-\theta)} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\ &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 1 & -\cos(\theta-\phi) & 0 \\ -\cos(\theta-\phi) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\ &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} \rho - a\cos(\theta-\phi) \\ -\rho\cos(\theta-\phi) + a \\ z-h \end{bmatrix}\end{aligned} \tag{86.15}$$

So we have

$$\begin{aligned}\mathbf{R}^2 &= \rho^2 + a^2 + (z-h)^2 - 2a\rho\cos(\theta-\phi) \\ R &= \sqrt{\rho^2 + a^2 + (z-h)^2 - 2a\rho\cos(\theta-\phi)}\end{aligned} \tag{86.16}$$

Now consider the velocity of the charged particle. We can write this as

$$\frac{d\mathbf{x}_c}{dt} = \begin{bmatrix} \mathbf{e}_3 & \mathbf{e}_1 e^{i\theta} & \mathbf{e}_2 e^{i\theta} \end{bmatrix} \begin{bmatrix} \dot{h} \\ \dot{a} \\ a\dot{\theta} \end{bmatrix} \tag{86.17}$$

To compute $\mathbf{v}_c \cdot \mathbf{R}$ we have to extract scalar grades of the matrix product

$$\begin{aligned}
\left\langle \begin{bmatrix} \mathbf{e}_1 e^{i\phi} \\ -\mathbf{e}_1 e^{i\theta} \\ \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_3 & \mathbf{e}_1 e^{i\theta} & \mathbf{e}_2 e^{i\theta} \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} \mathbf{e}_1 e^{i\phi} \\ -\mathbf{e}_1 e^{i\theta} \\ \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_3 & \mathbf{e}_1 e^{i\theta} & \mathbf{e}_2 e^{i\theta} \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \mathbf{e}_1 e^{i\phi}\mathbf{e}_3 & \mathbf{e}_1 e^{i\phi}\mathbf{e}_1 e^{i\theta} & \mathbf{e}_1 e^{i\phi}\mathbf{e}_2 e^{i\theta} \\ -\mathbf{e}_1 e^{i\theta}\mathbf{e}_3 & -\mathbf{e}_1 e^{i\theta}\mathbf{e}_1 e^{i\theta} & -\mathbf{e}_1 e^{i\theta}\mathbf{e}_2 e^{i\theta} \\ \mathbf{e}_3\mathbf{e}_3 & \mathbf{e}_3\mathbf{e}_1 e^{i\theta} & \mathbf{e}_3\mathbf{e}_2 e^{i\theta} \end{bmatrix} \right\rangle \\
&= \begin{bmatrix} 0 & \cos(\theta-\phi) & -\sin(\theta-\phi) \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{86.18}$$

So the dot product is

$$\begin{aligned}
\mathbf{R}\cdot\mathbf{v} &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 0 & \cos(\theta-\phi) & -\sin(\theta-\phi) \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{h} \\ \dot{a} \\ a\dot{\theta} \end{bmatrix} \\
&= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} \dot{a}\cos(\theta-\phi) - a\dot{\theta}\sin(\theta-\phi) \\ -\dot{a} \\ \dot{h} \end{bmatrix} \\
&= (z-h)\dot{h} - \dot{a}a + \rho\dot{a}\cos(\theta-\phi) - \rho a\dot{\theta}\sin(\theta-\phi)
\end{aligned} \tag{86.19}$$

This is the last of what we needed for the potentials, so we have

$$\begin{aligned}
A^0 &= \frac{q}{R - (z-h)\dot{h}/c + a\dot{a}/c + \rho\cos(\theta-\phi)\dot{a}/c - \rho a\sin(\theta-\phi)\dot{\theta}/c} \\
\mathbf{A} &= \frac{\dot{h}\mathbf{e}_3 + (\dot{a}\mathbf{e}_1 + a\dot{\theta}\mathbf{e}_2)e^{i\theta}}{c} A^0,
\end{aligned} \tag{86.20}$$

where all the time dependent terms in the potentials are evaluated at the retarded time t_r, defined implicitly by the messy relationship

$$c(t - t_r) = \sqrt{(\rho(t_r))^2 + (a(t_r))^2 + (z - h(t_r))^2 - 2a(t_r)\rho\cos(\theta(t_r) - \phi)}. \tag{86.21}$$

86.4 DOING THIS CALCULATION WITH PLAIN OLD CYLINDRICAL COORDINATES

It is worth trying this same calculation without any geometric algebra to contrast it. I had expect that the same sort of factorization could also be performed. Let us try it

$$\begin{aligned}
\mathbf{x}_c &= \begin{bmatrix} a\cos\theta \\ a\sin\theta \\ h \end{bmatrix} \\
\mathbf{r} &= \begin{bmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{bmatrix}
\end{aligned} \tag{86.22}$$

$$\begin{aligned}
\mathbf{R} &= \mathbf{r} - \mathbf{x}_c \\
&= \begin{bmatrix} \rho\cos\phi - a\cos\theta \\ \rho\sin\phi - a\sin\theta \\ z - h \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi & -\cos\theta & 0 \\ \sin\phi & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z - h \end{bmatrix}
\end{aligned} \tag{86.23}$$

So for $\mathbf{R}^2$ we really just need to multiply out two matrices

$$\begin{aligned}
&\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\cos\theta & 0 \\ \sin\phi & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\phi + \sin^2\phi & -(\cos\phi\cos\phi + \sin\phi\sin\theta) & 0 \\ -(\cos\phi\cos\theta + \sin\theta\sin\phi) & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\cos(\phi-\theta) & 0 \\ -\cos(\phi-\theta) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{86.24}$$

So for $\mathbf{R}^2$ we have

$$\begin{aligned}
\mathbf{R}^2 &= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} 1 & -\cos(\phi-\theta) & 0 \\ -\cos(\phi-\theta) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \rho & a & (z-h) \end{bmatrix} \begin{bmatrix} \rho - a\cos(\phi-\theta) \\ -\rho\cos(\phi-\theta) + a \\ z-h \end{bmatrix} \\
&= (z-h)^2 + \rho^2 + a^2 - 2a\rho\cos(\phi-\theta)
\end{aligned} \tag{86.25}$$

We get the same result this way, as expected. The matrices of multivector products provide a small computational savings, since we do not have to look up the $\cos\phi\cos\phi + \sin\phi\sin\theta = \cos(\phi-\theta)$ identity, but other than that minor detail, we get the same result.

For the particle velocity we have

$$\begin{aligned}
\mathbf{v}_c &= \begin{bmatrix} \dot{a}\cos\theta - a\dot{\theta}\sin\theta \\ \dot{a}\sin\theta + a\dot{\theta}\cos\theta \\ \dot{h} \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a} \\ a\dot{\theta} \\ \dot{h} \end{bmatrix}
\end{aligned} \tag{86.26}$$

So the dot product is

$$
\begin{aligned}
\mathbf{v}_c \cdot \mathbf{R} &= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\cos\theta & 0 \\ \sin\phi & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & -\cos^2\theta - \sin^2\theta & 0 \\ -\cos\phi\sin\theta + \cos\theta\sin\phi & 0 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \begin{bmatrix} \dot{a} & a\dot{\theta} & \dot{h} \end{bmatrix} \begin{bmatrix} \cos(\phi-\theta) & -1 & 0 \\ \sin(\phi-\theta) & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ a \\ z-h \end{bmatrix} \\
&= \dot{h}(z-h) - \dot{a}a + \rho\dot{a}\cos(\phi-\theta) + \rho a\dot{\theta}\sin(\phi-\theta)
\end{aligned}
\tag{86.27}
$$

86.5 REFLECTING ON TWO THE CALCULATION METHODS

With a learning curve to both Geometric Algebra, and overhead required for this new multivector matrix formalism, it is definitely not a clear winner as a calculation method. Having worked a couple examples now this way, the first being the N spherical pendulum problem, and now this potentials problem, I will keep my eye out for new opportunities. If nothing else this can be a useful private calculation tool, and the translation into more pedestrian matrix methods has been seen in both cases to not be too difficult.

87

PLANE WAVE SOLUTIONS OF MAXWELL'S EQUATION

88

PLANE WAVE SOLUTIONS OF MAXWELL'S EQUATION USING GEOMETRIC ALGEBRA

88.1 MOTIVATION

Study of reflection and transmission of radiation in isotropic, charge and current free, linear matter utilizes the plane wave solutions to Maxwell's equations. These have the structure of phasor equations, with some specific constraints on the components and the exponents.

These constraints are usually derived starting with the plain old vector form of Maxwell's equations, and it is natural to wonder how this is done directly using Geometric Algebra. [10] provides one such derivation, using the covariant form of Maxwell's equations. Here's a slightly more pedestrian way of doing the same.

88.2 MAXWELL'S EQUATIONS IN MEDIA

We start with Maxwell's equations for linear matter as found in [17]

$$\nabla \cdot \mathbf{E} = 0 \tag{88.1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{88.1b}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{88.1c}$$

$$\nabla \times \mathbf{B} = \mu\epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{88.1d}$$

We merge these using the geometric identity

$$\nabla \cdot \mathbf{a} + I\nabla \times \mathbf{a} = \nabla \mathbf{a}, \tag{88.2}$$

where I is the 3D pseudoscalar $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, to find

$$\nabla \mathbf{E} = -I\frac{\partial \mathbf{B}}{\partial t} \tag{88.3a}$$

$$\nabla \mathbf{B} = I\mu\epsilon\frac{\partial \mathbf{E}}{\partial t}. \tag{88.3b}$$

We want dimensions of $1/L$ for the derivative operator on the RHS of eq. (88.3b), so we divide through by $\sqrt{\mu\epsilon}I$ for

$$-I\frac{1}{\sqrt{\mu\epsilon}}\nabla \mathbf{B} = \sqrt{\mu\epsilon}\frac{\partial \mathbf{E}}{\partial t}. \tag{88.4}$$

This can now be added to eq. (88.3a) for

$$\left(\nabla + \sqrt{\mu\epsilon}\frac{\partial}{\partial t}\right)\left(\mathbf{E} + \frac{I}{\sqrt{\mu\epsilon}}\mathbf{B}\right) = 0. \tag{88.5}$$

This is Maxwell's equation in linear isotropic charge and current free matter in Geometric Algebra form.

88.3 PHASOR SOLUTIONS

We write the electromagnetic field as

$$F = \left(\mathbf{E} + \frac{I}{\sqrt{\mu\epsilon}}\mathbf{B}\right), \tag{88.6}$$

so that for vacuum where $1/\sqrt{\mu\epsilon} = c$ we have the usual $F = \mathbf{E} + Ic\mathbf{B}$. Assuming a phasor solution of

$$\tilde{F} = F_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{88.7}$$

where F_0 is allowed to be complex, and the actual field is obtained by taking the real part

$$F = \operatorname{Re}\tilde{F} = \operatorname{Re}(F_0)\cos(\mathbf{k}\cdot\mathbf{x}-\omega t) - \operatorname{Im}(F_0)\sin(\mathbf{k}\cdot\mathbf{x}-\omega t). \tag{88.8}$$

Note carefully that we are using a scalar imaginary i, as well as the multivector (pseudoscalar) I, despite the fact that both have the square to scalar minus one property.

We now seek the constraints on $\mathbf{k}$, ω, and F_0 that allow $\tilde{F}$ to be a solution to eq. (88.5)

$$0 = \left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\frac{\partial}{\partial t}\right)\tilde{F}. \tag{88.9}$$

As usual in the non-geometric algebra treatment, we observe that any such solution $\tilde{F}$ to Maxwell's equation is also a wave equation solution. In GA we can do so by right multiplying an operator that has a conjugate form,

$$\begin{aligned}
0 &= \left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\frac{\partial}{\partial t}\right)\tilde{F} \\
&= \left(\boldsymbol{\nabla} - \sqrt{\mu\epsilon}\frac{\partial}{\partial t}\right)\left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\frac{\partial}{\partial t}\right)\tilde{F} \\
&= \left(\boldsymbol{\nabla}^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right)\tilde{F} \\
&= \left(\boldsymbol{\nabla}^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\tilde{F},
\end{aligned} \tag{88.10}$$

where $v = 1/\sqrt{\mu\epsilon}$ is the speed of the wave described by this solution.

Inserting the exponential form of our assumed solution eq. (88.7) we find

$$0 = -(\mathbf{k}^2 - \omega^2/v^2)F_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{88.11}$$

which implies that the wave number vector $\mathbf{k}$ and the angular frequency ω are related by

$$v^2\mathbf{k}^2 = \omega^2. \tag{88.12}$$

Our assumed solution must also satisfy the first order system eq. (88.9)

$$\begin{aligned}
0 &= \left(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\frac{\partial}{\partial t}\right)F_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
&= i\left(\mathbf{e}_m k_m - \frac{\omega}{v}\right)F_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
&= ik(\hat{\mathbf{k}} - 1)F_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}.
\end{aligned} \tag{88.13}$$

The constraints on F_0 must then be given by

$$0 = \left(\hat{\mathbf{k}} - 1\right) F_0. \tag{88.14}$$

With

$$F_0 = \mathbf{E}_0 + Iv\mathbf{B}_0, \tag{88.15}$$

we must then have all grades of the multivector equation equal to zero

$$0 = (\hat{\mathbf{k}} - 1)\left(\mathbf{E}_0 + Iv\mathbf{B}_0\right). \tag{88.16}$$

Writing out all the geometric products, grouping into columns by grade, we have

$$\begin{aligned} 0 \quad &= \hat{\mathbf{k}} \cdot \mathbf{E}_0 \quad -\mathbf{E}_0 \quad && +\hat{\mathbf{k}} \wedge \mathbf{E}_0 \quad Iv\hat{\mathbf{k}} \cdot \mathbf{B}_0 \\ & \qquad\qquad +Iv\hat{\mathbf{k}} \wedge \mathbf{B}_0 && +Iv\mathbf{B}_0 \end{aligned} \tag{88.17}$$

We've made use of the fact that I commutes with all of $\hat{\mathbf{k}}$, $\mathbf{E}_0$, and $\mathbf{B}_0$ and employed the identity $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$.

Collecting the scalar, vector, bivector, and pseudoscalar grades and using $\mathbf{a} \wedge \mathbf{b} = I\mathbf{a} \times \mathbf{b}$ again, we have a set of constraints resulting from the first order system

$$0 = \hat{\mathbf{k}} \cdot \mathbf{E}_0 \tag{88.18a}$$

$$\mathbf{E}_0 = -\hat{\mathbf{k}} \times v\mathbf{B}_0 \tag{88.18b}$$

$$v\mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0 \tag{88.18c}$$

$$0 = \hat{\mathbf{k}} \cdot \mathbf{B}_0. \tag{88.18d}$$

This and eq. (88.12) describe all the constraints on our phasor that are required for it to be a solution. Note that only one of the two cross product equations in eq. (88.18) are required because the two are not independent (problem 88.1).

Writing out the complete expression for F_0 we have

$$\begin{aligned} F_0 &= \mathbf{E}_0 + I v \mathbf{B}_0 \\ &= \mathbf{E}_0 + I\hat{\mathbf{k}} \times \mathbf{E}_0 \\ &= \mathbf{E}_0 + \hat{\mathbf{k}} \wedge \mathbf{E}_0. \end{aligned} \tag{88.19}$$

Since $\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0$, this is

$$F_0 = (1 + \hat{\mathbf{k}})\mathbf{E}_0. \tag{88.20}$$

Had we been clever enough this could have been deduced directly from the eq. (88.14) directly, since we require a product that is killed by left multiplication with $\hat{\mathbf{k}} - 1$. Our complete plane wave solution to Maxwell's equation is therefore given by

$$\begin{aligned} F &= \mathrm{Re}(\tilde{F}) = \mathbf{E} + \frac{I}{\sqrt{\mu\epsilon}}\mathbf{B} \\ \tilde{F} &= (1 \pm \hat{\mathbf{k}})\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} \mp \omega t)} \\ 0 &= \hat{\mathbf{k}} \cdot \mathbf{E}_0 \\ \mathbf{k}^2 &= \omega^2 \mu\epsilon. \end{aligned} \tag{88.21}$$

88.4 PROBLEMS

Exercise 88.1 Electrodynamic plane wave constraints

It was claimed that

$$\mathbf{E}_0 = -\hat{\mathbf{k}} \times v\mathbf{B}_0 \tag{88.22a}$$

$$v\mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0 \tag{88.22b}$$

relating the electric and magnetic field of electrodynamic plane waves were dependent. Show this.

Exercise 88.2 Proving that the wavevectors are all coplanar

[17] poses the following simple but excellent problem, related to the relationship between the incident, transmission and reflection phasors, which he states has the following form

$$()e^{i(\mathbf{k}_i\cdot\mathbf{x}-\omega t)} + ()e^{i(\mathbf{k}_r\cdot\mathbf{x}-\omega t)} = ()e^{i(\mathbf{k}_t\cdot\mathbf{x}-\omega t)}, \tag{88.25}$$

He poses the problem (9.15)

Suppose $Ae^{iax} + Be^{ibx} = Ce^{icx}$ for some nonzero constants A, B, C, a, b, c, and for all x. Prove that $a = b = c$ and $A + B = C$.

88.5 SOLUTIONS

Answer for Exercise 44.1

Part 1. Two parameter volume, curl of vector

$$\begin{aligned}
d^2\mathbf{x}\cdot(\boldsymbol{\partial}\wedge\mathbf{f}) &= d^2u\big((\mathbf{x}_1\wedge\mathbf{x}_2)\cdot\mathbf{x}^i\big)\cdot\partial_i\mathbf{f} \\
&= d^2u\,(\mathbf{x}_1\cdot\partial_2\mathbf{f} - \mathbf{x}_2\cdot\partial_1\mathbf{f}) \\
&= d^2u\,(\partial_2 f_1 - \partial_1 f_2) \\
&= -d^2u\epsilon^{ab}\partial_a f_b. \qquad \square
\end{aligned} \tag{44.146}$$

Part 2. Three parameter volume, curl of vector

$$\begin{aligned}
d^3\mathbf{x}\cdot(\boldsymbol{\partial}\wedge\mathbf{f}) &= d^3u\big((\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3)\cdot\mathbf{x}^i\big)\cdot\partial_i\mathbf{f} \\
&= d^3u\big((\mathbf{x}_1\wedge\mathbf{x}_2)\cdot\partial_3\mathbf{f} + (\mathbf{x}_3\wedge\mathbf{x}_1)\cdot\partial_2\mathbf{f} + (\mathbf{x}_2\wedge\mathbf{x}_3)\cdot\partial_1\mathbf{f}\big) \\
&= d^3u\big((\mathbf{x}_1\partial_3\mathbf{f}\cdot\mathbf{x}_2 - \mathbf{x}_2\partial_3\mathbf{f}\cdot\mathbf{x}_1) + (\mathbf{x}_3\partial_2\mathbf{f}\cdot\mathbf{x}_1 - \mathbf{x}_1\partial_2\mathbf{f}\cdot\mathbf{x}_3) + (\mathbf{x}_2\partial_1\mathbf{f}\cdot\mathbf{x}_3 - \mathbf{x}_3\partial_1\mathbf{f}\cdot\mathbf{x}_2)\big) \\
&= d^3u\big(\mathbf{x}_1\,(-\partial_2\mathbf{f}\cdot\mathbf{x}_3 + \partial_3\mathbf{f}\cdot\mathbf{x}_2) + \mathbf{x}_2\,(-\partial_3\mathbf{f}\cdot\mathbf{x}_1 + \partial_1\mathbf{f}\cdot\mathbf{x}_3) + \mathbf{x}_3\,(-\partial_1\mathbf{f}\cdot\mathbf{x}_2 + \partial_2\mathbf{f}\cdot\mathbf{x}_1)\big) \\
&= d^3u\big(\mathbf{x}_1\,(-\partial_2 f_3 + \partial_3 f_2) + \mathbf{x}_2\,(-\partial_3 f_1 + \partial_1 f_3) + \mathbf{x}_3\,(-\partial_1 f_2 + \partial_2 f_1)\big) \\
&= -d^3u\epsilon^{abc}\partial_b f_c. \qquad \square
\end{aligned} \tag{44.147}$$

Part 3. Four parameter volume, curl of vector

$$
\begin{aligned}
d^4\mathbf{x}\cdot(\boldsymbol{\partial}\wedge\mathbf{f}) &= d^4u\left(\left(\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3\wedge\mathbf{x}_4\right)\cdot\mathbf{x}^i\right)\cdot\partial_i\mathbf{f} \\
&= d^4u\left(\left(\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3\right)\cdot\partial_4\mathbf{f} - \left(\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_4\right)\cdot\partial_3\mathbf{f} + \left(\mathbf{x}_1\wedge\mathbf{x}_3\wedge\mathbf{x}_4\right)\cdot\partial_2\mathbf{f} - \left(\mathbf{x}_2\wedge\mathbf{x}_3\wedge\mathbf{x}_4\right)\cdot\partial_1\mathbf{f}\right) \\
&= d^4u(\\
&\qquad \left(\mathbf{x}_1\wedge\mathbf{x}_2\right)\mathbf{x}_3\cdot\partial_4\mathbf{f} - \left(\mathbf{x}_1\wedge\mathbf{x}_3\right)\mathbf{x}_2\cdot\partial_4\mathbf{f} + \left(\mathbf{x}_2\wedge\mathbf{x}_3\right)\mathbf{x}_1\cdot\partial_4\mathbf{f} \\
&\quad - \left(\mathbf{x}_1\wedge\mathbf{x}_2\right)\mathbf{x}_4\cdot\partial_3\mathbf{f} + \left(\mathbf{x}_1\wedge\mathbf{x}_4\right)\mathbf{x}_2\cdot\partial_3\mathbf{f} - \left(\mathbf{x}_2\wedge\mathbf{x}_4\right)\mathbf{x}_1\cdot\partial_3\mathbf{f} \\
&\quad + \left(\mathbf{x}_1\wedge\mathbf{x}_3\right)\mathbf{x}_4\cdot\partial_2\mathbf{f} - \left(\mathbf{x}_1\wedge\mathbf{x}_4\right)\mathbf{x}_3\cdot\partial_2\mathbf{f} + \left(\mathbf{x}_3\wedge\mathbf{x}_4\right)\mathbf{x}_1\cdot\partial_2\mathbf{f} \\
&\quad - \left(\mathbf{x}_2\wedge\mathbf{x}_3\right)\mathbf{x}_4\cdot\partial_1\mathbf{f} + \left(\mathbf{x}_2\wedge\mathbf{x}_4\right)\mathbf{x}_3\cdot\partial_1\mathbf{f} - \left(\mathbf{x}_3\wedge\mathbf{x}_4\right)\mathbf{x}_2\cdot\partial_1\mathbf{f} \\
&\qquad) \\
&= d^4u\left(\mathbf{x}_1\wedge\mathbf{x}_2\partial_{[4}f_{3]} + \mathbf{x}_1\wedge\mathbf{x}_3\partial_{[2}f_{4]} + \mathbf{x}_1\wedge\mathbf{x}_4\partial_{[3}f_{2]} + \mathbf{x}_2\wedge\mathbf{x}_3\partial_{[4}f_{1]} + \mathbf{x}_2\wedge\mathbf{x}_4\partial_{[1}f_{3]} + \mathbf{x}_3\wedge\mathbf{x}_4\partial_{[2}f_{1]}\right) \\
&= -\frac{1}{2}d^4u\epsilon^{abcd}\mathbf{x}_a\wedge\mathbf{x}_b\partial_c f_d. \qquad \square
\end{aligned}
\tag{44.148}
$$

Part 4. Three parameter volume, curl of bivector

$$
\begin{aligned}
d^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge B) &= d^3u\big((\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \mathbf{x}^i\big) \cdot \partial_i B \\
&= d^3u\big((\mathbf{x}_1 \wedge \mathbf{x}_2) \cdot \partial_3 B + (\mathbf{x}_3 \wedge \mathbf{x}_1) \cdot \partial_2 B + (\mathbf{x}_2 \wedge \mathbf{x}_3) \cdot \partial_1 B\big) \\
&= \frac{1}{2}d^3u(\mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot \partial_3 B) - \mathbf{x}_2 \cdot (\mathbf{x}_1 \cdot \partial_3 B) \\
&\qquad + \mathbf{x}_3 \cdot (\mathbf{x}_1 \cdot \partial_2 B) - \mathbf{x}_1 \cdot (\mathbf{x}_3 \cdot \partial_2 B) \\
&\qquad + \mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot \partial_1 B) - \mathbf{x}_3 \cdot (\mathbf{x}_2 \cdot \partial_1 B)) \\
&= \frac{1}{2}d^3u(\mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot \partial_3 B - \mathbf{x}_3 \cdot \partial_2 B) \\
&\qquad + \mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot \partial_1 B - \mathbf{x}_1 \cdot \partial_3 B) \\
&\qquad + \mathbf{x}_3 \cdot (\mathbf{x}_1 \cdot \partial_2 B - \mathbf{x}_2 \cdot \partial_1 B)) \\
&= \frac{1}{2}d^3u(\mathbf{x}_1 \cdot (\partial_3 (\mathbf{x}_2 \cdot B) - \partial_2 (\mathbf{x}_3 \cdot B)) \\
&\qquad + \mathbf{x}_2 \cdot (\partial_1 (\mathbf{x}_3 \cdot B) - \partial_3 (\mathbf{x}_1 \cdot B)) \\
&\qquad + \mathbf{x}_3 \cdot (\partial_2 (\mathbf{x}_1 \cdot B) - \partial_1 (\mathbf{x}_2 \cdot B))) \\
&= \frac{1}{2}d^3u(\partial_2 (\mathbf{x}_3 \cdot (\mathbf{x}_1 \cdot B)) - \partial_3 (\mathbf{x}_2 \cdot (\mathbf{x}_1 \cdot B)) \\
&\qquad + \partial_3 (\mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot B)) - \partial_1 (\mathbf{x}_3 \cdot (\mathbf{x}_2 \cdot B)) \\
&\qquad + \partial_1 (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot B)) - \partial_2 (\mathbf{x}_1 \cdot (\mathbf{x}_3 \cdot B))) \\
&= \frac{1}{2}d^3u(\partial_2 B_{13} - \partial_3 B_{12} + \partial_3 B_{21} - \partial_1 B_{23} + \partial_1 B_{32} - \partial_2 B_{31}) \\
&= d^3u(\partial_2 B_{13} + \partial_3 B_{21} + \partial_1 B_{32}) \\
&= -\frac{1}{2}d^3u\epsilon^{abc}\partial_a B_{bc}. \qquad \square
\end{aligned}
\tag{44.149}
$$

Part 5. Four parameter volume, curl of bivector To start, we require theorem B.3. For convenience lets also write our wedge products as a single indexed quantity, as in $\mathbf{x}_{abc}$ for $\mathbf{x}_a \wedge \mathbf{x}_b \wedge \mathbf{x}_c$. The expansion is

$$\begin{aligned}
d^4\mathbf{x} \cdot (\boldsymbol{\partial} \wedge B) &= d^4 u \left(\mathbf{x}_{1234} \cdot \mathbf{x}^i\right) \cdot \partial_i B \\
&= d^4 u \left(\mathbf{x}_{123} \cdot \partial_4 B - \mathbf{x}_{124} \cdot \partial_3 B + \mathbf{x}_{134} \cdot \partial_2 B - \mathbf{x}_{234} \cdot \partial_1 B\right) \\
&= d^4 u \big(\mathbf{x}_1 \left(\mathbf{x}_{23} \cdot \partial_4 B\right) + \mathbf{x}_2 \left(\mathbf{x}_{32} \cdot \partial_4 B\right) + \mathbf{x}_3 \left(\mathbf{x}_{12} \cdot \partial_4 B\right) \\
&\quad - \mathbf{x}_1 \left(\mathbf{x}_{24} \cdot \partial_3 B\right) - \mathbf{x}_2 \left(\mathbf{x}_{41} \cdot \partial_3 B\right) - \mathbf{x}_4 \left(\mathbf{x}_{12} \cdot \partial_3 B\right) \\
&\quad + \mathbf{x}_1 \left(\mathbf{x}_{34} \cdot \partial_2 B\right) + \mathbf{x}_3 \left(\mathbf{x}_{41} \cdot \partial_2 B\right) + \mathbf{x}_4 \left(\mathbf{x}_{13} \cdot \partial_2 B\right) \\
&\quad - \mathbf{x}_2 \left(\mathbf{x}_{34} \cdot \partial_1 B\right) - \mathbf{x}_3 \left(\mathbf{x}_{42} \cdot \partial_1 B\right) - \mathbf{x}_4 \left(\mathbf{x}_{23} \cdot \partial_1 B\right)\big) \\
&= d^4 u \big(\mathbf{x}_1 \left(\mathbf{x}_{23} \cdot \partial_4 B + \mathbf{x}_{42} \cdot \partial_3 B + \mathbf{x}_{34} \cdot \partial_2 B\right) \\
&\quad + \mathbf{x}_2 \left(\mathbf{x}_{32} \cdot \partial_4 B + \mathbf{x}_{14} \cdot \partial_3 B + \mathbf{x}_{43} \cdot \partial_1 B\right) \\
&\quad + \mathbf{x}_3 \left(\mathbf{x}_{12} \cdot \partial_4 B + \mathbf{x}_{41} \cdot \partial_2 B + \mathbf{x}_{24} \cdot \partial_1 B\right) \\
&\quad + \mathbf{x}_4 \left(\mathbf{x}_{21} \cdot \partial_3 B + \mathbf{x}_{13} \cdot \partial_2 B + \mathbf{x}_{32} \cdot \partial_1 B\right)\big) \\
&= -\frac{1}{2} d^4 u \epsilon^{abcd} \mathbf{x}_a \partial_b B_{cd}. \qquad \square
\end{aligned} \tag{44.150}$$

This last step uses an intermediate result from the **??** expansion above, since each of the four terms has the same structure we have previously observed.

Part 6. Four parameter volume, curl of trivector Using the $\mathbf{x}_{ijk}$ shorthand again, the initial expansion gives

$$d^4\mathbf{x} \cdot (\boldsymbol{\partial} \wedge T) = d^4 u \left(\mathbf{x}_{123} \cdot \partial_4 T - \mathbf{x}_{124} \cdot \partial_3 T + \mathbf{x}_{134} \cdot \partial_2 T - \mathbf{x}_{234} \cdot \partial_1 T\right). \tag{44.151}$$

Applying theorem B.4 to expand the inner products within the braces we have

$$
\begin{aligned}
\mathbf{x}_{123} \cdot \partial_4 T - &\mathbf{x}_{124} \cdot \partial_3 T + \mathbf{x}_{134} \cdot \partial_2 T - \mathbf{x}_{234} \cdot \partial_1 T \\
&= \mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot \partial_4 T)) - \mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\mathbf{x}_4 \cdot \partial_3 T)) \\
&\qquad\qquad\qquad\qquad\qquad \text{Apply cyclic permutations} \\
&\quad + \boxed{\mathbf{x}_1 \cdot (\mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot \partial_2 T)) - \mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot \partial_1 T))} \\
&= \mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot \partial_4 T)) - \mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\mathbf{x}_4 \cdot \partial_3 T)) \\
&\quad + \mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot (\mathbf{x}_1 \cdot \partial_2 T)) - \mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot (\mathbf{x}_2 \cdot \partial_1 T)) \\
&= \mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot \partial_4 T - \mathbf{x}_4 \cdot \partial_3 T)) \\
&\quad + \mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot (\mathbf{x}_1 \cdot \partial_2 T - \mathbf{x}_2 \cdot \partial_1 T)) \\
&= \mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\partial_4 (\mathbf{x}_3 \cdot T) - \partial_3 (\mathbf{x}_4 \cdot T))) \\
&\quad + \mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot (\partial_2 (\mathbf{x}_1 \cdot T) - \partial_1 (\mathbf{x}_2 \cdot T))) \\
&= \mathbf{x}_1 \cdot \partial_4 (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot T)) + \mathbf{x}_2 \cdot \partial_3 (\mathbf{x}_1 \cdot (\mathbf{x}_4 \cdot T)) \\
&\quad + \mathbf{x}_3 \cdot \partial_2 (\mathbf{x}_4 \cdot (\mathbf{x}_1 \cdot T)) + \mathbf{x}_4 \cdot \partial_1 (\mathbf{x}_3 \cdot (\mathbf{x}_2 \cdot T)) \\
&- \mathbf{x}_1 \cdot ((\partial_4 \mathbf{x}_2) \cdot (\mathbf{x}_3 \cdot T)) - \mathbf{x}_2 \cdot ((\partial_3 \mathbf{x}_1) \cdot (\mathbf{x}_4 \cdot T)) \\
&\quad - \mathbf{x}_3 \cdot ((\partial_2 \mathbf{x}_4) \cdot (\mathbf{x}_1 \cdot T)) - \mathbf{x}_4 \cdot ((\partial_1 \mathbf{x}_3) \cdot (\mathbf{x}_2 \cdot T)) \\
&= \mathbf{x}_1 \cdot \partial_4 (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot T)) + \mathbf{x}_2 \cdot \partial_3 (\mathbf{x}_1 \cdot (\mathbf{x}_4 \cdot T)) \\
&\quad + \mathbf{x}_3 \cdot \partial_2 (\mathbf{x}_4 \cdot (\mathbf{x}_1 \cdot T)) + \mathbf{x}_4 \cdot \partial_1 (\mathbf{x}_3 \cdot (\mathbf{x}_2 \cdot T)) \\
&+ \frac{\partial^2 \mathbf{x}}{\partial u^4 \partial u^2} \cdot \cancel{(\mathbf{x}_1 \cdot (\mathbf{x}_3 \cdot T) + \mathbf{x}_3 \cdot (\mathbf{x}_1 \cdot T))} \\
&\quad + \frac{\partial^2 \mathbf{x}}{\partial u^1 \partial u^3} \cdot \cancel{(\mathbf{x}_2 \cdot (\mathbf{x}_4 \cdot T) + \mathbf{x}_4 \cdot (\mathbf{x}_2 \cdot T))}.
\end{aligned}
\tag{44.152}
$$

We can cancel those last terms using theorem B.5. Using the same reverse chain rule expansion once more we have

$$
\begin{aligned}
\mathbf{x}_{123} \cdot \partial_4 T - &\mathbf{x}_{124} \cdot \partial_3 T + \mathbf{x}_{134} \cdot \partial_2 T - \mathbf{x}_{234} \cdot \partial_1 T \\
&= \partial_4 (\mathbf{x}_1 \cdot (\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot T))) + \partial_3 (\mathbf{x}_2 \cdot (\mathbf{x}_1 \cdot (\mathbf{x}_4 \cdot T))) + \partial_2 (\mathbf{x}_3 \cdot (\mathbf{x}_4 \cdot (\mathbf{x}_1 \cdot T))) + \partial_1 (\mathbf{x}_4 \cdot (\mathbf{x}_3 \cdot (\mathbf{x}_2 \cdot T))) \\
&- (\partial_4 \mathbf{x}_1) \cdot \cancel{(\mathbf{x}_2 \cdot (\mathbf{x}_3 \cdot T) + \mathbf{x}_3 \cdot (\mathbf{x}_2 \cdot T))} - (\partial_3 \mathbf{x}_2) \cdot \cancel{(\mathbf{x}_1 \cdot (\mathbf{x}_4 \cdot T) \mathbf{x}_4 \cdot (\mathbf{x}_1 \cdot T))},
\end{aligned}
\tag{44.153}
$$

or

$$
d^4 \mathbf{x} \cdot (\boldsymbol{\partial} \wedge T) = d^4 u (\partial_4 T_{321} + \partial_3 T_{412} + \partial_2 T_{143} + \partial_1 T_{234}). \tag{44.154}
$$

The final result follows after permuting the indices slightly.

Answer for Exercise 88.1

This can be shown by crossing $\hat{\mathbf{k}}$ with eq. (88.22a) and using the identity

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = -\mathbf{a}^2\mathbf{b} + \mathbf{a}(\mathbf{a} \cdot \mathbf{b}). \tag{88.23}$$

This gives

$$\begin{aligned}
\hat{\mathbf{k}} \times \mathbf{E}_0 &= -\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times v\mathbf{B}_0) \\
&= \hat{\mathbf{k}}^2 v\mathbf{B}_0 - \hat{\mathbf{k}}(\cancel{\hat{\mathbf{k}} \cdot \mathbf{E}_0}) \\
&= v\mathbf{B}_0
\end{aligned} \tag{88.24}$$

Answer for Exercise 88.2

If this relation holds for all x, then for $x = 0$, we have $A + B = C$. We are left to show that

$$A\left(e^{iax} - e^{icx}\right) + B\left(e^{ibx} - e^{icx}\right) = 0. \tag{88.26}$$

Let $a = c + \delta$ and $b = c + \epsilon$, so that

$$A\left(e^{i\delta x} - 1\right) + B\left(e^{i\epsilon x} - 1\right) = 0. \tag{88.27}$$

Now consider some special values of x. For $x = 2\pi/\epsilon$ we have

$$A\left(e^{2\pi i\delta/\epsilon} - 1\right) = 0, \tag{88.28}$$

and because $A \neq 0$, we must conclude that δ/ϵ is an integer.
Similarily, for $x = 2\pi/\delta$, we have

$$B\left(e^{2\pi i\epsilon/\delta} - 1\right) = 0, \tag{88.29}$$

and this time must conclude that ϵ/δ is an integer. These ratios must therefore take one of the values $0, 1, -1$. Consider the points $x = 2n\pi/\epsilon$ or $x = 2m\pi/\delta$ we find that $n\delta/\epsilon$ and $m\epsilon/\delta$ must be integers for any integers m, n. This only leaves $\epsilon = \delta = 0$, or $a = b = c$ as possibilities.

Part VIII

LORENTZ FORCE

89

LORENTZ BOOST OF LORENTZ FORCE EQUATIONS

89.1 MOTIVATION

Reading of [3] is a treatment of the Lorentz transform properties of the Lorentz force equation. This is not clear to me without working through it myself, so do this.

I also have the urge to try this with the GA formulation of the Lorentz transformation. That may not end up being simpler if one works with the non-covariant form of the Lorentz force equation, but only trying it will tell.

89.2 COMPARE FORMS OF THE LORENTZ BOOST

Working from the Geometric Algebra form of the Lorentz boost, show equivalence to the standard coordinate matrix form and the vector form from Bohm.

89.2.1 *Exponential form*

Write the Lorentz boost of a four vector $x = x^\mu \gamma_\mu = ct\gamma_0 + x^k \gamma_k$ as

$$L(x) = e^{-\alpha\hat{\mathbf{v}}/2} x e^{\alpha\hat{\mathbf{v}}/2} \tag{89.1}$$

89.2.2 *Invariance property*

A Lorentz transformation (boost or rotation) can be defined as those transformation that leave the four vector square unchanged.

Following [10], work with a $+---$ metric signature ($1 = \gamma_0^2 = -\gamma_k^2$), and $\sigma_k = \gamma_k\gamma_0$. Our four vector square in this representation has the familiar invariant form

$$
\begin{aligned}
x^2 &= (ct\gamma_0 + x^m\gamma_m)\left(ct\gamma_0 + x^k\gamma_k\right) \\
&= (ct\gamma_0 + x^m\gamma_m)\,\gamma_0^2\left(ct\gamma_0 + x^k\gamma_k\right) \\
&= (ct + x^m\sigma_m)\left(ct - x^k\sigma_k\right) \\
&= (ct + \mathbf{x})(ct - \mathbf{x}) \\
&= (ct)^2 - \mathbf{x}^2
\end{aligned}
\tag{89.2}
$$

and we expect this of the Lorentz boost of eq. (89.1). To verify we have

$$
\begin{aligned}
L(x)^2 &= e^{-\alpha\hat{\mathbf{v}}/2} x e^{\alpha\hat{\mathbf{v}}/2} e^{-\alpha\hat{\mathbf{v}}/2} x e^{\alpha\hat{\mathbf{v}}/2} \\
&= e^{-\alpha\hat{\mathbf{v}}/2} x x e^{\alpha\hat{\mathbf{v}}/2} \\
&= x^2 e^{-\alpha\hat{\mathbf{v}}/2} e^{\alpha\hat{\mathbf{v}}/2} \\
&= x^2
\end{aligned}
\tag{89.3}
$$

89.2.3 *Sign of the rapidity angle*

The factor α will be the rapidity angle, but what sign do we want for a boost along the positive $\hat{\mathbf{v}}$ direction?

Dropping to coordinates is an easy way to determine the sign convention in effect. Write $\hat{\mathbf{v}} = \sigma_1$

$$
\begin{aligned}
L(x) &= e^{-\alpha\hat{\mathbf{v}}/2} x e^{\alpha\hat{\mathbf{v}}/2} \\
&= (\cosh(\alpha/2) - \sigma_1 \sinh(\alpha/2))(x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3)(\cosh(\alpha/2) + \sigma_1 \sinh(\alpha/2))
\end{aligned}
\tag{89.4}
$$

σ_1 commutes with γ_2 and γ_3 and anticommutes otherwise, so we have

$$
\begin{aligned}
L(x) &= \left(x^2\gamma_2 + x^3\gamma_3\right) e^{-\alpha\hat{\mathbf{v}}/2} e^{\alpha\hat{\mathbf{v}}/2} + \left(x^0\gamma_0 + x^1\gamma_1\right) e^{\alpha\hat{\mathbf{v}}} \\
&= x^2\gamma_2 + x^3\gamma_3 + \left(x^0\gamma_0 + x^1\gamma_1\right) e^{\alpha\hat{\mathbf{v}}} \\
&= x^2\gamma_2 + x^3\gamma_3 + \left(x^0\gamma_0 + x^1\gamma_1\right)(\cosh(\alpha) + \sigma_1 \sinh(\alpha))
\end{aligned}
\tag{89.5}
$$

Expanding out just the 0, 1 terms changed by the transformation we have

$$
\begin{aligned}
&\left(x^0\gamma_0 + x^1\gamma_1\right)(\cosh(\alpha) + \sigma_1 \sinh(\alpha)) \\
&\quad = x^0\gamma_0 \cosh(\alpha) + x^1\gamma_1 \cosh(\alpha) + x^0\gamma_0\sigma_1 \sinh(\alpha) + x^1\gamma_1\sigma_1 \sinh(\alpha) \\
&\quad = x^0\gamma_0 \cosh(\alpha) + x^1\gamma_1 \cosh(\alpha) + x^0\gamma_0\gamma_1\gamma_0 \sinh(\alpha) + x^1\gamma_1\gamma_1\gamma_0 \sinh(\alpha) \\
&\quad = x^0\gamma_0 \cosh(\alpha) + x^1\gamma_1 \cosh(\alpha) - x^0\gamma_1 \sinh(\alpha) - x^1\gamma_0 \sinh(\alpha) \\
&\quad = \gamma_0(x^0 \cosh(\alpha) - x^1 \sinh(\alpha)) + \gamma_1(x^1 \cosh(\alpha) - x^0 \sinh(\alpha))
\end{aligned}
$$

Writing $x^{\mu\prime} = L(x) \cdot \gamma^\mu$, and $x^\mu = x \cdot \gamma^\mu$, and a substitution of $\cosh(\alpha) = 1/\sqrt{1 - \mathbf{v}^2/c^2}$, and $\alpha\hat{\mathbf{v}} = \tanh^{-1}(\mathbf{v}/c)$, we have the traditional coordinate expression for the one directional Lorentz boost

$$
\begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix}
=
\begin{bmatrix}
\cosh\alpha & -\sinh\alpha & 0 & 0 \\
-\sinh\alpha & \cosh\alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}
\tag{89.6}
$$

Performing this expansion showed initially showed that I had the wrong sign for α in the exponentials and I went back and adjusted it all accordingly.

89.2.4 *Expanding out the Lorentz boost for projective and rejective directions*

Two forms of Lorentz boost representations have been compared above. An additional one is used in the Bohm text (a vector form of the Lorentz transformation not using coordinates). Let us see if we can derive that from the exponential form.

Start with computation of components of a four vector relative to an observer timelike unit vector γ_0.

$$
\begin{aligned}
x &= x\gamma_0\gamma_0 \\
&= (x\gamma_0)\gamma_0 \\
&= \left(x \cdot \gamma_0 + x \wedge \gamma_0\right)\gamma_0
\end{aligned}
\tag{89.7}
$$

For the spatial vector factor above write $\mathbf{x} = x \wedge \gamma_0$, for

$$
\begin{aligned}
x &= \left(x \cdot \gamma_0\right)\gamma_0 + \mathbf{x}\gamma_0 \\
&= \left(x \cdot \gamma_0\right)\gamma_0 + \mathbf{x}\hat{\mathbf{v}}\hat{\mathbf{v}}\gamma_0 \\
&= \left(x \cdot \gamma_0\right)\gamma_0 + (\mathbf{x} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + (\mathbf{x} \wedge \hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0
\end{aligned}
\tag{89.8}
$$

We have the following commutation relations for the various components

$$\begin{aligned}\hat{\mathbf{v}}(\gamma_0) &= -\gamma_0\hat{\mathbf{v}} \\ \hat{\mathbf{v}}(\hat{\mathbf{v}}\gamma_0) &= -(\hat{\mathbf{v}}\gamma_0)\hat{\mathbf{v}} \\ \hat{\mathbf{v}}((\mathbf{x}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0) &= ((\mathbf{x}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0)\hat{\mathbf{v}}\end{aligned} \tag{89.9}$$

For a four vector u that commutes with $\hat{\mathbf{v}}$ we have $e^{-\alpha\hat{\mathbf{v}}/2}u = ue^{-\alpha\hat{\mathbf{v}}/2}$, and if it anticommutes we have the conjugate relation $e^{-\alpha\hat{\mathbf{v}}/2}u = ue^{\alpha\hat{\mathbf{v}}/2}$. This gives us

$$L(x) = (\mathbf{x}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + \big((x\cdot\gamma_0)\gamma_0 + (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0\big)\, e^{\alpha\hat{\mathbf{v}}} \tag{89.10}$$

Now write the exponential as a scalar and spatial vector sum

$$\begin{aligned}e^{\alpha\hat{\mathbf{v}}} &= \cosh\alpha + \hat{\mathbf{v}}\sinh\alpha \\ &= \gamma(1+\hat{\mathbf{v}}\tanh\alpha) \\ &= \gamma(1+\hat{\mathbf{v}}\beta) \\ &= \gamma(1+\mathbf{v}/c)\end{aligned} \tag{89.11}$$

Expanding out the exponential product above, also writing $x^0 = ct = x\cdot\gamma_0$, we have

$$\begin{aligned}&(x^0\gamma_0 + (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0)e^{\alpha\hat{\mathbf{v}}} \\ &\quad= \gamma(x^0\gamma_0 + (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0)(1+\mathbf{v}/c) \\ &\quad= \gamma(x^0\gamma_0 + (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + x^0\gamma_0\mathbf{v}/c + (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0\mathbf{v}/c)\end{aligned} \tag{89.12}$$

So for the total Lorentz boost in vector form we have

$$L(x) = (\mathbf{x}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}\gamma_0 + \gamma\left(x^0 - \mathbf{x}\cdot\frac{\mathbf{v}}{c}\right)\gamma_0 + \gamma\left(\mathbf{x}\cdot\frac{1}{\mathbf{v}/c} - x^0\right)\frac{\mathbf{v}}{c}\gamma_0 \tag{89.13}$$

Now a visual inspection shows that this does match equation (15-12) from the text:

$$\begin{aligned}\mathbf{x}' &= \mathbf{x} - (\hat{\mathbf{v}}\cdot\mathbf{x})\hat{\mathbf{v}} + \frac{(\hat{\mathbf{v}}\cdot\mathbf{x})\hat{\mathbf{v}} - \mathbf{v}t}{\sqrt{1-(v^2/c^2)}} \\ t' &= \frac{t - (\mathbf{v}\cdot\mathbf{x})/c^2}{\sqrt{1-(v^2/c^2)}}\end{aligned} \tag{89.14}$$

but the equivalence of these is perhaps not so obvious without familiarity with the GA constructs.

89.2.5 *differential form*

Bohm utilizes a vector differential form of the Lorentz transformation for both the spacetime and energy-momentum vectors. From equation eq. (89.14) we can derive the expressions used. In particular for the transformed spatial component we have

$$\begin{aligned}
\mathbf{x}' &= \mathbf{x} + \gamma\left(-(\hat{\mathbf{v}}\cdot\mathbf{x})\hat{\mathbf{v}}\frac{1}{\gamma} + (\hat{\mathbf{v}}\cdot\mathbf{x})\hat{\mathbf{v}} - \mathbf{v}t\right) \\
&= \mathbf{x} + \gamma\left((\hat{\mathbf{v}}\cdot\mathbf{x})\hat{\mathbf{v}}\left(1 - \frac{1}{\gamma}\right) - \mathbf{v}t\right) \\
&= \mathbf{x} + (\gamma - 1)(\hat{\mathbf{v}}\cdot\mathbf{x})\hat{\mathbf{v}} - \gamma\mathbf{v}t
\end{aligned} \tag{89.15}$$

So in differential vector form we have

$$\begin{aligned}
d\mathbf{x}' &= d\mathbf{x} + (\gamma - 1)(\hat{\mathbf{v}}\cdot d\mathbf{x})\hat{\mathbf{v}} - \gamma\mathbf{v}dt \\
dt' &= \gamma(dt - \left(\mathbf{v}\cdot d\mathbf{x}\right)/c^2)
\end{aligned} \tag{89.16}$$

and by analogy with $dx^0 = cdt \rightarrow dE/c$, and $d\mathbf{x} \rightarrow d\mathbf{p}$, we also have the energy momentum transformation

$$\begin{aligned}
d\mathbf{p}' &= d\mathbf{p} + (\gamma - 1)(\hat{\mathbf{v}}\cdot d\mathbf{p})\hat{\mathbf{v}} - \gamma\mathbf{v}dE/c^2 \\
dE' &= \gamma(dE - \mathbf{v}\cdot d\mathbf{p})
\end{aligned} \tag{89.17}$$

Reflecting on these forms of the Lorentz transformation, they are quite natural ways to express the vector results. The terms with γ factors are exactly what we are used to in the coordinate representation (transformation of only the time component and the projection of the spatial vector in the velocity direction), while the -1 part of the $(\gamma - 1)$ term just subtracts off the projection unaltered, leaving $d\mathbf{x} - (d\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} = (d\mathbf{x}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}$, the rejection from the $\hat{\mathbf{v}}$ direction.

89.3 LORENTZ FORCE TRANSFORMATION

Preliminaries out of the way, now we want to examine the transform of the electric and magnetic field as used in the Lorentz force equation. In CGS units as in the text we have

$$\begin{aligned}
\frac{d\mathbf{p}}{dt} &= q\left(\boldsymbol{\mathcal{E}} + \frac{\mathbf{v}}{c}\times\boldsymbol{\mathcal{H}}\right) \\
\frac{dE}{dt} &= q\boldsymbol{\mathcal{E}}\cdot\mathbf{v}
\end{aligned} \tag{89.18}$$

After writing this in differential form

$$
\begin{aligned}
d\mathbf{p} &= q\left(\mathcal{E}dt + \frac{d\mathbf{x}}{c} \times \mathcal{H}\right) \\
dE &= q\mathcal{E} \cdot d\mathbf{x}
\end{aligned}
\tag{89.19}
$$

and the transformed variation of this equation, also in differential form

$$
\begin{aligned}
d\mathbf{p}' &= q\left(\mathcal{E}'dt' + \frac{d\mathbf{x}'}{c} \times \mathcal{H}'\right) \\
dE' &= q\mathcal{E}' \cdot d\mathbf{x}'
\end{aligned}
\tag{89.20}
$$

A brute force insertion of the transform results of equations eq. (89.16), and eq. (89.17) into these is performed. This is mostly a mess of algebra.

While the Bohm book covers some of this, other parts are left for the reader. Do the whole thing here as an exercise.

89.3.1 *Transforming the Lorentz power equation*

Let us start with the energy rate equation in its entirety without interleaving the momentum calculation.

$$
\begin{aligned}
\frac{1}{q}dE' &= \mathcal{E}' \cdot d\mathbf{x}' \\
&= \mathcal{E}' \cdot \left(d\mathbf{x} + (\gamma - 1)(\hat{\mathbf{V}} \cdot d\mathbf{x})\hat{\mathbf{V}} - \gamma\mathbf{V}dt\right) \\
&= \mathcal{E}' \cdot d\mathbf{x} + (\gamma - 1)(\hat{\mathbf{V}} \cdot d\mathbf{x})\mathcal{E}' \cdot \hat{\mathbf{V}} - \gamma\mathcal{E}' \cdot \mathbf{V}dt \\
\frac{1}{q}\gamma(dE - \mathbf{V} \cdot d\mathbf{p}) &= \\
\gamma\mathcal{E} \cdot d\mathbf{x} - \gamma\mathbf{V} \cdot \left(\mathcal{E}dt + \frac{d\mathbf{x}}{c} \times \mathcal{H}\right) &= \\
\gamma\mathcal{E} \cdot d\mathbf{x} - \gamma\mathbf{V} \cdot \mathcal{E}dt - \gamma\frac{1}{c}d\mathbf{x} \cdot (\mathcal{H} \times \mathbf{V}) &=
\end{aligned}
\tag{89.21}
$$

Grouping dt and $d\mathbf{x}$ terms we have

$$
0 = d\mathbf{x} \cdot \left(\mathcal{E}' + (\gamma - 1)\hat{\mathbf{V}}(\mathcal{E}' \cdot \hat{\mathbf{V}}) - \gamma\mathcal{E} + \gamma(\mathcal{H} \times \mathbf{V}/c)\right) + dt\gamma\mathbf{V} \cdot (\mathcal{E} - \mathcal{E}')
\tag{89.22}
$$

Now the argument is that both the dt and $d\mathbf{x}$ factors must separately equal zero. Assuming that for now (but come back to this and think it through), and writing $\mathcal{E} = \mathcal{E}_\parallel + \mathcal{E}_\perp$ for the projective and rejective components of the field relative to the boost direction $\mathbf{V}$ (same for $\mathcal{H}$ and the transformed fields) we have from the dt term

$$\begin{aligned} 0 &= \mathbf{V} \cdot (\mathcal{E}_\parallel + \mathcal{E}_\perp - \mathcal{E}'_\parallel - \mathcal{E}'_\perp) \\ &= \mathbf{V} \cdot (\mathcal{E}_\parallel - \mathcal{E}'_\parallel) \end{aligned} \tag{89.23}$$

So we can conclude

$$\mathcal{E}'_\parallel = \mathcal{E}_\parallel \tag{89.24}$$

Now from the $d\mathbf{x}$ coefficient, we have

$$\begin{aligned} 0 &= \mathcal{E}'_\parallel + \mathcal{E}'_\perp + (\gamma - 1)\hat{\mathbf{V}}(\mathcal{E}'_\parallel \cdot \hat{\mathbf{V}}) - \gamma\mathcal{E}_\parallel - \gamma\mathcal{E}_\perp + \gamma(\mathcal{H}_\perp \times \mathbf{V}/c) \\ &= \underbrace{\boxed{\left(\mathcal{E}'_\parallel - \hat{\mathbf{V}}(\mathcal{E}'_\parallel \cdot \hat{\mathbf{V}})\right)}}_{\mathcal{E}'_\parallel - \mathcal{E}'_\parallel} + \mathcal{E}'_\perp - \gamma \underbrace{\boxed{\left(\mathcal{E}_\parallel - \hat{\mathbf{V}}(\mathcal{E}'_\parallel \cdot \hat{\mathbf{V}})\right)}}_{\mathcal{E}_\parallel - \mathcal{E}_\parallel} - \gamma\mathcal{E}_\perp + \gamma(\mathcal{H}_\perp \times \mathbf{V}/c) \end{aligned} \tag{89.25}$$

This now completely specifies the transformation properties of the electric field under a $\mathbf{V}$ boost

$$\begin{aligned} \mathcal{E}'_\perp &= \gamma\left(\mathcal{E}_\perp + \frac{\mathbf{V}}{c} \times \mathcal{H}_\perp\right) \\ \mathcal{E}'_\parallel &= \mathcal{E}_\parallel \end{aligned} \tag{89.26}$$

(it also confirms the typos in the text).

89.3.2 *Transforming the Lorentz momentum equation*

Now we do the exercise for the reader part, and express the transformed momentum differential of equation eq. (89.20) in terms of eq. (89.16)

$$\begin{aligned} \frac{1}{q}d\mathbf{p}' &= \mathcal{E}'dt' + \frac{d\mathbf{x}'}{c} \times \mathcal{H}' \\ &= \gamma\mathcal{E}'dt - \gamma\mathcal{E}'(\mathbf{V} \cdot d\mathbf{x})/c^2 + d\mathbf{x} \times \mathcal{H}'/c + (\gamma - 1)(\hat{\mathbf{V}} \cdot d\mathbf{x})\hat{\mathbf{V}} \times \mathcal{H}'/c - \gamma\mathbf{V} \times \mathcal{H}'/cdt \end{aligned} \tag{89.27}$$

Now for the LHS using eq. (89.17) and eq. (89.19) we have

$$\begin{aligned}
\frac{1}{q} d\mathbf{p}' &= d\mathbf{p}/q + (\gamma - 1)(\hat{\mathbf{V}} \cdot d\mathbf{p}/q)\hat{\mathbf{V}} - \gamma \mathbf{V} dE/qc^2 \\
&= \mathcal{E} dt + \frac{d\mathbf{x}}{c} \times \mathcal{H} + (\gamma - 1)(\hat{\mathbf{V}} \cdot \mathcal{E} dt + \hat{\mathbf{V}} \cdot (d\mathbf{x} \times \mathcal{H}/c))\hat{\mathbf{V}} - \gamma \mathbf{V}(\mathcal{E} \cdot d\mathbf{x})/c^2 \\
&= \mathcal{E} dt + \frac{d\mathbf{x}}{c} \times \mathcal{H} + (\gamma - 1)(\hat{\mathbf{V}} \cdot \mathcal{E})\hat{\mathbf{V}} dt + (\gamma - 1)(d\mathbf{x} \cdot (\mathcal{H} \times \hat{\mathbf{V}}/c))\hat{\mathbf{V}} - \gamma \mathbf{V}(\mathcal{E} \cdot d\mathbf{x})/c^2
\end{aligned} \tag{89.28}$$

Combining these and grouping by dt and $d\mathbf{x}$ we have

$$\begin{aligned}
&dt\left(-(\mathcal{E} - (\hat{\mathbf{V}} \cdot \mathcal{E})\hat{\mathbf{V}}) + \gamma(\mathcal{E}' - (\hat{\mathbf{V}} \cdot \mathcal{E})\hat{\mathbf{V}}) - \gamma \mathbf{V} \times \mathcal{H}'/c\right) \\
&\quad = \frac{\gamma}{c^2}\left(\mathcal{E}'(\mathbf{V} \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E} \cdot d\mathbf{x})\right) + \frac{d\mathbf{x}}{c} \times (\mathcal{H} - \mathcal{H}') \\
&\qquad + \frac{\gamma - 1}{c}\left((d\mathbf{x} \cdot (\mathcal{H} \times \hat{\mathbf{V}}))\hat{\mathbf{V}} - (\hat{\mathbf{V}} \cdot d\mathbf{x})(\hat{\mathbf{V}} \times \mathcal{H}')\right)
\end{aligned} \tag{89.29}$$

What a mess, and this is after some initial grouping! From the power result we have $\hat{\mathbf{V}} \cdot \mathcal{E} = \hat{\mathbf{V}} \cdot \mathcal{E}'$ so we can write the LHS of this mess as

$$\begin{aligned}
&dt\left(-(\mathcal{E} - (\hat{\mathbf{V}} \cdot \mathcal{E})\hat{\mathbf{V}}) + \gamma(\mathcal{E}' - (\hat{\mathbf{V}} \cdot \mathcal{E})\hat{\mathbf{V}}) - \gamma \mathbf{V} \times \mathcal{H}'/c\right) \\
&\quad = dt\left(-(\mathcal{E} - (\hat{\mathbf{V}} \cdot \mathcal{E})\hat{\mathbf{V}}) + \gamma(\mathcal{E}' - (\hat{\mathbf{V}} \cdot \mathcal{E}')\hat{\mathbf{V}}) - \gamma \mathbf{V} \times \mathcal{H}'/c\right) \\
&\quad = dt\left(-\mathcal{E}_\perp + \gamma \mathcal{E}'_\perp - \gamma \mathbf{V} \times \mathcal{H}'/c\right) \\
&\quad = dt\left(-\mathcal{E}_\perp + \gamma \mathcal{E}'_\perp - \gamma \mathbf{V} \times \mathcal{H}'_\perp/c\right)
\end{aligned} \tag{89.30}$$

If this can separately equal zero independent of the $d\mathbf{x}$ terms we have

$$\mathcal{E}_\perp = \gamma\left(\mathcal{E}'_\perp - \frac{\mathbf{V}}{c} \times \mathcal{H}'_\perp\right) \tag{89.31}$$

Contrast this to the result for $\mathcal{E}'_\perp$ in the first of eq. (89.26). It differs only by a sign which has an intuitive relativistic (anti)symmetry that is not entirely unsurprising. If a boost along $\mathbf{V}$ takes $\mathcal{E}$ to $\mathcal{E}'$, then an boost with opposing direction makes sense for the reverse.

Despite being reasonable seeming, a relation like $\mathcal{H}_\parallel = \mathcal{H}'_\parallel$ was expected ... does that follow from this somehow? Perhaps things will become more clear after examining the mess on the RHS involving all the $d\mathbf{x}$ terms?

The first part of this looks amenable to some algebraic manipulation. Using $(\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} = \mathcal{E}'(\mathbf{V} \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E}' \cdot d\mathbf{x})$, we have

$$\begin{aligned}
\mathcal{E}'(\mathbf{V} \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E} \cdot d\mathbf{x}) &= (\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} + \mathbf{V}(\mathcal{E}' \cdot d\mathbf{x}) - \mathbf{V}(\mathcal{E} \cdot d\mathbf{x}) \\
&= (\mathcal{E}' \wedge \mathbf{V}) \cdot d\mathbf{x} + \mathbf{V}((\mathcal{E}' - \mathcal{E}) \cdot d\mathbf{x})
\end{aligned} \tag{89.32}$$

and

$$
\begin{aligned}
(\boldsymbol{\mathcal{E}}' \wedge \mathbf{V}) \cdot d\mathbf{x} &= \left\langle (\boldsymbol{\mathcal{E}}' \wedge \mathbf{V}) d\mathbf{x} \right\rangle_1 \\
&= \left\langle i(\boldsymbol{\mathcal{E}}' \times \mathbf{V}) d\mathbf{x} \right\rangle_1 \\
&= \left\langle i((\boldsymbol{\mathcal{E}}' \times \mathbf{V}) \wedge d\mathbf{x}) \right\rangle_1 \\
&= \left\langle i^2((\boldsymbol{\mathcal{E}}' \times \mathbf{V}) \times d\mathbf{x}) \right\rangle_1 \\
&= d\mathbf{x} \times (\boldsymbol{\mathcal{E}}' \times \mathbf{V})
\end{aligned}
\tag{89.33}
$$

Putting things back together, does it improve things?

$$
\begin{aligned}
0 &= d\mathbf{x} \times \left(\gamma \left(\boldsymbol{\mathcal{E}}' \times \frac{\mathbf{V}}{c} \right) + (\boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}') \right) \\
&\quad + \frac{\gamma}{c} \mathbf{V}((\boldsymbol{\mathcal{E}}' - \boldsymbol{\mathcal{E}}) \cdot d\mathbf{x}) \\
&\quad + (\gamma - 1) \left((d\mathbf{x} \cdot (\boldsymbol{\mathcal{H}} \times \hat{\mathbf{V}})) \hat{\mathbf{V}} - (\hat{\mathbf{V}} \cdot d\mathbf{x})(\hat{\mathbf{V}} \times \boldsymbol{\mathcal{H}}') \right)
\end{aligned}
\tag{89.34}
$$

Perhaps the last bit can be factored into $d\mathbf{x}$ crossed with some function of $\boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}'$?

90

LORENTZ FORCE LAW

90.1 SOME NOTES ON GAFP 5.5.3 THE LORENTZ FORCE LAW

Expand on treatment of [10].

The idea behind this derivation, is to express the vector part of the proper force in covariant form, and then do the same for the energy change part of the proper momentum. That first part is:

$$
\begin{aligned}
\frac{dp}{d\tau} \wedge \gamma_0 &= \frac{d(\gamma \mathbf{p})}{d\tau} \\
&= \frac{d(\gamma \mathbf{p})}{dt} \frac{dt}{d\tau} \\
&= \frac{dt}{d\tau} q \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)
\end{aligned}
\tag{90.1}
$$

Now, the spacetime split of velocity is done in the normal fashion:

$$
\begin{aligned}
x &= ct\gamma_0 + \sum x^i \gamma_i \\
v &= \frac{dx}{d\tau} = c\frac{dt}{d\tau}\gamma_0 + \sum \frac{dx^i}{d\tau}\gamma_i \\
v \cdot \gamma_0 &= c\frac{dt}{d\tau} = c\gamma \\
v \wedge \gamma_0 &= \sum \frac{dx^i}{dt}\frac{dt}{d\tau}\gamma_i\gamma_0 \\
&= (v \cdot \gamma_0)/c \sum v^i \sigma_i \\
&= (v \cdot \gamma_0)\mathbf{v}/c.
\end{aligned}
\tag{90.2}
$$

Writing $\dot{p} = dp/d\tau$, substitute the gamma factor into the force equation:

$$
\dot{p} \wedge \gamma_0 = (v/c \cdot \gamma_0) q \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)
$$

Now, GAFP goes on to show that the $\gamma\mathbf{E}$ term can be reduced to the form $(\mathbf{E}\cdot v)\wedge\gamma_0$. Their method is not exactly obvious, for example writing $\mathbf{E} = (1/2)(\mathbf{E}+\mathbf{E})$ to start. Let us just do this backwards instead, expanding $\mathbf{E}\cdot v$ to see the form of that term:

$$
\begin{aligned}
\mathbf{E}\cdot v &= \left(\sum E^i\gamma_{i0}\right)\cdot\left(\sum v^\mu\gamma_\mu\right) \\
&= \sum E^i v^\mu \left\langle \gamma_{i0\mu} \right\rangle_1 \\
&= v^0 \sum E^i\gamma_i + \sum E^i v^j \overbrace{\boxed{\left\langle \gamma_{i0j} \right\rangle_1}}^{-\delta_{ij}\gamma_0} \\
&= v^0 \sum E^i\gamma_i - \sum E^i v^i \gamma_0.
\end{aligned}
\tag{90.3}
$$

Wedging with γ_0 we have the desired result:

$$
(\mathbf{E}\cdot v)\wedge\gamma_0 = v^0\sum E^i\gamma_{i0} = (v\cdot\gamma_0)\mathbf{E} = c\gamma\mathbf{E}
$$

Now, for equation 5.164 there are not any surprising steps, but lets try this backwards too:

$$
\begin{aligned}
(I\mathbf{B})\cdot v &= \left(\sum B^i \overbrace{\boxed{\gamma_{102030i0}}}^{\gamma_{123i}}\right)\cdot\left(\sum v^\mu\gamma_\mu\right) \\
&= \sum B^i v^\mu \left\langle \gamma_{123i\mu} \right\rangle_1
\end{aligned}
\tag{90.4}
$$

That vector selection does yield the cross product as expected:

$$
\left\langle \gamma_{123i\mu} \right\rangle_1 = \begin{cases} 0 & \mu = 0 \\ 0 & i = \mu \\ \gamma_1 & i\mu = 32 \\ -\gamma_2 & i\mu = 31 \\ \gamma_3 & i\mu = 21 \end{cases}
$$

(with alternation for the missing set of index pairs).
This gives:

$$
(I\mathbf{B})\cdot v = (B^3v^2 - B^2v^3)\gamma_1 + (B^1v^3 - B^3v^1)\gamma_2 + (B^2v^1 - B^1v^2)\gamma_3,
\tag{90.5}
$$

thus, since $v^i = \gamma dx^i/dt$, this yields the desired result

$$((I\mathbf{B}) \cdot v) \wedge \gamma_0 = \gamma \mathbf{v} \times \mathbf{B}$$

In retrospect, for this magnetic field term, the GAFP approach is cleaner and easier than to try to do it the dumb way.

Combining the results we have:

$$\begin{aligned}
\dot{p} \wedge \gamma_0 &= q\gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= q((\mathbf{E} + cI\mathbf{B}) \cdot (v/c)) \wedge \gamma_0
\end{aligned} \tag{90.6}$$

Or with $F = \mathbf{E} + cI\mathbf{B}$, we have:

$$\dot{p} \wedge \gamma_0 = q(F \cdot v/c) \wedge \gamma_0 \tag{90.7}$$

It is tempting here to attempt to cancel the $\wedge\gamma_0$ parts of this equation, but that cannot be done until one also shows:

$$\dot{p} \cdot \gamma_0 = q(F \cdot v/c) \cdot \gamma_0$$

I follow most of the details of GAFP on this fine. I found they omitted a couple steps that would have been helpful.

For the four momentum we have:

$$p_0 = p \cdot \gamma_0 = E/c$$

The rate of change work done on the particle by the force is:

$$\begin{aligned}
dW &= q\mathbf{E} \cdot d\mathbf{x} \\
\frac{dW}{dt} &= q\mathbf{E} \cdot \frac{d\mathbf{x}}{dt} = c\frac{dp_0}{dt} \\
\frac{dp_0}{dt} &= q\mathbf{E} \cdot \mathbf{v}/c \\
\frac{dp_0}{d\tau} &= \overset{v/c \cdot \gamma_0}{\boxed{\frac{dt}{d\tau}}} q\mathbf{E} \cdot \left(\frac{v \wedge \gamma_0}{v \cdot \gamma_0}\right) \\
&= q\mathbf{E} \cdot (v/c \wedge \gamma_0) \\
&= q\left(\mathbf{E} + cI\mathbf{B}\right) \cdot (v/c \wedge \gamma_0)
\end{aligned} \tag{90.8}$$

$I\mathbf{B}$ has only purely spatial bivectors, γ_{12}, γ_{13}, and γ_{23}. On the other hand $v \wedge \gamma_0 = \sum v^i \gamma_{i0}$ has only spacetime bivectors, so $I\mathbf{B} \cdot (v/c \wedge \gamma_0) = 0$, which is why it can be added above to complete the field.

That leaves:

$$\frac{dp_0}{d\tau} = qF \cdot \left(v/c \wedge \gamma_0\right), \tag{90.9}$$

but we want to put this in the same form as eq. (90.7). To do so, note how we can reduce the dot product of two bivectors:

$$\begin{aligned}(a \wedge b) \cdot (c \wedge d) &= \langle (a \wedge b)(c \wedge d) \rangle \\ &= \langle (a \wedge b)(cd - c \cdot d) \rangle \\ &= \langle ((a \wedge b) \cdot c)d + ((a \wedge b) \wedge c)d \rangle \\ &= ((a \wedge b) \cdot c) \cdot d.\end{aligned} \tag{90.10}$$

Using this, and adding the result to eq. (90.7) we have:

$$\dot{p} \cdot \gamma_0 + \dot{p} \wedge \gamma_0 = q(F \cdot v/c) \cdot \gamma_0 + q(F \cdot v/c) \wedge \gamma_0$$

Or

$$\dot{p}\gamma_0 = q(F \cdot v/c)\gamma_0$$

Right multiplying by γ_0 on both sides to cancel those terms we have our end result, the covariant form of the Lorentz proper force equation:

$$\dot{p} = q(F \cdot v/c) \tag{90.11}$$

90.2 LORENTZ FORCE IN TERMS OF FOUR POTENTIAL

If one expresses the Faraday bivector in terms of a spacetime curl of a potential vector:

$$F = \nabla \wedge A, \tag{90.12}$$

then inserting into eq. (90.11) we have:

$$\begin{aligned}\dot{p} &= q(F \cdot v/c) \\ &= q(\nabla \wedge A) \cdot v/c \\ &= q\left(\nabla(A \cdot v/c) - A(\nabla \cdot v/c)\right)\end{aligned} \tag{90.13}$$

Let us look at that proper velocity divergence term:

$$\begin{aligned}
\nabla \cdot v/c &= \frac{1}{c}\left(\nabla \cdot \frac{dx}{d\tau}\right) \\
&= \frac{1}{c}\frac{d}{d\tau}\nabla \cdot x \\
&= \frac{1}{c}\frac{d}{d\tau}\sum \frac{\partial x^\mu}{\partial x^\mu} \\
&= \frac{1}{c}\frac{d4}{d\tau} \\
&= 0
\end{aligned} \tag{90.14}$$

This leaves the proper Lorentz force expressible as the (spacetime) gradient of a scalar quantity:

$$\dot{p} = q\nabla(A \cdot v/c) \tag{90.15}$$

I believe this dot product is likely an invariant of electromagnetism. Looking from the rest frame one has:

$$\dot{p} = q\nabla A^0 = q\sum \gamma^\mu \partial_\mu A^0 = \sum E^i \gamma_i \tag{90.16}$$

Wedging with γ_0 to calculate $\mathbf{E} = \sum E^i \gamma_i$, we have:

$$q\sum -\gamma_{i0}\partial_i A^0 = -q\boldsymbol{\nabla} A^0$$

So we want to identify this component of the four vector potential with electrostatic potential:

$$A^0 = \phi \tag{90.17}$$

90.3 EXPLICIT EXPANSION OF POTENTIAL SPACETIME CURL IN COMPONENTS

Having used the gauge condition $\nabla \cdot A = 0$, to express the Faraday bivector as a gradient, we should be able to verify that this produces the familiar equations for $\mathbf{E}$, and $\mathbf{B}$ in terms of ϕ, and $\mathbf{A}$.

First lets do the electric field components, which are easier.

With $F = E + icB = \nabla \wedge A$, we calculate $\mathbf{E} = \sum \sigma_i E^i = \sum \gamma_{i0} E^i$.

$$\begin{aligned}
E^i &= F \cdot \left(\gamma^0 \wedge \gamma^i\right) = F \cdot \gamma^{0i} \\
&= \left(\sum \gamma^\mu \partial_\mu \wedge \gamma_\nu A^\nu\right) \cdot \gamma^{0i} \\
&= \sum \partial_\mu A^\nu \gamma^\mu{}_\nu \cdot \gamma^{0i} \\
&= \partial_0 A^i \gamma^0{}_i \cdot \gamma^{0i} + \partial_i A^0 \gamma^i{}_0 \cdot \gamma^{0i} \\
&= -\left(\partial_0 A^i + \partial_i A^0\right) \\
\sum E^i \sigma_i &= -\left(\partial_{ct} \sum \sigma_i A^i + \sum \sigma_i \partial_i A^0\right) \\
&= -\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla} A^0\right)
\end{aligned} \tag{90.18}$$

Again we see that we should identify $A^0 = \phi$, and write:

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\phi \tag{90.19}$$

Now, let us calculate the magnetic field components (setting $c = 1$ temporarily):

$$\begin{aligned}
i\mathbf{B} &= \sigma_{123} \sum \sigma_i B^i \\
&= \sum \sigma_{123i} B^i \\
&= \sigma_{1231} B^1 + \sigma_{1232} B^2 + \sigma_{1233} B^3 \\
&= \sigma_{23} B^1 + \sigma_{31} B^2 + \sigma_{12} B^3 \\
&= \gamma_{2030} B^1 + \gamma_{3010} B^2 + \gamma_{1020} B^3 \\
&= \gamma_{32} B^1 + \gamma_{13} B^2 + \gamma_{21} B^3
\end{aligned} \tag{90.20}$$

Thus, we can calculate the magnetic field components with:

$$\begin{aligned}
B^1 &= F \cdot \gamma^{23} \\
B^2 &= F \cdot \gamma^{31} \\
B^3 &= F \cdot \gamma^{12}
\end{aligned} \tag{90.21}$$

Here the components of F of interest are: $\gamma^i \wedge \gamma_j \partial_i A^j = -\gamma_{ij}\partial_i A^j$.

$$
\begin{aligned}
B^1 &= -\partial_2 A^3 \gamma_{23} \cdot \gamma^{23} - \partial_3 A^2 \gamma_{32} \cdot \gamma^{23} \\
B^2 &= -\partial_3 A^1 \gamma_{31} \cdot \gamma^{31} - \partial_1 A^3 \gamma_{13} \cdot \gamma^{31} \\
B^3 &= -\partial_1 A^2 \gamma_{12} \cdot \gamma^{12} - \partial_2 A^1 \gamma_{21} \cdot \gamma^{12} \\
\Longrightarrow & \\
B^1 &= \partial_2 A^3 - \partial_3 A^2 \\
B^2 &= \partial_3 A^1 - \partial_1 A^3 \\
B^3 &= \partial_1 A^2 - \partial_2 A^1
\end{aligned}
\tag{90.22}
$$

Or, with $\mathbf{A} = \sum \sigma_i A^i$ and $\boldsymbol{\nabla} = \sum \sigma_i \partial_i$, this is our familiar:

$$
\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}
\tag{90.23}
$$

91 LORENTZ FORCE ROTOR FORMULATION

91.1 MOTIVATION

Both [1] and [10] cover rotor formulations of the Lorentz force equation. Work through some of this on my own to better understand it.

91.2 IN TERMS OF GA

An active Lorentz transformation can be used to translate from the rest frame of a particle with worldline x to an observer frame, as in

$$y = \Lambda x \tilde{\Lambda} \tag{91.1}$$

Here Lorentz transformation is used in the general sense, and can include both spatial rotation and boost effects, but satisfies $\Lambda\tilde{\Lambda} = 1$. Taking proper time derivatives we have

$$\begin{aligned}\dot{y} &= \dot{\Lambda} x \tilde{\Lambda} + \Lambda x \dot{\tilde{\Lambda}} \\ &= \Lambda \left(\tilde{\Lambda}\dot{\Lambda}\right) x \tilde{\Lambda} + \Lambda x \left(\dot{\tilde{\Lambda}}\Lambda\right) \tilde{\Lambda}\end{aligned} \tag{91.2}$$

Since $\tilde{\Lambda}\Lambda = \Lambda\tilde{\Lambda} = 1$ we also have

$$\begin{aligned}0 &= \dot{\Lambda}\tilde{\Lambda} + \Lambda\dot{\tilde{\Lambda}} \\ 0 &= \tilde{\Lambda}\dot{\Lambda} + \dot{\tilde{\Lambda}}\Lambda\end{aligned} \tag{91.3}$$

Here is where a bivector variable

$$\Omega/2 = \tilde{\Lambda}\dot{\Lambda} \tag{91.4}$$

is introduced, from which we have $\dot{\tilde{\Lambda}}\Lambda = -\Omega/2$, and

$$\dot{y} = \frac{1}{2}\left(\Lambda\Omega x \tilde{\Lambda} - \Lambda x \Omega \tilde{\Lambda}\right) \tag{91.5}$$

Or

$$\tilde{\Lambda}\dot{y}\Lambda = \frac{1}{2}(\Omega x - x\Omega) \tag{91.6}$$

The inclusion of the factor of two in the definition of Ω was cheating, so that we get the bivector vector dot product above. Presuming Ω is really a bivector (return to this in a bit), we then have

$$\tilde{\Lambda}\dot{y}\Lambda = \Omega \cdot x \tag{91.7}$$

We can express the time evolution of y using this as a stepping stone, since we have

$$\tilde{\Lambda}y\Lambda = x \tag{91.8}$$

Using this we have

$$\begin{aligned}
0 &= \left\langle \tilde{\Lambda}\dot{y}\Lambda - \Omega \cdot x \right\rangle_1 \\
&= \left\langle \tilde{\Lambda}\dot{y}\Lambda - \Omega x \right\rangle_1 \\
&= \left\langle \tilde{\Lambda}\dot{y}\Lambda - \Omega\tilde{\Lambda}y\Lambda \right\rangle_1 \\
&= \left\langle \left(\tilde{\Lambda}\dot{y} - \tilde{\Lambda}\Lambda\Omega\tilde{\Lambda}y\right)\Lambda \right\rangle_1 \\
&= \left\langle \tilde{\Lambda}\left(\dot{y} - \Lambda\Omega\tilde{\Lambda}y\right)\Lambda \right\rangle_1
\end{aligned} \tag{91.9}$$

So we have the complete time evolution of our observer frame worldline for the particle, as a sort of an eigenvalue equation for the proper time differential operator

$$\dot{y} = \left(\Lambda\Omega\tilde{\Lambda}\right) \cdot y = \left(2\dot{\Lambda}\tilde{\Lambda}\right) \cdot y \tag{91.10}$$

Now, what Baylis did in his lecture, and what Doran/Lasenby did as well in the text (but I did not understand it then when I read it the first time) was to identify this time evolution in terms of Lorentz transform change with the Lorentz force.

Recall that the Lorentz force equation is

$$\dot{v} = \frac{e}{mc} F \cdot v \tag{91.11}$$

where $F = \mathbf{E} + ic\mathbf{B}$, like $\dot{\Lambda}\tilde{\Lambda}$ is also a bivector. If we write the velocity worldline of the particle in the lab frame in terms of the rest frame particle worldline as

$$v = \Lambda c t \gamma_0 \tilde{\Lambda} \tag{91.12}$$

Then for the field F observed in the lab frame we are left with a differential equation $2\dot{\Lambda}\tilde{\Lambda} = eF/mc$ for the Lorentz transformation that produces the observed motion of the particle given the field that acts on it

$$\dot{\Lambda} = \frac{e}{2mc} F\Lambda \tag{91.13}$$

Okay, good. I understand now well enough what they have done to reproduce the end result (with the exception of my result including a factor of c since they have worked with $c = 1$).

91.2.1 *Omega bivector*

It has been assumed above that $\Omega = 2\tilde{\Lambda}\dot{\Lambda}$ is a bivector. One way to confirm this is by examining the grades of this product. Two bivectors, not necessarily related can only have grades 0, 2, and 4. Because $\Omega = -\tilde{\Omega}$, as seen above, it can have no grade 0 or grade 4 parts.

While this is a powerful way to verify the bivector nature of this object it is fairly abstract. To get a better feel for this, let us consider this object in detail for a purely spatial rotation, such as

$$\begin{aligned} R_\theta(x) &= \Lambda x \tilde{\Lambda} \\ \Lambda &= \exp(-in\theta/2) = \cos(\theta/2) - in\sin(\theta/2) \end{aligned} \tag{91.14}$$

where n is a spatial unit bivector, $n^2 = 1$, in the span of $\{\sigma_k = \gamma_k\gamma_0\}$.

91.2.1.1 *Verify rotation form*

To verify that this has the appropriate action, by linearity two two cases must be considered. First is the action on n or the components of any vector in this direction.

$$\begin{aligned} R_\theta(n) &= \Lambda n \tilde{\Lambda} \\ &= \left(\cos(\theta/2) - in\sin(\theta/2)\right) n\tilde{\Lambda} \\ &= n\left(\cos(\theta/2) - in\sin(\theta/2)\right)\tilde{\Lambda} \\ &= n\Lambda\tilde{\Lambda} \\ &= n \end{aligned} \tag{91.15}$$

The rotation operator does not change any vector colinear with the axis of rotation (the normal). For a vector m that is perpendicular to axis of rotation n (ie: $2(m \cdot n) = mn + nm = 0$), we have

$$
\begin{aligned}
R_\theta(m) &= \Lambda m \tilde{\Lambda} \\
&= \left(\cos(\theta/2) - in \sin(\theta/2)\right) m \tilde{\Lambda} \\
&= \left(m \cos(\theta/2) - i(nm) \sin(\theta/2)\right) \tilde{\Lambda} \\
&= \left(m \cos(\theta/2) + i(mn) \sin(\theta/2)\right) \tilde{\Lambda} \\
&= m(\tilde{\Lambda})^2 \\
&= m \exp(in\theta)
\end{aligned}
\tag{91.16}
$$

This is a rotation of the vector m that lies in the in plane by θ as desired.

91.2.1.2 *The rotation bivector*

We want derivatives of the Λ object.

$$
\begin{aligned}
\dot{\Lambda} &= \frac{\dot{\theta}}{2} \left(-\sin(\theta/2) - in \cos(\theta/2)\right) - i\dot{n} \cos(\theta/2) \\
&= \frac{in\dot{\theta}}{2} \left(in \sin(\theta/2) - \cos(\theta/2)\right) - i\dot{n} \cos(\theta/2) \\
&= -\frac{1}{2} \exp(-in\theta/2) in\dot{\theta} - i\dot{n} \cos(\theta/2)
\end{aligned}
\tag{91.17}
$$

So we have

$$
\begin{aligned}
\Omega &= 2\tilde{\Lambda}\dot{\Lambda} \\
&= -in\dot{\theta} - 2 \exp(in\theta/2) i\dot{n} \cos(\theta/2) \\
&= -in\dot{\theta} - 2 \cos(\theta/2) \left(\cos(\theta/2) - in \sin(\theta/2)\right) i\dot{n} \\
&= -in\dot{\theta} - 2 \cos(\theta/2) \left(\cos(\theta/2) i\dot{n} + n\dot{n} \sin(\theta/2)\right)
\end{aligned}
\tag{91.18}
$$

Since $n \cdot \dot{n} = 0$, we have $n\dot{n} = n \wedge \dot{n}$, and sure enough all the terms are bivectors. Specifically we have

$$
\Omega = -\dot{\theta}(in) - (1 + \cos\theta)(i\dot{n}) - \sin\theta (n \wedge \dot{n})
\tag{91.19}
$$

91.2.2 *Omega bivector for boost*

TODO.

91.3 TENSOR VARIATION OF THE ROTOR LORENTZ FORCE RESULT

There is not anything in the initial Lorentz force rotor result that intrinsically requires geometric algebra. At least until one actually wants to express the Lorentz transformation concisely in terms of half angle or boost rapidity exponentials.

In fact the logic above is not much different than the approach used in [42] for rigid body motion. Let us try this in matrix or tensor form and see how it looks.

91.3.1 *Tensor setup*

Before anything else some notation for the tensor work must be established. Similar to eq. (91.1) write a Lorentz transformed vector as a linear transformation. Since we want only the matrix of this linear transformation with respect to a specific observer frame, the details of the transformation can be omitted for now. Write

$$y = \mathcal{L}(x) \tag{91.20}$$

and introduce an orthonormal frame $\{\gamma_\mu\}$, and the corresponding reciprocal frame $\{\gamma^\mu\}$, where $\gamma_\mu \cdot \gamma^\nu = {\delta_\mu}^\nu$. In this basis, the relationship between the vectors becomes

$$\begin{aligned} y^\mu \gamma_\mu &= \mathcal{L}(x^\nu \gamma_\nu) \\ &= x^\nu \mathcal{L}(\gamma_\nu) \end{aligned} \tag{91.21}$$

Or

$$y^\mu = x^\nu \mathcal{L}(\gamma_\nu) \cdot \gamma^\mu \tag{91.22}$$

The matrix of the linear transformation can now be written as

$${\Lambda_\nu}^\mu = \mathcal{L}(\gamma_\nu) \cdot \gamma^\mu \tag{91.23}$$

and this can now be used to express the coordinate transformation in abstract index notation

$$y^\mu = x^\nu {\Lambda_\nu}^\mu \tag{91.24}$$

Similarly, for the inverse transformation, we can write

$$\begin{aligned} x &= \mathcal{L}^{-1}(y) \\ {\Pi_\nu}^\mu &= \mathcal{L}^{-1}(\gamma_\nu) \cdot \gamma^\mu \\ x^\mu &= y^\nu {\Pi_\nu}^\mu \end{aligned} \tag{91.25}$$

I have seen this expressed using primed indices and the same symbol Λ used for both the forward and inverse transformation ... lacking skill in tricky index manipulation I have avoided such a notation because I will probably get it wrong. Instead different symbols for the two different matrices will be used here and Π was picked for the inverse rather arbitrarily.

With substitution

$$\begin{aligned} y^\mu &= x^\nu {\Lambda_\nu}^\mu = (y^\alpha {\Pi_\alpha}^\nu){\Lambda_\nu}^\mu \\ x^\mu &= y^\nu {\Pi_\nu}^\mu = (x^\alpha {\Lambda_\alpha}^\nu){\Pi_\nu}^\mu \end{aligned} \tag{91.26}$$

the pair of explicit inverse relationships between the two matrices can be read off as

$$ {\delta_\alpha}^\mu = {\Pi_\alpha}^\nu {\Lambda_\nu}^\mu = {\Lambda_\alpha}^\nu {\Pi_\nu}^\mu \tag{91.27}$$

91.3.2 *Lab frame velocity of particle in tensor form*

In tensor form we want to express the worldline of the particle in the lab frame coordinates. That is

$$\begin{aligned} v &= \mathcal{L}(ct\gamma_0) \\ &= \mathcal{L}(x^0\gamma_0) \\ &= x^0\mathcal{L}(\gamma_0) \end{aligned} \tag{91.28}$$

Or

$$\begin{aligned} v^\mu &= x^0\mathcal{L}(\gamma_0)\cdot\gamma^\mu \\ &= x^0{\Lambda_0}^\mu \end{aligned} \tag{91.29}$$

91.3.3 *Lorentz force in tensor form*

The Lorentz force equation eq. (91.11) in tensor form will also be needed. The bivector F is

$$ F = \frac{1}{2}F_{\mu\nu}\gamma^\mu \wedge \gamma^\nu \tag{91.30}$$

So we can write

$$
\begin{aligned}
F \cdot v &= \frac{1}{2} F_{\mu\nu} (\gamma^\mu \wedge \gamma^\nu) \cdot \gamma_\alpha v^\alpha \\
&= \frac{1}{2} F_{\mu\nu} (\gamma^\mu {\delta^\nu}_\alpha - \gamma^\nu {\delta^\mu}_\alpha) v^\alpha \\
&= \frac{1}{2} (v^\alpha F_{\mu\alpha} \gamma^\mu - v^\alpha F_{\alpha\nu} \gamma^\nu)
\end{aligned}
\tag{91.31}
$$

And

$$
\begin{aligned}
\dot{v}_\sigma &= \frac{e}{mc} (F \cdot v) \cdot \gamma_\sigma \\
&= \frac{e}{2mc} (v^\alpha F_{\mu\alpha} \gamma^\mu - v^\alpha F_{\alpha\nu} \gamma^\nu) \cdot \gamma_\sigma \\
&= \frac{e}{2mc} v^\alpha (F_{\sigma\alpha} - F_{\alpha\sigma}) \\
&= \frac{e}{mc} v^\alpha F_{\sigma\alpha}
\end{aligned}
\tag{91.32}
$$

Or

$$
\dot{v}^\sigma = \frac{e}{mc} v^\alpha {F^\sigma}_\alpha \tag{91.33}
$$

91.3.4 *Evolution of Lab frame vector*

Given a lab frame vector with all the (proper) time evolution expressed via the Lorentz transformation

$$
y^\mu = x^\nu {\Lambda_\nu}^\mu \tag{91.34}
$$

we want to calculate the derivatives as in the GA procedure

$$
\begin{aligned}
\dot{y}^\mu &= x^\nu \dot{\Lambda}_\nu^\mu \\
&= x^\alpha {\delta_\alpha}^\nu \dot{\Lambda}_\nu^\mu \\
&= x^\alpha {\Lambda_\alpha}^\beta {\Pi_\beta}^\nu \dot{\Lambda}_\nu^\mu
\end{aligned}
\tag{91.35}
$$

With $y = v$, this is

$$
\begin{aligned}
\dot{v}^\sigma &= v^\alpha {\Pi_\alpha}^\nu \dot{\Lambda}_\nu^\sigma \\
&= v^\alpha \frac{e}{mc} {F^\sigma}_\alpha
\end{aligned}
\tag{91.36}
$$

So we can make the identification of the bivector field with the Lorentz transformation matrix

$$\Pi_\alpha{}^\nu \dot{\Lambda}^\sigma_\nu = \frac{e}{mc} F^\sigma{}_\alpha \tag{91.37}$$

With an additional summation to invert we have

$$\Lambda_\beta{}^\alpha \Pi_\alpha{}^\nu \dot{\Lambda}^\sigma_\nu = \Lambda_\beta{}^\alpha \frac{e}{mc} F^\sigma{}_\alpha \tag{91.38}$$

This leaves a tensor differential equation that will provide the complete time evolution of the lab frame worldline for the particle in the field

$$\dot{\Lambda}^\nu_\mu = \frac{e}{mc} \Lambda_\mu{}^\alpha F^\nu{}_\alpha \tag{91.39}$$

This is the equivalent of the GA equation eq. (91.13). However, while the GA equation is directly integrable for constant F, how to do this in the equivalent tensor formulation is not so clear.

Want to revisit this, and try to perform this integral in both forms, ideally for both the simpler constant field case, as well as for a more general field. Even better would be to be able to express F in terms of the current density vector, and then treat the proper interaction of two charged particles.

91.4 GAUGE TRANSFORMATION FOR SPIN

In the Baylis article eq. (91.13) is transformed as $\Lambda \to \Lambda_{\omega_0} \exp(-i\mathbf{e}_3 \omega_0 \tau)$.

Using this we have

$$\begin{aligned}\dot{\Lambda} &\to \frac{d}{d\tau}\left(\Lambda_{\omega_0} \exp(-i\mathbf{e}_3\omega_0\tau)\right) \\ &= \dot{\Lambda}_{\omega_0} \exp(-i\mathbf{e}_3\omega_0\tau) - \Lambda_{\omega_0}(i\mathbf{e}_3\omega_0) \exp(-i\mathbf{e}_3\omega_0\tau)\end{aligned} \tag{91.40}$$

For the transformed eq. (91.13) this gives

$$\dot{\Lambda}_{\omega_0} \exp(-i\mathbf{e}_3\omega_0\tau) - \Lambda_{\omega_0}(i\mathbf{e}_3\omega_0) \exp(-i\mathbf{e}_3\omega_0\tau) = \frac{e}{2mc} F \Lambda_{\omega_0} \exp(-i\mathbf{e}_3\omega_0\tau) \tag{91.41}$$

Canceling the exponentials, and shuffling

$$\dot{\Lambda}_{\omega_0} = \frac{e}{2mc} F \Lambda_{\omega_0} + \Lambda_{\omega_0}(i\mathbf{e}_3\omega_0) \tag{91.42}$$

How does he commute the $i\mathbf{e}_3$ term with the Lorentz transform? How about instead transforming as $\Lambda \rightarrow \exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0}$.

Using this we have

$$\begin{aligned}\dot{\Lambda} &\rightarrow \frac{d}{d\tau}\left(\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0}\right) \\ &= \exp(-i\mathbf{e}_3\omega_0\tau)\dot{\Lambda}_{\omega_0} - (i\mathbf{e}_3\omega_0)\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0}\end{aligned} \tag{91.43}$$

then, the transformed eq. (91.13) gives

$$\exp(-i\mathbf{e}_3\omega_0\tau)\dot{\Lambda}_{\omega_0} - (i\mathbf{e}_3\omega_0)\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0} = \frac{e}{2mc}F\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0} \tag{91.44}$$

Multiplying by the inverse exponential, and shuffling, noting that $\exp(i\mathbf{e}_3\alpha)$ commutes with $i\mathbf{e}_3$, we have

$$\begin{aligned}\dot{\Lambda}_{\omega_0} &= (i\mathbf{e}_3\omega_0)\Lambda_{\omega_0} + \frac{e}{2mc}\exp(i\mathbf{e}_3\omega_0\tau)F\exp(-i\mathbf{e}_3\omega_0\tau)\Lambda_{\omega_0} \\ &= \frac{e}{2mc}\left(\frac{2mc}{e}(i\mathbf{e}_3\omega_0) + \exp(i\mathbf{e}_3\omega_0\tau)F\exp(-i\mathbf{e}_3\omega_0\tau)\right)\Lambda_{\omega_0}\end{aligned} \tag{91.45}$$

So, if one writes $F_{\omega_0} = \exp(i\mathbf{e}_3\omega_0\tau)F\exp(-i\mathbf{e}_3\omega_0\tau)$, then the transformed differential equation for the Lorentz transformation takes the form

$$\dot{\Lambda}_{\omega_0} = \frac{e}{2mc}\left(\frac{2mc}{e}(i\mathbf{e}_3\omega_0) + F_{\omega_0}\right)\Lambda_{\omega_0} \tag{91.46}$$

This is closer to Baylis's equation 31. Dropping ω_0 subscripts this is

$$\dot{\Lambda} = \frac{e}{2mc}\left(\frac{2mc}{e}(i\mathbf{e}_3\omega_0) + F\right)\Lambda \tag{91.47}$$

A phase change in the Lorentz transformation rotor has introduced an additional term, one that Baylis appears to identify with the spin vector $\mathbf{S}$. My way of getting there seems fishy, so I think that I am missing something.

Ah, I see. If we go back to eq. (91.42), then with $\mathbf{S} = \Lambda_{\omega_0}(i\mathbf{e}_3)\tilde{\Lambda}_{\omega_0}$ (an application of a Lorentz transform to the unit bivector for the $\mathbf{e}_2\mathbf{e}_3$ plane), one has

$$\dot{\Lambda}_{\omega_0} = \frac{1}{2}\left(\frac{e}{mc}F + 2\omega_0\mathbf{S}\right)\Lambda_{\omega_0} \tag{91.48}$$

92

(INCOMPLETE) GEOMETRY OF MAXWELL RADIATION SOLUTIONS

92.1 MOTIVATION

We have in GA multiple possible ways to parametrize an oscillatory time dependence for a radiation field.

This was going to be an attempt to systematically solve the resulting eigen-multivector problem, starting with the a $I\hat{\mathbf{z}}\omega t$ exponential time parametrization, but I got stuck part way. Perhaps using a plain old $I\omega t$ would work out better, but I have spent more time on this than I want for now.

92.2 SETUP. THE EIGENVALUE PROBLEM

Again following Jackson [22], we use CGS units. Maxwell's equation in these units, with $F = \mathbf{E} + I\mathbf{B}/\sqrt{\mu\epsilon}$ is

$$0 = (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F \tag{92.1}$$

With an assumed oscillatory time dependence

$$F = \mathcal{F}e^{i\omega t} \tag{92.2}$$

Maxwell's equation reduces to a multivariable eigenvalue problem

$$\begin{aligned} \boldsymbol{\nabla}\mathcal{F} &= -\mathcal{F}i\lambda \\ \lambda &= \sqrt{\mu\epsilon}\frac{\omega}{c} \end{aligned} \tag{92.3}$$

We have some flexibility in picking the imaginary. As well as a non-geometric imaginary i typically used for a phasor representation where we take real parts of the field, we have additional possibilities, two of which are

$$\begin{aligned} i &= \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}} = I \\ i &= \hat{\mathbf{x}}\hat{\mathbf{y}} = I\hat{\mathbf{z}} \end{aligned} \tag{92.4}$$

The first is the spatial pseudoscalar, which commutes with all vectors and bivectors. The second is the unit bivector for the transverse plane, here parametrized by duality using the perpendicular to the plane direction $\hat{\mathbf{z}}$.

Let us examine the geometry required of the object $\mathcal{F}$ for each of these two geometric modeling choices.

92.3 USING THE TRANSVERSE PLANE BIVECTOR FOR THE IMAGINARY

Assuming no prior assumptions about $\mathcal{F}$ let us allow for the possibility of scalar, vector, bivector and pseudoscalar components

$$F = e^{-I\hat{\mathbf{z}}\omega t}(F_0 + F_1 + F_2 + F_3) \tag{92.5}$$

Writing $e^{-I\hat{\mathbf{z}}\omega t} = \cos(\omega t) - I\hat{\mathbf{z}}\sin(\omega t) = C_\omega - I\hat{\mathbf{z}}S_\omega$, an expansion of this product separated into grades is

$$\begin{aligned}
F &= C_\omega F_0 - IS_\omega(\hat{\mathbf{z}} \wedge F_2) \\
&\quad + C_\omega F_1 - \hat{\mathbf{z}}S_\omega(IF_3) + S_\omega(\hat{\mathbf{z}} \times F_1) \\
&\quad + C_\omega F_2 - I\hat{\mathbf{z}}S_\omega F_0 - IS_\omega(\hat{\mathbf{z}} \cdot F_2) \\
&\quad + C_\omega F_3 - IS_\omega(\hat{\mathbf{z}} \cdot F_1)
\end{aligned} \tag{92.6}$$

By construction F has only vector and bivector grades, so a requirement for zero scalar and pseudoscalar for all t means that we have four immediate constraints (with $\mathbf{n} \perp \hat{\mathbf{z}}$.)

$$\begin{aligned}
F_0 &= 0 \\
F_3 &= 0 \\
F_2 &= \hat{\mathbf{z}} \wedge \mathbf{m} \\
F_1 &= \mathbf{n}
\end{aligned} \tag{92.7}$$

Since we have the flexibility to add or subtract any scalar multiple of $\hat{\mathbf{z}}$ to $\mathbf{m}$ we can write $F_2 = \hat{\mathbf{z}}\mathbf{m}$ where $\mathbf{m} \perp \hat{\mathbf{z}}$. Our field can now be written as just

$$\begin{aligned}
F &= C_\omega \mathbf{n} - IS_\omega(\hat{\mathbf{z}} \wedge \mathbf{n}) \\
&\quad + C_\omega \hat{\mathbf{z}}\mathbf{m} - IS_\omega(\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}}\mathbf{m}))
\end{aligned} \tag{92.8}$$

We can similarly require $\mathbf{n} \perp \hat{\mathbf{z}}$, leaving

$$F = (C_\omega - I\hat{\mathbf{z}}S_\omega)\mathbf{n} + (C_\omega - I\hat{\mathbf{z}}S_\omega)\mathbf{m}\hat{\mathbf{z}} \tag{92.9}$$

So, just the geometrical constraints give us

$$F = e^{-I\hat{\mathbf{z}}\omega t}(\mathbf{n} + \mathbf{m}\hat{\mathbf{z}}) \tag{92.10}$$

The first thing to be noted is that this phasor representation utilizing for the imaginary the transverse plane bivector $I\hat{\mathbf{z}}$ cannot be the most general. This representation allows for only transverse fields! This can be seen two ways. Computing the transverse and propagation field components we have

$$\begin{aligned}
F_z &= \frac{1}{2}(F + \hat{\mathbf{z}}F\hat{\mathbf{z}}) \\
&= \frac{1}{2}e^{-I\hat{\mathbf{z}}\omega t}(\mathbf{n} + \mathbf{m}\hat{\mathbf{z}} + \hat{\mathbf{z}}\mathbf{n}\hat{\mathbf{z}} + \hat{\mathbf{z}}\mathbf{m}\hat{\mathbf{z}}\hat{\mathbf{z}}) \\
&= \frac{1}{2}e^{-I\hat{\mathbf{z}}\omega t}(\mathbf{n} + \mathbf{m}\hat{\mathbf{z}} - \mathbf{n} - \mathbf{m}\hat{\mathbf{z}}) \\
&= 0
\end{aligned} \tag{92.11}$$

The computation for the transverse field $F_t = (F - \hat{\mathbf{z}}F\hat{\mathbf{z}})/2$ shows that $F = F_t$ as expected since the propagation component is zero.

Another way to observe this is from the split of F into electric and magnetic field components. From eq. (92.9) we have

$$\begin{aligned}
\mathbf{E} &= \cos(\omega t)\mathbf{m} + \sin(\omega t)(\hat{\mathbf{z}} \times \mathbf{m}) \\
\mathbf{B} &= \cos(\omega t)(\hat{\mathbf{z}} \times \mathbf{n}) - \sin(\omega t)\mathbf{n}
\end{aligned} \tag{92.12}$$

The space containing each of the $\mathbf{E}$ and $\mathbf{B}$ vectors lies in the span of the transverse plane. We also see that there is some potential redundancy in the representation visible here since we have four vectors describing this span $\mathbf{m}$, $\mathbf{n}$, $\hat{\mathbf{z}} \times \mathbf{m}$, and $\hat{\mathbf{z}} \times \mathbf{n}$, instead of just two.

92.3.1 *General wave packet*

If eq. (92.1) were a scalar equation for $F(\mathbf{x}, t)$ it can be readily shown using Fourier transforms the field propagation in time given initial time description of the field is

$$F(\mathbf{x}, t) = \int \left(\frac{1}{(2\pi)^3} \int F(\mathbf{x}', 0) e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})} d^3x \right) e^{ic\mathbf{k}t/\sqrt{\mu\epsilon}} d^3k \tag{92.13}$$

In traditional complex algebra the vector exponentials would not be well formed. We do not have the problem in the GA formalism, but this does lead to a contraction since the resulting

$F(\mathbf{x}, t)$ cannot be scalar valued. However, by using this as a motivational tool, and also using assumed structure for the discrete frequency infinite wavetrain phasor, we can guess that a transverse only (to z-axis) wave packet may be described by a single direction variant of the Fourier result above. That is

$$F(\mathbf{x}, t) = \frac{1}{\sqrt{2\pi}} \int e^{-I\hat{\mathbf{z}}\omega t} \mathcal{F}(\mathbf{x}, \omega) d\omega \tag{92.14}$$

Since eq. (92.14) has the same form as the earlier single frequency phasor test solution, we now know that $\mathcal{F}$ is required to anticommute with $\hat{\mathbf{z}}$. Application of Maxwell's equation to this test solution gives us

$$\begin{aligned}(\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)F(\mathbf{x}, t) &= (\boldsymbol{\nabla} + \sqrt{\mu\epsilon}\partial_0)\frac{1}{\sqrt{2\pi}} \int \mathcal{F}(\mathbf{x}, \omega) e^{I\hat{\mathbf{z}}\omega t} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int \left(\boldsymbol{\nabla}\mathcal{F} + \mathcal{F} I\hat{\mathbf{z}} \sqrt{\mu\epsilon}\frac{\omega}{c} \right) e^{I\hat{\mathbf{z}}\omega t} d\omega\end{aligned} \tag{92.15}$$

This means that $\mathcal{F}$ must satisfy the gradient eigenvalue equation for all ω

$$\boldsymbol{\nabla}\mathcal{F} = -\mathcal{F} I\hat{\mathbf{z}} \sqrt{\mu\epsilon}\frac{\omega}{c} \tag{92.16}$$

Observe that this is the single frequency problem of equation eq. (92.3), so for mono-directional light we can consider the infinite wave train instead of a wave packet with no loss of generality.

92.3.2 *Applying separation of variables*

While this may not lead to the most general solution to the radiation problem, the transverse only propagation problem is still one of interest. Let us see where this leads. In order to reduce the scope of the problem by one degree of freedom, let us split out the $\hat{\mathbf{z}}$ component of the gradient, writing

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_t + \hat{\mathbf{z}}\partial_z \tag{92.17}$$

Also introduce a product split for separation of variables for the z dependence. That is

$$\mathcal{F} = G(x, y)Z(z) \tag{92.18}$$

Again we are faced with the problem of too many choices for the grades of each of these factors. We can pick one of these, say Z, to have only scalar and pseudoscalar grades so that the two factors commute. Then we have

$$(\boldsymbol{\nabla}_t + \boldsymbol{\nabla}_z)\mathcal{F} = (\boldsymbol{\nabla}_t G)Z + \hat{\mathbf{z}}G\partial_z Z = -GZI\hat{\mathbf{z}}\lambda \tag{92.19}$$

With Z in an algebra isomorphic to the complex numbers, it is necessarily invertible (and commutes with it is derivative). Similar arguments to the grade fixing for $\mathcal{F}$ show that G has only vector and bivector grades, but does G have the inverse required to do the separation of variables? Let us blindly suppose that we can do this (and if we can not we can probably fudge it since we multiply again soon after). With some rearranging we have

$$-\frac{1}{G}\hat{\mathbf{z}}(\boldsymbol{\nabla}_t G + GI\hat{\mathbf{z}}\lambda) = (\partial_z Z)\frac{1}{Z} = \text{constant} \tag{92.20}$$

We want to separately equate these to a constant. In order to commute these factors we have only required that Z have only scalar and pseudoscalar grades, so for the constant let us pick an arbitrary element in this subspace. That is

$$(\partial_z Z)\frac{1}{Z} = \alpha + kI \tag{92.21}$$

The solution for the Z factor in the separation of variables is thus

$$Z \propto e^{(\alpha+kI)z} \tag{92.22}$$

For G the separation of variables gives us

$$\boldsymbol{\nabla}_t G + (G\hat{\mathbf{z}}\lambda + \hat{\mathbf{z}}Gk)I + \hat{\mathbf{z}}G\alpha = 0 \tag{92.23}$$

We have now reduced the problem to something like a two variable eigenvalue problem, where the differential operator to find eigenvectors for is the transverse gradient $\boldsymbol{\nabla}_t$. We unfortunately have an untidy split of the eigenvalue into left and right hand factors.

While the product GZ was transverse only, we have now potentially lost that nice property for G itself, and do not know if G is strictly commuting or anticommuting with $\hat{\mathbf{z}}$. Assuming either possibility for now, we can split this multivector into transverse and propagation direction fields $G = G_t + G_z$

$$\begin{aligned}
G_t &= \frac{1}{2}(G - \hat{\mathbf{z}}G\hat{\mathbf{z}}) \\
G_z &= \frac{1}{2}(G + \hat{\mathbf{z}}G\hat{\mathbf{z}})
\end{aligned} \tag{92.24}$$

With this split, noting that $\hat{\mathbf{z}}G_t = -G_t\hat{\mathbf{z}}$, and $\hat{\mathbf{z}}G_z = G_z\hat{\mathbf{z}}$ a rearrangement of eq. (92.23) produces

$$(\nabla_t + \hat{\mathbf{z}}((k-\lambda)I + \alpha))G_t = -(\nabla_t + \hat{\mathbf{z}}((k+\lambda)I + \alpha))G_z \tag{92.25}$$

How do we find the eigen multivectors G_t and G_z? A couple possibilities come to mind (perhaps not encompassing all solutions). One is for one of G_t or G_z to be zero, and the other to separately require both halves of eq. (92.25) equal a constant, very much like separation of variables despite the fact that both of these functions G_t and G_z are functions of x and y. The easiest non-trivial path is probably letting both sides of eq. (92.25) separately equal zero, so that we are left with two independent eigen-multivector problems to solve

$$\begin{aligned}
\nabla_t G_t &= -\hat{\mathbf{z}}((k-\lambda)I + \alpha))G_t \\
\nabla_t G_z &= -\hat{\mathbf{z}}((k+\lambda)I + \alpha))G_z
\end{aligned} \tag{92.26}$$

Damn. have to mull this over. Do not know where to go with it.

93

RELATIVISTIC CLASSICAL PROTON ELECTRON INTERACTION

93.1 MOTIVATION

The problem of a solving for the relativistically correct trajectories of classically interacting proton and electron is one that I have wanted to try for a while. Conceptually this is just about the simplest interaction problem in electrodynamics (other than motion of a particle in a field), but it is not obvious to me how to even set up the right equations to solve. I should have the tools now to at least write down the equations to solve, and perhaps solve them too.

Familiarity with Geometric Algebra, and the STA form of the Maxwell and Lorentz force equation will be assumed. Writing $F = \mathbf{E} + cI\mathbf{B}$ for the Faraday bivector, these equations are respectively

$$\begin{aligned}
\nabla F &= J/\epsilon_0 c \\
m\frac{d^2 X}{d\tau} &= \frac{q}{c} F \cdot \frac{dX}{d\tau}
\end{aligned} \tag{93.1}$$

The possibility of self interaction will also be ignored here. From what I have read this self interaction is more complex than regular two particle interaction.

93.2 WITH ONLY COULOMB INTERACTION

With just Coulomb (non-relativistic) interaction setup of the equations of motion for the relative vector difference between the particles is straightforward. Let us write this out as a reference. Whatever we come up with for the relativistic case should reduce to this at small velocities.

Fixing notation, lets write the proton and electron positions respectively by $\mathbf{r}_p$ and $\mathbf{r}_e$, the proton charge as Ze, and the electron charge $-e$. For the forces we have

FIXME: picture

$$\begin{aligned}
\text{Force on electron} = m_e \frac{d^2\mathbf{r}_e}{dt^2} &= -\frac{1}{4\pi\epsilon_0} Z e^2 \frac{\mathbf{r}_e - \mathbf{r}_p}{\left|\mathbf{r}_e - \mathbf{r}_p\right|^3} \\
\text{Force on proton} = m_p \frac{d^2\mathbf{r}_p}{dt^2} &= \frac{1}{4\pi\epsilon_0} Z e^2 \frac{\mathbf{r}_e - \mathbf{r}_p}{\left|\mathbf{r}_e - \mathbf{r}_p\right|^3}
\end{aligned} \tag{93.2}$$

Subtracting the two after mass division yields the reduced mass equation for the relative motion

$$\frac{d^2(\mathbf{r}_e - \mathbf{r}_p)}{dt^2} = -\frac{1}{4\pi\epsilon_0} Ze^2 \left(\frac{1}{m_e} + \frac{1}{m_p}\right) \frac{\mathbf{r}_e - \mathbf{r}_p}{\left|\mathbf{r}_e - \mathbf{r}_p\right|^3} \tag{93.3}$$

This is now of the same form as the classical problem of two particle gravitational interaction, with the well known conic solutions.

93.3 USING THE DIVERGENCE EQUATION INSTEAD

While use of the Coulomb force above provides the equation of motion for the relative motion of the charges, how to generalize this to the relativistic case is not entirely clear. For the relativistic case we need to consider all of Maxwell's equations, and not just the divergence equation. Let us back up a step and setup the problem using the divergence equation instead of Coulomb's law. This is a bit closer to the use of all of Maxwell's equations.

To start off we need a discrete charge expression for the charge density, and can use the delta distribution to express this.

$$0 = \int d^3x \left(\boldsymbol{\nabla} \cdot \mathbf{E} - \frac{1}{\epsilon_0}\left(Ze\delta^3(\mathbf{x} - \mathbf{r}_p) - e\delta^3(\mathbf{x} - \mathbf{r}_e)\right)\right) \tag{93.4}$$

Picking a volume element that only encloses one of the respective charges gives us the Coulomb law for the field produced by those charges as above

$$\begin{aligned}
0 &= \int_{\text{Volume around proton only}} d^3x \left(\boldsymbol{\nabla} \cdot \mathbf{E}_p - \frac{1}{\epsilon_0} Ze\delta^3(\mathbf{x} - \mathbf{r}_p)\right) \\
0 &= \int_{\text{Volume around electron only}} d^3x \left(\boldsymbol{\nabla} \cdot \mathbf{E}_e + \frac{1}{\epsilon_0} e\delta^3(\mathbf{x} - \mathbf{r}_e)\right)
\end{aligned} \tag{93.5}$$

Here $\mathbf{E}_p$ and $\mathbf{E}_e$ denote the electric fields due to the proton and electron respectively. Ignoring the possibility of self interaction the Lorentz forces on the particles are

$$\text{Force on proton/electron} = \text{charge of proton/electron times field due to electron/proton} \tag{93.6}$$

In symbols, this is

$$\begin{aligned}
m_p \frac{d^2\mathbf{r}_p}{dt^2} &= Ze\mathbf{E}_e \\
m_e \frac{d^2\mathbf{r}_e}{dt^2} &= -e\mathbf{E}_p
\end{aligned} \tag{93.7}$$

If we were to substitute back into the volume integrals we would have

$$
\begin{aligned}
0 &= \int_{\text{Volume around proton only}} d^3x \left(-\frac{m_e}{e} \boldsymbol{\nabla} \cdot \frac{d^2 \mathbf{r}_e}{dt^2} - \frac{1}{\epsilon_0} Ze\delta^3(\mathbf{x} - \mathbf{r}_p) \right) \\
0 &= \int_{\text{Volume around electron only}} d^3x \left(\frac{m_p}{Ze} \boldsymbol{\nabla} \cdot \frac{d^2 \mathbf{r}_p}{dt^2} + \frac{1}{\epsilon_0} e\delta^3(\mathbf{x} - \mathbf{r}_e) \right)
\end{aligned}
\tag{93.8}
$$

It is tempting to take the differences of these two equations so that we can write this in terms of the relative acceleration $d^2(\mathbf{r}_e - \mathbf{r}_p)/dt^2$. I did just this initially, and was surprised by a mass term of the form $1/m_e - 1/m_p$ instead of reduced mass, which cannot be right. The key to avoiding this mistake is the proper considerations of the integration volumes. Since the volumes are different and can in fact be entirely disjoint, subtracting these is not possible. For this reason we have to be especially careful if a differential form of the divergence integrals eq. (93.7) were to be used, as in

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E}_p &= \frac{1}{\epsilon_0} Ze\delta^3(\mathbf{x} - \mathbf{r}_p) \\
\boldsymbol{\nabla} \cdot \mathbf{E}_e &= -\frac{1}{\epsilon_0} e\delta^3(\mathbf{x} - \mathbf{r}_e)
\end{aligned}
\tag{93.9}
$$

The domain of applicability of these equations is no longer explicit, since each has to omit a neighborhood around the other charge. When using a delta distribution to express the point charge density it is probably best to stick with an explicit integral form.

Comparing how far we can get starting with the Gauss's law instead of the Coulomb force, and looking forward to the relativistic case, it seems likely that solving the field equations due to the respective current densities will be the first required step. Only then can we substitute that field solution back into the Lorentz force equation to complete the search for the particle trajectories.

93.4 RELATIVISTIC INTERACTION

First order of business is an expression for a point charge current density four vector. Following Jackson [22], but switching to vector notation from coordinates, we can apparently employ an arbitrary parametrization for the four-vector particle trajectory $R = R^\mu \gamma_\mu$, as measured in the observer frame, and write

$$
J(X) = qc \int d\lambda \frac{dX}{d\lambda} \delta^4(X - R(\lambda))
\tag{93.10}
$$

Here $X = X^\mu \gamma_\mu$ is the four vector event specifying the spacetime position of the current, also as measured in the observer frame. Reparameterizating in terms of time should get us back something more familiar looking

$$\begin{aligned} J(X) &= qc \int dt \frac{dX}{dt} \delta^4(X - R(t)) \\ &= qc \int dt \frac{d}{dt}(ct\gamma_0 + \gamma_k X^k)\delta^4(X - R(t)) \\ &= qc \int dt \frac{d}{dt}(ct + \mathbf{x})\delta^4(X - R(t))\gamma_0 \\ &= qc \int dt(c + \mathbf{v})\delta^4(X - R(t))\gamma_0 \\ &= qc \int dt'(c + \mathbf{v}(t'))\delta^3(\mathbf{x} - \mathbf{r}(t'))\delta(ct' - ct)\gamma_0 \end{aligned} \tag{93.11}$$

Note that the scaling property of the delta function implies $\delta(ct) = \delta(t)/c$. With the split of the four-volume delta function $\delta^4(X - R(t)) = \delta^3(\mathbf{x} - \mathbf{r}(t))\delta(x^{0\prime} - x^0)$, where $x^0 = ct$, we have an explanation for why Jackson had a factor of c in his representation. I initially thought this factor of c was due to CGS vs SI units! One more Jackson equation decoded. We are left with the following spacetime split for a point charge current density four vector

$$J(X) = q(c + \mathbf{v}(t))\delta^3(\mathbf{x} - \mathbf{r}(t))\gamma_0 \tag{93.12}$$

Comparing to the continuous case where we have $J = \rho(c + \mathbf{v})\gamma_0$, it appears that this works out right. One thing worth noting is that in this time reparameterization I accidentally mixed up X, the observation event coordinates of $J(X)$, and R, the spacetime trajectory of the particle itself. Despite this, I am saved by the delta function since no contributions to the current can occur on trajectories other than R, the worldline of the particle itself. So in the final result it should be correct to interpret $\mathbf{v}$ as the spatial particle velocity as I did accidentally.

With the time reparameterization of the current density, we have for the field due to our proton and electron

$$0 = \int d^3x \left(\epsilon_0 c \nabla F - Ze(c + \mathbf{v}_p(t))\delta^3(\mathbf{x} - \mathbf{r}_p(t)) + e(c + \mathbf{v}_e(t))\delta^3(\mathbf{x} - \mathbf{r}_e(t))\gamma_0\right) \tag{93.13}$$

How to write this in a more tidy covariant form? If we reparametrize with any of the other spatial coordinates, say x we end up having to integrate the field gradient with a spacetime three form ($dtdydz$ if parametrizing the current density with x). Since the entire equation must be zero I suppose we can just integrate that once more, and simply write

$$\text{constant} = \int d^4x \left(\nabla F - \frac{e}{\epsilon_0 c} \int d\tau \frac{dX}{d\tau} \left(Z\delta^4(X - R_p(\tau)) - \delta^4(X - R_e(\tau)) \right) \right) \tag{93.14}$$

Like eq. (93.5) we can pick spacetime volumes that surround just the individual particle worldlines, in which case we have a Coulomb's law like split where the field depends on just the enclosed current. That is

$$\begin{aligned}
\text{constant} &= \int_{\text{spacetime volume around only the proton}} d^4x \left(\nabla F_p - \frac{Ze}{\epsilon_0 c} \int d\tau \frac{dX}{d\tau} \delta^4(X - R_e(\tau)) \right) \\
\text{constant} &= \int_{\text{spacetime volume around only the electron}} d^4x \left(\nabla F_e + \frac{e}{\epsilon_0 c} \int d\tau \frac{dX}{d\tau} \delta^4(X - R_e(\tau)) \right)
\end{aligned} \tag{93.15}$$

Here F_e is the field due to only the electron charge, whereas F_p would be that part of the total field due to the proton charge.

FIXME: attempt to draw a picture (one or two spatial dimensions) to develop some comfort with tossing out a phrase like "spacetime volume surrounding a particle worldline".

Having expressed the equation for the total field eq. (93.14), we are tracking a nice parallel to the setup for the non-relativistic treatment. Next is the pair of Lorentz force equations. As in the non-relativistic setup, if we only consider the field due to the other charge we have in in covariant Geometric Algebra form, the following pair of proper force equations in terms of the particle worldline trajectories

$$\begin{aligned}
\text{proper Force on electron} &= m_e \frac{d^2 R_e}{d\tau^2} = -eF_p \cdot \frac{dR_e}{cd\tau} \\
\text{proper Force on proton} &= m_p \frac{d^2 R_p}{d\tau^2} = ZeF_e \cdot \frac{dR_p}{cd\tau}
\end{aligned} \tag{93.16}$$

We have the four sets of coupled multivector equations to be solved, so the question remains how to do so. Each of the two Lorentz force equations supplies four equations with four unknowns, and the field equations are really two sets of eight equations with six unknown field variables each. Then they are all tied up together is a big coupled mess. Wow. How do we solve this?

With eq. (93.15), and eq. (93.16) committed to pdf at least the first goal of writing down the equations is done.

As for the actual solution. Well, that is a problem for another night. TO BE CONTINUED (if I can figure out an attack).

Part IX

ELECTRODYNAMICS STRESS ENERGY

94 POYNTING VECTOR AND ELECTROMAGNETIC ENERGY CONSERVATION

94.1 MOTIVATION

Clarify Poynting discussion from [10].

Equation 7.59 and 7.60 derives a $\mathbf{E} \times \mathbf{B}$ quantity, the Poynting vector, as a sort of energy flux through the surface of the containing volume.

There are a couple of magic steps here that were not at all obvious to me. Go through this in enough detail that it makes sense to me.

94.2 CHARGE FREE CASE

In SI units the Energy density is given as

$$U = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \tag{94.1}$$

In 96 the electrostatic energy portion of this energy was observed. FIXME: A magnetostatics derivation (ie: unchanging currents) is possible for the $\mathbf{B}^2$ term, but I have not done this myself yet.

It is somewhat curious that the total field energy is just this sum without any cross terms (all those cross terms show up in the field momentum). A logical confirmation of this in a general non-electrostatics and non-magnetostatics context will not be done here. Instead it will be assumed that eq. (94.1) has been correctly identified as the field energy (density), and a mechanical calculation of the time rate of change of this quantity (the power density) will be performed. In doing so we can find the analogue of the momentum. How to truly identify this quantity with momentum will hopefully become clear as we work with it.

Given this energy density the rate of change of energy in a volume is then

$$\begin{aligned}
\frac{dU}{dt} &= \frac{d}{dt}\frac{\epsilon_0}{2}\int dV\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\
&= \epsilon_0 \int dV\left(\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + c^2\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\right)
\end{aligned} \tag{94.2}$$

The next (omitted in the text) step is to utilize Maxwell's equation to eliminate the time derivatives. Since this is the charge and current free case, we can write Maxwell's as

$$\begin{aligned}
0 &= \gamma_0 \nabla F \\
&= \gamma_0 (\gamma^0 \partial_0 + \gamma^k \partial_k) F \\
&= (\partial_0 + \gamma_k \gamma_0 \partial_k) F \\
&= (\partial_0 + \sigma_k \partial_k) F \\
&= (\partial_0 + \boldsymbol{\nabla}) F \\
&= (\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + ic\mathbf{B}) \\
&= \partial_0 \mathbf{E} + ic\partial_0 \mathbf{B} + \boldsymbol{\nabla}\mathbf{E} + ic\boldsymbol{\nabla}\mathbf{B}
\end{aligned} \tag{94.3}$$

In the spatial (σ) basis we can separate this into even and odd grades, which are separately equal to zero

$$\begin{aligned}
0 &= \partial_0 \mathbf{E} + ic\boldsymbol{\nabla}\mathbf{B} \\
0 &= ic\partial_0 \mathbf{B} + \boldsymbol{\nabla}\mathbf{E}
\end{aligned} \tag{94.4}$$

A selection of just the vector parts is

$$\begin{aligned}
\partial_t \mathbf{E} &= -ic^2 \boldsymbol{\nabla} \wedge \mathbf{B} \\
\partial_t \mathbf{B} &= i\boldsymbol{\nabla} \wedge \mathbf{E}
\end{aligned} \tag{94.5}$$

Which can be back substituted into the energy flux

$$\begin{aligned}
\frac{dU}{dt} &= \epsilon_0 \int dV \left(\mathbf{E} \cdot (-ic^2 \boldsymbol{\nabla} \wedge \mathbf{B}) + c^2 \mathbf{B} \cdot (i\boldsymbol{\nabla} \wedge \mathbf{E}) \right) \\
&= \epsilon_0 c^2 \int dV \langle \mathbf{B} i \boldsymbol{\nabla} \wedge \mathbf{E} - \mathbf{E} i \boldsymbol{\nabla} \wedge \mathbf{B} \rangle
\end{aligned} \tag{94.6}$$

Since the two divergence terms are zero we can drop the wedges here for

$$\begin{aligned}
\frac{dU}{dt} &= \epsilon_0 c^2 \int dV \langle \mathbf{B} i \boldsymbol{\nabla} \mathbf{E} - \mathbf{E} i \boldsymbol{\nabla} \mathbf{B} \rangle \\
&= \epsilon_0 c^2 \int dV \langle (i\mathbf{B}) \boldsymbol{\nabla} \mathbf{E} - \mathbf{E} \boldsymbol{\nabla} (i\mathbf{B}) \rangle \\
&= \epsilon_0 c^2 \int dV \boldsymbol{\nabla} \cdot ((i\mathbf{B}) \cdot \mathbf{E})
\end{aligned} \tag{94.7}$$

Justification for this last step can be found below in the derivation of eq. (94.30).

We can now use Stokes theorem to change this into a surface integral for a final energy flux

$$\frac{dU}{dt} = \epsilon_0 c^2 \int d\mathbf{A} \cdot ((i\mathbf{B}) \cdot \mathbf{E}) \tag{94.8}$$

This last bivector/vector dot product is the Poynting vector

$$\begin{aligned}
(i\mathbf{B}) \cdot \mathbf{E} &= \langle (i\mathbf{B}) \cdot \mathbf{E} \rangle_1 \\
&= \langle i\mathbf{B}\mathbf{E} \rangle_1 \\
&= \langle i(\mathbf{B} \wedge \mathbf{E}) \rangle_1 \\
&= i(\mathbf{B} \wedge \mathbf{E}) \\
&= i^2(\mathbf{B} \times \mathbf{E}) \\
&= \mathbf{E} \times \mathbf{B}
\end{aligned} \tag{94.9}$$

So, we can identity the quantity

$$\mathbf{P} = \epsilon_0 c^2 \mathbf{E} \times \mathbf{B} = \epsilon_0 c (ic\mathbf{B}) \cdot \mathbf{E} \tag{94.10}$$

as a directed energy density flux through the surface of a containing volume.

94.3 WITH CHARGES AND CURRENTS

To calculate time derivatives we want to take Maxwell's equation and put into a form with explicit time derivatives, as was done before, but this time be more careful with the handling of the four vector current term. Starting with left factoring out of a γ_0 from the spacetime gradient.

$$\begin{aligned}
\nabla &= \gamma^0 \partial_0 + \gamma^k \partial_k \\
&= \gamma^0 (\partial_0 - \gamma^k \gamma_0 \partial_k) \\
&= \gamma^0 (\partial_0 + \sigma_k \partial_k)
\end{aligned} \tag{94.11}$$

Similarly, the γ_0 can be factored from the current density

$$\begin{aligned}
J &= \gamma_0 c\rho + \gamma_k J^k \\
&= \gamma_0 (c\rho - \gamma_k \gamma_0 J^k) \\
&= \gamma_0 (c\rho - \sigma_k J^k) \\
&= \gamma_0 (c\rho - \mathbf{j})
\end{aligned} \tag{94.12}$$

With this Maxwell's equation becomes

$$\begin{aligned}\gamma_0 \nabla F &= \gamma_0 J/\epsilon_0 c \\ (\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + ic\mathbf{B}) &= \rho/\epsilon_0 - \mathbf{j}/\epsilon_0 c\end{aligned} \tag{94.13}$$

A split into even and odd grades including current and charge density is thus

$$\begin{aligned}\boldsymbol{\nabla}\mathbf{E} + \partial_t(i\mathbf{B}) &= \rho/\epsilon_0 \\ \boldsymbol{\nabla}(i\mathbf{B})c^2 + \partial_t\mathbf{E} &= -\mathbf{j}/\epsilon_0\end{aligned} \tag{94.14}$$

Now, taking time derivatives of the energy density gives

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{2}\epsilon_0\left(\mathbf{E}^2 - (ic\mathbf{B})^2\right) \\ &= \epsilon_0\left(\mathbf{E}\cdot\partial_t\mathbf{E} - c^2(i\mathbf{B})\cdot\partial_t(i\mathbf{B})\right) \\ &= \epsilon_0\left\langle\mathbf{E}(-\mathbf{j}/\epsilon_0 - \boldsymbol{\nabla}(i\mathbf{B})c^2) - c^2(i\mathbf{B})(-\boldsymbol{\nabla}\mathbf{E} + \rho/\epsilon_0)\right\rangle \\ &= -\mathbf{E}\cdot\mathbf{j} + c^2\epsilon_0\langle i\mathbf{B}\boldsymbol{\nabla}\mathbf{E} - \mathbf{E}\boldsymbol{\nabla}(i\mathbf{B})\rangle \\ &= -\mathbf{E}\cdot\mathbf{j} + c^2\epsilon_0\left((i\mathbf{B})\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\cdot(i\mathbf{B}))\right)\end{aligned} \tag{94.15}$$

Using eq. (94.30), we now have the rate of change of field energy for the general case including currents. That is

$$\frac{\partial U}{\partial t} = -\mathbf{E}\cdot\mathbf{j} + c^2\epsilon_0\boldsymbol{\nabla}\cdot(\mathbf{E}\cdot(i\mathbf{B})) \tag{94.16}$$

Written out in full, and in terms of the Poynting vector this is

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) + c^2\epsilon_0\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot\mathbf{j} \tag{94.17}$$

94.4 POYNTING VECTOR IN TERMS OF COMPLETE FIELD

In eq. (94.10) the individual parts of the complete Faraday bivector $F = \mathbf{E} + ic\mathbf{B}$ stand out. How would the Poynting vector be expressed in terms of F or in tensor form?

One possibility is to write $\mathbf{E}\times\mathbf{B}$ in terms of F using a conjugate split of the Maxwell bivector

$$F\gamma_0 = -\gamma_0(\mathbf{E} - ic\mathbf{B}) \tag{94.18}$$

we have

$$\gamma^0 F \gamma_0 = -(\mathbf{E} - ic\mathbf{B}) \tag{94.19}$$

and

$$\begin{aligned} ic\mathbf{B} &= \frac{1}{2}(F + \gamma^0 F \gamma_0) \\ \mathbf{E} &= \frac{1}{2}(F - \gamma^0 F \gamma_0) \end{aligned} \tag{94.20}$$

However [10] has the answer more directly in terms of the electrodynamic stress tensor.

$$T(a) = -\frac{\epsilon_0}{2} F a F \tag{94.21}$$

In particular for $a = \gamma_0$, this is

$$\begin{aligned} T(\gamma_0) &= -\frac{\epsilon_0}{2} F \gamma_0 F \\ &= \frac{\epsilon_0}{2}(\mathbf{E} + ic\mathbf{B})(\mathbf{E} - ic\mathbf{B})\gamma_0 \\ &= \frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2 + ic(\mathbf{B}\mathbf{E} - \mathbf{B}\mathbf{E}))\gamma_0 \\ &= \frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2)\gamma_0 + ic\epsilon_0(\mathbf{B} \wedge \mathbf{E})\gamma_0 \\ &= \frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2)\gamma_0 + c\epsilon_0(\mathbf{E} \times \mathbf{B})\gamma_0 \end{aligned} \tag{94.22}$$

So one sees that the energy and the Poynting vector are components of an energy density momentum four vector

$$T(\gamma_0) = U\gamma_0 + \frac{1}{c}\mathbf{P}\gamma_0 \tag{94.23}$$

Writing $U^0 = U$ and $U^k = P^k/c$, this is $T(\gamma_0) = U^\mu \gamma_\mu$.

(inventing such a four vector is how Doran/Lasenby started, so this is not be too surprising). This relativistic context helps justify the Poynting vector as a momentum like quantity, but is not quite satisfactory. It would make sense to do some classical comparisons, perhaps of interacting wave functions or something like that, to see how exactly this quantity is momentum like. Also how exactly is this energy momentum tensor used, how does it transform, ...

94.5 ENERGY DENSITY FROM LAGRANGIAN?

I did not get too far trying to calculate the electrodynamic Hamiltonian density for the general case, so I tried it for a very simple special case, with just an electric field component in one direction:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}(E_x)^2 \\
&= \frac{1}{2}(F_{01})^2 \\
&= \frac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2
\end{aligned}
\tag{94.24}
$$

[16] gives the Hamiltonian density as

$$
\begin{aligned}
\pi &= \frac{\partial \mathcal{L}}{\partial \dot{n}} \\
\mathcal{H} &= \dot{n}\pi - \mathcal{L}
\end{aligned}
\tag{94.25}
$$

If I try calculating this I get

$$
\begin{aligned}
\pi &= \frac{\partial}{\partial(\partial_0 A_1)}\left(\frac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2\right) \\
&= \partial_0 A_1 - \partial_1 A_0 \\
&= F_{01}
\end{aligned}
\tag{94.26}
$$

So this gives a Hamiltonian of

$$
\begin{aligned}
\mathcal{H} &= \partial_0 A_1 F_{01} - \frac{1}{2}(\partial_0 A_1 - \partial_1 A_0)F_{01} \\
&= \frac{1}{2}(\partial_0 A_1 + \partial_1 A_0)F_{01} \\
&= \frac{1}{2}((\partial_0 A_1)^2 - (\partial_1 A_0)^2)
\end{aligned}
\tag{94.27}
$$

For a Lagrangian density of $E^2 - B^2$ we have an energy density of $E^2 + B^2$, so I had have expected the Hamiltonian density here to stay equal to $E_x^2/2$, but it does not look like that is what I get (what I calculated is not at all familiar seeming).

If I have not made a mistake here, perhaps I am incorrect in assuming that the Hamiltonian density of the electrodynamic Lagrangian should be the energy density?

94.6 APPENDIX. MESSY DETAILS

For both the charge and the charge free case, we need a proof of

$$(i\mathbf{B}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \cdot (i\mathbf{B})) = \boldsymbol{\nabla} \cdot (\mathbf{E} \cdot (i\mathbf{B})) \tag{94.28}$$

This is relativity straightforward, albeit tedious, to do backwards.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot ((i\mathbf{B}) \cdot \mathbf{E}) &= \langle \boldsymbol{\nabla}((i\mathbf{B}) \cdot \mathbf{E}) \rangle \\
&= \frac{1}{2}\langle \boldsymbol{\nabla}(i\mathbf{B}\mathbf{E} - \mathbf{E}i\mathbf{B}) \rangle \\
&= \frac{1}{2}\left\langle \dot{\boldsymbol{\nabla}} i\dot{\mathbf{B}}\mathbf{E} + \dot{\boldsymbol{\nabla}} i\mathbf{B}\dot{\mathbf{E}} - \dot{\boldsymbol{\nabla}}\dot{\mathbf{E}}i\mathbf{B} - \dot{\boldsymbol{\nabla}}\mathbf{E}i\dot{\mathbf{B}} \right\rangle \\
&= \frac{1}{2}\left\langle \mathbf{E}\boldsymbol{\nabla}(i\mathbf{B}) - (i\dot{\mathbf{B}})\dot{\boldsymbol{\nabla}}\mathbf{E} + \dot{\mathbf{E}}\dot{\boldsymbol{\nabla}}i\mathbf{B} - i\mathbf{B}\boldsymbol{\nabla}\mathbf{E} \right\rangle \\
&= \frac{1}{2}\left(\mathbf{E} \cdot (\boldsymbol{\nabla} \cdot (i\mathbf{B})) - ((i\dot{\mathbf{B}}) \cdot \dot{\boldsymbol{\nabla}}) \cdot \mathbf{E} + (\dot{\mathbf{E}} \wedge \dot{\boldsymbol{\nabla}}) \cdot (i\mathbf{B}) - (i\mathbf{B}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{E}) \right)
\end{aligned} \tag{94.29}$$

Grouping the two sets of repeated terms after reordering and the associated sign adjustments we have

$$\boldsymbol{\nabla} \cdot ((i\mathbf{B}) \cdot \mathbf{E}) = \mathbf{E} \cdot (\boldsymbol{\nabla} \cdot (i\mathbf{B})) - (i\mathbf{B}) \cdot (\boldsymbol{\nabla} \wedge \mathbf{E}) \tag{94.30}$$

which is the desired identity (in negated form) that was to be proved.

There is likely some theorem that could be used to avoid some of this algebra.

94.7 REFERENCES FOR FOLLOWUP STUDY

Some of the content available in the article Energy Conservation looks like it will also be useful to study (in particular it goes through some examples that convert this from a math treatment to a physics story).

95

TIME RATE OF CHANGE OF THE POYNTING VECTOR, AND ITS CONSERVATION LAW

95.1 MOTIVATION

Derive the conservation laws for the time rate of change of the Poynting vector, which appears to be a momentum density like quantity.

The Poynting conservation relationship has been derived previously. Additionally a starting exploration 97 of the related four vector quantity has been related to a subset of the energy momentum stress tensor. This was incomplete since the meaning of the T_{kj} terms of the tensor were unknown and the expected Lorentz transform relationships had not been determined. The aim here is to try to figure out this remainder.

95.2 CALCULATION

Repeating again from 94, the electrodynamic energy density U and momentum flux density vectors are related as follows

$$\begin{aligned}
U &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\
\mathbf{P} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0}(i\mathbf{B})\cdot\mathbf{E} \\
0 &= \frac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{P} + \mathbf{E}\cdot\mathbf{j}
\end{aligned} \tag{95.1}$$

We want to now calculate the time rate of change of this Poynting (field momentum density) vector.

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{1}{\mu_0}(i\mathbf{B})\cdot\mathbf{E}\right) \\
&= \partial_0\left(\frac{1}{\mu_0}(ic\mathbf{B})\cdot\mathbf{E}\right) \\
&= \frac{1}{\mu_0}\left(\partial_0(ic\mathbf{B})\cdot\mathbf{E} + (ic\mathbf{B})\cdot\partial_0\mathbf{E}\right)
\end{aligned} \tag{95.2}$$

We will want to express these time derivatives in terms of the current and spatial derivatives to determine the conservation identity. To do this let us go back to Maxwell's equation once more, with a premultiplication by γ_0 to provide us with an observer dependent spacetime split

$$\begin{aligned}\gamma_0 \nabla F &= \gamma_0 J/\epsilon_0 c \\ (\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + ic\mathbf{B}) &= \rho/\epsilon_0 - \mathbf{j}/\epsilon_0 c\end{aligned} \tag{95.3}$$

We want the grade one and grade two components for the time derivative terms. For grade one we have

$$\begin{aligned}-\mathbf{j}/\epsilon_0 c &= \langle (\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + ic\mathbf{B}) \rangle_1 \\ &= \partial_0 \mathbf{E} + \boldsymbol{\nabla} \cdot (ic\mathbf{B})\end{aligned} \tag{95.4}$$

and for grade two

$$\begin{aligned}0 &= \langle (\partial_0 + \boldsymbol{\nabla})(\mathbf{E} + ic\mathbf{B}) \rangle_2 \\ &= \partial_0 (ic\mathbf{B}) + \boldsymbol{\nabla} \wedge \mathbf{E}\end{aligned} \tag{95.5}$$

Using these we can express the time derivatives for back substitution

$$\begin{aligned}\partial_0 \mathbf{E} &= -\mathbf{j}/\epsilon_0 c - \boldsymbol{\nabla} \cdot (ic\mathbf{B}) \\ \partial_0 (ic\mathbf{B}) &= -\boldsymbol{\nabla} \wedge \mathbf{E}\end{aligned} \tag{95.6}$$

yielding

$$\begin{aligned}\mu_0 \frac{\partial \mathbf{P}}{\partial t} &= \partial_0 (ic\mathbf{B}) \cdot \mathbf{E} + (ic\mathbf{B}) \cdot \partial_0 \mathbf{E} \\ &= -(\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} - (ic\mathbf{B}) \cdot \big(\mathbf{j}/\epsilon_0 c + \boldsymbol{\nabla} \cdot (ic\mathbf{B})\big)\end{aligned} \tag{95.7}$$

Or

$$\begin{aligned}0 &= \partial_0 ((ic\mathbf{B}) \cdot \mathbf{E}) + (\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + (ic\mathbf{B}) \cdot (\boldsymbol{\nabla} \cdot (ic\mathbf{B})) + (ic\mathbf{B}) \cdot \mathbf{j}/\epsilon_0 c \\ &= \langle \partial_0 (ic\mathbf{B}\mathbf{E}) + (\boldsymbol{\nabla} \wedge \mathbf{E})\mathbf{E} + ic\mathbf{B}(\boldsymbol{\nabla} \cdot (ic\mathbf{B})) + ic\mathbf{B}\mathbf{j}/\epsilon_0 c \rangle_1 \\ &= \langle \partial_0 (ic\mathbf{B}\mathbf{E}) + (\boldsymbol{\nabla} \wedge \mathbf{E})\mathbf{E} + (\boldsymbol{\nabla} \wedge (c\mathbf{B}))c\mathbf{B} + ic\mathbf{B}\mathbf{j}/\epsilon_0 c \rangle_1 \\ 0 &= i\partial_0 (c\mathbf{B} \wedge \mathbf{E}) + (\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + (\boldsymbol{\nabla} \wedge (c\mathbf{B})) \cdot (c\mathbf{B}) + i(c\mathbf{B} \wedge \mathbf{j})/\epsilon_0 c\end{aligned} \tag{95.8}$$

This appears to be the conservation law that is expected for the change in vector field momentum density.

$$\partial_t (\mathbf{E} \times \mathbf{B}) + (\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + c^2 (\boldsymbol{\nabla} \wedge \mathbf{B}) \cdot \mathbf{B} = (\mathbf{B} \times \mathbf{j})/\epsilon_0 \tag{95.9}$$

In terms of the original Poynting vector this is

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{1}{\mu_0}(\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + c^2 \frac{1}{\mu_0}(\boldsymbol{\nabla} \wedge \mathbf{B}) \cdot \mathbf{B} = c^2(\mathbf{B} \times \mathbf{j}) \tag{95.10}$$

Now, there are a few things to pursue here.

- How to or can we put this in four vector divergence form.
- Relate this to the wikipedia result which is very different looking.
- Find the relation to the stress energy tensor.
- Lorentz transformation relation to Poynting energy momentum conservation law.

95.2.1 *Four vector form?*

If $\mathbf{P} = P^m \sigma_m$, then each of the P^m coordinates could be thought of as the zero coordinate of a four vector. Can we get a four vector divergence out of eq. (95.9)?

Let us expand the wedge-dot term in coordinates.

$$\begin{aligned}
((\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E}) \cdot \sigma_m &= ((\sigma^a \wedge \sigma_b) \cdot \sigma_k) \cdot \sigma_m (\partial_a E^b) E^k \\
&= (\delta^a_m \delta_{bk} - \delta_{bm} \delta^a_k)(\partial_a E^b) E^k \\
&= \sum_k (\partial_m E^k - \partial_k E^m) E^k \\
&= \partial_m \frac{\mathbf{E}^2}{2} - (\mathbf{E} \cdot \boldsymbol{\nabla}) E^m
\end{aligned} \tag{95.11}$$

So we have three equations, one for each $m = \{1, 2, 3\}$

$$\frac{\partial P^m}{\partial t} + c^2 \frac{\partial U}{\partial x^m} - \frac{1}{\mu_0}((\mathbf{E} \cdot \boldsymbol{\nabla}) E^m + c^2 (\mathbf{B} \cdot \boldsymbol{\nabla}) B^m) = c^2 (\mathbf{B} \times \mathbf{j})_m \tag{95.12}$$

Damn. This does not look anything like the four vector divergence that we had with the Poynting conservation equation. In the second last line of the wedge dot expansion we do see that we only have to sum over the $k \neq m$ terms. Can that help simplify this?

95.2.2 *Compare to wikipedia form*

To compare eq. (95.10) with the wikipedia article , the first thing we have to do is eliminate the wedge products.

This can be done in a couple different ways. One, is conversion to cross products

$$\begin{aligned}
(\boldsymbol{\nabla} \wedge \mathbf{a}) \cdot \mathbf{a} &= \langle (\boldsymbol{\nabla} \wedge \mathbf{a})\mathbf{a} \rangle_1 \\
&= \langle i(\boldsymbol{\nabla} \times \mathbf{a})\mathbf{a} \rangle_1 \\
&= \langle i((\boldsymbol{\nabla} \times \mathbf{a}) \cdot \mathbf{a}) + i((\boldsymbol{\nabla} \times \mathbf{a}) \wedge \mathbf{a}) \rangle_1 \\
&= \langle i((\boldsymbol{\nabla} \times \mathbf{a}) \wedge \mathbf{a}) \rangle_1 \\
&= i^2((\boldsymbol{\nabla} \times \mathbf{a}) \times \mathbf{a})
\end{aligned} \tag{95.13}$$

So we have

$$(\boldsymbol{\nabla} \wedge \mathbf{a}) \cdot \mathbf{a} = \mathbf{a} \times (\boldsymbol{\nabla} \times \mathbf{a}) \tag{95.14}$$

so we can rewrite the Poynting time change eq. (95.10) as

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{1}{\mu_0}\Big(\mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) + c^2 \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})\Big) = c^2(\mathbf{B} \times \mathbf{j}) \tag{95.15}$$

However, the wikipedia article has $\rho\mathbf{E}$ terms, which suggests that a $\boldsymbol{\nabla} \cdot \mathbf{E}$ based expansion has been used. Take II.

Let us try expanding this wedge dot differently, and to track what is being operated on write $\mathbf{x}$ as a variable vector, and $\mathbf{a}$ as a constant vector. Now expand

$$\begin{aligned}
(\boldsymbol{\nabla} \wedge \mathbf{x}) \cdot \mathbf{a} &= -\mathbf{a} \cdot (\boldsymbol{\nabla} \wedge \mathbf{x}) \\
&= \boldsymbol{\nabla}(\mathbf{a} \cdot \mathbf{x}) - (\mathbf{a} \cdot \boldsymbol{\nabla}) \wedge \mathbf{x}
\end{aligned} \tag{95.16}$$

What we really want is an expansion of $(\boldsymbol{\nabla} \wedge \mathbf{x}) \cdot \mathbf{x}$. To get there consider

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{x}^2 &= \dot{\boldsymbol{\nabla}}\dot{\mathbf{x}} \cdot \mathbf{x} + \dot{\boldsymbol{\nabla}}\mathbf{x} \cdot \dot{\mathbf{x}} \\
&= 2\dot{\boldsymbol{\nabla}}\mathbf{x} \cdot \dot{\mathbf{x}}
\end{aligned} \tag{95.17}$$

This has the same form as the first term above. We take the gradient and apply it to a dot product where one of the vectors is kept constant, so we can write

$$\boldsymbol{\nabla}\mathbf{x} \cdot \dot{\mathbf{x}} = \frac{1}{2}\boldsymbol{\nabla}\mathbf{x}^2 \tag{95.18}$$

and finally

$$(\boldsymbol{\nabla} \wedge \mathbf{x}) \cdot \mathbf{x} = \frac{1}{2}\boldsymbol{\nabla}\mathbf{x}^2 - (\mathbf{x} \cdot \boldsymbol{\nabla})\mathbf{x} \tag{95.19}$$

We can now reassemble the equations and write

$$
\begin{aligned}
(\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + c^2 (\boldsymbol{\nabla} \wedge \mathbf{B}) \cdot \mathbf{B} &= \frac{1}{2} \boldsymbol{\nabla} \mathbf{E}^2 - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} + c^2 \left(\frac{1}{2} \boldsymbol{\nabla} \mathbf{B}^2 - (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B} \right) \\
&= \frac{1}{\epsilon_0} \boldsymbol{\nabla} U - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} - c^2 (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}
\end{aligned}
\tag{95.20}
$$

Now, we have the time derivative of momentum and the spatial derivative of the energy grouped together in a nice relativistic seeming pairing. For comparison let us also put the energy density rate change equation with this to observe them together

$$
\begin{aligned}
&\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{P} = -\mathbf{j} \cdot \mathbf{E} \\
&\frac{\partial \mathbf{P}}{\partial t} + c^2 \boldsymbol{\nabla} U = -c^2 (\mathbf{j} \times \mathbf{B}) + \frac{1}{\mu_0} \Big((\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} + c^2 (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B} \Big)
\end{aligned}
\tag{95.21}
$$

The second equation here is exactly what we worked out above by coordinate expansion when looking for a four vector formulation of this equation. This however, appears much closer to the desired result, which was not actually clear looking at the coordinate expansion.

These equations are not tidy enough seeming, so one can intuit that there is some more natural way to express those misfit seeming $(\mathbf{x} \cdot \boldsymbol{\nabla})\mathbf{x}$ terms. It would be logically tidier if we could express those both in terms of charge and current densities.

Now, it is too bad that it is not true that

$$
(\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} = \mathbf{E}(\boldsymbol{\nabla} \cdot \mathbf{E})
\tag{95.22}
$$

If that were the case then we would have on the right hand side

$$
\begin{aligned}
-c^2 (\mathbf{j} \times \mathbf{B}) + \frac{1}{\mu} \Big(\mathbf{E}(\boldsymbol{\nabla} \cdot \mathbf{E}) + c^2 \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{B}) \Big) &= -c^2 (\mathbf{j} \times \mathbf{B}) + \frac{1}{\mu_0} (\mathbf{E}\rho + c^2 \mathbf{B}(0)) \\
&= -c^2 (\mathbf{j} \times \mathbf{B}) + \frac{1}{\mu_0} \rho \mathbf{E}
\end{aligned}
\tag{95.23}
$$

This has a striking similarity to the Lorentz force law, and is also fairly close to the wikipedia equation, with the exception that the $\mathbf{j} \times \mathbf{B}$ and $\rho \mathbf{E}$ terms have opposing signs.

Lets instead adding and subtracting this term so that the conservation equation remains correct

$$
\begin{aligned}
&\frac{1}{c^2} \frac{\partial \mathbf{P}}{\partial t} + \boldsymbol{\nabla} U - \epsilon_0 \Big(\mathbf{E}(\boldsymbol{\nabla} \cdot \mathbf{E}) + (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} + c^2 \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{B}) + c^2 (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B} \Big) \\
&\qquad = -(\mathbf{j} \times \mathbf{B}) - \epsilon_0 \rho \mathbf{E}
\end{aligned}
\tag{95.24}
$$

Now we are left with quantities of the following form.

$$\mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{x}) + (\mathbf{x} \cdot \boldsymbol{\nabla})\mathbf{x} \tag{95.25}$$

The sum of these for the electric and magnetic fields appears to be what the wiki article calls $\boldsymbol{\nabla} \cdot \sigma$, although it appears there that σ is a scalar so this does not quite make sense.

It appears that we should therefore be looking to express these in terms of a gradient of the squared fields? We have such $\mathbf{E}^2$ and $\mathbf{B}^2$ terms in the energy so it would make some logical sense if this could be done.

The essence of the desired reduction is to see if we can find a scalar function $\sigma(\mathbf{x})$ such that

$$\boldsymbol{\nabla}\sigma(\mathbf{x}) = \frac{1}{2}\boldsymbol{\nabla}\mathbf{x}^2 - \left(\mathbf{x}(\boldsymbol{\nabla} \cdot \mathbf{x}) + (\mathbf{x} \cdot \boldsymbol{\nabla})\mathbf{x}\right) \tag{95.26}$$

95.2.3 *stress tensor*

From [10] we expect that there is a relationship between the equations eq. (95.12), and $F\gamma_k F$. Let us see if we can find exactly how these relate.

TODO: ...

95.3 TAKE II

After going in circles and having a better idea now where I am going, time to restart and make sure that errors are not compounding.

The starting point will be

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial t} &= \frac{1}{\mu_0}\left(\partial_0(ic\mathbf{B}) \cdot \mathbf{E} + (ic\mathbf{B}) \cdot \partial_0 \mathbf{E}\right) \\
\partial_0 \mathbf{E} &= -\mathbf{j}/\epsilon_0 c - \boldsymbol{\nabla} \cdot (ic\mathbf{B}) \\
\partial_0(ic\mathbf{B}) &= -\boldsymbol{\nabla} \wedge \mathbf{E}
\end{aligned} \tag{95.27}$$

Assembling we have

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{1}{\mu_0}\left((\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + (ic\mathbf{B}) \cdot (\mathbf{j}/\epsilon_0 c + \boldsymbol{\nabla} \cdot (ic\mathbf{B}))\right) = 0 \tag{95.28}$$

This is

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{1}{\mu_0}\left((\boldsymbol{\nabla} \wedge \mathbf{E}) \cdot \mathbf{E} + (ic\mathbf{B}) \cdot (\boldsymbol{\nabla} \cdot (ic\mathbf{B}))\right) = -c^2(i\mathbf{B}) \cdot \mathbf{j}. \tag{95.29}$$

Now get rid of the pseudoscalars

$$\begin{aligned}(i\mathbf{B})\cdot\mathbf{j} &= \langle i\mathbf{B}\mathbf{j}\rangle_1 \\ &= i(\mathbf{B}\wedge\mathbf{j}) \\ &= i^2(\mathbf{B}\times\mathbf{j}) \\ &= -(\mathbf{B}\times\mathbf{j})\end{aligned} \tag{95.30}$$

and

$$\begin{aligned}(ic\mathbf{B})\cdot(\boldsymbol{\nabla}\cdot(ic\mathbf{B})) &= c^2\langle i\mathbf{B}(\boldsymbol{\nabla}\cdot(i\mathbf{B}))\rangle_1 \\ &= c^2\langle i\mathbf{B}\langle\boldsymbol{\nabla}i\mathbf{B}\rangle_1\rangle_1 \\ &= c^2\langle i\mathbf{B}i(\boldsymbol{\nabla}\wedge\mathbf{B})\rangle_1 \\ &= -c^2\langle\mathbf{B}(\boldsymbol{\nabla}\wedge\mathbf{B})\rangle_1 \\ &= -c^2\mathbf{B}\cdot(\boldsymbol{\nabla}\wedge\mathbf{B})\end{aligned} \tag{95.31}$$

So we have

$$\frac{\partial\mathbf{P}}{\partial t} - \frac{1}{\mu_0}\Big(\mathbf{E}\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) + c^2\mathbf{B}\cdot(\boldsymbol{\nabla}\wedge\mathbf{B})\Big) = c^2(\mathbf{B}\times\mathbf{j}) \tag{95.32}$$

Now we subtract $(\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + c^2\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}))/\mu_0 = \mathbf{E}\rho/\epsilon_0\mu_0$ from both sides yielding

$$\frac{\partial\mathbf{P}}{\partial t} - \frac{1}{\mu_0}\Big(\mathbf{E}\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) + \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + c^2\mathbf{B}\cdot(\boldsymbol{\nabla}\wedge\mathbf{B}) + c^2\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B})\Big) = -c^2(\mathbf{j}\times\mathbf{B} + \rho\mathbf{E}) \tag{95.33}$$

Regrouping slightly

$$\begin{aligned}0 = \frac{1}{c^2}\frac{\partial\mathbf{P}}{\partial t} &+ (\mathbf{j}\times\mathbf{B} + \rho\mathbf{E}) \\ &- \epsilon_0\Big(\mathbf{E}\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) + \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + c^2\mathbf{B}\cdot(\boldsymbol{\nabla}\wedge\mathbf{B}) + c^2\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B})\Big)\end{aligned} \tag{95.34}$$

Now, let us write the **E** gradient terms here explicitly in coordinates.

$$\begin{aligned}-\mathbf{E}\cdot(\boldsymbol{\nabla}\wedge\mathbf{E}) - \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) &= -\sigma_k\cdot(\sigma^m\wedge\sigma_n)E^k\partial_m E^n - E^k\sigma_k\partial_m E^m \\ &= -\delta_k^m\sigma_n E^k\partial_m E^n + \delta_{kn}\sigma^m E^k\partial_m E^n - E^k\sigma_k\partial_m E^m \\ &= -\sigma_n E^k\partial_k E^n + \sigma^m E^k\partial_m E^k - E^k\sigma_k\partial_m E^m \\ &= \sum_{k,m}\sigma_k\Big(-E^m\partial_m E^k + E^m\partial_k E^m - E^k\partial_m E^m\Big)\end{aligned} \tag{95.35}$$

We could do the **B** terms too, but they will have the same form. Now [39] contains a relativistic treatment of the stress tensor that would take some notation study to digest, but the end result appears to have the divergence result that is desired. It is a second rank tensor which probably explains the $\nabla \cdot \sigma$ notation in wikipedia.

For the x coordinate of the $\partial \mathbf{P}/\partial t$ vector the book says we have a vector of the form

$$\mathbf{T}_x = \frac{1}{2}(-E_x^2 + E_y^2 + E_z^2)\sigma_1 - E_x E_y \sigma_2 - E_x E_z \sigma_3 \tag{95.36}$$

and it looks like the divergence of this should give us our desired mess. Let us try this, writing k, m, n as distinct indices.

$$\mathbf{T}_k = \frac{1}{2}(-(E^k)^2 + (E^m)^2 + (E^n)^2)\sigma_k - E^k E^m \sigma_m - E^k E^n \sigma_n \tag{95.37}$$

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{T}_k &= \frac{1}{2}\partial_k(-(E^k)^2 + (E^m)^2 + (E^n)^2) - \partial_m(E^k E^m) - \partial_n(E^k E^n) \\
&= -E^k \partial_k E^k + E^m \partial_k E^m + E^n \partial_k E^n - E^k \partial_m E^m - E^m \partial_m E^k - E^k \partial_n E^n - E^n \partial_n E^k \\
&= -E^k \partial_k E^k - E^k \partial_m E^m - E^k \partial_n E^n \\
&- E^m \partial_m E^k + E^m \partial_k E^m \\
&- E^n \partial_n E^k + E^n \partial_k E^n
\end{aligned} \tag{95.38}$$

Does this match? Let us expand our k term above to see if it looks the same. That is

$$\begin{aligned}
\sum_m (-E^m \partial_m E^k + E^m \partial_k E^m - E^k \partial_m E^m) &= -E^k \partial_k E^k + E^k \partial_k E^k - E^k \partial_k E^k \\
&- E^m \partial_m E^k + E^m \partial_k E^m - E^k \partial_m E^m \\
&- E^n \partial_n E^k + E^n \partial_k E^n - E^k \partial_n E^n \\
&= -E^k \partial_k E^k - E^k \partial_m E^m - E^k \partial_n E^n \\
&- E^m \partial_m E^k + E^m \partial_k E^m \\
&- E^n \partial_n E^k + E^n \partial_k E^n
\end{aligned} \tag{95.39}$$

Yeah! Finally have a form of the momentum conservation equation that is strictly in terms of gradients and time partials. Summarizing the results, this is

$$\frac{1}{c^2}\frac{\partial \mathbf{P}}{\partial t} + \mathbf{j} \times \mathbf{B} + \rho \mathbf{E} + \sum_k \sigma_k \boldsymbol{\nabla} \cdot \mathbf{T}_k = 0 \tag{95.40}$$

Where

$$\sum_k \sigma_k \boldsymbol{\nabla} \cdot \mathbf{T}_k = -\epsilon_0 \Big(\mathbf{E} \cdot (\boldsymbol{\nabla} \wedge \mathbf{E}) + \mathbf{E}(\boldsymbol{\nabla} \cdot \mathbf{E}) + c^2 \mathbf{B} \cdot (\boldsymbol{\nabla} \wedge \mathbf{B}) + c^2 \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{B})\Big) \tag{95.41}$$

For $\mathbf{T}_k$ itself, with $k \neq m \neq n$ we have

$$\begin{aligned}
\mathbf{T}_k = \epsilon_0 &\left(\frac{1}{2}(-(E^k)^2 + (E^m)^2 + (E^n)^2)\sigma_k - E^k E^m \sigma_m - E^k E^n \sigma_n\right) \\
&+ \frac{1}{\mu_0}\left(\frac{1}{2}(-(B^k)^2 + (B^m)^2 + (B^n)^2)\sigma_k - B^k B^m \sigma_m - B^k B^n \sigma_n\right)
\end{aligned} \tag{95.42}$$

96

FIELD AND WAVE ENERGY AND MOMENTUM

96.1 MOTIVATION

The concept of energy in the electric and magnetic fields I am getting closer to understanding, but there is a few ways that I would like to approach it.

I have now explored the Poynting vector energy conservation relationships in 94, and 97 , but hhad not understood fully where the energy expressions in the electro and magneto statics cases came from separately. I also do not yet know where the $F\gamma_k F$ terms of the stress tensor fit in the big picture? I suspect that they can be obtained by Lorentz transforming the rest frame expression $F\gamma_0 F$ (the energy density, Poynting momentum density four vector).

It also ought to be possible to relate the field energies to a Lagrangian and Hamiltonian, but I have not had success doing so.

The last thing that I had like to understand is how the energy and momentum of a wave can be expressed, both in terms of the abstract conjugate field momentum concept and with a concrete example such as the one dimensional oscillating rod that can be treated in a limiting coupled oscillator approach as in [16].

Once I have got all these down I think I will be ready to revisit Bohm's Rayleigh-Jeans law treatment in [2]. Unfortunately, each time I try perusing some interesting aspect of QM I find that I end up back studying electrodynamics, and suspect that needs to be my focus for the foreseeable future (perhaps working thoroughly through Feynman's volume II).

96.2 ELECTROSTATIC ENERGY IN A FIELD

Feynman's treatment in [12] of the energy $\frac{\epsilon_0}{2}\mathbf{E}^2$ associated with the electrostatic $\mathbf{E}$ field is very easy to understand. Here is a write up of this myself without looking at the book to see if I really understood the ideas.

The first step is consideration of the force times distance for two charges gives you the energy required (or gained) by moving one of those charges from infinity to some given separation

$$\begin{aligned}
W &= \frac{1}{4\pi\epsilon_0}\int_\infty^r \frac{q_1 q_2}{x^2}\mathbf{e}_1 \cdot (-\mathbf{e}_1 dx) \\
&= \frac{q_1 q_2}{4\pi\epsilon_0 r}
\end{aligned} \tag{96.1}$$

This provides a quantization for an energy in a field concept. A distribution of charge requires energy to assemble and it is possible to enumerate that energy separately by considering all the charges, or alternatively, by not looking at the final charge distribution, but only considering the net field associated with this charge distribution. This is a pretty powerful, but somewhat abstract seeming idea.

The generalization to continuous charge distribution from there was pretty straightforward, requiring a double integration over all space

$$\begin{aligned} W &= \frac{1}{2}\int \frac{1}{4\pi\epsilon_0}\frac{\rho_1 dV_1 \rho_2 dV_2}{r_{12}} \\ &= \frac{1}{2}\int \rho_1 \phi_2 dV_1 \end{aligned} \tag{96.2}$$

The $1/2$ factor was due to double counting all "pairs" of charge elements. The next step was to rewrite the charge density by using Maxwell's equations. In terms of the four vector potential Maxwell's equation (with the $\nabla \cdot A = 0$ gauge) is

$$\nabla^2 A = \frac{1}{\epsilon_0 c}(c\rho\gamma_0 + J^k\gamma_k) \tag{96.3}$$

So, to write ρ in terms of potential $A^0 = \phi$, we have

$$\left(\frac{1}{c^2}\frac{\partial^2}{(\partial t)^2} - \boldsymbol{\nabla}^2\right)\phi = \frac{1}{\epsilon_0}\rho \tag{96.4}$$

In the statics case, where $\frac{\partial\phi}{\partial t} = 0$, we can thus write the charge density in terms of the potential

$$\rho = -\epsilon_0 \boldsymbol{\nabla}^2 \phi \tag{96.5}$$

and substitute back into the energy summation

$$\begin{aligned} W &= \frac{1}{2}\int \rho\phi dV \\ &= \frac{-\epsilon_0}{2}\int \phi\boldsymbol{\nabla}^2\phi dV \end{aligned} \tag{96.6}$$

Now, Feynman's last step was a bit sneaky, which was to convert the $\phi\boldsymbol{\nabla}^2\phi$ term into a divergence integral. Working backwards to derive the identity that he used

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\phi\boldsymbol{\nabla}\phi) &= \langle \boldsymbol{\nabla}(\phi\boldsymbol{\nabla}\phi)\rangle \\ &= \langle (\boldsymbol{\nabla}\phi)\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi)\rangle \\ &= (\boldsymbol{\nabla}\phi)^2 + \phi\boldsymbol{\nabla}^2\phi \end{aligned} \tag{96.7}$$

This can then be used with Stokes theorem in its dual form to convert our $\phi\boldsymbol{\nabla}^2\phi$ the into plain volume and surface integral

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\int \Big((\boldsymbol{\nabla}\phi)^2 - \boldsymbol{\nabla}\cdot(\phi\boldsymbol{\nabla}\phi)\Big)\,dV \\
&= \frac{\epsilon_0}{2}\int (\boldsymbol{\nabla}\phi)^2 dV - \frac{\epsilon_0}{2}\int_{\partial V} (\phi\boldsymbol{\nabla}\phi)\cdot\hat{\mathbf{n}} dA
\end{aligned}
\tag{96.8}
$$

Letting the surface go to infinity and employing a limiting argument on the magnitudes of the ϕ and $\boldsymbol{\nabla}\phi$ terms was enough to produce the final electrostatics energy result

$$
\begin{aligned}
W &= \frac{\epsilon_0}{2}\int (\boldsymbol{\nabla}\phi)^2 dV \\
&= \frac{\epsilon_0}{2}\int \mathbf{E}^2 dV
\end{aligned}
\tag{96.9}
$$

96.3 MAGNETOSTATIC FIELD ENERGY

Feynman's energy discussion of the magnetic field for a constant current loop (magnetostatics), is not so easy to follow. He considers the dipole moment of a small loop, obtained by comparison to previous electrostatic results (that I had have to go back and read or re-derive) and some subtle seeming arguments about the mechanical vs. total energy of the system.

96.3.1 *Biot Savart*

As an attempt to understand all this, let us break it up into pieces. First, is calculation of the field for a current loop. Let us also use this as an opportunity to see how one would work directly and express the Biot-Savart law in the STA formulation.

Going back to Maxwell's equation (with the $\nabla \cdot A$ gauge again), we have

$$
\begin{aligned}
\nabla F &= \nabla(\nabla \wedge A) \\
&= \nabla^2 A^\mu \\
&= J^\mu/\epsilon_0 c
\end{aligned}
\tag{96.10}
$$

For a static current configuration with $J^0 = c\rho = 0$, we have $\partial A^\mu/\partial t = 0$, and our vector potential equations are

$$
\boldsymbol{\nabla}^2 A^k = -J^k/\epsilon_0 c
\tag{96.11}
$$

Recall that the solution of A^k can be expressed as the impulse response of a function of the following form

$$A^k = C\frac{1}{r} \tag{96.12}$$

and that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}(1/r))$ is zero for all $r \neq 0$. Performing a volume integral of the expected Laplacian we can integrate over an infinitesimal spherical volume of radius R

$$\begin{aligned}
\int \boldsymbol{\nabla}^2 A^k dV &= C \int \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\frac{1}{r} dV \\
&= C \int \boldsymbol{\nabla} \cdot \left(-\hat{\mathbf{r}}\frac{1}{r^2}\right) dV \\
&= -C \int_{\partial V} \hat{\mathbf{r}}\frac{1}{r^2} \cdot \hat{\mathbf{r}} dA \\
&= -C \frac{1}{R^2} 4\pi R^2 \\
&= -4\pi C
\end{aligned} \tag{96.13}$$

Equating we can solve for C

$$\begin{aligned}
-4\pi C &= -J^k/\epsilon_0 c \\
C &= \frac{1}{4\pi\epsilon_0 c} J^k
\end{aligned} \tag{96.14}$$

Note that this is cheating slightly since C was kind of treated as a constant, whereas this equality makes it a function. It works because the infinitesimal volume can be made small enough that J^k can be treated as a constant. This therefore provides our potential function in terms of this impulse response

$$A^k = \frac{1}{4\pi\epsilon_0 c} \int \frac{J^k}{r} dV \tag{96.15}$$

Now, this could have all been done with a comparison to the electrostatic result. Regardless, it now leaves us in the position to calculate the field bivector

$$\begin{aligned}
F &= \nabla \wedge A \\
&= (\gamma^\mu \wedge \gamma_k)\partial_\mu A^k \\
&= -(\gamma_m \wedge \gamma_k)\partial_m A^k
\end{aligned} \tag{96.16}$$

So our field in terms of components is

$$F = (\sigma_m \wedge \sigma_k)\partial_m A^k \tag{96.17}$$

Which in terms of spatial vector potential $\mathbf{A} = A^k \sigma_k$ is also

$$F = \boldsymbol{\nabla} \wedge \mathbf{A} \tag{96.18}$$

From eq. (96.17) we can calculate the field in terms of our potential directly

$$\begin{aligned}
\partial_m A^k &= \frac{1}{4\pi\epsilon_0 c} \int dV \partial_m \frac{J^k}{r} \\
&= \frac{1}{4\pi\epsilon_0 c} \int dV \left(J^k \partial_m \frac{1}{r} + \frac{1}{r} \partial_m J^k \right) \\
&= \frac{1}{4\pi\epsilon_0 c} \int dV \left(J^k \partial_m \left(\sum_j ((x^j)^2)^{-1/2} \right) + \frac{1}{r} \partial_m J^k \right) \\
&= \frac{1}{4\pi\epsilon_0 c} \int dV \left(J^k \left(-\frac{1}{2} \right) 2x^m \frac{1}{r^3} + \frac{1}{r} \partial_m J^k \right) \\
&= \frac{1}{4\pi\epsilon_0 c} \int \frac{1}{r^3} dV \left(-x^m J^k + r^2 \partial_m J^k \right)
\end{aligned} \tag{96.19}$$

So with $\mathbf{j} = J^k \sigma_k$ we have

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0 c} \int \frac{1}{r^3} dV \left(-\mathbf{r} \wedge \mathbf{j} + r^2 (\boldsymbol{\nabla} \wedge \mathbf{j}) \right) \\
&= \frac{1}{4\pi\epsilon_0 c} \int dV \left(\frac{\mathbf{j} \wedge \hat{\mathbf{r}}}{r^2} + \frac{1}{r} (\boldsymbol{\nabla} \wedge \mathbf{j}) \right)
\end{aligned} \tag{96.20}$$

The first term here is essentially the Biot Savart law once the current density is converted to current $\int \mathbf{j} dV = I \int \hat{\mathbf{j}} dl$, so we expect the second term to be zero.

To calculate the current density divergence we first need the current density in vector form

$$\begin{aligned}
\mathbf{j} &= -\epsilon_0 c \boldsymbol{\nabla}^2 \mathbf{A} \\
&= -\epsilon_0 c \langle \boldsymbol{\nabla} (\boldsymbol{\nabla} \mathbf{A}) \rangle_1 \\
&= -\epsilon_0 c \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{A}) + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A})
\end{aligned} \tag{96.21}$$

Now, recall the gauge choice was

$$\begin{aligned}
0 &= \nabla \cdot A \\
&= \partial_0 A^0 + \partial_k A^k \\
&= \frac{1}{c} \frac{\partial A^0}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A}
\end{aligned} \tag{96.22}$$

So, provided we also have $\partial A^0/\partial t = 0$, we also have $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, which is true due to the assumed static conditions, we are left with

$$\mathbf{j} = -\epsilon_0 c \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A}) \tag{96.23}$$

Now we can take the curl of $\mathbf{j}$, also writing this magnetic field F in its dual form $F = ic\mathbf{B}$, we see that the curl of our static current density vector is zero:

$$\begin{aligned}
\boldsymbol{\nabla} \wedge \mathbf{j} &= \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \cdot F) \\
&= c\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \cdot (i\mathbf{B})) \\
&= \frac{c}{2}\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla}(i\mathbf{B}) - i\dot{\mathbf{B}}\dot{\boldsymbol{\nabla}}) \\
&= c\boldsymbol{\nabla} \wedge (i\boldsymbol{\nabla} \wedge \mathbf{B}) \\
&= c\boldsymbol{\nabla} \wedge (i^2\boldsymbol{\nabla} \times \mathbf{B}) \\
&= -ci\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) \\
&= 0
\end{aligned} \tag{96.24}$$

This leaves us with

$$F = \frac{1}{4\pi\epsilon_0 c} \int \frac{\mathbf{j} \wedge \hat{\mathbf{r}}}{r^2} dV \tag{96.25}$$

Which with the current density expressed in terms of current is the desired Biot-Savart law

$$F = \frac{1}{4\pi\epsilon_0 c} \int \frac{I d\mathbf{s} \wedge \hat{\mathbf{r}}}{r^2} \tag{96.26}$$

Much shorter derivations are possible than this one which was essentially done from first principles. The one in [10], which also uses the STA formulation, is the shortest I have ever seen, utilizing a vector Green's function for the Laplacian. However, that requires understanding the geometric calculus chapter of that book, which is a battle for a different day.

96.3.2 *Magnetic field torque and energy*

TODO: work out on paper and write up.

I created a PF thread, electric and magnetic field energy , to followup on these ideas, and now have an idea how to proceed.

96.4 COMPLETE FIELD ENERGY

Can a integral of the Lorentz force coupled with Maxwell's equations in their entirety produce the energy expression $\frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)$? It seems like cheating to add these arbitrarily and then follow the Poynting derivation by taking derivatives. That shows this quantity is a conserved quantity, but does it really show that it is the energy? One could imagine that there could be other terms in a total energy expression such as $\mathbf{E}\cdot\mathbf{B}$.

Looking in more detail at the right hand side of the energy/Poynting relationship is the key. That is

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) + c^2\epsilon_0\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot\mathbf{j} \tag{96.27}$$

Two questions to ask. The first is that if the left hand side is to be a conserved quantity then we need the right hand side to be one too? Is that really the case? Second, how can this be related to work done (a line integral of the Lorentz force).

The second question is easiest, and the result actually follows directly.

$$\begin{aligned}
\text{Work done moving a charge against the Lorentz force} &= \int \mathbf{F}\cdot(-d\mathbf{x}) \\
&= \int q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot(-d\mathbf{x}) \\
&= -\int q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v}dt \\
&= -\int q\mathbf{E}\cdot\mathbf{v}dt \\
&= -\int \mathbf{E}\cdot\mathbf{j}dtdV
\end{aligned} \tag{96.28}$$

From this we see that $-\mathbf{E}\cdot\mathbf{j}$ is the rate of change of power density in an infinitesimal volume! Let us write

$$\begin{aligned}
U &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\
\mathbf{P} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})
\end{aligned} \tag{96.29}$$

and now take eq. (96.27) and integrate over a (small) volume

$$\int_V \frac{\partial U}{\partial t}dV + \int_{\partial V}\mathbf{P}\cdot\hat{\mathbf{n}}dA = -\int_V(\mathbf{E}\cdot\mathbf{j})dV \tag{96.30}$$

So, for a small time increment $\Delta t = t_1 - t_0$, corresponding to the start and end times of the particle at the boundaries of the work line integral, we have

$$\begin{aligned}
\text{Work done on particle against field} &= \int_{t_0}^{t_1} \int_V \frac{\partial U}{\partial t} dV dt + \int_{t_0}^{t_1} \int_{\partial V} \mathbf{P} \cdot \hat{\mathbf{n}} dA dt \\
&= \int_V (U(t_1) - U(t_0)) dV + \int_{t_0}^{t_1} \int_{\partial V} \mathbf{P} \cdot \hat{\mathbf{n}} dA dt \\
&= \int_V \Delta U dV + \int_{t_0}^{t_1} \int_{\partial V} \mathbf{P} \cdot \hat{\mathbf{n}} dA dt
\end{aligned} \tag{96.31}$$

Roughly speaking, it appears that the energy provided to move a charge against the field is absorbed into the field in one of two parts, one of which is what gets identified as the energy of the field $\int U dV$. The other part is the time integral of the flux through the surface of the volume of this Poynting vector $\mathbf{P}$.

96.4.1 *Dimensional analysis*

That is a funny looking term though? Where would we see momentum integrated over time in classical mechanics?

$$\int mvdt = mx \tag{96.32}$$

Let us look at the dimensions of all the terms in the conservation equation. We have identified the $\mathbf{j} \cdot \mathbf{E}$ term with energy density, and should see this

$$\begin{aligned}
[\mathbf{jE}] &= [(qv/x^3)(F/q)] \\
&= [(x/(x^3 t))(mx/t^2)] \\
&= [m(x^2/t^2)/(x^3 t)] \\
&= \frac{\text{Energy}}{\text{Volume} \times \text{Time}}
\end{aligned} \tag{96.33}$$

Good. That is what should have been the case.

Now, for the U term we must then have

$$[U] = \frac{\text{Energy}}{Volume} \tag{96.34}$$

Okay, that is good too, since we were calling U energy density. Now for the Poynting term we have

$$[\boldsymbol{\nabla} \cdot \mathbf{P}] = [1/x][\mathbf{P}] \tag{96.35}$$

So we have

$$\begin{aligned}
[\mathbf{P}] &= [1/x][\mathbf{P}] \\
&= \frac{\text{Energy} \times \text{velocity}}{\text{Volume}}
\end{aligned} \tag{96.36}$$

For uniform dimensions of all the terms this suggests that it is perhaps more natural to work with velocity scaled quantity, with

$$\frac{[\mathbf{P}]}{\text{Velocity}} = \frac{\text{Energy}}{\text{Volume}} \tag{96.37}$$

Rewriting the conservation equation scaling by a velocity, for which the obvious generic velocity choice is naturally c, we have

$$\frac{1}{c}\frac{\partial}{\partial t}U + \boldsymbol{\nabla} \cdot \frac{\mathbf{P}}{c} = -\frac{\mathbf{j}}{c} \cdot \mathbf{E} \tag{96.38}$$

Written this way we have $1/ct$ with dimensions of inverse distance matching the divergence, and the dimensions of U, and $\mathbf{P}/c$ are both energy density. Now it makes a bit more sense to say that the work done moving the charge against the field supplies energy to the field in some fashion between these two terms.

96.4.2 *A note on the scalar part of the covariant Lorentz force*

The covariant formulation of the Lorentz force equation, when wedged with γ_0 has been seen to recover the traditional Lorentz force equation (with a required modification to use relativistic momentum), but there was a scalar term that was unaccounted for.

Recall that the covariant Lorentz force, with derivatives all in terms of proper time, was

$$\begin{aligned}
m\dot{p} &= qF \cdot (v/c) \\
&= \frac{q}{2c}(Fv - vF) \\
&= \frac{q}{2c}((\mathbf{E} + ic\mathbf{B})\gamma_0(\dot{x^0} - \dot{x^k}\sigma_k) - \gamma_0(\dot{x^0} - \dot{x^k}\sigma_k)(\mathbf{E} + ic\mathbf{B}))
\end{aligned} \tag{96.39}$$

In terms of time derivatives, where factors of γ can be canceled on each side, we have

$$m\frac{dp}{dt} = \frac{q}{2}\gamma_0((-\mathbf{E} + ic\mathbf{B})(1 - \mathbf{v}/c) - (1 - \mathbf{v}/c)(\mathbf{E} + ic\mathbf{B})) \tag{96.40}$$

After some reduction this is

$$m\frac{dp}{dt} = q(-\mathbf{E} \cdot \mathbf{v}/c + (\mathbf{E} + \mathbf{v} \times \mathbf{B}))\gamma_0 \tag{96.41}$$

Or, with an explicit spacetime split for all components

$$\begin{aligned} mc\frac{d\gamma}{dt} &= -q\mathbf{E} \cdot \mathbf{v}/c \\ m\frac{d\gamma\mathbf{v}}{dt} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B})) \end{aligned} \tag{96.42}$$

We have got the spatial vector Lorentz force in the second term, and now have an idea what this $-\mathbf{j} \cdot \mathbf{E}$ term is in the energy momentum vector. It is not a random thing, but an intrinsic part (previously ignored) of the covariant Lorentz force.

Now recall that when the time variation of the Poynting was studied in 95 we had what looked like the Lorentz force components in all the right hand side terms. Let us reiterate that here, putting all the bits together

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}U + \boldsymbol{\nabla} \cdot \frac{\mathbf{P}}{c} &= -\frac{\mathbf{j}}{c} \cdot \mathbf{E} \\ \frac{1}{c^2}\frac{\partial \mathbf{P}}{\partial t} + \sum_k \sigma_k \boldsymbol{\nabla} \cdot \mathbf{T}_k &= -(\mathbf{j} \times \mathbf{B} + \rho\mathbf{E}) \end{aligned} \tag{96.43}$$

We have four scalar equations, where each one contains exactly one of the four vector components of the Lorentz force. This makes the stress energy tensor seem a lot less random. Now the interesting thing about this is that each of these equations required nothing more than a bunch of algebra applied to the Maxwell equation. Doing so required no use of the Lorentz force, but it shows up magically as an intrinsic quantity associated with the Maxwell equation. Before this I thought that one really needed both Maxwell's equation and the Lorentz force equation (or their corresponding Lagrangians), but looking at this result the Lorentz force seems to more of a property of the field than a fundamental quantity in its own right (although some means to relate this stress energy tensor to force is required).

97

ENERGY MOMENTUM TENSOR

97.1 EXPANDING OUT THE STRESS ENERGY VECTOR IN TENSOR FORM

[10] defines (with ϵ_0 omitted), the energy momentum stress tensor as a vector to vector mapping of the following form:

$$T(a) = \frac{\epsilon_0}{2} F a \tilde{F} = -\frac{\epsilon_0}{2} F a F \tag{97.1}$$

This quantity can only have vector, trivector, and five vector grades.Â The grade five term must be zero

$$\begin{aligned} \langle T(a) \rangle_5 &= \frac{\epsilon_0}{2} F \wedge a \wedge \tilde{F} \\ &= \frac{\epsilon_0}{2} a \wedge (F \wedge \tilde{F}) \\ &= 0 \end{aligned} \tag{97.2}$$

Since $(T(a))\tilde{} = T(a)$, the grade three term is also zero (trivectors invert on reversion), so this must therefore be a vector.

As a vector this can be expanded in coordinates

$$\begin{aligned} T(a) &= \left(T(a) \cdot \gamma^\nu\right) \gamma_\nu \\ &= \left(T(a^\mu \gamma_\mu) \cdot \gamma^\nu\right) \gamma_\nu \\ &= a^\mu \gamma_\nu \left(T(\gamma_\mu) \cdot \gamma^\nu\right) \end{aligned} \tag{97.3}$$

It is this last bit that has the form of a traditional tensor, so we can write

$$\begin{aligned} T(a) &= a^\mu \gamma_\nu {T_\mu}^\nu \\ {T_\mu}^\nu &= T(\gamma_\mu) \cdot \gamma^\nu \end{aligned} \tag{97.4}$$

Let us expand this tensor ${T_\mu}^\nu$ explicitly to verify its form.

We want to expand, and dot with γ^ν, the following

$$
\begin{aligned}
-2\frac{1}{\epsilon_0}\left(T(\gamma_\mu)\cdot\gamma^\nu\right)\gamma_\nu &= \Big\langle(\nabla\wedge A)\gamma_\mu(\nabla\wedge A)\Big\rangle_1 \\
&= \Big\langle(\nabla\wedge A)\cdot\gamma_\mu(\nabla\wedge A) + (\nabla\wedge A)\wedge\gamma^\mu(\nabla\wedge A)\Big\rangle_1 \\
&= ((\nabla\wedge A)\cdot\gamma_\mu)\cdot(\nabla\wedge A) + ((\nabla\wedge A)\wedge\gamma_\mu)\cdot(\nabla\wedge A)
\end{aligned}
\tag{97.5}
$$

Both of these will get temporarily messy, so let us do them in parts.Â Starting with

$$
\begin{aligned}
(\nabla\wedge A)\cdot\gamma_\mu &= (\gamma^\alpha\wedge\gamma^\beta)\cdot\gamma_\mu\partial_\alpha A_\beta \\
&= (\gamma^\alpha{\delta^\beta}_\mu - \gamma^\beta{\delta^\alpha}_\mu)\partial_\alpha A_\beta \\
&= \gamma^\alpha\partial_\alpha A_\mu - \gamma^\beta\partial_\mu A_\beta \\
&= \gamma^\alpha(\partial_\alpha A_\mu - \partial_\mu A_\alpha) \\
&= \gamma^\alpha F_{\alpha\mu}
\end{aligned}
\tag{97.6}
$$

$$
\begin{aligned}
((\nabla\wedge A)\cdot\gamma_\mu)\cdot(\nabla\wedge A) &= (\gamma^\alpha F_{\alpha\mu})\cdot(\gamma_\beta\wedge\gamma_\lambda)\partial^\beta A^\lambda \\
&= \partial^\beta A^\lambda F_{\alpha\mu}({\delta^\alpha}_\beta\gamma_\lambda - {\delta^\alpha}_\lambda\gamma_\beta) \\
&= (\partial^\alpha A^\beta F_{\alpha\mu} - \partial^\beta A^\alpha F_{\alpha\mu})\gamma_\beta \\
&= F^{\alpha\beta}F_{\alpha\mu}\gamma_\beta
\end{aligned}
\tag{97.7}
$$

So, by dotting with γ^ν we have

$$
((\nabla\wedge A)\cdot\gamma_\mu)\cdot(\nabla\wedge A)\cdot\gamma^\nu = F^{\alpha\nu}F_{\alpha\mu}
\tag{97.8}
$$

Moving on to the next bit, $(((\nabla\wedge A)\wedge\gamma^\mu)\cdot(\nabla\wedge A))\cdot\gamma^\nu$. By inspection the first part of this is

$$
(\nabla\wedge A)\wedge\gamma_\mu = (\gamma_\mu)^2(\gamma^\alpha\wedge\gamma^\beta)\wedge\gamma^\mu\partial_\alpha A_\beta
\tag{97.9}
$$

so dotting with $\nabla\wedge A$, we have

$$
\begin{aligned}
((\nabla\wedge A)\wedge\gamma_\mu)\cdot(\nabla\wedge A) &= (\gamma_\mu)^2\partial_\alpha A_\beta\partial^\lambda A^\delta(\gamma^\alpha\wedge\gamma^\beta\wedge\gamma^\mu)\cdot(\gamma_\lambda\wedge\gamma_\delta) \\
&= (\gamma_\mu)^2\partial_\alpha A_\beta\partial^\lambda A^\delta((\gamma^\alpha\wedge\gamma^\beta\wedge\gamma^\mu)\cdot\gamma_\lambda)\cdot\gamma_\delta
\end{aligned}
\tag{97.10}
$$

Expanding just the dot product parts of this we have

$$
\begin{aligned}
&(((\gamma^\alpha\wedge\gamma^\beta)\wedge\gamma^\mu)\cdot\gamma_\lambda)\cdot\gamma_\delta \\
&= (\gamma^\alpha\wedge\gamma^\beta){\delta^\mu}_\lambda - (\gamma^\alpha\wedge\gamma^\mu){\delta^\beta}_\lambda + (\gamma^\beta\wedge\gamma^\mu){\delta^\alpha}_\lambda)\cdot\gamma_\delta \\
&= \gamma^\alpha({\delta^\beta}_\delta{\delta^\mu}_\lambda - {\delta^\mu}_\delta{\delta^\beta}_\lambda) + \gamma^\beta({\delta^\mu}_\delta{\delta^\alpha}_\lambda - {\delta^\alpha}_\delta{\delta^\mu}_\lambda) + \gamma^\mu({\delta^\alpha}_\delta{\delta^\beta}_\lambda - {\delta^\beta}_\delta{\delta^\alpha}_\lambda)
\end{aligned}
\tag{97.11}
$$

This can now be applied to $\partial^\lambda A^\delta$

$$
\begin{aligned}
&\partial^\lambda A^\delta(((\gamma^\alpha \wedge \gamma^\beta) \wedge \gamma^\mu) \cdot \gamma_\lambda) \cdot \gamma_\delta \\
&\quad = \partial^\mu A^\beta \gamma^\alpha - \partial^\beta A^\mu \gamma^\alpha + \partial^\alpha A^\mu \gamma^\beta - \partial^\mu A^\alpha \gamma^\beta + \partial^\beta A^\alpha \gamma^\mu - \partial^\alpha A^\beta \gamma^\mu \\
&\quad = (\partial^\mu A^\beta - \partial^\beta A^\mu)\gamma^\alpha + (\partial^\alpha A^\mu - \partial^\mu A^\alpha)\gamma^\beta + (\partial^\beta A^\alpha - \partial^\alpha A^\beta)\gamma^\mu \\
&\quad = F^{\mu\beta}\gamma^\alpha + F^{\alpha\mu}\gamma^\beta + F^{\beta\alpha}\gamma^\mu
\end{aligned}
\tag{97.12}
$$

This is getting closer, and we can now write

$$
\begin{aligned}
((\nabla \wedge A) \wedge \gamma_\mu) \cdot (\nabla \wedge A) &= (\gamma_\mu)^2 \partial_\alpha A_\beta (F^{\mu\beta}\gamma^\alpha + F^{\alpha\mu}\gamma^\beta + F^{\beta\alpha}\gamma^\mu) \\
&= (\gamma_\mu)^2 \partial_\beta A_\alpha F^{\mu\alpha}\gamma^\beta + (\gamma_\mu)^2 \partial_\alpha A_\beta F^{\alpha\mu}\gamma^\beta + (\gamma_\mu)^2 \partial_\alpha A_\beta F^{\beta\alpha}\gamma^\mu \\
&= F^{\beta\alpha} F_{\mu\alpha}\gamma_\beta + \partial_\alpha A_\beta F^{\beta\alpha}\gamma_\mu
\end{aligned}
\tag{97.13}
$$

This can now be dotted with γ^ν,

$$
((\nabla \wedge A) \wedge \gamma_\mu) \cdot (\nabla \wedge A) \cdot \gamma^\nu = F^{\beta\alpha} F_{\mu\alpha} \delta_\beta{}^\nu + \partial_\alpha A_\beta F^{\beta\alpha} \delta_\mu{}^\nu
\tag{97.14}
$$

which is

$$
((\nabla \wedge A) \wedge \gamma_\mu) \cdot (\nabla \wedge A) \cdot \gamma^\nu = F^{\nu\alpha} F_{\mu\alpha} + \frac{1}{2} F_{\alpha\beta} F^{\beta\alpha} \delta_\mu{}^\nu
\tag{97.15}
$$

The final combination of results eq. (97.8), and eq. (97.15) gives

$$
(F\gamma_\mu F) \cdot \gamma^\nu = 2F^{\alpha\nu} F_{\alpha\mu} + \frac{1}{2} F_{\alpha\beta} F^{\beta\alpha} \delta_\mu{}^\nu
\tag{97.16}
$$

Yielding the tensor

$$
T_\mu{}^\nu = \epsilon_0 \left(\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta_\mu{}^\nu - F_{\alpha\mu} F^{\alpha\nu} \right)
\tag{97.17}
$$

97.2 VALIDATE AGAINST PREVIOUSLY CALCULATED POYNTING RESULT

In 94, the electrodynamic energy density U and momentum flux density vectors were related as follows

$$
\begin{aligned}
U &= \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) \\
\mathbf{P} &= \epsilon_0 c^2 \mathbf{E} \times \mathbf{B} = \epsilon_0 c (ic\mathbf{B}) \cdot \mathbf{E} \\
0 &= \frac{\partial}{\partial t} \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) + c^2 \epsilon_0 \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \cdot \mathbf{j}
\end{aligned}
\tag{97.18}
$$

Additionally the energy and momentum flux densities are components of this stress tensor four vector

$$T(\gamma_0) = U\gamma_0 + \frac{1}{c}\mathbf{P}\gamma_0 \tag{97.19}$$

From this we can read the first row of the tensor elements

$$\begin{aligned} {T_0}^0 &= U = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\ {T_0}^k &= \frac{1}{c}(\mathbf{P}\gamma_0)\cdot\gamma^k = \epsilon_0 c E^a B^b \epsilon_{kab} \end{aligned} \tag{97.20}$$

Let us compare these to eq. (97.17), which gives

$$\begin{aligned} {T_0}^0 &= \epsilon_0\left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - F_{\alpha 0}F^{\alpha 0}\right) \\ &= \frac{\epsilon_0}{4}\left(F_{\alpha j}F^{\alpha j} - 3F_{j0}F^{j0}\right) \\ &= \frac{\epsilon_0}{4}\left(F_{mj}F^{mj} + F_{0j}F^{0j} - 3F_{j0}F^{j0}\right) \\ &= \frac{\epsilon_0}{4}\left(F_{mj}F^{mj} - 2F_{j0}F^{j0}\right) \\ {T_0}^k &= -\epsilon_0 F_{\alpha 0}F^{\alpha k} \\ &= -\epsilon_0 F_{j0}F^{jk} \end{aligned} \tag{97.21}$$

Now, our field in terms of electric and magnetic coordinates is

$$\begin{aligned} F &= \mathbf{E} + ic\mathbf{B} \\ &= E^k\gamma_k\gamma_0 + icB^k\gamma_k\gamma_0 \\ &= E^k\gamma_k\gamma_0 - c\epsilon_{abk}B^k\gamma_a\gamma_b \end{aligned} \tag{97.22}$$

so the electric field tensor components are

$$\begin{aligned} F^{j0} &= (F\cdot\gamma^0)\cdot\gamma^j \\ &= E^k{\delta_k}^j \\ &= E^j \end{aligned} \tag{97.23}$$

and

$$\begin{aligned} F_{j0} &= (\gamma_j)^2(\gamma_0)^2F^{j0} \\ &= -E^j \end{aligned} \tag{97.24}$$

and the magnetic tensor components are

$$\begin{aligned} F^{mj} &= F_{mj} \\ &= -c\epsilon_{abk} B^k ((\gamma_a \gamma_b) \cdot \gamma_j) \cdot \gamma_m \\ &= -c\epsilon_{mjk} B^k \end{aligned} \tag{97.25}$$

This gives

$$\begin{aligned} {T_0}^0 &= \frac{\epsilon_0}{4} \left(2c^2 B^k B^k + 2E^j E^j \right) \\ &= \frac{\epsilon_0}{2} \left(c^2 \mathbf{B}^2 + \mathbf{E}^2 \right) \\ {T_0}^k &= \epsilon_0 E^j F^{jk} \\ &= \epsilon_0 c \epsilon_{kef} E^e B^f \\ &= \epsilon_0 (c\mathbf{E} \times \mathbf{B})_k \\ &= \frac{1}{c} (\mathbf{P} \cdot \sigma_k) \end{aligned} \tag{97.26}$$

Okay, good. This checks 4 of the elements of eq. (97.17) against the explicit **E** and **B** based representation of $T(\gamma_0)$ in eq. (97.18), leaving only 6 unique elements in the remaining parts of the (symmetric) tensor to verify.

97.3 FOUR VECTOR FORM OF ENERGY MOMENTUM CONSERVATION RELATIONSHIP

One can observe that there is a spacetime divergence hiding there directly in the energy conservation equation of eq. (97.18). In particular, writing the last of those as

$$0 = \partial_0 \frac{\epsilon_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) + \boldsymbol{\nabla} \cdot \mathbf{P}/c + \mathbf{E} \cdot \mathbf{j}/c \tag{97.27}$$

We can then write the energy-momentum parts as a four vector divergence

$$\nabla \cdot \left(\frac{\epsilon_0 \gamma_0}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2 \right) + \frac{1}{c} P^k \gamma_k \right) = -\mathbf{E} \cdot \mathbf{j}/c \tag{97.28}$$

Since we have a divergence relationship, it should also be possible to convert a spacetime hypervolume integration of this quantity into a time-surface integral or a pure volume integral. Pursing this will probably clarify how the tensor is related to the hypersurface flux as mentioned in the text here, but making this concrete will take a bit more thought.

Having seen that we have a divergence relationship for the energy momentum tensor in the rest frame, it is clear that the Poynting energy momentum flux relationship should follow much more directly if we play it backwards in a relativistic setting.

This is a very sneaky way to do it since we have to have seen the answer to get there, but it should avoid the complexity of trying to factor out the spacial gradients and recover the divergence relationship that provides the Poynting vector. Our sneaky starting point is to compute

$$\begin{aligned}\nabla \cdot (F\gamma_0\tilde{F}) &= \left\langle \nabla(F\gamma_0\tilde{F}) \right\rangle \\ &= \left\langle (\nabla F)\gamma_0\tilde{F} + \dot{\nabla} F\gamma_0\dot{\tilde{F}} \right\rangle \\ &= \left\langle (\nabla F)\gamma_0\tilde{F} + \dot{\tilde{F}}\dot{\nabla} F\gamma_0 \right\rangle\end{aligned} \tag{97.29}$$

Since this is a scalar quantity, it is equal to its own reverse and we can reverse all factors in this second term to convert the left acting gradient to a more regular right acting form. This is

$$\nabla \cdot (F\gamma_0\tilde{F}) = \left\langle (\nabla F)\gamma_0\tilde{F} + \gamma_0\tilde{F}(\nabla F) \right\rangle \tag{97.30}$$

Now using Maxwell's equation $\nabla F = J/\epsilon_0 c$, we have

$$\begin{aligned}\nabla \cdot (F\gamma_0\tilde{F}) &= \frac{1}{\epsilon_0 c}\left\langle J\gamma_0\tilde{F} + \gamma_0\tilde{F}J \right\rangle \\ &= \frac{2}{\epsilon_0 c}\left\langle J\gamma_0\tilde{F} \right\rangle \\ &= \frac{2}{\epsilon_0 c}(J \wedge \gamma_0) \cdot \tilde{F}\end{aligned} \tag{97.31}$$

Now, $J = \gamma_0 c\rho + \gamma_k J^k$, so $J \wedge \gamma_0 = J^k \gamma_k \gamma_0 = J^k \sigma_k = \mathbf{j}$, and dotting this with $\tilde{F} = -\mathbf{E} - ic\mathbf{B}$ will pick up only the (negated) electric field components, so we have

$$(J \wedge \gamma_0) \cdot \tilde{F} = \mathbf{j} \cdot (-\mathbf{E}) \tag{97.32}$$

Although done in 94, for completeness let us re-expand $F\gamma_0\tilde{F}$ in terms of the electric and magnetic field vectors.

$$\begin{aligned}F\gamma_0\tilde{F} &= -(\mathbf{E} + ic\mathbf{B})\gamma_0(\mathbf{E} + ic\mathbf{B}) \\ &= \gamma_0(\mathbf{E} - ic\mathbf{B})(\mathbf{E} + ic\mathbf{B}) \\ &= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2 + ic(\mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E})) \\ &= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2 + 2ic(\mathbf{E} \wedge \mathbf{B})) \\ &= \gamma_0(\mathbf{E}^2 + c^2\mathbf{B}^2 - 2c(\mathbf{E} \times \mathbf{B}))\end{aligned} \tag{97.33}$$

Next, we want an explicit spacetime split of the gradient

$$\begin{aligned}
\nabla\gamma_0 &= (\gamma^0\partial_0 + \gamma^k\partial_k)\gamma_0 \\
&= \partial_0 - \gamma_k\gamma_0\partial_k \\
&= \partial_0 - \sigma_k\partial_k \\
&= \partial_0 - \boldsymbol{\nabla}
\end{aligned} \tag{97.34}$$

We are now in shape to assemble all the intermediate results for the left hand side

$$\begin{aligned}
\nabla \cdot (F\gamma_0\tilde{F}) &= \left\langle \nabla(F\gamma_0\tilde{F}) \right\rangle \\
&= \left\langle (\partial_0 - \boldsymbol{\nabla})(\mathbf{E}^2 + c^2\mathbf{B}^2 - 2c(\mathbf{E}\times\mathbf{B})) \right\rangle \\
&= \partial_0(\mathbf{E}^2 + c^2\mathbf{B}^2) + 2c\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})
\end{aligned} \tag{97.35}$$

With a final reassembly of the left and right hand sides of $\nabla \cdot T(\gamma_0)$, the spacetime divergence of the rest frame stress vector we have

$$\frac{1}{c}\partial_t(\mathbf{E}^2 + c^2\mathbf{B}^2) + 2c\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}) = -\frac{2}{c\epsilon_0}\mathbf{j}\cdot\mathbf{E} \tag{97.36}$$

Multiplying through by $\epsilon_0 c/2$ we have the classical Poynting vector energy conservation relationship.

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = -\mathbf{j}\cdot\mathbf{E} \tag{97.37}$$

Observe that the momentum flux density, the Poynting vector $\mathbf{P} = (\mathbf{E}\times\mathbf{B})/\mu_0$, is zero in the rest frame, which makes sense since there is no magnetic field for a static charge distribution. So with no currents and therefore no magnetic fields the field energy is a constant.

97.3.1 *Transformation properties*

Equation (97.37) is the explicit spacetime expansion of the equivalent relativistic equation

$$\nabla \cdot (cT(\gamma_0)) = \nabla \cdot \left(\frac{c\epsilon_0}{2}F\gamma_0\tilde{F}\right) = \left\langle J\gamma_0\tilde{F} \right\rangle \tag{97.38}$$

This has all the same content, but in relativistic form seems almost trivial. While the stress vector $T(\gamma_0)$ is not itself a relativistic invariant, this divergence equation is.

Suppose we form a Lorentz transformation $\mathcal{L}(x) = Rx\tilde{R}$, applied to this equation we have

$$\begin{aligned} F' &= (R\nabla\tilde{R}) \wedge (RA\tilde{R}) \\ &= \left\langle R\nabla\tilde{R}RA\tilde{R} \right\rangle_2 \\ &= \left\langle R\nabla A\tilde{R} \right\rangle_2 \\ &= R(\nabla \wedge A)\tilde{R} \\ &= RF\tilde{R} \end{aligned} \tag{97.39}$$

Transforming all the objects in the equation we have

$$\begin{aligned} \nabla' \cdot \left(\frac{c\epsilon_0}{2} F'\gamma_0'\tilde{F}' \right) &= \left\langle J'\gamma_0'\tilde{F}' \right\rangle \\ (R\nabla\tilde{R}) \cdot \left(\frac{c\epsilon_0}{2} RF\tilde{R}R\gamma_0 R\tilde{R}\widetilde{(RF\tilde{R})} \right) &= \left\langle RJ\tilde{R}R\gamma_0\tilde{R}\widetilde{(RF\tilde{R})} \right\rangle \end{aligned} \tag{97.40}$$

This is nothing more than the original untransformed quantity

$$\nabla \cdot \left(\frac{c\epsilon_0}{2} F\gamma_0\tilde{F} \right) = \left\langle J\gamma_0\tilde{F} \right\rangle \tag{97.41}$$

97.4 VALIDATE WITH RELATIVISTIC TRANSFORMATION

As a relativistic quantity we should be able to verify the messy tensor relationship by Lorentz transforming the energy density from a rest frame to a moving frame.

Now let us try the Lorentz transformation of the energy density.

FIXME: TODO.

98

LORENTZ FORCE RELATION TO THE ENERGY MOMENTUM TENSOR

98.1 MOTIVATION

Have now made a few excursions related to the concepts of electrodynamic field energy and momentum.

In 94 the energy density rate and Poynting divergence relationship was demonstrated using Maxwell's equation. That was:

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = -\mathbf{E}\cdot\mathbf{j} \tag{98.1}$$

In terms of the field energy density U, and Poynting vector $\mathbf{P}$, this is

$$\begin{aligned}
U &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\
\mathbf{P} &= \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \\
\frac{\partial U}{\partial t} &+ \boldsymbol{\nabla}\cdot\mathbf{P} = -\mathbf{E}\cdot\mathbf{j}
\end{aligned} \tag{98.2}$$

In 97 this was related to the energy momentum four vectors

$$T(a) = \frac{\epsilon_0}{2} Fa\tilde{F} \tag{98.3}$$

as defined in [10], but the big picture view of things was missing.

Later in 95 the rate of change of Poynting vector was calculated, with an additional attempt to relate this to $T(\gamma_\mu)$.

These relationships, and the operations required to factoring out the divergence were considerably messier.

Finally, in 96 the four vector $T(\gamma_\mu)$ was related to the Lorentz force and the work done moving a charge against a field. This provides the natural context for the energy momentum tensor, since it appears that the spacetime divergence of each of the $T(\gamma_\mu)$ four vectors appears to be a component of the four vector Lorentz force (density).

In these notes the divergences will be calculated to confirm the connection between the Lorentz force and energy momentum tensor directly. This is actually expected to be simpler than the previous calculations.

It is also potentially of interest, as shown in 119, and 120 that the energy density and Poynting vectors, and energy momentum four vector, were seen to be naturally expressible as Hermitian conjugate operations

$$F^\dagger = \gamma_0 \tilde{F} \gamma_0 \tag{98.4}$$

$$T(\gamma_0) = \frac{\epsilon_0}{2} F F^\dagger \gamma_0 \tag{98.5}$$

$$\begin{aligned} U &= T(\gamma_0) \cdot \gamma_0 = \frac{\epsilon_0}{4} \left(F F^\dagger + F^\dagger F \right) \\ \mathbf{P}/c &= T(\gamma_0) \wedge \gamma_0 = \frac{\epsilon_0}{4} \left(F F^\dagger - F^\dagger F \right) \end{aligned} \tag{98.6}$$

It is conceivable that a generalization of Hermitian conjugation, where the spatial basis vectors are used instead of γ_0, will provide a mapping and driving structure from the Four vector quantities and the somewhat scrambled seeming set of relationships observed in the split spatial and time domain. That will also be explored here.

98.2 SPACETIME DIVERGENCE OF THE ENERGY MOMENTUM FOUR VECTORS

The spacetime divergence of the field energy momentum four vector $T(\gamma_0)$ has been calculated previously. Let us redo this calculation for the other components.

$$\begin{aligned} \nabla \cdot T(\gamma_\mu) &= \frac{\epsilon_0}{2} \left\langle \nabla (F \gamma_\mu \tilde{F}) \right\rangle \\ &= \frac{\epsilon_0}{2} \left\langle (\nabla F) \gamma_\mu \tilde{F} + (\tilde{F} \nabla) F \gamma_\mu \right\rangle \\ &= \frac{\epsilon_0}{2} \left\langle (\nabla F) \gamma_\mu \tilde{F} + \gamma_\mu \tilde{F} (\nabla F) \right\rangle \\ &= \epsilon_0 \left\langle (\nabla F) \gamma_\mu \tilde{F} \right\rangle \\ &= \frac{1}{c} \left\langle J \gamma_\mu \tilde{F} \right\rangle \end{aligned} \tag{98.7}$$

The ability to perform cyclic reordering of terms in a scalar product has been used above. Application of one more reverse operation (which does not change a scalar), gives us

$$\nabla \cdot T(\gamma_\mu) = \frac{1}{c} \left\langle F \gamma_\mu J \right\rangle \tag{98.8}$$

Let us expand the right hand size first.

$$\frac{1}{c}\left\langle F\gamma_\mu J\right\rangle = \frac{1}{c}\left\langle(\mathbf{E}+ic\mathbf{B})\gamma_\mu(c\rho\gamma_0+\mathbf{j}\gamma_0)\right\rangle \tag{98.9}$$

The $\mu = 0$ term looks the easiest, and for that one we have

$$\frac{1}{c}\langle(\mathbf{E}+ic\mathbf{B})(c\rho-\mathbf{j})\rangle = -\mathbf{j}\cdot\mathbf{E} \tag{98.10}$$

Now, for the other terms, say $\mu = k$, we have

$$\begin{aligned}
\frac{1}{c}\langle(\mathbf{E}+ic\mathbf{B})(c\rho\sigma_k-\sigma_k\mathbf{j})\rangle &= E^k\rho - \langle i\mathbf{B}\sigma_k\mathbf{j}\rangle \\
&= E^k\rho - J^aB^b\langle\sigma_1\sigma_2\sigma_3\sigma_b\sigma_k\sigma_a\rangle \\
&= E^k\rho - J^aB^b\epsilon_{akb} \\
&= E^k\rho + J^aB^b\epsilon_{kab} \\
&= (\rho\mathbf{E}+\mathbf{j}\times\mathbf{B})\cdot\sigma_k
\end{aligned} \tag{98.11}$$

Summarizing the two results we have

$$\begin{aligned}
\frac{1}{c}\langle F\gamma_0 J\rangle &= -\mathbf{j}\cdot\mathbf{E} \\
\frac{1}{c}\langle F\gamma_k J\rangle &= (\rho\mathbf{E}+\mathbf{j}\times\mathbf{B})\cdot\sigma_k
\end{aligned} \tag{98.12}$$

The second of these is easily recognizable as components of the Lorentz force for an element of charge (density). The first of these is actually the energy component of the four vector Lorentz force, so expanding that in terms of spacetime quantities is the next order of business.

98.3 FOUR VECTOR LORENTZ FORCE

The Lorentz force in covariant form is

$$m\ddot{x} = qF\cdot\dot{x}/c \tag{98.13}$$

Two verifications of this are in order. One is that we get the traditional vector form of the Lorentz force equation from this and the other is that we can get the traditional tensor form from this equation.

98.3.1 *Lorentz force in tensor form*

Recovering the tensor form is probably the easier of the two operations. We have

$$
\begin{aligned}
m\ddot{x}_\mu\gamma^\mu &= \frac{q}{2} F_{\alpha\beta}\dot{x}_\sigma(\gamma^\alpha \wedge \gamma^\beta) \cdot \gamma^\sigma \\
&= \frac{q}{2} F_{\alpha\beta}\dot{x}^\sigma(\gamma^\alpha {\delta^\beta}_\sigma - \gamma^\beta {\delta^\alpha}_\sigma) \\
&= \frac{q}{2} F_{\alpha\beta}\dot{x}^\beta\gamma^\alpha - \frac{q}{2} F_{\alpha\beta}\dot{x}^\alpha\gamma^\beta
\end{aligned}
\tag{98.14}
$$

Dotting with γ_μ the right hand side is

$$
\frac{q}{2} F_{\mu\beta}\dot{x}^\beta - \frac{q}{2} F_{\alpha\mu}\dot{x}^\alpha = qF_{\mu\alpha}\dot{x}^\alpha
\tag{98.15}
$$

Which recovers the tensor form of the equation

$$
m\ddot{x}_\mu = qF_{\mu\alpha}\dot{x}^\alpha
\tag{98.16}
$$

98.3.2 *Lorentz force components in vector form*

$$
\begin{aligned}
m\gamma\frac{d}{dt}\gamma\left(c + \sigma_k\frac{dx^k}{dt}\right)\gamma_0 &= \frac{q}{2c}(Fv - vF) \\
&= \frac{q\gamma}{2c}(\mathbf{E} + ic\mathbf{B})\left(c + \sigma_k\frac{dx^k}{dt}\right)\gamma_0 \\
&\quad - \frac{q\gamma}{2c}\left(c + \sigma_k\frac{dx^k}{dt}\right)\gamma_0(\mathbf{E} + ic\mathbf{B})
\end{aligned}
\tag{98.17}
$$

Right multiplication by γ_0/γ we have

$$
\begin{aligned}
m\frac{d}{dt}\gamma\left(c + \mathbf{v}\right) &= \frac{q}{2c}\left((\mathbf{E} + ic\mathbf{B})\left(c + \mathbf{v}\right) - \left(c + \mathbf{v}\right)\left(-\mathbf{E} + ic\mathbf{B}\right)\right) \\
&= \frac{q}{2c}\left(+2\mathbf{E}c + \mathbf{E}\mathbf{v} + \mathbf{v}\mathbf{E} + ic(\mathbf{B}\mathbf{v} - \mathbf{v}\mathbf{B})\right)
\end{aligned}
\tag{98.18}
$$

After a last bit of reduction this is

$$
m\frac{d}{dt}\gamma\left(c + \mathbf{v}\right) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q\mathbf{E} \cdot \mathbf{v}/c
\tag{98.19}
$$

In terms of four vector momentum this is

$$
\dot{p} = q(\mathbf{E} \cdot \mathbf{v}/c + \mathbf{E} + \mathbf{v} \times \mathbf{B})\gamma_0
\tag{98.20}
$$

98.3.3 *Relation to the energy momentum tensor*

It appears that to relate the energy momentum tensor to the Lorentz force we have to work with the upper index quantities rather than the lower index stress tensor vectors. Doing so our four vector force per unit volume is

$$\begin{aligned}\frac{\partial \dot{p}}{\partial V} &= (\mathbf{j} \cdot \mathbf{E} + \rho \mathbf{E} + \mathbf{j} \times \mathbf{B})\gamma_0 \\ &= -\frac{1}{c}\langle F \gamma^\mu J \rangle \gamma_\mu \\ &= -(\nabla \cdot T(\gamma^\mu))\gamma_\mu\end{aligned} \tag{98.21}$$

The term $\langle F \gamma^\mu J \rangle \gamma_\mu$ appears to be expressed simply has $F \cdot J$ in [10]. Understanding that simple statement is now possible now that an exploration of some of the underlying ideas has been made. In retrospect having seen the bivector product form of the Lorentz force equation, it should have been clear, but some of the associated trickiness in their treatment obscured this fact (Although their treatment is only two pages, I still only understand half of what they are doing!)

98.4 EXPANSION OF THE ENERGY MOMENTUM TENSOR

While all the components of the divergence of the energy momentum tensor have been expanded explicitly, this has not been done here for the tensor itself. A mechanical expansion of the tensor in terms of field tensor components $F^{\mu\nu}$ has been done previously and is not particularly enlightening. Let us work it out here in terms of electric and magnetic field components. In particular for the $T^{0\mu}$ and $T^{\mu 0}$ components of the tensor in terms of energy density and the Poynting vector.

98.4.1 *In terms of electric and magnetic field components*

Here we want to expand

$$T(\gamma^\mu) = \frac{-\epsilon_0}{2}(\mathbf{E} + ic\mathbf{B})\gamma^\mu(\mathbf{E} + ic\mathbf{B}) \tag{98.22}$$

It will be convenient here to temporarily work with $\epsilon_0 = c = 1$, and put them back in afterward.

98.4.1.1 *First row*

First expanding $T(\gamma^0)$ we have

$$\begin{aligned} T(\gamma^0) &= \frac{1}{2}(\mathbf{E} + i\mathbf{B})(\mathbf{E} - i\mathbf{B})\gamma^0 \\ &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2 + i(\mathbf{B}\mathbf{E} - \mathbf{E}\mathbf{B}))\gamma^0 \\ &= \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\gamma^0 + i(\mathbf{B} \wedge \mathbf{E})\gamma^0 \end{aligned} \tag{98.23}$$

Using the wedge product dual $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b})$, and putting back in the units, we have our first stress energy four vector,

$$T(\gamma^0) = \left(\frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) + \frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})\right)\gamma^0 \tag{98.24}$$

In particular the energy density and the components of the Poynting vector can be picked off by dotting with each of the γ^μ vectors. That is

$$\begin{aligned} U &= T(\gamma^0) \cdot \gamma^0 \\ \mathbf{P}/c \cdot \sigma_k &= T(\gamma^0) \cdot \gamma^k \end{aligned} \tag{98.25}$$

98.4.1.2 *First column*

We have Poynting vector terms in the T^{0k} elements of the matrix. Let us quickly verify that we have them in the T^{k0} positions too.

To do so, again with $c = \epsilon_0 = 1$ temporarily this is a computation of

$$\begin{aligned} T(\gamma^k) \cdot \gamma^0 &= \frac{1}{2}(T(\gamma^k)\gamma^0 + \gamma^0 T(\gamma^k)) \\ &= \frac{-1}{4}(F\gamma^k F\gamma^0 + \gamma^0 F\gamma^k F) \\ &= \frac{1}{4}(F\sigma_k\gamma_0 F\gamma^0 - \gamma^0 F\gamma_0\sigma_k F) \\ &= \frac{1}{4}(F\sigma_k(-\mathbf{E} + i\mathbf{B}) - (-\mathbf{E} + i\mathbf{B})\sigma_k F) \\ &= \frac{1}{4}\langle\sigma_k(-\mathbf{E} + i\mathbf{B})(\mathbf{E} + i\mathbf{B}) - \sigma_k(\mathbf{E} + i\mathbf{B})(-\mathbf{E} + i\mathbf{B})\rangle \\ &= \frac{1}{4}\left\langle\sigma_k(-\mathbf{E}^2 - \mathbf{B}^2 + 2(\mathbf{E} \times \mathbf{B})) - \sigma_k(-\mathbf{E}^2 - \mathbf{B}^2 - 2(\mathbf{E} \times \mathbf{B}))\right\rangle \end{aligned} \tag{98.26}$$

Adding back in the units we have

$$T(\gamma^k) \cdot \gamma^0 = \epsilon_0 c(\mathbf{E} \times \mathbf{B}) \cdot \sigma_k = \frac{1}{c}\mathbf{P} \cdot \sigma_k \tag{98.27}$$

As expected, these are the components of the Poynting vector (scaled by $1/c$ for units of energy density).

98.4.1.3 *Diagonal and remaining terms*

$$\begin{aligned}
T(\gamma^a) \cdot \gamma^b &= \frac{1}{2}(T(\gamma^a)\gamma^b + \gamma^b T(\gamma^a)) \\
&= \frac{-1}{4}(F\gamma^a F\gamma^b + \gamma^a F\gamma^b F) \\
&= \frac{1}{4}(F\sigma_a\gamma_0 F\gamma^b - \gamma^a F\gamma_0\sigma_b F) \\
&= \frac{1}{4}(F\sigma_a(-\mathbf{E} + i\mathbf{B})\sigma_b + \sigma_a(-\mathbf{E} + i\mathbf{B})\sigma_b F) \\
&= \frac{1}{2}\langle\sigma_a(-\mathbf{E} + i\mathbf{B})\sigma_b(\mathbf{E} + i\mathbf{B})\rangle
\end{aligned} \tag{98.28}$$

From this point is there any particularly good or clever way to do the remaining reduction? Doing it with coordinates looks like it would be easy, but also messy. A decomposition of $\mathbf{E}$ and $\mathbf{B}$ that are parallel and perpendicular to the spatial basis vectors also looks feasible.

Let us try the dumb way first

$$\begin{aligned}
T(\gamma^a) \cdot \gamma^b &= \frac{1}{2}\Big\langle\sigma_a(-E^k\sigma_k + iB^k\sigma_k)\sigma_b(E^m\sigma_m + iB^m\sigma_m)\Big\rangle \\
&= \frac{1}{2}(B^kE^m - E^kB^m)\langle i\sigma_a\sigma_k\sigma_b\sigma_m\rangle - \frac{1}{2}(E^kE^m + B^kB^m)\langle\sigma_a\sigma_k\sigma_b\sigma_m\rangle
\end{aligned} \tag{98.29}$$

Reducing the scalar operations is going to be much different for the $a = b$, and $a \neq b$ cases. For the diagonal case we have

$$\begin{aligned}
T(\gamma^a) \cdot \gamma^a &= \frac{1}{2}(B^kE^m - E^kB^m)\langle i\sigma_a\sigma_k\sigma_a\sigma_m\rangle - \frac{1}{2}(E^kE^m + B^kB^m)\langle\sigma_a\sigma_k\sigma_a\sigma_m\rangle \\
&= -\frac{1}{2}\sum_{m,k\neq a}\frac{1}{2}(B^kE^m - E^kB^m)\langle i\sigma_k\sigma_m\rangle + \frac{1}{2}\sum_{m,k\neq a}(E^kE^m + B^kB^m)\langle\sigma_k\sigma_m\rangle \\
&+ \frac{1}{2}\sum_m(B^aE^m - E^aB^m)\langle i\sigma_a\sigma_m\rangle - \frac{1}{2}\sum_m(E^aE^m + B^aB^m)\langle\sigma_a\sigma_m\rangle
\end{aligned} \tag{98.30}$$

Inserting the units again we have

$$T(\gamma^a)\cdot\gamma^a = \frac{\epsilon_0}{2}\left(\sum_{k\neq a}\left((E^k)^2 + c^2(B^k)^2\right) - \left((E^a)^2 + c^2(B^a)^2\right)\right) \tag{98.31}$$

Or, adding and subtracting, we have the diagonal in terms of energy density (minus a fudge)

$$T(\gamma^a)\cdot\gamma^a = U - \epsilon_0\left((E^a)^2 + c^2(B^a)^2\right) \tag{98.32}$$

Now, for the off diagonal terms. For $a \neq b$ this is

$$\begin{aligned}
T(\gamma^a)\cdot\gamma^b =& \frac{1}{2}\sum_m (B^aE^m - E^aB^m)\langle i\sigma_b\sigma_m\rangle + \frac{1}{2}\sum_m (B^bE^m - E^bB^m)\langle i\sigma_a\sigma_m\rangle \\
&- \frac{1}{2}\sum_m (E^aE^m + B^aB^m)\langle \sigma_b\sigma_m\rangle - \frac{1}{2}\sum_m (E^bE^m + B^bB^m)\langle \sigma_a\sigma_m\rangle \\
&+ \frac{1}{2}\sum_{m,k\neq a,b} (B^kE^m - E^kB^m)\langle i\sigma_a\sigma_k\sigma_b\sigma_m\rangle - \frac{1}{2}\sum_{m,k\neq a,b} (E^kE^m + B^kB^m)\langle \sigma_a\sigma_k\sigma_b\sigma_m\rangle
\end{aligned} \tag{98.33}$$

The first two scalar filters that include i will be zero, and we have deltas $\langle\sigma_b\sigma_m\rangle = \delta_{bm}$ in the next two. The remaining two terms have only vector and bivector terms, so we have zero scalar parts. That leaves (restoring units)

$$T(\gamma^a)\cdot\gamma^b = -\frac{\epsilon_0}{2}\left(E^aE^b + E^bE^a + c^2(B^aB^b + B^bB^a)\right) \tag{98.34}$$

98.4.2 *Summarizing*

Collecting all the results, with $T^{\mu\nu} = T(\gamma^\mu)\cdot\gamma^\nu$, we have

$$\begin{aligned}
T^{00} &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\
T^{aa} &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) - \epsilon_0\left((E^a)^2 + c^2(B^a)^2\right) \\
T^{k0} = T^{0k} &= \frac{1}{c}\left(\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\right)\cdot\sigma_k \\
T^{ab} = T^{ba} &= -\frac{\epsilon_0}{2}\left(E^aE^b + E^bE^a + c^2(B^aB^b + B^bB^a)\right)
\end{aligned} \tag{98.35}$$

98.4.3 *Assembling a four vector*

Let us see what one of the $T^{a\mu}\gamma_\mu$ rows of the tensor looks like in four vector form. Let $f \neq g \neq h$ represent an even permutation of the integers $1, 2, 3$. Then we have

$$\begin{aligned}
T^f &= T^{f\mu}\gamma_\mu \\
&= \frac{\epsilon_0}{2} c(E^g B^h - E^h B^g)\gamma_0 \\
&+ \frac{\epsilon_0}{2}\Big(-(E^f)^2 + (E^g)^2 + (E^h)^2 + c^2(-(B^f)^2 + (B^g)^2 + (B^h)^2)\Big)\gamma_f \\
&- \frac{\epsilon_0}{2}\Big(E^f E^g + E^g E^f + c^2(B^f B^g + B^g B^f)\Big)\gamma_g \\
&- \frac{\epsilon_0}{2}\Big(E^f E^h + E^h E^f + c^2(B^f B^h + B^h B^f)\Big)\gamma_h
\end{aligned} \tag{98.36}$$

It is pretty amazing that the divergence of this produces the f component of the Lorentz force (density)

$$\partial_\mu T^{f\mu} = (\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}) \cdot \sigma_f \tag{98.37}$$

Demonstrating this directly without having STA as an available tool would be quite tedious, and looking at this expression inspires no particular attempt to try!

98.5 CONJUGATION?

98.5.1 *Followup: energy momentum tensor*

This also suggests a relativistic generalization of conjugation, since the time basis vector should perhaps not have a distinguishing role. Something like this:

$$F^{\dagger_\mu} = \gamma_\mu \tilde{F} \gamma_\mu \tag{98.38}$$

Or perhaps:

$$F^{\dagger_\mu} = \gamma_\mu \tilde{F} \gamma^\mu \tag{98.39}$$

may make sense for consideration of the other components of the general energy momentum tensor, which had roughly the form:

$$T^{\mu\nu} \propto T(\gamma_\mu) \cdot \gamma^\nu \tag{98.40}$$

(with some probable adjustments to index positions). Think this through later.

99

ENERGY MOMENTUM TENSOR RELATION TO LORENTZ FORCE

99.1 MOTIVATION

In 98 the energy momentum tensor was related to the Lorentz force in STA form. Work the same calculation strictly in tensor form, to develop more comfort with tensor manipulation. This should also serve as a translation aid to compare signs due to metric tensor differences in other reading.

99.1.1 *Definitions*

The energy momentum "tensor", really a four vector, is defined in [10] as

$$T(a) = \frac{\epsilon_0}{2} F a \tilde{F} = -\frac{\epsilon_0}{2} F a F \tag{99.1}$$

We have seen that the divergence of the $T(\gamma^\mu)$ vectors generate the Lorentz force relations.

Let us expand this with respect to index lower basis vectors for use in the divergence calculation.

$$T(\gamma^\mu) = \left(T(\gamma^\mu) \cdot \gamma^\nu\right) \gamma_\nu \tag{99.2}$$

So we define

$$T^{\mu\nu} = T(\gamma^\mu) \cdot \gamma^\nu \tag{99.3}$$

and can write these four vectors in tensor form as

$$T(\gamma^\mu) = T^{\mu\nu} \gamma_\nu \tag{99.4}$$

99.1.2 *Expanding out the tensor*

An expansion of $T^{\mu\nu}$ was done in 97, but looking back that seems a peculiar way, using the four vector potential.

Let us try again in terms of $F^{\mu\nu}$ instead. Our field is

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \tag{99.5}$$

So our tensor components are

$$\begin{aligned} T^{\mu\nu} &= T(\gamma^\mu) \cdot \gamma^\nu \\ &= -\frac{\epsilon_0}{8} F^{\lambda\sigma} F^{\alpha\beta} \Big\langle (\gamma_\lambda \wedge \gamma_\sigma) \gamma^\mu (\gamma_\alpha \wedge \gamma_\beta) \gamma^\nu \Big\rangle \end{aligned} \tag{99.6}$$

Or

$$\begin{aligned} -8 \frac{1}{\epsilon_0} T^{\mu\nu} = F^{\lambda\sigma} F^{\alpha\beta} &\Big\langle (\gamma_\lambda {\delta_\sigma}^\mu - \gamma_\sigma {\delta_\lambda}^\mu)(\gamma_\alpha {\delta_\beta}^\nu - \gamma_\beta {\delta_\alpha}^\nu) \Big\rangle \\ &+ F^{\lambda\sigma} F^{\alpha\beta} \Big\langle (\gamma_\lambda \wedge \gamma_\sigma \wedge \gamma^\mu)(\gamma_\alpha \wedge \gamma_\beta \wedge \gamma^\nu) \Big\rangle \end{aligned} \tag{99.7}$$

Expanding only the first term to start with

$$\begin{aligned} &F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\lambda {\delta_\sigma}^\mu) \cdot (\gamma_\alpha {\delta_\beta}^\nu) + F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\sigma {\delta_\lambda}^\mu) \cdot (\gamma_\beta {\delta_\alpha}^\nu) \\ &- F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\lambda {\delta_\sigma}^\mu) \cdot (\gamma_\beta {\delta_\alpha}^\nu) - F^{\lambda\sigma} F^{\alpha\beta} (\gamma_\sigma {\delta_\lambda}^\mu) \cdot (\gamma_\alpha {\delta_\beta}^\nu) \\ &= F^{\lambda\mu} F^{\alpha\nu} \gamma_\lambda \cdot \gamma_\alpha + F^{\mu\sigma} F^{\nu\beta} \gamma_\sigma \cdot \gamma_\beta - F^{\lambda\mu} F^{\nu\beta} \gamma_\lambda \cdot \gamma_\beta - F^{\mu\sigma} F^{\alpha\nu} \gamma_\sigma \cdot \gamma_\alpha \\ &= \eta_{\alpha\beta} (F^{\lambda\mu} F^{\alpha\nu} \gamma_\lambda \cdot \gamma^\beta + F^{\mu\sigma} F^{\nu\alpha} \gamma_\sigma \cdot \gamma^\beta - F^{\lambda\mu} F^{\nu\alpha} \gamma_\lambda \cdot \gamma^\beta - F^{\mu\sigma} F^{\alpha\nu} \gamma_\sigma \cdot \gamma^\beta) \\ &= \eta_{\alpha\lambda} F^{\lambda\mu} F^{\alpha\nu} + \eta_{\alpha\sigma} F^{\mu\sigma} F^{\nu\alpha} - \eta_{\alpha\lambda} F^{\lambda\mu} F^{\nu\alpha} - \eta_{\alpha\sigma} F^{\mu\sigma} F^{\alpha\nu} \\ &= 2(\eta_{\alpha\lambda} F^{\lambda\mu} F^{\alpha\nu} + \eta_{\alpha\sigma} F^{\mu\sigma} F^{\nu\alpha}) \\ &= 2(\eta_{\alpha\beta} F^{\beta\mu} F^{\alpha\nu} + \eta_{\alpha\beta} F^{\mu\beta} F^{\nu\alpha}) \\ &= 4 \eta_{\alpha\beta} F^{\beta\mu} F^{\alpha\nu} \\ &= 4 F^{\beta\mu} {F_\beta}^\nu \\ &= 4 F^{\alpha\mu} {F_\alpha}^\nu \end{aligned} \tag{99.8}$$

For the second term after a shuffle of indices we have

$$F^{\lambda\sigma} F_{\alpha\beta} \eta^{\mu\mu'} \Big\langle (\gamma_\lambda \wedge \gamma_\sigma \wedge \gamma_\mu)(\gamma^\alpha \wedge \gamma^\beta \wedge \gamma^\nu) \Big\rangle \tag{99.9}$$

This dot product is reducible with the identity

$$(a \wedge b \wedge c) \cdot (d \wedge e \wedge f) = (((a \wedge b \wedge c) \cdot d) \cdot e) \cdot f \tag{99.10}$$

leaving a completely antisymmetized sum

$$\begin{aligned} &F^{\lambda\sigma} F_{\alpha\beta} \eta^{\mu\mu'} ({\delta_\lambda}^\nu {\delta_\sigma}^\beta {\delta_{\mu'}}^\alpha - {\delta_\lambda}^\nu {\delta_\sigma}^\alpha {\delta_{\mu'}}^\beta - {\delta_\lambda}^\beta {\delta_\sigma}^\nu {\delta_{\mu'}}^\alpha + {\delta_\lambda}^\alpha {\delta_\sigma}^\nu {\delta_{\mu'}}^\beta + {\delta_\lambda}^\beta {\delta_\sigma}^\alpha {\delta_{\mu'}}^\nu - {\delta_\lambda}^\alpha {\delta_\sigma}^\beta {\delta_{\mu'}}^\nu) \\ &= F^{\nu\beta} F_{\mu'\beta} \eta^{\mu\mu'} - F^{\nu\alpha} F_{\alpha\mu'} \eta^{\mu\mu'} - F^{\beta\nu} F_{\mu'\beta} \eta^{\mu\mu'} + F^{\alpha\nu} F_{\alpha\mu'} \eta^{\mu\mu'} + F^{\beta\alpha} F_{\alpha\beta} \eta^{\mu\mu'} {\delta_{\mu'}}^\nu - F^{\alpha\beta} F_{\alpha\beta} \eta^{\mu\mu'} {\delta_{\mu'}}^\nu \\ &= 4 F^{\nu\alpha} F_{\mu'\alpha} \eta^{\mu\mu'} + 2 F^{\beta\alpha} F_{\alpha\beta} \eta^{\mu\mu'} {\delta_{\mu'}}^\nu \\ &= 4 F^{\nu\alpha} {F^\mu}_\alpha + 2 F^{\beta\alpha} F_{\alpha\beta} \eta^{\mu\nu} \end{aligned}$$

$$\tag{99.11}$$

Combining these we have

$$\begin{aligned}
T^{\mu\nu} &= -\frac{\epsilon_0}{8}\left(4F^{\alpha\mu}F_\alpha{}^\nu + 4F^{\nu\alpha}F^\mu{}_\alpha + 2F^{\beta\alpha}F_{\alpha\beta}\eta^{\mu\nu}\right) \\
&= \frac{\epsilon_0}{8}\left(-4F^{\alpha\mu}F_\alpha{}^\nu + 4F^{\alpha\mu}F^\nu{}_\alpha + 2F^{\alpha\beta}F_{\alpha\beta}\eta^{\mu\nu}\right)
\end{aligned}\tag{99.12}$$

If by some miracle all the index manipulation worked out, we have

$$T^{\mu\nu} = \epsilon_0\left(F^{\alpha\mu}F^\nu{}_\alpha + \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\eta^{\mu\nu}\right)\tag{99.13}$$

99.1.2.1 *Justifying some of the steps*

For justification of some of the index manipulations of the F tensor components it is helpful to think back to the definitions in terms of four vector potentials

$$\begin{aligned}
F &= \nabla \wedge A \\
&= \partial^\mu A^\nu \gamma_\mu \wedge \gamma_\nu \\
&= \partial_\mu A_\nu \gamma^\mu \wedge \gamma^\nu \\
&= \partial_\mu A^\nu \gamma^\mu \wedge \gamma_\nu \\
&= \partial^\mu A_\nu \gamma_\mu \wedge \gamma^\nu \\
&= \frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu)\gamma_\mu \wedge \gamma_\nu \\
&= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\gamma^\mu \wedge \gamma^\nu \\
&= \frac{1}{2}(\partial_\mu A^\nu - \partial^\nu A_\mu)\gamma^\mu \wedge \gamma_\nu \\
&= \frac{1}{2}(\partial^\mu A_\nu - \partial_\nu A^\mu)\gamma_\mu \wedge \gamma^\nu
\end{aligned}\tag{99.14}$$

So with the shorthand

$$\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
F_\mu{}^\nu &= \partial_\mu A^\nu - \partial^\nu A_\mu \\
F^\mu{}_\nu &= \partial^\mu A_\nu - \partial_\nu A^\mu
\end{aligned}\tag{99.15}$$

We have

$$\begin{aligned}F &= \frac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu \\ &= \frac{1}{2}F_{\mu\nu}\gamma^\mu \wedge \gamma^\nu \\ &= \frac{1}{2}F_\mu{}^\nu\gamma^\mu \wedge \gamma_\nu \\ &= \frac{1}{2}F^\mu{}_\nu\gamma_\mu \wedge \gamma^\nu\end{aligned} \tag{99.16}$$

In particular, and perhaps not obvious without the definitions handy, the following was used above

$$F^\mu{}_\nu = -F_\nu{}^\mu \tag{99.17}$$

99.1.3 *The divergence*

What is our divergence in tensor form? This would be

$$\nabla \cdot T(\gamma^\mu) = (\gamma^\alpha\partial_\alpha) \cdot (T^{\mu\nu}\gamma_\nu) \tag{99.18}$$

So we have

$$\nabla \cdot T(\gamma^\mu) = \partial_\nu T^{\mu\nu} \tag{99.19}$$

Ignoring the ϵ_0 factor for now, chain rule gives us

$$\begin{aligned}&(\partial_\nu F^{\alpha\mu})F^\nu{}_\alpha + F^{\alpha\mu}(\partial_\nu F^\nu{}_\alpha) + \frac{1}{2}(\partial_\nu F^{\alpha\beta})F_{\alpha\beta}\eta^{\mu\nu} \\ &\quad = (\partial_\nu F^{\alpha\mu})F^\nu{}_\alpha + F_\alpha{}^\mu(\partial_\nu F^{\nu\alpha}) + \frac{1}{2}(\partial_\nu F^{\alpha\beta})F_{\alpha\beta}\eta^{\mu\nu}\end{aligned} \tag{99.20}$$

Only this center term is recognizable in terms of current since we have

$$\nabla \cdot F = J/\epsilon_0 c \tag{99.21}$$

Where the LHS is

$$\begin{aligned}\nabla \cdot F &= \gamma^\alpha\partial_\alpha \cdot \left(\frac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu\right) \\ &= \frac{1}{2}\partial_\alpha F^{\mu\nu}(\delta^\alpha{}_\mu\gamma_\nu - \delta^\alpha{}_\nu\gamma_\mu) \\ &= \partial_\mu F^{\mu\nu}\gamma_\nu\end{aligned} \tag{99.22}$$

So we have

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= (J \cdot \gamma^\nu)/\epsilon_0 c \\
&= ((J^\alpha \gamma_\alpha) \cdot \gamma^\nu)/\epsilon_0 c \\
&= J^\nu/\epsilon_0 c
\end{aligned}
\tag{99.23}
$$

Or

$$
\partial_\mu F^{\mu\nu} = J^\nu/\epsilon_0 c \tag{99.24}
$$

So we have

$$
\nabla \cdot T(\gamma^\mu) = \epsilon_0 \left((\partial_\nu F^{\alpha\mu}) F^\nu{}_\alpha + \frac{1}{2} (\partial_\nu F^{\alpha\beta}) F_{\alpha\beta} \eta^{\mu\nu} \right) + F_\alpha{}^\mu J^\alpha / c \tag{99.25}
$$

So, to get the expected result the remaining two derivative terms must somehow cancel. How to reduce these? Let us look at twice that

$$
\begin{aligned}
&2(\partial_\nu F^{\alpha\mu}) F^\nu{}_\alpha + (\partial_\nu F^{\alpha\beta}) F_{\alpha\beta} \eta^{\mu\nu} \\
&\quad = 2(\partial^\nu F^{\alpha\mu}) F_{\nu\alpha} + (\partial^\mu F^{\alpha\beta}) F_{\alpha\beta} \\
&\quad = (\partial^\nu F^{\alpha\mu})(F_{\nu\alpha} - F_{\alpha\nu}) + (\partial^\mu F^{\alpha\beta}) F_{\alpha\beta} \\
&\quad = (\partial^\alpha F^{\beta\mu}) F_{\alpha\beta} + (\partial^\beta F^{\mu\alpha}) F_{\alpha\beta} + (\partial^\mu F^{\alpha\beta}) F_{\alpha\beta} \\
&\quad = (\partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} + \partial^\mu F^{\alpha\beta}) F_{\alpha\beta}
\end{aligned}
\tag{99.26}
$$

Ah, there is the trivector term of Maxwell's equation hiding in there.

$$
\begin{aligned}
0 &= \nabla \wedge F \\
&= \gamma_\mu \partial^\mu \wedge \left(\frac{1}{2} F^{\alpha\beta} (\gamma_\alpha \wedge \gamma_\beta) \right) \\
&= \frac{1}{2} (\partial^\mu F^{\alpha\beta}) (\gamma_\mu \wedge \gamma_\alpha \wedge \gamma_\beta) \\
&= \frac{1}{3!} \left(\partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} \right) (\gamma_\mu \wedge \gamma_\alpha \wedge \gamma_\beta)
\end{aligned}
\tag{99.27}
$$

Since this is zero, each component of this trivector must separately equal zero, and we have

$$
\partial^\mu F^{\alpha\beta} + \partial^\alpha F^{\beta\mu} + \partial^\beta F^{\mu\alpha} = 0 \tag{99.28}
$$

So, where $T^{\mu\nu}$ is defined by eq. (99.13), the final result is

$$
\partial_\nu T^{\mu\nu} = F^{\alpha\mu} J_\alpha / c \tag{99.29}
$$

100

DC POWER CONSUMPTION FORMULA FOR RESISTIVE LOAD

100.1 MOTIVATION

Despite a lot of recent study of electrodynamics, faced with a simple electrical problem:

"What capacity generator would be required for an arc welder on a 30 Amp breaker using a 220 volt circuit".

I could not think of how to answer this off the top of my head. Back in school without hesitation I would have been able to plug into $P = IV$ to get a capacity estimation for the generator.

Having forgotten the formula, let us see how we get that $P = IV$ relationship from Maxwell's equations.

100.2

Having just derived the Poynting energy momentum density relationship from Maxwell's equations, let that be the starting point

$$\frac{d}{dt}\left(\frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)\right) = -\frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right) - \mathbf{E}\cdot\mathbf{j} \tag{100.1}$$

The left hand side is the energy density time variation, which is power per unit volume, so we can integrate this over a volume to determine the power associated with a change in the field.

$$P = -\int dV\left(\frac{1}{\mu_0}\left(\mathbf{E}\times\mathbf{B}\right) + \mathbf{E}\cdot\mathbf{j}\right) \tag{100.2}$$

As a reminder, let us write the magnetic and electric fields in terms of potentials.

In terms of the "native" four potential our field is

$$\begin{aligned}
F &= \mathbf{E} + ic\mathbf{B} \\
&= \nabla\wedge A \\
&= \gamma^0\gamma_k\partial_0 A^k + \gamma^j\gamma_0\partial_j A^0 + \gamma^m\wedge\gamma_n\partial_m A^n
\end{aligned} \tag{100.3}$$

The electric field is

$$\mathbf{E} = \sum_k (\nabla \wedge A) \cdot (\gamma^0 \gamma^k) \gamma_k \gamma_0 \tag{100.4}$$

From this, with $\phi = A^0$, and $\mathbf{A} = \sigma_k A^k$ we have

$$\begin{aligned}\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \\ i\mathbf{B} &= \boldsymbol{\nabla} \wedge \mathbf{A}\end{aligned} \tag{100.5}$$

Now, the arc welder is (I think) a DC device, and to get a rough idea of what it requires lets just assume that its a rectifier that outputs RMS DC. So if we make this simplification, and assume that we have a purely resistive load (ie: no inductance and therefore no magnetic fields) and a DC supply and constant current, then we eliminate the vector potential terms.

This wipes out the **B** and the Poynting vector, and leaves our electric field specified in terms of the potential difference across the load $\mathbf{E} = -\boldsymbol{\nabla}\phi$.

That is

$$P = \int dV (\boldsymbol{\nabla}\phi) \cdot \mathbf{j} \tag{100.6}$$

Suppose we are integrating over the length of a uniformly resistive load with some fixed cross sectional area. $\mathbf{j}dV$ is then the magnitude of the current directed along the wire for its length. This basically leaves us with a line integral over the length of the wire that we are measuring our potential drop over so we are left with just

$$P = (\delta\phi) I \tag{100.7}$$

This $\delta\phi$ is just our voltage drop V, and this gives us the desired $P = IV$ equation. Now, I also recall from school now that I think about it that $P = IV$ also applied to inductive loads, but it required that I and V be phasors that represented the sinusoidal currents and sources. A good followup exercise would be to show from Maxwell's equations that this is in fact valid. Eventually I will know the origin of all the formulas from my old engineering courses.

101

RAYLEIGH-JEANS LAW NOTES

101.1 MOTIVATION

Fill in the gaps for a reading of the initial parts of the Rayleigh-Jeans discussion of [2].

101.2 2. ELECTROMAGNETIC ENERGY

Energy of the field given to be:

$$E = \frac{1}{8\pi} \int (\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2) \tag{101.1}$$

I still do not really know where this comes from. Could perhaps justify this with a Hamiltonian of a field (although this is uncomfortably abstract).

With the particle Hamiltonian we have

$$H = \dot{q}_i p_i - \mathcal{L} \tag{101.2}$$

What is the field equivalent of this? Try to get the feel for this with some simple fields (such as the one dimensional vibrating string), and the Coulomb field. For the physical case, do this with both the Hamiltonian approach and a physical limiting argument.

101.3 3. ELECTROMAGNETIC POTENTIALS

Bohm writes Maxwell's equations in non-SI units, and also, naturally, not in STA form which would be somewhat more natural for a gauge discussion.

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} &= -\frac{1}{c}\partial_t \boldsymbol{\mathcal{H}} \\
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} &= 4\pi\rho \\
\boldsymbol{\nabla} \times \boldsymbol{\mathcal{H}} &= \frac{1}{c}\partial_t \boldsymbol{\mathcal{E}} + 4\pi\mathbf{j} \\
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{H}} &= 0
\end{aligned} \tag{101.3}$$

In STA form this is

$$
\begin{aligned}
\boldsymbol{\nabla}\boldsymbol{\mathcal{E}} &= -\partial_0 i\boldsymbol{\mathcal{H}} + 4\pi\rho \\
\boldsymbol{\nabla} i\boldsymbol{\mathcal{H}} &= -\partial_0\boldsymbol{\mathcal{E}} - 4\pi\mathbf{j}
\end{aligned}
\tag{101.4}
$$

Or

$$
\boldsymbol{\nabla}(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) + \partial_0(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) = 4\pi(\rho - \mathbf{j})
\tag{101.5}
$$

Left multiplying by γ_0 gives

$$
\begin{aligned}
\gamma_0\boldsymbol{\nabla} &= \gamma_0 \sum_k \sigma_k\partial_k \\
&= \gamma_0 \sum_k \gamma_k\gamma_0\partial_k \\
&= -\sum_k \gamma_k\partial_k \\
&= \gamma^k\partial_k
\end{aligned}
\tag{101.6}
$$

and

$$
\begin{aligned}
\gamma_0\mathbf{j} &= \sum_k \gamma_0\sigma_k j^k \\
&= -\sum_k \gamma_k j^k,
\end{aligned}
\tag{101.7}
$$

so with $J^0 = \rho$, $J^k = j^k$ and $J = \gamma_\mu J^\mu$, we have

$$
\gamma^\mu\partial_\mu(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) = 4\pi J
\tag{101.8}
$$

and finally with $F = \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}$, we have Maxwell's equation in covariant form

$$
\nabla F = 4\pi J.
\tag{101.9}
$$

Next it is stated that general solutions can be expressed as

$$
\begin{aligned}
\boldsymbol{\mathcal{H}} &= \boldsymbol{\nabla} \times \mathbf{a} \\
\boldsymbol{\mathcal{E}} &= -\frac{1}{c}\frac{\partial \mathbf{a}}{\partial t} - \boldsymbol{\nabla}\phi
\end{aligned}
\tag{101.10}
$$

Let us double check that this jives with the bivector potential solution $F = \nabla \wedge A = \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}$. Let us split our bivector into spacetime and spatial components by the conjugate operation

$$
\begin{aligned}
F^* &= \gamma_0 F \gamma_0 \\
&= \gamma_0 \gamma^\mu \wedge \gamma^\nu \partial_\mu A_\mu \gamma_0 \\
&= \begin{cases} 0 & \text{if } \mu = \nu \\ \gamma^\mu \gamma^\nu \partial_\mu A_\nu & \text{if } \mu \in \{1,2,3\}, \text{ and } \nu \in \{1,2,3\} \\ -\gamma^\mu \gamma^\nu \partial_\mu A_\nu & \text{one of } \mu = 0 \text{ or } \nu = 0 \end{cases}
\end{aligned}
\tag{101.11}
$$

$$
\begin{aligned}
F &= \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}} \\
&= \frac{1}{2}(F - F^*) + \frac{1}{2}(F + F^*) \\
&= \left(\gamma^k \wedge \gamma^0 \partial_k A_0 + \gamma^0 \wedge \gamma^k \partial_0 A_k\right) + \left(\gamma^a \wedge \gamma^b \partial_a A_b\right) \\
&= -\left(\sum_k \sigma_k \partial_k A^0 + \partial_0 \sigma_k A^k\right) + i\left(\epsilon_{abc} \sigma_a \partial_b A^c\right)
\end{aligned}
\tag{101.12}
$$

So, with $\mathbf{a} = \sigma_k A^k$, and $\phi = A^0$, we do have equations eq. (101.10) as identical to $F = \nabla \wedge A$.

Now how about the gauge variations of the fields? Bohm writes that we can alter the potentials by

$$
\begin{aligned}
\mathbf{a}' &= \mathbf{a} - \boldsymbol{\nabla}\psi \\
\phi' &= \phi + \frac{1}{c}\frac{\partial \psi}{\partial t}
\end{aligned}
\tag{101.13}
$$

How does this translate to an alteration of the four potential? For the vector potential we have

$$
\begin{aligned}
\sigma_k A^{k\prime} &= \sigma_k A^k - \sigma_k \partial \psi \\
\gamma_k \gamma_0 A^{k\prime} &= \gamma_k \gamma_0 A^k - \gamma_k \gamma_0 \partial_k \psi \\
-\gamma_0 \gamma_k A^{k\prime} &= -\gamma_0 \gamma_k A^k - \gamma_0 \gamma^k \partial_k \psi \\
\gamma_k A^{k\prime} &= \gamma_k A^k + \gamma^k \partial_k \psi
\end{aligned}
\tag{101.14}
$$

with $\phi = A^0$, add in the ϕ term

$$
\begin{aligned}
\gamma_0 \phi' &= \gamma_0 \phi + \gamma_0 \frac{\partial \psi}{\partial x^0} \\
\gamma_0 \phi' &= \gamma_0 \phi + \gamma^0 \frac{\partial \psi}{\partial x^0}
\end{aligned}
\tag{101.15}
$$

For

$$\gamma_\mu A^{\mu\prime} = \gamma_\mu A^\mu + \gamma^\mu \partial_\mu \psi \tag{101.16}$$

Which is just a statement that we can add a spacetime gradient to our vector potential without altering the field equation:

$$A' = A + \nabla\psi \tag{101.17}$$

Let us verify that this does in fact not alter Maxwell's equation.

$$\nabla(\nabla \wedge (A + \nabla\psi) = 4\pi J \nabla(\nabla \wedge A) + \nabla(\nabla \wedge \nabla\psi) \quad = \tag{101.18}$$

Since $\nabla \wedge \nabla = 0$ we have

$$\nabla(\nabla \wedge A') = \nabla(\nabla \wedge A) \tag{101.19}$$

Now the statement that $\nabla \wedge \nabla$ as an operator equals zero, just by virtue of ∇ being a vector is worth explicit confirmation. Let us expand that to verify

$$\begin{aligned}
\nabla \wedge \nabla\psi &= \gamma^\mu \wedge \gamma^\nu \partial_\mu \partial_\nu \psi \\
&= \left(\sum_{\mu<\nu} + \sum_{\nu<\mu} \right) \gamma^\mu \wedge \gamma^\nu \partial_\mu \partial_\nu \psi \\
&= \sum_{\mu<\nu} \gamma^\mu \wedge \gamma^\nu (\partial_\mu \partial_\nu \psi - \partial_\nu \partial_\mu \psi)
\end{aligned} \tag{101.20}$$

So, we see that we additionally need the field variable ψ to be sufficiently continuous for mixed partial equality for the statement that $\nabla \wedge \nabla = 0$ to be valid. Assuming that continuity is taken as a given the confirmation of the invariance under this transformation is thus complete.

Now, Bohm says it is possible to pick $\boldsymbol{\nabla} \cdot \mathbf{a}' = 0$. From eq. (101.13) that implies

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{a}' &= \boldsymbol{\nabla} \cdot \mathbf{a} - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\psi \\
&= \boldsymbol{\nabla} \cdot \mathbf{a} - \boldsymbol{\nabla}^2\psi = 0
\end{aligned} \tag{101.21}$$

So, provided we can find a solution to the Poisson equation

$$\boldsymbol{\nabla}^2\psi = \boldsymbol{\nabla} \cdot \mathbf{a} \tag{101.22}$$

one can find a ψ, $\mathbf{a}$ gauge transformation that has the particular quality that $\boldsymbol{\nabla} \cdot \mathbf{a}' = 0$. That solution, from eq. (101.53) is

$$\psi(\mathbf{r}) = -\frac{1}{4\pi} \int (\boldsymbol{\nabla}' \cdot \mathbf{a}(\mathbf{r}')) dV' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{101.23}$$

The corollary to this is that one may similarly impose a requirement that $\boldsymbol{\nabla} \cdot \mathbf{a} = 0$, since if that is not the case, some $\mathbf{a}'$ can be added to the vector potential to make that the case.

FIXME: handwaving description here. Show with a math statement with $\mathbf{a} \rightarrow \mathbf{a}'$.

101.3.1 *Free space solutions*

From eq. (101.5) and eq. (101.10) the free space solution to Maxwell's equation must satisfy

$$\begin{aligned}
0 &= \left(\boldsymbol{\nabla} + \partial_0\right)\left(\mathcal{E} + i\mathcal{H}\right) \\
&= \left(\boldsymbol{\nabla} + \partial_0\right)\left(-\partial_0 \mathbf{a} - \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \wedge \mathbf{a}\right) \\
&= -\boldsymbol{\nabla}\partial_0 \mathbf{a} - \boldsymbol{\nabla}^2 \phi + \boldsymbol{\nabla}(\boldsymbol{\nabla} \wedge \mathbf{a}) - \partial_{00}\mathbf{a} - \partial_0 \boldsymbol{\nabla}\phi + \partial_0(\boldsymbol{\nabla} \wedge \mathbf{a}) \\
&= -\boldsymbol{\nabla} \cdot \partial_0 \mathbf{a} - \boldsymbol{\nabla}^2 \phi + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{a}) - \partial_{00}\mathbf{a} - \partial_0 \boldsymbol{\nabla}\phi
\end{aligned} \tag{101.24}$$

Since the scalar and vector parts of this equation must separately equal zero we have

$$\begin{aligned}
0 &= -\partial_0 \boldsymbol{\nabla} \cdot \mathbf{a} - \boldsymbol{\nabla}^2 \phi \\
0 &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{a}) - \partial_{00}\mathbf{a} - \partial_0 \boldsymbol{\nabla}\phi
\end{aligned} \tag{101.25}$$

If one picks a gauge transformation such that $\boldsymbol{\nabla} \cdot \mathbf{a} = 0$ we then have

$$\begin{aligned}
0 &= \boldsymbol{\nabla}^2 \phi \\
0 &= \boldsymbol{\nabla}^2 \mathbf{a} - \partial_{00}\mathbf{a} - \partial_0 \boldsymbol{\nabla}\phi
\end{aligned} \tag{101.26}$$

For the first Bohm argues that "It is well known that the only solution of this equation that is regular over all space is $\phi = 0$", and anything else implies charge in the region. What does regular mean here? I suppose this seems like a reasonable enough statement, but I think the proper way to think about this is really that one has picked the covariant gauge $\nabla \cdot A = 0$ (that is simpler anyhow). With an acceptance of the $\phi = 0$ argument one is left with the vector potential wave equation which was the desired goal of that section.

Note: The following physicsforums thread discusses some of the confusion I had in this bit of text.

101.3.2 *Doing this all directly*

Now, the whole point of the gauge transformation appears to be to show that one can find the four wave equation solutions for Maxwell's equation by picking a specific gauge. This is actually trivial to do from the STA Maxwell equation:

$$\nabla(\nabla \wedge A) = \nabla(\nabla A - \nabla \cdot A) = \nabla^2 A - \nabla(\nabla \cdot A) = 4\pi J \tag{101.27}$$

So, if one picks a gauge transformation with $\nabla \cdot A = 0$, one has

$$\nabla^2 A = 4\pi J \tag{101.28}$$

This is precisely the four wave equations desired

$$\partial_\nu \partial^\nu A^\mu = 4\pi J^\mu \tag{101.29}$$

FIXME: show the precise gauge transformation $A \rightarrow A'$ that leads to $\nabla \cdot A = 0$.

101.4 ENERGY DENSITY. GET THE UNITS RIGHT WITH THESE CGS EQUATIONS

We will want to calculate the equivalent of

$$U = \frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) \tag{101.30}$$

but are faced with the alternate units of Bohm's text. Let us repeat the derivation of the electric field energy from 96 in the CGS units directly from Maxwell's equation

$$\begin{aligned}
F &= \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}} \\
J &= (\rho + \mathbf{j})\gamma_0 \\
\nabla F &= 4\pi J
\end{aligned} \tag{101.31}$$

to ensure we get it right.

To start with we our spacetime split of eq. (101.31) is

$$(\partial_0 + \boldsymbol{\nabla})(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{H}}) = 4\pi(\rho - \mathbf{j}) \tag{101.32}$$

The scalar part gives us Coulomb's law

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} = 4\pi\rho \tag{101.33}$$

Gauss's theorem applied to a spherical constant density charge distribution gives us

$$\begin{aligned}
\int \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} dV &= 4\pi \int \rho dV \\
&\implies \\
\int \boldsymbol{\mathcal{E}} \cdot \hat{\mathbf{n}} dA &= 4\pi Q \\
&\implies \\
|\boldsymbol{\mathcal{E}}| 4\pi r^2 &= 4\pi Q
\end{aligned} \tag{101.34}$$

so we have the expected "unitless" Coulomb law force equation

$$\mathbf{F} = q\boldsymbol{\mathcal{E}} = \frac{qQ}{r^2}\hat{\mathbf{r}} \tag{101.35}$$

So far so good. Next introduction of a potential. For statics we do not care about the four vectors and stick with the old fashion definition of the potential ϕ indirectly in terms of $\boldsymbol{\mathcal{E}}$. That is

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\phi \tag{101.36}$$

A line integral of this gives us ϕ in terms of $\boldsymbol{\mathcal{E}}$

$$\begin{aligned}
-\int \boldsymbol{\mathcal{E}} \cdot \mathbf{r} &= \int \boldsymbol{\nabla}\phi \cdot d\mathbf{r} \\
&= \phi - \phi_0
\end{aligned} \tag{101.37}$$

With $\phi(\infty) = 0$ this is

$$\begin{aligned}
\phi(d) &= -\int_{r=\infty}^{d} \boldsymbol{\mathcal{E}} \cdot d\mathbf{r} \\
&= -\int_{r=\infty}^{d} \frac{Q}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{r} \\
&= -\int_{r=\infty}^{d} \frac{Q}{r^2} dr \\
&= \frac{Q}{d}
\end{aligned} \tag{101.38}$$

Okay. Now onto the electrostatic energy. The work done to move one charge from infinite to some separation d of another like sign charge is

$$
\begin{aligned}
\int_{r=\infty}^{d} F \cdot d\mathbf{r} &= \int_{r=\infty}^{d} \frac{qQ}{r^2}\hat{\mathbf{r}} \cdot (-d\mathbf{r}) \\
&= -\int_{r=\infty}^{d} \frac{qQ}{r^2} dr \\
&= \frac{qQ}{d} \\
&= q_1\phi_2(d)
\end{aligned}
\tag{101.39}
$$

For a distribution of discrete charges we have to sum over all pairs

$$
\begin{aligned}
W &= \sum_{i \neq j} \frac{q_i q_j}{d_{ij}} \\
&= \sum_{i,j} \frac{1}{2} \frac{q_i q_j}{d_{ij}}
\end{aligned}
\tag{101.40}
$$

In a similar fashion we can do a continuous variation, employing a double summation over all space. Note first that we can also write one of the charge densities in terms of the potential

$$
\begin{aligned}
\boldsymbol{\mathcal{E}} &= -\boldsymbol{\nabla}\phi \\
\implies& \\
\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{E}} &= -\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi \\
&= -\boldsymbol{\nabla}^2\phi \\
&= 4\pi\rho
\end{aligned}
\tag{101.41}
$$

$$
\begin{aligned}
W &= \frac{1}{2}\int \rho\phi(r) dV \\
&= -\frac{1}{8\pi}\int \phi\boldsymbol{\nabla}^2\phi dV \\
&= \frac{1}{8\pi}\int ((\boldsymbol{\nabla}\phi)^2 - \boldsymbol{\nabla} \cdot (\phi\boldsymbol{\nabla}\phi)) dV \\
&= \frac{1}{8\pi}\int (-\boldsymbol{\mathcal{E}})^2 - \frac{1}{8\pi}\int (\phi\boldsymbol{\nabla}\phi) \cdot \hat{\mathbf{n}} dA
\end{aligned}
\tag{101.42}
$$

Here the one and two subscripts could be dropped with a switch to the total charge density and the potential from this complete charge superposition. For our final result we have an energy density of

$$\frac{dW}{dV} = \frac{1}{8\pi}\boldsymbol{\mathcal{E}}^2 \tag{101.43}$$

101.5 AUXILIARY DETAILS

101.5.1 *Confirm Poisson solution to Laplacian*

Bohm lists the solution for eq. (101.22) (a Poisson integral), but I forget how one shows this. I can not figure out how to integrate this Laplacian, but it is simple enough to confirm this by back substitution.

Suppose one has

$$\psi = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' \tag{101.44}$$

We can take the Laplacian by direct differentiation under the integration sign

$$\boldsymbol{\nabla}^2\psi = \int \rho(\mathbf{r}') dV' \boldsymbol{\nabla}^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{101.45}$$

To evaluate the Laplacian we need

$$\begin{aligned}
\frac{\partial |\mathbf{r}-\mathbf{r}'|^k}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\sum_j (x_j - x_j')^2\right)^{k/2} \\
&= \frac{k}{2}\left|\mathbf{r}-\mathbf{r}'\right|^{k-2} \frac{\partial}{\partial x_i}\left(\sum_j (x_j - x_j')^2\right) \\
&= k\left|\mathbf{r}-\mathbf{r}'\right|^{k-2}(x_i - x_i')
\end{aligned} \tag{101.46}$$

So we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\left|\mathbf{r}-\mathbf{r}'\right|^{-1} &= -(x_i - x_i')\frac{\partial}{\partial x_i}\frac{1}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}\frac{\partial(x_i - x_i')}{\partial x_i} \\
&= 3(x_i - x_i')^2\frac{1}{|\mathbf{r}-\mathbf{r}'|^5} - \frac{1}{|\mathbf{r}-\mathbf{r}'|^3}
\end{aligned} \tag{101.47}$$

So, provided $\mathbf{r} \neq \mathbf{r}'$ we have

$$
\begin{aligned}
\boldsymbol{\nabla}^2 \psi &= 3(\mathbf{r} - \mathbf{r}')^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|^5} - 3 \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \\
&= 0
\end{aligned}
\tag{101.48}
$$

Observe that this is true only for $\mathbb{R}^3$. Now, one is left with only an integral around a neighborhood around the point $\mathbf{r}$ which can be made small enough that $\rho(\mathbf{r}') = \rho(\mathbf{r})$ in that volume can be taken as constant.

$$
\begin{aligned}
\boldsymbol{\nabla}^2 \psi &= \rho(\mathbf{r}) \int dV' \boldsymbol{\nabla}^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\
&= \rho(\mathbf{r}) \int dV' \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\
&= -\rho(\mathbf{r}) \int dV' \boldsymbol{\nabla} \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}
\end{aligned}
\tag{101.49}
$$

Now, if the divergence in this integral was with respect to the primed variable that ranges over the infinitesimal volume, then this could be converted to a surface integral. Observe that a radial expansion of this divergence allows for convenient change of variables to the primed x_i' coordinates

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} &= \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial}{\partial |\mathbf{r} - \mathbf{r}'|} \right) \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \right) \\
&= \frac{\partial}{\partial |\mathbf{r}' - \mathbf{r}|} \left| \mathbf{r}' - \mathbf{r} \right|^{-2} \\
&= \left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|} \frac{\partial}{\partial |\mathbf{r}' - \mathbf{r}|} \right) \cdot \left(\frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|} \frac{1}{|\mathbf{r}' - \mathbf{r}|^2} \right) \\
&= \boldsymbol{\nabla}' \cdot \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3}
\end{aligned}
\tag{101.50}
$$

Now, since $\mathbf{r}' - \mathbf{r}$ is in the direction of the outwards normal the divergence theorem can be used

$$
\begin{aligned}
\boldsymbol{\nabla}^2 \psi &= -\rho(\mathbf{r}) \int dV' \boldsymbol{\nabla}' \cdot \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \\
&= -\rho(\mathbf{r}) \int_{\partial V'} dA' \frac{1}{|\mathbf{r}' - \mathbf{r}|^2}
\end{aligned}
\tag{101.51}
$$

Picking a spherical integration volume, for which the radius is constant $R = |\mathbf{r}' - \mathbf{r}|$, we have

$$\boldsymbol{\nabla}^2\psi = -\rho(\mathbf{r})4\pi R^2 \frac{1}{R^2} \tag{101.52}$$

In summary this is

$$\begin{aligned}\psi &= \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' \\ \boldsymbol{\nabla}^2\psi &= -4\pi\rho(\mathbf{r})\end{aligned} \tag{101.53}$$

Having written this out I recall that the same approach was used in [39] (there it was to calculate $\boldsymbol{\nabla} \cdot \mathbf{E}$ in terms of the charge density, but the ideas are all the same.)

102

ENERGY AND MOMENTUM FOR COMPLEX ELECTRIC AND MAGNETIC FIELD PHASORS

102.1 MOTIVATION

In [22] a complex phasor representations of the electric and magnetic fields is used

$$\begin{aligned}\mathbf{E} &= \boldsymbol{\mathcal{E}} e^{-i\omega t} \\ \mathbf{B} &= \boldsymbol{\mathcal{B}} e^{-i\omega t}.\end{aligned} \tag{102.1}$$

Here the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are allowed to take on complex values. Jackson uses the real part of these complex vectors as the true fields, so one is really interested in just these quantities

$$\begin{aligned}\operatorname{Re} \mathbf{E} &= \boldsymbol{\mathcal{E}}_r \cos(\omega t) + \boldsymbol{\mathcal{E}}_i \sin(\omega t) \\ \operatorname{Re} \mathbf{B} &= \boldsymbol{\mathcal{B}}_r \cos(\omega t) + \boldsymbol{\mathcal{B}}_i \sin(\omega t),\end{aligned} \tag{102.2}$$

but carry the whole thing in manipulations to make things simpler. It is stated that the energy for such complex vector fields takes the form (ignoring constant scaling factors and units)

$$\text{Energy} \propto \mathbf{E} \cdot \mathbf{E}^* + \mathbf{B} \cdot \mathbf{B}^*. \tag{102.3}$$

In some ways this is an obvious generalization. Less obvious is how this and the Poynting vector are related in their corresponding conservation relationships.

Here I explore this, employing a Geometric Algebra representation of the energy momentum tensor based on the real field representation found in [10]. Given the complex valued fields and a requirement that both the real and imaginary parts of the field satisfy Maxwell's equation, it should be possible to derive the conservation relationship between the energy density and Poynting vector from first principles.

102.2 REVIEW OF GA FORMALISM FOR REAL FIELDS

In SI units the Geometric algebra form of Maxwell's equation is

$$\nabla F = J/\epsilon_0 c, \tag{102.4}$$

where one has for the symbols

$$
\begin{aligned}
F &= \mathbf{E} + cI\mathbf{B} \\
I &= \gamma_0\gamma_1\gamma_2\gamma_3 \\
\mathbf{E} &= E^k\gamma_k\gamma_0 \\
\mathbf{B} &= B^k\gamma_k\gamma_0 \\
(\gamma^0)^2 &= -(\gamma^k)^2 = 1 \\
\gamma^\mu\cdot\gamma_\nu &= \delta^\mu{}_\nu \\
J &= c\rho\gamma_0 + J^k\gamma_k \\
\nabla &= \gamma^\mu\partial_\mu = \gamma^\mu\partial/\partial x^\mu .
\end{aligned}
\tag{102.5}
$$

The symmetric electrodynamic energy momentum tensor for real fields $\mathbf{E}$ and $\mathbf{B}$ is

$$
\begin{aligned}
T(a) &= \frac{-\epsilon_0}{2} F a F \\
&= \frac{\epsilon_0}{2} F a \tilde{F}.
\end{aligned}
\tag{102.6}
$$

It may not be obvious that this is in fact a four vector, but this can be seen since it can only have grade one and three components, and also equals its reverse implying that the grade three terms are all zero. To illustrate this explicitly consider the components of $T^{\mu 0}$

$$
\begin{aligned}
\frac{2}{\epsilon_0} T\left(\gamma^0\right) &= -\left(\mathbf{E} + cI\mathbf{B}\right)\gamma^0\left(\mathbf{E} + cI\mathbf{B}\right) \\
&= \left(\mathbf{E} + cI\mathbf{B}\right)\left(\mathbf{E} - cI\mathbf{B}\right)\gamma^0 \\
&= \left(\mathbf{E}^2 + c^2\mathbf{B}^2 + cI\left(\mathbf{B}\mathbf{E} - \mathbf{E}\mathbf{B}\right)\right)\gamma^0 \\
&= \left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)\gamma^0 + 2cI\left(\mathbf{B}\wedge\mathbf{E}\right)\gamma^0 \\
&= \left(\mathbf{E}^2 + c^2\mathbf{B}^2\right)\gamma^0 + 2c\left(\mathbf{E}\times\mathbf{B}\right)\gamma^0
\end{aligned}
\tag{102.7}
$$

Our result is a four vector in the Dirac basis as expected

$$
\begin{aligned}
T\left(\gamma^0\right) &= T^{\mu 0}\gamma_\mu \\
T^{00} &= \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) \\
T^{k0} &= c\epsilon_0\left(\mathbf{E}\times\mathbf{B}\right)_k
\end{aligned}
\tag{102.8}
$$

Similar expansions are possible for the general tensor components $T^{\mu\nu}$ but lets defer this more general expansion until considering complex valued fields. The main point here is to remind oneself how to express the energy momentum tensor in a fashion that is natural in a GA context. We also know that one has a conservation relationship associated with the divergence of this tensor $\nabla \cdot T(a)$ (ie. $\partial_\mu T^{\mu\nu}$), and want to rederive this relationship after guessing what form the GA expression for the energy momentum tensor takes when one allows the field vectors to take complex values.

102.3 COMPUTING THE CONSERVATION RELATIONSHIP FOR COMPLEX FIELD VECTORS

As in eq. (102.3), if one wants

$$T^{00} \propto \mathbf{E} \cdot \mathbf{E}^* + c^2 \mathbf{B} \cdot \mathbf{B}^*, \tag{102.9}$$

it is reasonable to assume that our energy momentum tensor will take the form

$$T(a) = \frac{\epsilon_0}{4} \left(F^* a \tilde{F} + \tilde{F} a F^* \right) = \frac{\epsilon_0}{2} \operatorname{Re} \left(F^* a \tilde{F} \right) \tag{102.10}$$

For real vector fields this reduces to the previous results and should produce the desired mix of real and imaginary dot products for the energy density term of the tensor. This is also a real four vector even when the field is complex, so the energy density and power density terms will all be real valued, which seems desirable.

102.3.1 *Expanding the tensor. Easy parts*

As with real fields expansion of $T(a)$ in terms of $\mathbf{E}$ and $\mathbf{B}$ is simplest for $a = \gamma^0$. Let us start with that.

$$\begin{aligned}
\frac{4}{\epsilon_0} T(\gamma^0)\gamma_0 &= -(\mathbf{E}^* + cI\mathbf{B}^*)\gamma^0(\mathbf{E} + cI\mathbf{B})\gamma_0 - (\mathbf{E} + cI\mathbf{B})\gamma^0(\mathbf{E}^* + cI\mathbf{B}^*)\gamma_0 \\
&= (\mathbf{E}^* + cI\mathbf{B}^*)(\mathbf{E} - cI\mathbf{B}) + (\mathbf{E} + cI\mathbf{B})(\mathbf{E}^* - cI\mathbf{B}^*) \\
&= \mathbf{E}^*\mathbf{E} + \mathbf{E}\mathbf{E}^* + c^2(\mathbf{B}^*\mathbf{B} + \mathbf{B}\mathbf{B}^*) + cI(\mathbf{B}^*\mathbf{E} - \mathbf{E}^*\mathbf{B} + \mathbf{B}\mathbf{E}^* - \mathbf{E}\mathbf{B}^*) \\
&= 2\mathbf{E} \cdot \mathbf{E}^* + 2c^2\mathbf{B} \cdot \mathbf{B}^* + 2c(\mathbf{E} \times \mathbf{B}^* + \mathbf{E}^* \times \mathbf{B}).
\end{aligned} \tag{102.11}$$

This gives

$$T(\gamma^0) = \frac{\epsilon_0}{2} \left(\mathbf{E} \cdot \mathbf{E}^* + c^2 \mathbf{B} \cdot \mathbf{B}^* \right) \gamma^0 + \frac{\epsilon_0 c}{2} (\mathbf{E} \times \mathbf{B}^* + \mathbf{E}^* \times \mathbf{B}) \gamma^0 \tag{102.12}$$

The sum of $F^* a F$ and its conjugate has produced the desired energy density expression. An implication of this is that one can form and take real parts of a complex Poynting vector $\mathbf{S} \propto \mathbf{E} \times \mathbf{B}^*$ to calculate the momentum density. This is stated but not demonstrated in Jackson, perhaps considered too obvious or messy to derive.

Observe that the a choice to work with complex valued vector fields gives a nice consistency, and one has the same factor of $1/2$ in both the energy and momentum terms. While the energy term is obviously real, the momentum terms can be written in an explicitly real notation as well since one has a quantity plus its conjugate. Using a more conventional four vector notation (omitting the explicit Dirac basis vectors), one can write this out as a strictly real quantity.

$$T(\gamma^0) = \epsilon_0 \left(\frac{1}{2} \left(\mathbf{E} \cdot \mathbf{E}^* + c^2 \mathbf{B} \cdot \mathbf{B}^* \right), c \operatorname{Re}(\mathbf{E} \times \mathbf{B}^*) \right) \tag{102.13}$$

Observe that when the vector fields are restricted to real quantities, the conjugate and real part operators can be dropped and the real vector field result **??** is recovered.

102.3.2 *Expanding the tensor. Messier parts*

I intended here to compute $T(\gamma^k)$, and my starting point was a decomposition of the field vectors into components that anticommute or commute with γ^k

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_\parallel + \mathbf{E}_\perp \\ \mathbf{B} &= \mathbf{B}_\parallel + \mathbf{B}_\perp . \end{aligned} \tag{102.14}$$

The components parallel to the spatial vector $\sigma_k = \gamma_k \gamma_0$ are anticommuting $\gamma^k \mathbf{E}_\parallel = -\mathbf{E}_\parallel \gamma^k$, whereas the perpendicular components commute $\gamma^k \mathbf{E}_\perp = \mathbf{E}_\perp \gamma^k$. The expansion of the tensor products is then

$$\begin{aligned} (F^* \gamma^k \tilde{F} + \tilde{F} \gamma^k F^*) \gamma_k &= -(\mathbf{E}^* + I c \mathbf{B}^*) \gamma^k (\mathbf{E}_\parallel + \mathbf{E}_\perp + c I (\mathbf{B}_\parallel + \mathbf{B}_\perp)) \gamma_k \\ &\quad - (\mathbf{E} + I c \mathbf{B}) \gamma^k ({\mathbf{E}_\parallel}^* + {\mathbf{E}_\perp}^* + c I ({\mathbf{B}_\parallel}^* + {\mathbf{B}_\perp}^*)) \gamma_k \\ &= (\mathbf{E}^* + I c \mathbf{B}^*)(\mathbf{E}_\parallel - \mathbf{E}_\perp + c I (-\mathbf{B}_\parallel + \mathbf{B}_\perp)) \\ &\quad + (\mathbf{E} + I c \mathbf{B})({\mathbf{E}_\parallel}^* - {\mathbf{E}_\perp}^* + c I (-{\mathbf{B}_\parallel}^* + {\mathbf{B}_\perp}^*)) \end{aligned} \tag{102.15}$$

This is not particularly pretty to expand out. I did attempt it, but my result looked wrong. For the application I have in mind I do not actually need anything more than $T^{\mu 0}$, so rather than show something wrong, I will just omit it (at least for now).

102.3.3 *Calculating the divergence*

Working with eq. (102.10), let us calculate the divergence and see what one finds for the corresponding conservation relationship.

$$
\begin{aligned}
\frac{4}{\epsilon_0}\nabla\cdot T(a) &= \left\langle \nabla(F^* a\tilde{F} + \tilde{F} a F^*)\right\rangle \\
&= -\left\langle F\overset{\leftrightarrow}{\nabla}F^* a + F^*\overset{\leftrightarrow}{\nabla}F a\right\rangle \\
&= -\left\langle F\overset{\leftrightarrow}{\nabla}F^* + F^*\overset{\leftrightarrow}{\nabla}F\right\rangle_1 \cdot a \\
&= -\left\langle F\overset{\rightarrow}{\nabla}F^* + F\overset{\leftarrow}{\nabla}F^* + F^*\overset{\leftarrow}{\nabla}F + F^*\overset{\rightarrow}{\nabla}F\right\rangle_1 \cdot a \\
&= -\frac{1}{\epsilon_0 c}\langle F J^* - J F^* - J^* F + F^* J\rangle_1 \cdot a \\
&= \frac{2}{\epsilon_0 c} a\cdot(J\cdot F^* + J^*\cdot F) \\
&= \frac{4}{\epsilon_0 c} a\cdot \mathrm{Re}(J\cdot F^*).
\end{aligned}
\tag{102.16}
$$

We have then for the divergence

$$
\nabla\cdot T(a) = a\cdot\frac{1}{c}\,\mathrm{Re}\left(J\cdot F^*\right). \tag{102.17}
$$

Lets write out $J\cdot F^*$ in the (stationary) observer frame where $J = (c\rho + \mathbf{J})\gamma_0$. This is

$$
\begin{aligned}
J\cdot F^* &= \langle (c\rho + \mathbf{J})\gamma_0(\mathbf{E}^* + Ic\mathbf{B}^*)\rangle_1 \\
&= -(\mathbf{J}\cdot\mathbf{E}^*)\gamma_0 - c\left(\rho\mathbf{E}^* + \mathbf{J}\times\mathbf{B}^*\right)\gamma_0
\end{aligned}
\tag{102.18}
$$

Writing out the four divergence relationships in full one has

$$
\begin{aligned}
\nabla\cdot T(\gamma^0) &= -\frac{1}{c}\,\mathrm{Re}(\mathbf{J}\cdot\mathbf{E}^*) \\
\nabla\cdot T(\gamma^k) &= -\,\mathrm{Re}\left(\rho(E^k)^* + (\mathbf{J}\times\mathbf{B}^*)_k\right)
\end{aligned}
\tag{102.19}
$$

Just as in the real field case one has a nice relativistic split into energy density and force (momentum change) components, but one has to take real parts and conjugate half the terms appropriately when one has complex fields.

Combining the divergence relation for $T(\gamma^0)$ with eq. (102.13) the conservation relation for this subset of the energy momentum tensor becomes

$$\frac{1}{c}\frac{\partial}{\partial t}\frac{\epsilon_0}{2}(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*) + c\epsilon_0 \operatorname{Re} \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B}^*) = -\frac{1}{c}\operatorname{Re}(\mathbf{J}\cdot\mathbf{E}^*) \tag{102.20}$$

Or

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*) + \operatorname{Re} \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}^*) + \operatorname{Re}(\mathbf{J}\cdot\mathbf{E}^*) = 0 \tag{102.21}$$

It is this last term that puts some meaning behind Jackson's treatment since we now know how the energy and momentum are related as a four vector quantity in this complex formalism.

While I have used geometric algebra to get to this final result, I would be interested to compare how the intermediate mess compares with the same complex field vector result obtained via traditional vector techniques. I am sure I could try this myself, but am not interested enough to attempt it.

Instead, now that this result is obtained, proceeding on to application is now possible. My intention is to try the vacuum electromagnetic energy density example from [2] using complex exponential Fourier series instead of the doubled sum of sines and cosines that Bohm used.

103

ELECTRODYNAMIC FIELD ENERGY FOR VACUUM

103.1 MOTIVATION

From 102 how to formulate the energy momentum tensor for complex vector fields (ie. phasors) in the Geometric Algebra formalism is now understood. To recap, for the field $F = \mathbf{E} + Ic\mathbf{B}$, where $\mathbf{E}$ and $\mathbf{B}$ may be complex vectors we have for Maxwell's equation

$$\nabla F = J/\epsilon_0 c. \tag{103.1}$$

This is a doubly complex representation, with the four vector pseudoscalar $I = \gamma_0\gamma_1\gamma_2\gamma_3$ acting as a non-commutatitive imaginary, as well as real and imaginary parts for the electric and magnetic field vectors. We take the real part (not the scalar part) of any bivector solution F of Maxwell's equation as the actual solution, but allow ourself the freedom to work with the complex phasor representation when convenient. In these phasor vectors, the imaginary i, as in $\mathbf{E} = \mathrm{Re}(\mathbf{E}) + i\,\mathrm{Im}(\mathbf{E})$, is a commuting imaginary, commuting with all the multivector elements in the algebra.

The real valued, four vector, energy momentum tensor $T(a)$ was found to be

$$T(a) = \frac{\epsilon_0}{4}\left(F^* a\tilde{F} + \tilde{F} a F^*\right) = -\frac{\epsilon_0}{2}\,\mathrm{Re}\left(F^* a F\right). \tag{103.2}$$

To supply some context that gives meaning to this tensor the associated conservation relationship was found to be

$$\nabla \cdot T(a) = a \cdot \frac{1}{c}\,\mathrm{Re}\left(J \cdot F^*\right). \tag{103.3}$$

and in particular for $a = \gamma^0$, this four vector divergence takes the form

$$\frac{\partial}{\partial t}\frac{\epsilon_0}{2}(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*) + \boldsymbol{\nabla}\cdot\frac{1}{\mu_0}\,\mathrm{Re}(\mathbf{E}\times\mathbf{B}^*) + \mathrm{Re}(\mathbf{J}\cdot\mathbf{E}^*) = 0, \tag{103.4}$$

relating the energy term $T^{00} = T(\gamma^0)\cdot\gamma^0$ and the Poynting spatial vector $T(\gamma^0)\wedge\gamma^0$ with the current density and electric field product that constitutes the energy portion of the Lorentz force density.

Let us apply this to calculating the energy associated with the field that is periodic within a rectangular prism as done by Bohm in [2]. We do not necessarily need the Geometric Algebra formalism for this calculation, but this will be a fun way to attempt it.

103.2 SETUP

Let us assume a Fourier representation for the four vector potential A for the field $F = \nabla \wedge A$. That is

$$A = \sum_{\mathbf{k}} A_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{103.5}$$

where summation is over all angular wave number triplets $\mathbf{k} = 2\pi(k_1/\lambda_1, k_2/\lambda_2, k_3/\lambda_3)$. The Fourier coefficients $A_{\mathbf{k}} = {A_{\mathbf{k}}}^\mu \gamma_\mu$ are allowed to be complex valued, as is the resulting four vector A, and the associated bivector field F.

Fourier inversion, with $V = \lambda_1\lambda_2\lambda_3$, follows from

$$\delta_{\mathbf{k}',\mathbf{k}} = \frac{1}{V}\int_0^{\lambda_1}\int_0^{\lambda_2}\int_0^{\lambda_3} e^{i\mathbf{k}'\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} dx^1 dx^2 dx^3, \tag{103.6}$$

but only this orthogonality relationship and not the Fourier coefficients themselves

$$A_{\mathbf{k}} = \frac{1}{V}\int_0^{\lambda_1}\int_0^{\lambda_2}\int_0^{\lambda_3} A(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} dx^1 dx^2 dx^3, \tag{103.7}$$

will be of interest here. Evaluating the curl for this potential yields

$$F = \nabla \wedge A = \sum_{\mathbf{k}} \left(\frac{1}{c}\gamma^0 \wedge \dot{A}_{\mathbf{k}} + \gamma^m \wedge A_{\mathbf{k}} \frac{2\pi i k_m}{\lambda_m} \right) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{103.8}$$

Since the four vector potential has been expressed using an explicit split into time and space components it will be natural to re express the bivector field in terms of scalar and (spatial) vector potentials, with the Fourier coefficients. Writing $\sigma_m = \gamma_m\gamma_0$ for the spatial basis vectors, ${A_{\mathbf{k}}}^0 = \phi_{\mathbf{k}}$, and $\mathbf{A} = A^k \sigma_k$, this is

$$A_{\mathbf{k}} = (\phi_{\mathbf{k}} + \mathbf{A}_{\mathbf{k}})\gamma_0. \tag{103.9}$$

The Faraday bivector field F is then

$$F = \sum_{\mathbf{k}} \left(-\frac{1}{c}\dot{\mathbf{A}}_{\mathbf{k}} - i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{103.10}$$

This is now enough to express the energy momentum tensor $T(\gamma^\mu)$

$$\begin{aligned}
&T(\gamma^\mu) \\
&= -\frac{\epsilon_0}{2} \sum_{\mathbf{k},\mathbf{k}'} \mathrm{Re} \left(\left(-\frac{1}{c}(\dot{\mathbf{A}}_{\mathbf{k}'})^* + i\mathbf{k}'{\phi_{\mathbf{k}'}}^* - i\mathbf{k}' \wedge {\mathbf{A}_{\mathbf{k}'}}^* \right) \gamma^\mu \left(-\frac{1}{c}\dot{\mathbf{A}}_{\mathbf{k}} - i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \right).
\end{aligned} \tag{103.11}$$

It will be more convenient to work with a scalar plus bivector (spatial vector) form of this tensor, and right multiplication by γ_0 produces such a split

$$T(\gamma^\mu)\gamma_0 = \langle T(\gamma^\mu)\gamma_0 \rangle + \sigma_a \langle \sigma_a T(\gamma^\mu)\gamma_0 \rangle \tag{103.12}$$

The primary object of this treatment will be consideration of the $\mu = 0$ components of the tensor, which provide a split into energy density $T(\gamma^0) \cdot \gamma_0$, and Poynting vector (momentum density) $T(\gamma^0) \wedge \gamma_0$.

Our first step is to integrate Equation 103.12 over the volume V. This integration and the orthogonality relationship Equation 103.6, removes the exponentials, leaving

$$\begin{aligned}
\int T(\gamma^\mu) \cdot \gamma_0 &= -\frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \mathrm{Re} \left\langle \left(-\frac{1}{c}(\dot{\mathbf{A}}_{\mathbf{k}})^* + i\mathbf{k}{\phi_{\mathbf{k}}}^* - i\mathbf{k} \wedge {\mathbf{A}_{\mathbf{k}}}^* \right) \gamma^\mu \left(-\frac{1}{c}\dot{\mathbf{A}}_{\mathbf{k}} - i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) \gamma_0 \right\rangle \\
\int T(\gamma^\mu) \wedge \gamma_0 &= -\frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \mathrm{Re}\, \sigma_a \left\langle \sigma_a \left(-\frac{1}{c}(\dot{\mathbf{A}}_{\mathbf{k}})^* + i\mathbf{k}{\phi_{\mathbf{k}}}^* - i\mathbf{k} \wedge {\mathbf{A}_{\mathbf{k}}}^* \right) \gamma^\mu \left(-\frac{1}{c}\dot{\mathbf{A}}_{\mathbf{k}} - i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) \gamma_0 \right\rangle
\end{aligned} \tag{103.13}$$

Because γ_0 commutes with the spatial bivectors, and anticommutes with the spatial vectors, the remainder of the Dirac basis vectors in these expressions can be eliminated

$$\int T(\gamma^0) \cdot \gamma_0 = -\frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \operatorname{Re} \left\langle \left(-\frac{1}{c} (\dot{\mathbf{A}}_{\mathbf{k}})^* + i\mathbf{k}\phi_{\mathbf{k}}^* - i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}}^* \right) \left(\frac{1}{c} \dot{\mathbf{A}}_{\mathbf{k}} + i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) \right\rangle \tag{103.14a}$$

$$\int T(\gamma^0) \wedge \gamma_0 = -\frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \operatorname{Re} \sigma_a \left\langle \sigma_a \left(-\frac{1}{c} (\dot{\mathbf{A}}_{\mathbf{k}})^* + i\mathbf{k}\phi_{\mathbf{k}}^* - i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}}^* \right) \left(\frac{1}{c} \dot{\mathbf{A}}_{\mathbf{k}} + i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) \right\rangle \tag{103.14b}$$

$$\int T(\gamma^m) \cdot \gamma_0 = \frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \operatorname{Re} \left\langle \left(-\frac{1}{c} (\dot{\mathbf{A}}_{\mathbf{k}})^* + i\mathbf{k}\phi_{\mathbf{k}}^* - i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}}^* \right) \sigma_m \left(\frac{1}{c} \dot{\mathbf{A}}_{\mathbf{k}} + i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) \right\rangle \tag{103.14c}$$

$$\int T(\gamma^m) \wedge \gamma_0 = \frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \operatorname{Re} \sigma_a \left\langle \sigma_a \left(-\frac{1}{c} (\dot{\mathbf{A}}_{\mathbf{k}})^* + i\mathbf{k}\phi_{\mathbf{k}}^* - i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}}^* \right) \sigma_m \left(\frac{1}{c} \dot{\mathbf{A}}_{\mathbf{k}} + i\mathbf{k}\phi_{\mathbf{k}} + i\mathbf{k} \wedge \mathbf{A}_{\mathbf{k}} \right) \right\rangle. \tag{103.14d}$$

103.3 EXPANDING THE ENERGY MOMENTUM TENSOR COMPONENTS

103.3.1 *Energy*

In Equation 103.14a only the bivector-bivector and vector-vector products produce any scalar grades. Except for the bivector product this can be done by inspection. For that part we utilize the identity

$$\langle (\mathbf{k} \wedge \mathbf{a})(\mathbf{k} \wedge \mathbf{b}) \rangle = (\mathbf{a} \cdot \mathbf{k})(\mathbf{b} \cdot \mathbf{k}) - \mathbf{k}^2 (\mathbf{a} \cdot \mathbf{b}). \tag{103.15}$$

This leaves for the energy $H = \int T(\gamma^0) \cdot \gamma_0$ in the volume

$$H = \frac{\epsilon_0 V}{2} \sum_{\mathbf{k}} \left(\frac{1}{c^2} \left| \dot{\mathbf{A}}_{\mathbf{k}} \right|^2 + \mathbf{k}^2 \left(|\phi_{\mathbf{k}}|^2 + |\mathbf{A}_{\mathbf{k}}|^2 \right) - |\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}}|^2 + \frac{2}{c} \operatorname{Re} \left(i\phi_{\mathbf{k}}^* \cdot \dot{\mathbf{A}}_{\mathbf{k}} \right) \right) \tag{103.16}$$

We are left with a completely real expression, and one without any explicit Geometric Algebra. This does not look like the Harmonic oscillator Hamiltonian that was expected. A gauge transformation to eliminate $\phi_{\mathbf{k}}$ and an observation about when $\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}}$ equals zero will give us that, but first lets get the mechanical jobs done, and reduce the products for the field momentum.

103.3.2 *Momentum*

Now move on to Equation 103.14b. For the factors other than σ_a only the vector-bivector products can contribute to the scalar product. We have two such products, one of the form

$$\begin{aligned}\sigma_a \langle \sigma_a \mathbf{a}(\mathbf{k} \wedge \mathbf{c}) \rangle &= \sigma_a (\mathbf{c} \cdot \sigma_a)(\mathbf{a} \cdot \mathbf{k}) - \sigma_a (\mathbf{k} \cdot \sigma_a)(\mathbf{a} \cdot \mathbf{c}) \\ &= \mathbf{c}(\mathbf{a} \cdot \mathbf{k}) - \mathbf{k}(\mathbf{a} \cdot \mathbf{c}),\end{aligned} \tag{103.17}$$

and the other

$$\begin{aligned}\sigma_a \langle \sigma_a (\mathbf{k} \wedge \mathbf{c}) \mathbf{a} \rangle &= \sigma_a (\mathbf{k} \cdot \sigma_a)(\mathbf{a} \cdot \mathbf{c}) - \sigma_a (\mathbf{c} \cdot \sigma_a)(\mathbf{a} \cdot \mathbf{k}) \\ &= \mathbf{k}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{k}).\end{aligned} \tag{103.18}$$

The momentum $\mathbf{P} = \int T(\gamma^0) \wedge \gamma_0$ in this volume follows by computation of

$$\begin{aligned}&\sigma_a \left\langle \sigma_a \left(-\frac{1}{c} (\dot{\mathbf{A}}_\mathbf{k})^* + i\mathbf{k}{\phi_\mathbf{k}}^* - i\mathbf{k} \wedge {\mathbf{A}_\mathbf{k}}^* \right) \left(\frac{1}{c} \dot{\mathbf{A}}_\mathbf{k} + i\mathbf{k}\phi_\mathbf{k} + i\mathbf{k} \wedge \mathbf{A}_\mathbf{k} \right) \right\rangle \\ &= i\mathbf{A}_\mathbf{k} \left(\left(-\frac{1}{c} (\dot{\mathbf{A}}_\mathbf{k})^* + i\mathbf{k}{\phi_\mathbf{k}}^* \right) \cdot \mathbf{k} \right) - i\mathbf{k} \left(\left(-\frac{1}{c} (\dot{\mathbf{A}}_\mathbf{k})^* + i\mathbf{k}{\phi_\mathbf{k}}^* \right) \cdot \mathbf{A}_\mathbf{k} \right) \\ &- i\mathbf{k} \left(\left(\frac{1}{c} \dot{\mathbf{A}}_\mathbf{k} + i\mathbf{k}\phi_\mathbf{k} \right) \cdot {\mathbf{A}_\mathbf{k}}^* \right) + i{\mathbf{A}_\mathbf{k}}^* \left(\left(\frac{1}{c} \dot{\mathbf{A}}_\mathbf{k} + i\mathbf{k}\phi_\mathbf{k} \right) \cdot \mathbf{k} \right)\end{aligned} \tag{103.19}$$

All the products are paired in nice conjugates, taking real parts, and premultiplication with $-\epsilon_0 V/2$ gives the desired result. Observe that two of these terms cancel, and another two have no real part. Those last are

$$-\frac{\epsilon_0 V \mathbf{k}}{2c} \operatorname{Re}\left(i({\dot{\mathbf{A}}_\mathbf{k}}^* \cdot \mathbf{A}_\mathbf{k} + \dot{\mathbf{A}}_\mathbf{k} \cdot {\mathbf{A}_\mathbf{k}}^*) \right) = -\frac{\epsilon_0 V \mathbf{k}}{2c} \operatorname{Re}\left(i \frac{d}{dt} \mathbf{A}_\mathbf{k} \cdot {\mathbf{A}_\mathbf{k}}^* \right) \tag{103.20}$$

Taking the real part of this pure imaginary $i|\mathbf{A}_\mathbf{k}|^2$ is zero, leaving just

$$\mathbf{P} = \epsilon_0 V \sum_\mathbf{k} \operatorname{Re}\left(i\mathbf{A}_\mathbf{k} \left(\frac{1}{c} \dot{\mathbf{A}}^*_\mathbf{k} \cdot \mathbf{k} \right) + \mathbf{k}^2 \phi_\mathbf{k} {\mathbf{A}_\mathbf{k}}^* - \mathbf{k}{\phi_\mathbf{k}}^* (\mathbf{k} \cdot \mathbf{A}_\mathbf{k}) \right) \tag{103.21}$$

I am not sure why exactly, but I actually expected a term with $|\mathbf{A}_\mathbf{k}|^2$, quadratic in the vector potential. Is there a mistake above?

103.3.3 *Gauge transformation to simplify the Hamiltonian*

In Equation 103.16 something that looked like the Harmonic oscillator was expected. On the surface this does not appear to be such a beast. Exploitation of gauge freedom is required to make the simplification that puts things into the Harmonic oscillator form.

If we are to change our four vector potential $A \rightarrow A + \nabla\psi$, then Maxwell's equation takes the form

$$J/\epsilon_0 c = \nabla(\nabla \wedge (A + \nabla\psi) = \nabla(\nabla \wedge A) + \nabla(\overbrace{\nabla \wedge \nabla\psi}^{=0}), \tag{103.22}$$

which is unchanged by the addition of the gradient to any original potential solution to the equation. In coordinates this is a transformation of the form

$$A^\mu \rightarrow A^\mu + \partial_\mu \psi, \tag{103.23}$$

and we can use this to force any one of the potential coordinates to zero. For this problem, it appears that it is desirable to seek a ψ such that $A^0 + \partial_0\psi = 0$. That is

$$\sum_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{1}{c}\partial_t \psi = 0. \tag{103.24}$$

Or,

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}, 0) - \frac{1}{c} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \int_{\tau=0}^{t} \phi_{\mathbf{k}}(\tau). \tag{103.25}$$

With such a transformation, the $\phi_{\mathbf{k}}$ and $\dot{\mathbf{A}}_{\mathbf{k}}$ cross term in the Hamiltonian Equation 103.16 vanishes, as does the $\phi_{\mathbf{k}}$ term in the four vector square of the last term, leaving just

$$H = \frac{\epsilon_0}{c^2} V \sum_{\mathbf{k}} \left(\frac{1}{2} \left|\dot{\mathbf{A}}_{\mathbf{k}}\right|^2 + \frac{1}{2} \left((c\mathbf{k})^2 |\mathbf{A}_{\mathbf{k}}|^2 + |(c\mathbf{k}) \cdot \mathbf{A}_{\mathbf{k}}|^2 + |c\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}}|^2 \right) \right). \tag{103.26}$$

Additionally, wedging Equation 103.5 with γ_0 now does not loose any information so our potential Fourier series is reduced to just

$$\begin{aligned} \mathbf{A} &= \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{2\pi i \mathbf{k}\cdot\mathbf{x}} \\ \mathbf{A}_{\mathbf{k}} &= \frac{1}{V} \int_0^{\lambda_1} \int_0^{\lambda_2} \int_0^{\lambda_3} \mathbf{A}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} dx^1 dx^2 dx^3. \end{aligned} \tag{103.27a}$$

The desired harmonic oscillator form would be had in Equation 103.26 if it were not for the $\mathbf{k} \cdot \mathbf{A}_\mathbf{k}$ term. Does that vanish? Returning to Maxwell's equation should answer that question, but first it has to be expressed in terms of the vector potential. While $\mathbf{A} = A \wedge \gamma_0$, the lack of an A^0 component means that this can be inverted as

$$A = \mathbf{A}\gamma_0 = -\gamma_0\mathbf{A}. \tag{103.28}$$

The gradient can also be factored scalar and spatial vector components

$$\nabla = \gamma^0(\partial_0 + \boldsymbol{\nabla}) = (\partial_0 - \boldsymbol{\nabla})\gamma^0. \tag{103.29}$$

So, with this $A^0 = 0$ gauge choice the bivector field F is

$$F = \nabla \wedge A = \frac{1}{2}\left(\overrightarrow{\nabla}A - A\overleftarrow{\nabla}\right) \tag{103.30}$$

From the left the gradient action on A is

$$\begin{aligned}\overrightarrow{\nabla}A &= (\partial_0 - \boldsymbol{\nabla})\gamma^0(-\gamma_0\mathbf{A}) \\ &= (-\partial_0 + \overrightarrow{\boldsymbol{\nabla}})\mathbf{A},\end{aligned} \tag{103.31}$$

and from the right

$$\begin{aligned}A\overleftarrow{\nabla} &= \mathbf{A}\gamma_0\gamma^0(\partial_0 + \boldsymbol{\nabla}) \\ &= \mathbf{A}(\partial_0 + \boldsymbol{\nabla}) \\ &= \partial_0\mathbf{A} + \mathbf{A}\overleftarrow{\boldsymbol{\nabla}}\end{aligned} \tag{103.32}$$

Taking the difference we have

$$F = \frac{1}{2}(-\partial_0\mathbf{A} + \overrightarrow{\boldsymbol{\nabla}}\mathbf{A} - \partial_0\mathbf{A} - \mathbf{A}\overleftarrow{\boldsymbol{\nabla}}). \tag{103.33}$$

Which is just

$$F = -\partial_0\mathbf{A} + \boldsymbol{\nabla} \wedge \mathbf{A}. \tag{103.34}$$

For this vacuum case, premultiplication of Maxwell's equation by γ_0 gives

$$
\begin{aligned}
0 &= \gamma_0 \nabla (-\partial_0 \mathbf{A} + \boldsymbol{\nabla} \wedge \mathbf{A}) \\
&= (\partial_0 + \boldsymbol{\nabla})(-\partial_0 \mathbf{A} + \boldsymbol{\nabla} \wedge \mathbf{A}) \\
&= -\frac{1}{c^2}\partial_{tt}\mathbf{A} - \partial_0 \boldsymbol{\nabla} \cdot \mathbf{A} - \partial_0 \boldsymbol{\nabla} \wedge \mathbf{A} + \partial_0 (\boldsymbol{\nabla} \wedge \mathbf{A}) + \overset{\boldsymbol{\nabla}^2\mathbf{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})}{\boxed{\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A})}} + \underset{= 0}{\boxed{\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{A})}}
\end{aligned}
\tag{103.35}
$$

The spatial bivector and trivector grades are all zero. Equating the remaining scalar and vector components to zero separately yields a pair of equations in $\mathbf{A}$

$$
\begin{aligned}
0 &= \partial_t (\boldsymbol{\nabla} \cdot \mathbf{A}) \\
0 &= -\frac{1}{c^2}\partial_{tt}\mathbf{A} + \boldsymbol{\nabla}^2 \mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})
\end{aligned}
\tag{103.36a}
$$

If the divergence of the vector potential is constant we have just a wave equation. Let us see what that divergence is with the assumed Fourier representation

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{A} &= \sum_{\mathbf{k} \neq (0,0,0)} {\mathbf{A}_\mathbf{k}}^m 2\pi i \frac{k_m}{\lambda_m} e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= i \sum_{\mathbf{k} \neq (0,0,0)} (\mathbf{A}_\mathbf{k} \cdot \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= i \sum_{\mathbf{k}} (\mathbf{A}_\mathbf{k} \cdot \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}
\tag{103.37}
$$

Since $\mathbf{A}_\mathbf{k} = \mathbf{A}_\mathbf{k}(t)$, there are two ways for $\partial_t(\boldsymbol{\nabla} \cdot \mathbf{A}) = 0$. For each $\mathbf{k}$ there must be a requirement for either $\mathbf{A}_\mathbf{k} \cdot \mathbf{k} = 0$ or $\mathbf{A}_\mathbf{k} = \text{constant}$. The constant $\mathbf{A}_\mathbf{k}$ solution to the first equation appears to represent a standing spatial wave with no time dependence. Is that of any interest?

The more interesting seeming case is where we have some non-static time varying state. In this case, if $\mathbf{A}_\mathbf{k} \cdot \mathbf{k}$, the second of these Maxwell's equations is just the vector potential wave equation, since the divergence is zero. That is

$$
0 = -\frac{1}{c^2}\partial_{tt}\mathbf{A} + \boldsymbol{\nabla}^2 \mathbf{A}
\tag{103.38}
$$

Solving this is not really what is of interest, since the objective was just to determine if the divergence could be assumed to be zero. This shows then, that if the transverse solution to Maxwell's equation is picked, the Hamiltonian for this field, with this gauge choice, becomes

$$H = \frac{\epsilon_0}{c^2} V \sum_{\mathbf{k}} \left(\frac{1}{2} \left| \dot{\mathbf{A}}_{\mathbf{k}} \right|^2 + \frac{1}{2} (c\mathbf{k})^2 |\mathbf{A}_{\mathbf{k}}|^2 \right). \tag{103.39}$$

How does the gauge choice alter the Poynting vector? From Equation 103.21, all the $\phi_{\mathbf{k}}$ dependence in that integrated momentum density is lost

$$\mathbf{P} = \epsilon_0 V \sum_{\mathbf{k}} \mathrm{Re} \left(i \mathbf{A}_{\mathbf{k}} \left(\frac{1}{c} \dot{\mathbf{A}}_{\mathbf{k}}^* \cdot \mathbf{k} \right) \right). \tag{103.40}$$

The $\mathbf{A}_{\mathbf{k}} \cdot \mathbf{k}$ solutions to Maxwell's equation are seen to result in zero momentum for this infinite periodic field. My expectation was something of the form $c\mathbf{P} = H\hat{\mathbf{k}}$, so intuition is either failing me, or my math is failing me, or this contrived periodic field solution leads to trouble.

What do we really know about the energy and momentum components of $T(\gamma^0)$? For vacuum, we have

$$\frac{1}{c} \frac{\partial T(\gamma^0) \cdot \gamma_0}{\partial t} + \boldsymbol{\nabla} \cdot \left(T(\gamma^0) \wedge \gamma_0 \right) = 0. \tag{103.41}$$

However, integration over the volume has been performed. That is different than integrating this four divergence. What we can say is

$$\frac{1}{c} \int d^3\mathbf{x} \frac{\partial T(\gamma^0) \cdot \gamma_0}{\partial t} + \int d^3\mathbf{x} \boldsymbol{\nabla} \cdot \left(T(\gamma^0) \wedge \gamma_0 \right) = 0. \tag{103.42}$$

It is not obvious that the integration and differentiation order can be switched in order to come up with an expression containing H and $\mathbf{P}$. This is perhaps where intuition is failing me.

103.4 CONCLUSIONS AND FOLLOWUP

The objective was met, a reproduction of Bohm's Harmonic oscillator result using a complex exponential Fourier series instead of separate sine and cosines.

The reason for Bohm's choice to fix zero divergence as the gauge choice upfront is now clear. That automatically cuts complexity from the results. Figuring out how to work this problem with complex valued potentials and also using the Geometric Algebra formulation probably also made the work a bit more difficult since blundering through both simultaneously was required instead of just one at a time.

This was an interesting exercise though, since doing it this way I am able to understand all the intermediate steps. Bohm employed some subtler argumentation to eliminate the scalar potential ϕ upfront, and I have to admit I did not follow his logic, whereas blindly following where the math leads me all makes sense.

As a bit of followup, I had like to consider the constant $\mathbf{A_k}$ case in more detail, and any implications of the freedom to pick $\mathbf{A}_0$.

The general calculation of $T^{\mu\nu}$ for the assumed Fourier solution should be possible too, but was not attempted. Doing that general calculation with a four dimensional Fourier series is likely tidier than working with scalar and spatial variables as done here.

Now that the math is out of the way (except possibly for the momentum which does not seem right), some discussion of implications and applications is also in order. My preference is to let the math sink-in a bit first and mull over the momentum issues at leisure.

104

FOURIER TRANSFORM SOLUTIONS AND ASSOCIATED ENERGY AND MOMENTUM FOR THE HOMOGENEOUS MAXWELL EQUATION

104.1 MOTIVATION AND NOTATION

In 103, building on 102 a derivation for the energy and momentum density was derived for an assumed Fourier series solution to the homogeneous Maxwell's equation. Here we move to the continuous case examining Fourier transform solutions and the associated energy and momentum density.

A complex (phasor) representation is implied, so taking real parts when all is said and done is required of the fields. For the energy momentum tensor the Geometric Algebra form, modified for complex fields, is used

$$T(a) = -\frac{\epsilon_0}{2} \mathrm{Re}\big(F^* a F\big). \tag{104.1}$$

The assumed four vector potential will be written

$$A(\mathbf{x}, t) = A^\mu(\mathbf{x}, t)\gamma_\mu = \frac{1}{(\sqrt{2\pi})^3} \int A(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.2}$$

Subject to the requirement that A is a solution of Maxwell's equation

$$\nabla(\nabla \wedge A) = 0. \tag{104.3}$$

To avoid latex hell, no special notation will be used for the Fourier coefficients,

$$A(\mathbf{k}, t) = \frac{1}{(\sqrt{2\pi})^3} \int A(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{x}. \tag{104.4}$$

When convenient and unambiguous, this $(\mathbf{k}, t)$ dependence will be implied.

Having picked a time and space representation for the field, it will be natural to express both the four potential and the gradient as scalar plus spatial vector, instead of using the Dirac basis. For the gradient this is

$$\nabla = \gamma^\mu \partial_\mu = (\partial_0 - \boldsymbol{\nabla})\gamma_0 = \gamma_0(\partial_0 + \boldsymbol{\nabla}), \tag{104.5}$$

and for the four potential (or the Fourier transform functions), this is

$$A = \gamma_\mu A^\mu = (\phi + \mathbf{A})\gamma_0 = \gamma_0(\phi - \mathbf{A}). \tag{104.6}$$

104.2 SETUP

The field bivector $F = \nabla \wedge A$ is required for the energy momentum tensor. This is

$$\begin{aligned}
\nabla \wedge A &= \frac{1}{2}\left(\overrightarrow{\nabla} A - A\overleftarrow{\nabla}\right) \\
&= \frac{1}{2}\left((\overrightarrow{\partial}_0 - \overrightarrow{\boldsymbol{\nabla}})\gamma_0\gamma_0(\phi - \mathbf{A}) - (\phi + \mathbf{A})\gamma_0\gamma_0(\overleftarrow{\partial}_0 + \overleftarrow{\boldsymbol{\nabla}})\right) \\
&= -\boldsymbol{\nabla}\phi - \partial_0\mathbf{A} + \frac{1}{2}(\overrightarrow{\boldsymbol{\nabla}}\mathbf{A} - \mathbf{A}\overleftarrow{\boldsymbol{\nabla}})
\end{aligned} \tag{104.7}$$

This last term is a spatial curl and the field is then

$$F = -\boldsymbol{\nabla}\phi - \partial_0\mathbf{A} + \boldsymbol{\nabla} \wedge \mathbf{A} \tag{104.8}$$

Applied to the Fourier representation this is

$$F = \frac{1}{(\sqrt{2\pi})^3} \int \left(-\frac{1}{c}\dot{\mathbf{A}} - i\mathbf{k}\phi + i\mathbf{k} \wedge \mathbf{A}\right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.9}$$

It is only the real parts of this that we are actually interested in, unless physical meaning can be assigned to the complete complex vector field.

104.3 CONSTRAINTS SUPPLIED BY MAXWELL'S EQUATION

A Fourier transform solution of Maxwell's vacuum equation $\nabla F = 0$ has been assumed. Having expressed the Faraday bivector in terms of spatial vector quantities, it is more convenient to do this back substitution into after pre-multiplying Maxwell's equation by γ_0, namely

$$\begin{aligned}
0 &= \gamma_0 \nabla F \\
&= (\partial_0 + \boldsymbol{\nabla})F.
\end{aligned} \tag{104.10}$$

Applied to the spatially decomposed field as specified in Equation 104.8, this is

$$\begin{aligned}
0 &= -\partial_0\boldsymbol{\nabla}\phi - \partial_{00}\mathbf{A} + \partial_0\boldsymbol{\nabla} \wedge \mathbf{A} - \boldsymbol{\nabla}^2\phi - \boldsymbol{\nabla}\partial_0\mathbf{A} + \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A}) \\
&= -\partial_0\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}^2\phi - \partial_{00}\mathbf{A} - \boldsymbol{\nabla} \cdot \partial_0\mathbf{A} + \boldsymbol{\nabla}^2\mathbf{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})
\end{aligned} \tag{104.11}$$

All grades of this equation must simultaneously equal zero, and the bivector grades have canceled (assuming commuting space and time partials), leaving two equations of constraint for the system

$$0 = \boldsymbol{\nabla}^2 \phi + \boldsymbol{\nabla} \cdot \partial_0 \mathbf{A} \tag{104.12a}$$

$$0 = \partial_{00} \mathbf{A} - \boldsymbol{\nabla}^2 \mathbf{A} + \boldsymbol{\nabla} \partial_0 \phi + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) \tag{104.12b}$$

It is immediately evident that a gauge transformation could be immediately helpful to simplify things. In [2] the gauge choice $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ is used. From Equation 104.12a this implies that $\boldsymbol{\nabla}^2 \phi = 0$. Bohm argues that for this current and charge free case this implies $\phi = 0$, but he also has a periodicity constraint. Without a periodicity constraint it is easy to manufacture non-zero counterexamples. One is a linear function in the space and time coordinates

$$\phi = px + qy + rz + st \tag{104.13}$$

This is a valid scalar potential provided that the wave equation for the vector potential is also a solution. We can however, force $\phi = 0$ by making the transformation $A^\mu \rightarrow A^\mu + \partial^\mu \psi$, which in non-covariant notation is

$$\begin{aligned} \phi &\rightarrow \phi + \frac{1}{c} \partial_t \psi \\ \mathbf{A} &\rightarrow \phi - \boldsymbol{\nabla} \psi \end{aligned} \tag{104.14}$$

If the transformed field $\phi' = \phi + \partial_t \psi / c$ can be forced to zero, then the complexity of the associated Maxwell equations are reduced. In particular, antidifferentiation of $\phi = -(1/c)\partial_t \psi$, yields

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}, 0) - c \int_{\tau=0}^{t} \phi(\mathbf{x}, \tau) d\tau. \tag{104.15}$$

Dropping primes, the transformed Maxwell equations now take the form

$$0 = \partial_t (\boldsymbol{\nabla} \cdot \mathbf{A}) \tag{104.16a}$$

$$0 = \partial_{00} \mathbf{A} - \boldsymbol{\nabla}^2 \mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}). \tag{104.16b}$$

There are two classes of solutions that stand out for these equations. If the vector potential is constant in time $\mathbf{A}(\mathbf{x}, t) = \mathbf{A}(\mathbf{x})$, Maxwell's equations are reduced to the single equation

$$0 = -\boldsymbol{\nabla}^2 \mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}). \tag{104.17}$$

Observe that a gradient can be factored out of this equation

$$\begin{aligned} -\boldsymbol{\nabla}^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) &= \boldsymbol{\nabla}(-\boldsymbol{\nabla}\mathbf{A} + \boldsymbol{\nabla}\cdot\mathbf{A}) \\ &= -\boldsymbol{\nabla}(\boldsymbol{\nabla}\wedge\mathbf{A}). \end{aligned} \tag{104.18}$$

The solutions are then those **A**s that satisfy both

$$0 = \partial_t\mathbf{A} \tag{104.19a}$$

$$0 = \boldsymbol{\nabla}(\boldsymbol{\nabla}\wedge\mathbf{A}). \tag{104.19b}$$

In particular any non-time dependent potential **A** with constant curl provides a solution to Maxwell's equations. There may be other solutions to Equation 104.17 too that are more general. Returning to Equation 104.16 a second way to satisfy these equations stands out. Instead of requiring of **A** constant curl, constant divergence with respect to the time partial eliminates Equation 104.16a. The simplest resulting equations are those for which the divergence is a constant in time and space (such as zero). The solution set are then spanned by the vectors **A** for which

$$\text{constant} = \boldsymbol{\nabla}\cdot\mathbf{A} \tag{104.20a}$$

$$0 = \frac{1}{c^2}\partial_{tt}\mathbf{A} - \boldsymbol{\nabla}^2\mathbf{A}. \tag{104.20b}$$

Any **A** that both has constant divergence and satisfies the wave equation will via Equation 104.8 then produce a solution to Maxwell's equation.

104.4 MAXWELL EQUATION CONSTRAINTS APPLIED TO THE ASSUMED FOURIER SOLUTIONS

Let us consider Maxwell's equations in all three forms, Equation 104.12, Equation 104.19a, and Equation 104.20 and apply these constraints to the assumed Fourier solution.

In all cases the starting point is a pair of Fourier transform relationships, where the Fourier transforms are the functions to be determined

$$\phi(\mathbf{x}, t) = (2\pi)^{-3/2}\int \phi(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \tag{104.21a}$$

$$\mathbf{A}(\mathbf{x}, t) = (2\pi)^{-3/2}\int \mathbf{A}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \tag{104.21b}$$

104.4.1 *Case I. Constant time vector potential. Scalar potential eliminated by gauge transformation*

From Equation 104.21a we require

$$0 = (2\pi)^{-3/2} \int \partial_t \mathbf{A}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.22}$$

So the Fourier transform also cannot have any time dependence, and we have

$$\mathbf{A}(\mathbf{x}, t) = (2\pi)^{-3/2} \int \mathbf{A}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \tag{104.23}$$

What is the curl of this? Temporarily falling back to coordinates is easiest for this calculation

$$\begin{aligned} \boldsymbol{\nabla} \wedge \mathbf{A}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} &= \sigma_m \partial_m \wedge \sigma_n A^n(\mathbf{k}) e^{i\mathbf{x}\cdot\mathbf{x}} \\ &= \sigma_m \wedge \sigma_n A^n(\mathbf{k}) i k^m e^{i\mathbf{x}\cdot\mathbf{x}} \\ &= i\mathbf{k} \wedge \mathbf{A}(\mathbf{k}) e^{i\mathbf{x}\cdot\mathbf{x}} \end{aligned} \tag{104.24}$$

This gives

$$\boldsymbol{\nabla} \wedge \mathbf{A}(\mathbf{x}, t) = (2\pi)^{-3/2} \int i\mathbf{k} \wedge \mathbf{A}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.25}$$

We want to equate the divergence of this to zero. Neglecting the integral and constant factor this requires

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot \left(i\mathbf{k} \wedge \mathbf{A} e^{i\mathbf{k}\cdot\mathbf{x}} \right) \\ &= \left\langle \sigma_m \partial_m i(\mathbf{k} \wedge \mathbf{A}) e^{i\mathbf{k}\cdot\mathbf{x}} \right\rangle_1 \\ &= -\left\langle \sigma_m (\mathbf{k} \wedge \mathbf{A}) k^m e^{i\mathbf{k}\cdot\mathbf{x}} \right\rangle_1 \\ &= -\mathbf{k} \cdot (\mathbf{k} \wedge \mathbf{A}) e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned} \tag{104.26}$$

Requiring that the plane spanned by $\mathbf{k}$ and $\mathbf{A}(\mathbf{k})$ be perpendicular to $\mathbf{k}$ implies that $\mathbf{A} \propto \mathbf{k}$. The solution set is then completely described by functions of the form

$$\mathbf{A}(\mathbf{x}, t) = (2\pi)^{-3/2} \int \mathbf{k} \psi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}, \tag{104.27}$$

where $\psi(\mathbf{k})$ is an arbitrary scalar valued function. This is however, an extremely uninteresting solution since the curl is uniformly zero

$$\begin{aligned} F &= \boldsymbol{\nabla} \wedge \mathbf{A} \\ &= (2\pi)^{-3/2} \int (i\mathbf{k}) \wedge \mathbf{k}\psi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \end{aligned} \tag{104.28}$$

Since $\mathbf{k} \wedge \mathbf{k} = 0$, when all is said and done the $\phi = 0$, $\partial_t \mathbf{A} = 0$ case appears to have no non-trivial (zero) solutions. Moving on, ...

104.4.2 *Case II. Constant vector potential divergence. Scalar potential eliminated by gauge transformation*

Next in the order of complexity is consideration of the case Equation 104.20. Here we also have $\phi = 0$, eliminated by gauge transformation, and are looking for solutions with the constraint

$$\begin{aligned} \text{constant} &= \boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x}, t) \\ &= (2\pi)^{-3/2} \int i\mathbf{k} \cdot \mathbf{A}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \end{aligned} \tag{104.29}$$

How can this constraint be enforced? The only obvious way is a requirement for $\mathbf{k} \cdot \mathbf{A}(\mathbf{k}, t)$ to be zero for all $(\mathbf{k}, t)$, meaning that our to be determined Fourier transform coefficients are required to be perpendicular to the wave number vector parameters at all times.

The remainder of Maxwell's equations, Equation 104.20b impose the addition constraint on the Fourier transform $\mathbf{A}(\mathbf{k}, t)$

$$0 = (2\pi)^{-3/2} \int \left(\frac{1}{c^2} \partial_{tt} \mathbf{A}(\mathbf{k}, t) - i^2 \mathbf{k}^2 \mathbf{A}(\mathbf{k}, t) \right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.30}$$

For zero equality for all $\mathbf{x}$ it appears that we require the Fourier transforms $\mathbf{A}(\mathbf{k})$ to be harmonic in time

$$\partial_{tt} \mathbf{A}(\mathbf{k}, t) = -c^2 \mathbf{k}^2 \mathbf{A}(\mathbf{k}, t). \tag{104.31}$$

This has the familiar exponential solutions

$$\mathbf{A}(\mathbf{k}, t) = \mathbf{A}_\pm(\mathbf{k}) e^{\pm ic|\mathbf{k}|t}, \tag{104.32}$$

also subject to a requirement that $\mathbf{k} \cdot \mathbf{A}(\mathbf{k}) = 0$. Our field, where the $\mathbf{A}_\pm(\mathbf{k})$ are to be determined by initial time conditions, is by Equation 104.8 of the form

$$\begin{aligned} F(\mathbf{x}, t) &= \operatorname{Re} \frac{i}{(\sqrt{2\pi})^3} \int \left(-|\mathbf{k}|\mathbf{A}_+(\mathbf{k}) + \mathbf{k} \wedge \mathbf{A}_+(\mathbf{k}) \right) \exp(i\mathbf{k} \cdot \mathbf{x} + ic|\mathbf{k}|t) d^3\mathbf{k} \\ &+ \operatorname{Re} \frac{i}{(\sqrt{2\pi})^3} \int \left(|\mathbf{k}|\mathbf{A}_-(\mathbf{k}) + \mathbf{k} \wedge \mathbf{A}_-(\mathbf{k}) \right) \exp(i\mathbf{k} \cdot \mathbf{x} - ic|\mathbf{k}|t) d^3\mathbf{k}. \end{aligned} \tag{104.33}$$

Since $0 = \mathbf{k} \cdot \mathbf{A}_\pm(\mathbf{k})$, we have $\mathbf{k} \wedge \mathbf{A}_\pm(\mathbf{k}) = \mathbf{k}\mathbf{A}_\pm$. This allows for factoring out of $|\mathbf{k}|$. The structure of the solution is not changed by incorporating the $i(2\pi)^{-3/2}|\mathbf{k}|$ factors into $\mathbf{A}_\pm$, leaving the field having the general form

$$\begin{aligned} &F(\mathbf{x}, t) \\ &= \operatorname{Re} \int (\hat{\mathbf{k}} - 1)\mathbf{A}_+(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x} + ic|\mathbf{k}|t) d^3\mathbf{k} + \operatorname{Re} \int (\hat{\mathbf{k}} + 1)\mathbf{A}_-(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x} - ic|\mathbf{k}|t) d^3\mathbf{k}. \end{aligned} \tag{104.34}$$

The original meaning of $\mathbf{A}_\pm$ as Fourier transforms of the vector potential is obscured by the tidy up change to absorb $|\mathbf{k}|$, but the geometry of the solution is clearer this way.

It is also particularly straightforward to confirm that $\gamma_0 \nabla F = 0$ separately for either half of Equation 104.34.

104.4.3 *Case III. Non-zero scalar potential. No gauge transformation*

Now lets work from Equation 104.12. In particular, a divergence operation can be factored from Equation 104.12a, for

$$0 = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\phi + \partial_0 \mathbf{A}). \tag{104.35}$$

Right off the top, there is a requirement for

$$\text{constant} = \boldsymbol{\nabla}\phi + \partial_0 \mathbf{A}. \tag{104.36}$$

In terms of the Fourier transforms this is

$$\text{constant} = \frac{1}{(\sqrt{2\pi})^3} \int \left(i\mathbf{k}\phi(\mathbf{k}, t) + \frac{1}{c} \partial_t \mathbf{A}(\mathbf{k}, t) \right) e^{i\mathbf{k} \cdot \mathbf{x}} d^3\mathbf{k}. \tag{104.37}$$

Are there any ways for this to equal a constant for all $\mathbf{x}$ without requiring that constant to be zero? Assuming no for now, and that this constant must be zero, this implies a coupling between the ϕ and $\mathbf{A}$ Fourier transforms of the form

$$\phi(\mathbf{k}, t) = -\frac{1}{ic\mathbf{k}}\partial_t \mathbf{A}(\mathbf{k}, t) \tag{104.38}$$

A secondary implication is that $\partial_t \mathbf{A}(\mathbf{k}, t) \propto \mathbf{k}$ or else $\phi(\mathbf{k}, t)$ is not a scalar. We had a transverse solution by requiring via gauge transformation that $\phi = 0$, and here we have instead the vector potential in the propagation direction.

A secondary confirmation that this is a required coupling between the scalar and vector potential can be had by evaluating the divergence equation of Equation 104.35

$$0 = \frac{1}{(\sqrt{2\pi})^3} \int \left(-\mathbf{k}^2 \phi(\mathbf{k}, t) + \frac{i\mathbf{k}}{c} \cdot \partial_t \mathbf{A}(\mathbf{k}, t)\right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.39}$$

Rearranging this also produces Equation 104.38. We want to now substitute this relationship into Equation 104.12b.

Starting with just the $\partial_0 \phi - \boldsymbol{\nabla} \cdot \mathbf{A}$ part we have

$$\partial_0 \phi + \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{(\sqrt{2\pi})^3} \int \left(\frac{i}{c^2\mathbf{k}} \partial_{tt} \mathbf{A}(\mathbf{k}, t) + i\mathbf{k} \cdot \mathbf{A}\right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.40}$$

Taking the gradient of this brings down a factor of $i\mathbf{k}$ for

$$\boldsymbol{\nabla}(\partial_0 \phi + \boldsymbol{\nabla} \cdot \mathbf{A}) = -\frac{1}{(\sqrt{2\pi})^3} \int \left(\frac{1}{c^2} \partial_{tt} \mathbf{A}(\mathbf{k}, t) + \mathbf{k}(\mathbf{k} \cdot \mathbf{A})\right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.41}$$

Equation 104.12b in its entirety is now

$$0 = \frac{1}{(\sqrt{2\pi})^3} \int \left(-(i\mathbf{k})^2 \mathbf{A} + \mathbf{k}(\mathbf{k} \cdot \mathbf{A})\right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.42}$$

This is not terribly pleasant looking. Perhaps going the other direction. We could write

$$\phi = \frac{i}{c\mathbf{k}} \frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c} \frac{\partial \psi}{\partial t}, \tag{104.43}$$

so that

$$\mathbf{A}(\mathbf{k}, t) = \mathbf{k}\psi(\mathbf{k}, t). \tag{104.44}$$

$$0 = \frac{1}{(\sqrt{2\pi})^3} \int (\frac{1}{c^2} \mathbf{k}\psi_{tt} - \boldsymbol{\nabla}^2 \mathbf{k}\psi + \boldsymbol{\nabla} \frac{i}{c^2} \psi_{tt} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\mathbf{k}\psi)))e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \tag{104.45}$$

Note that the gradients here operate on everything to the right, including and especially the exponential. Each application of the gradient brings down an additional $i\mathbf{k}$ factor, and we have

$$\frac{1}{(\sqrt{2\pi})^3} \int \mathbf{k}(\frac{1}{c^2} \psi_{tt} - i^2 \mathbf{k}^2 \psi + \frac{i^2}{c^2} \psi_{tt} + i^2 \mathbf{k}^2 \psi) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.46}$$

This is identically zero, so we see that this second equation provides no additional information. That is somewhat surprising since there is not a whole lot of constraints supplied by the first equation. The function $\psi(\mathbf{k}, t)$ can be anything. Understanding of this curiosity comes from computation of the Faraday bivector itself. From Equation 104.8, that is

$$F = \frac{1}{(\sqrt{2\pi})^3} \int (-i\mathbf{k} \frac{i}{c} \psi_t - \frac{1}{c} \mathbf{k}\psi_t + i\mathbf{k} \wedge \mathbf{k}\psi) e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k}. \tag{104.47}$$

All terms cancel, so we see that a non-zero ϕ leads to $F = 0$, as was the case when considering Equation 104.21a (a case that also resulted in $\mathbf{A}(\mathbf{k}) \propto \mathbf{k}$).

Can this Fourier representation lead to a non-transverse solution to Maxwell's equation? If so, it is not obvious how.

104.5 THE ENERGY MOMENTUM TENSOR

The energy momentum tensor is then

$$\begin{aligned} T(a) = -\frac{\epsilon_0}{2(2\pi)^3} \operatorname{Re} \iint &\left(-\frac{1}{c} \dot{\mathbf{A}}^*(\mathbf{k}', t) + i\mathbf{k}' \phi^*(\mathbf{k}', t) - i\mathbf{k}' \wedge \mathbf{A}^*(\mathbf{k}', t) \right) \\ &a \left(-\frac{1}{c} \dot{\mathbf{A}}(\mathbf{k}, t) - i\mathbf{k}\phi(\mathbf{k}, t) + i\mathbf{k} \wedge \mathbf{A}(\mathbf{k}, t) \right) e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{k} d^3\mathbf{k}'. \end{aligned} \tag{104.48}$$

Observing that γ_0 commutes with spatial bivectors and anticommutes with spatial vectors, and writing $\sigma_\mu = \gamma_\mu \gamma_0$, the tensor splits neatly into scalar and spatial vector components

$$
\begin{aligned}
T(\gamma_\mu) \cdot \gamma_0 &= \frac{\epsilon_0}{2(2\pi)^3} \operatorname{Re} \iint e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{k} d^3\mathbf{k}' \\
&\left\langle \left(\frac{1}{c}\dot{\mathbf{A}}^*(\mathbf{k}',t) - i\mathbf{k}'\phi^*(\mathbf{k}',t) + i\mathbf{k}' \wedge \mathbf{A}^*(\mathbf{k}',t) \right) \sigma_\mu \left(\frac{1}{c}\dot{\mathbf{A}}(\mathbf{k},t) + i\mathbf{k}\phi(\mathbf{k},t) + i\mathbf{k} \wedge \mathbf{A}(\mathbf{k},t) \right) \right\rangle \\
T(\gamma_\mu) \wedge \gamma_0 &= \frac{\epsilon_0}{2(2\pi)^3} \operatorname{Re} \iint e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{k} d^3\mathbf{k}' \\
&\left\langle \left(\frac{1}{c}\dot{\mathbf{A}}^*(\mathbf{k}',t) - i\mathbf{k}'\phi^*(\mathbf{k}',t) + i\mathbf{k}' \wedge \mathbf{A}^*(\mathbf{k}',t) \right) \sigma_\mu \left(\frac{1}{c}\dot{\mathbf{A}}(\mathbf{k},t) + i\mathbf{k}\phi(\mathbf{k},t) + i\mathbf{k} \wedge \mathbf{A}(\mathbf{k},t) \right) \right\rangle_1 .
\end{aligned}
\tag{104.49}
$$

In particular for $\mu = 0$, we have

$$
\begin{aligned}
H \equiv T(\gamma_0) \cdot \gamma_0 &= \frac{\epsilon_0}{2(2\pi)^3} \operatorname{Re} \iint e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{k} d^3\mathbf{k}' \\
&\left(\left(\frac{1}{c}\dot{\mathbf{A}}^*(\mathbf{k}',t) - i\mathbf{k}'\phi^*(\mathbf{k}',t) \right) \cdot \left(\frac{1}{c}\dot{\mathbf{A}}(\mathbf{k},t) + i\mathbf{k}\phi(\mathbf{k},t) \right) - (\mathbf{k}' \wedge \mathbf{A}^*(\mathbf{k}',t)) \cdot (\mathbf{k} \wedge \mathbf{A}(\mathbf{k},t)) \right) \\
\mathbf{P} \equiv T(\gamma_\mu) \wedge \gamma_0 &= \frac{\epsilon_0}{2(2\pi)^3} \operatorname{Re} \iint e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} d^3\mathbf{k} d^3\mathbf{k}' \\
&\left(i \left(\frac{1}{c}\dot{\mathbf{A}}^*(\mathbf{k}',t) - i\mathbf{k}'\phi^*(\mathbf{k}',t) \right) \cdot (\mathbf{k} \wedge \mathbf{A}(\mathbf{k},t)) - i \left(\frac{1}{c}\dot{\mathbf{A}}(\mathbf{k},t) + i\mathbf{k}\phi(\mathbf{k},t) \right) \cdot (\mathbf{k}' \wedge \mathbf{A}^*(\mathbf{k}',t)) \right).
\end{aligned}
\tag{104.50}
$$

Integrating this over all space and identification of the delta function

$$
\delta(\mathbf{k}) \equiv \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{x},
\tag{104.51}
$$

reduces the tensor to a single integral in the continuous angular wave number space of $\mathbf{k}$.

$$
\int T(a) d^3\mathbf{x} = -\frac{\epsilon_0}{2} \operatorname{Re} \int \left(-\frac{1}{c}\dot{\mathbf{A}}^* + i\mathbf{k}\phi^* - i\mathbf{k} \wedge \mathbf{A}^* \right) a \left(-\frac{1}{c}\dot{\mathbf{A}} - i\mathbf{k}\phi + i\mathbf{k} \wedge \mathbf{A} \right) d^3\mathbf{k}.
\tag{104.52}
$$

Or,

$$
\int T(\gamma_\mu)\gamma_0 d^3\mathbf{x} = \frac{\epsilon_0}{2} \operatorname{Re} \int \left\langle \left(\frac{1}{c}\dot{\mathbf{A}}^* - i\mathbf{k}\phi^* + i\mathbf{k} \wedge \mathbf{A}^* \right) \sigma_\mu \left(\frac{1}{c}\dot{\mathbf{A}} + i\mathbf{k}\phi + i\mathbf{k} \wedge \mathbf{A} \right) \right\rangle_{0,1} d^3\mathbf{k}.
\tag{104.53}
$$

Multiplying out Equation 104.53 yields for $\int H$

$$\int H d^3\mathbf{x} = \frac{\epsilon_0}{2} \int d^3\mathbf{k} \left(\frac{1}{c^2} \left| \dot{\mathbf{A}} \right|^2 + \mathbf{k}^2 (|\phi|^2 + |\mathbf{A}|^2) - |\mathbf{k} \cdot \mathbf{A}|^2 + 2\frac{\mathbf{k}}{c} \cdot \mathrm{Re}(i\phi^* \dot{\mathbf{A}}) \right) \tag{104.54}$$

Recall that the only non-trivial solution we found for the assumed Fourier transform representation of F was for $\phi = 0$, $\mathbf{k} \cdot \mathbf{A}(\mathbf{k}, t) = 0$. Thus we have for the energy density integrated over all space, just

$$\int H d^3\mathbf{x} = \frac{\epsilon_0}{2} \int d^3\mathbf{k} \left(\frac{1}{c^2} \left| \dot{\mathbf{A}} \right|^2 + \mathbf{k}^2 |\mathbf{A}|^2 \right). \tag{104.55}$$

Observe that we have the structure of a Harmonic oscillator for the energy of the radiation system. What is the canonical momentum for this system? Will it correspond to the Poynting vector, integrated over all space?

Let us reduce the vector component of Equation 104.53, after first imposing the $\phi = 0$, and $\mathbf{k} \cdot \mathbf{A} = 0$ conditions used to above for our harmonic oscillator form energy relationship. This is

$$\begin{aligned}
\int \mathbf{P} d^3\mathbf{x} &= \frac{\epsilon_0}{2c} \mathrm{Re} \int d^3\mathbf{k} \left(i\mathbf{A}_t^* \cdot (\mathbf{k} \wedge \mathbf{A}) + i(\mathbf{k} \wedge \mathbf{A}^*) \cdot \mathbf{A}_t \right) \\
&= \frac{\epsilon_0}{2c} \mathrm{Re} \int d^3\mathbf{k} \left(-i(\mathbf{A}_t^* \cdot \mathbf{A})\mathbf{k} + i\mathbf{k}(\mathbf{A}^* \cdot \mathbf{A}_t) \right)
\end{aligned} \tag{104.56}$$

This is just

$$\int \mathbf{P} d^3\mathbf{x} = \frac{\epsilon_0}{c} \mathrm{Re}\, i \int \mathbf{k}(\mathbf{A}^* \cdot \mathbf{A}_t) d^3\mathbf{k}. \tag{104.57}$$

Recall that the Fourier transforms for the transverse propagation case had the form $\mathbf{A}(\mathbf{k}, t) = \mathbf{A}_\pm(\mathbf{k}) e^{\pm ic|\mathbf{k}|t}$, where the minus generated the advanced wave, and the plus the receding wave. With substitution of the vector potential for the advanced wave into the energy and momentum results of Equation 104.55 and Equation 104.57 respectively, we have

$$\begin{aligned}
\int H d^3\mathbf{x} &= \epsilon_0 \int \mathbf{k}^2 |\mathbf{A}(\mathbf{k})|^2 d^3\mathbf{k} \\
\int \mathbf{P} d^3\mathbf{x} &= \epsilon_0 \int \hat{\mathbf{k}} \mathbf{k}^2 |\mathbf{A}(\mathbf{k})|^2 d^3\mathbf{k}.
\end{aligned} \tag{104.58}$$

After a somewhat circuitous route, this has the relativistic symmetry that is expected. In particular the for the complete $\mu = 0$ tensor we have after integration over all space

$$\int T(\gamma_0)\gamma_0 d^3\mathbf{x} = \epsilon_0 \int (1 + \hat{\mathbf{k}}) \mathbf{k}^2 |\mathbf{A}(\mathbf{k})|^2 d^3\mathbf{k}. \tag{104.59}$$

The receding wave solution would give the same result, but directed as $1 - \hat{\mathbf{k}}$ instead.

Observe that we also have the four divergence conservation statement that is expected

$$\frac{\partial}{\partial t} \int H d^3\mathbf{x} + \boldsymbol{\nabla} \cdot \int c\mathbf{P} d^3\mathbf{x} = 0. \tag{104.60}$$

This follows trivially since both the derivatives are zero. If the integration region was to be more specific instead of a $0 + 0 = 0$ relationship, we would have the power flux $\partial H / \partial t$ equal in magnitude to the momentum change through a bounding surface. For a more general surface the time and spatial dependencies should not necessarily vanish, but we should still have this radiation energy momentum conservation.

Part X

QUANTUM MECHANICS

105

BOHR MODEL

105.1 MOTIVATION

The Bohr model is taught as early as high school chemistry when the various orbitals are discussed (or maybe it was high school physics). I recall that the first time I saw this I did not see where all the ideas came from. With a bit more math under my belt now, reexamine these ideas as a lead up to the proper wave mechanics.

105.2 CALCULATIONS

105.2.1 *Equations of motion*

A prerequisite to discussing electron orbits is first setting up the equations of motion for the two charged particles (ie: the proton and electron).

With the proton position at $\mathbf{r}_p$, and the electron at $\mathbf{r}_e$, we have two equations, one for the force on the proton from the electron and the other for the force on the proton from the electron. These are respectively

$$\begin{aligned} \frac{1}{4\pi\epsilon_0} e^2 \frac{\mathbf{r}_e - \mathbf{r}_p}{\left|\mathbf{r}_e - \mathbf{r}_p\right|^3} &= m_p \frac{d^2\mathbf{r}_p}{dt^2} \\ -\frac{1}{4\pi\epsilon_0} e^2 \frac{\mathbf{r}_e - \mathbf{r}_p}{\left|\mathbf{r}_e - \mathbf{r}_p\right|^3} &= m_e \frac{d^2\mathbf{r}_e}{dt^2} \end{aligned} \tag{105.1}$$

In lieu of a picture, setting $\mathbf{r}_p = 0$ works to check signs, leaving an inwards force on the electron as desired.

As usual for a two body problem, use of the difference vector and center of mass vector is desirable. That is

$$\begin{aligned} \mathbf{x} &= \mathbf{r}_e - \mathbf{r}_p \\ M &= m_e + m_p \\ \mathbf{R} &= \frac{1}{M}(m_e\mathbf{r}_e + m_p\mathbf{r}_p) \end{aligned} \tag{105.2}$$

Solving for $\mathbf{r}_p$ and $\mathbf{r}_e$ in terms of $\mathbf{R}$ and $\mathbf{x}$ we have

$$\begin{aligned}\mathbf{r}_e &= \frac{m_p}{M}\mathbf{x} + \mathbf{R} \\ \mathbf{r}_p &= \frac{-m_e}{M}\mathbf{x} + \mathbf{R}\end{aligned} \tag{105.3}$$

Substitution back into eq. (105.1) we have

$$\begin{aligned}\frac{1}{4\pi\epsilon_0}e^2\frac{\mathbf{x}}{|\mathbf{x}|^3} &= m_p\frac{d^2}{dt^2}\left(\frac{-m_e}{M}\mathbf{x} + \mathbf{R}\right) \\ -\frac{1}{4\pi\epsilon_0}e^2\frac{\mathbf{x}}{|\mathbf{x}|^3} &= m_e\frac{d^2}{dt^2}\left(\frac{m_p}{M}\mathbf{x} + \mathbf{R}\right),\end{aligned} \tag{105.4}$$

and sums and (scaled) differences of that give us our reduced mass equation and constant center-of-mass velocity equation

$$\begin{aligned}\frac{d^2\mathbf{x}}{dt^2} &= -\frac{1}{4\pi\epsilon_0}e^2\frac{\mathbf{x}}{|\mathbf{x}|^3}\left(\frac{1}{m_e} + \frac{1}{m_p}\right) \\ \frac{d^2\mathbf{R}}{dt^2} &= 0\end{aligned} \tag{105.5}$$

writing $1/\mu = 1/m_e + 1/m_p$, and $k = e^2/4\pi\epsilon_0$, our difference vector equation is thus

$$\mu\frac{d^2\mathbf{x}}{dt^2} = -k\frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{105.6}$$

105.2.2 *Circular solution*

The Bohr model postulates that electron orbits are circular. It is easy enough to verify that a circular orbit in the center of mass frame is a solution to equation eq. (105.6). Write the path in terms of the unit bivector for the plane of rotation i and an initial vector position $\mathbf{x}_0$

$$\mathbf{x} = \mathbf{x}_0 e^{i\omega t} \tag{105.7}$$

For constant i and ω, we have

$$\mu\mathbf{x}_0(i\omega)^2 e^{i\omega t} = -k\frac{\mathbf{x}_0}{|\mathbf{x}_0|^3}e^{i\omega t} \tag{105.8}$$

This provides the angular velocity in terms of the reduced mass of the system and the charge constants

$$\omega^2 = \frac{k}{\mu|\mathbf{x}_0|^3} = \frac{e^2}{4\pi\epsilon_0\mu|\mathbf{x}_0|^3}. \tag{105.9}$$

Although not relevant to the quantum theme, it is hard not to call out the observation that this is a Kepler's law like relation for the period of the circular orbit given the radial distance from the center of mass

$$T^2 = \frac{16\pi^3\epsilon_0\mu}{e^2}|\mathbf{x}_0|^3 \tag{105.10}$$

Kepler's law also holds for elliptical orbits, but this takes more work to show.

105.2.3 *Angular momentum conservation*

Now, the next step in the Bohr argument was that the angular momentum, a conserved quantity is also quantized. To give real meaning to the conservation statement we need the equivalent Lagrangian formulation of eq. (105.6). Anti-differentiation gives

$$\begin{aligned}
\nabla_{\mathbf{v}}\left(\frac{1}{2}\mu\mathbf{v}^2\right) &= k\hat{\mathbf{x}}\partial_x\frac{1}{x} \\
&= -\nabla_{\mathbf{x}}\underbrace{\boxed{\left(-k\frac{1}{|\mathbf{x}|}\right)}}_{=\phi}
\end{aligned} \tag{105.11}$$

So, our Lagrangian is

$$\mathcal{L} = K - \phi = \frac{1}{2}\mu\mathbf{v}^2 + k\frac{1}{|\mathbf{x}|} \tag{105.12}$$

The essence of the conservation argument, an application of Noether's theorem, is that a rotational transformation of the Lagrangian leaves this energy relationship unchanged. Repeating the angular momentum example from [23] (which was done for the more general case of any radial potential), we write $\hat{B}$ for the unit bivector associated with a rotational plane. The position vector is transformed by rotation in this plane as follows

$$\begin{aligned}
\mathbf{x} &\rightarrow \mathbf{x}' \\
\mathbf{x}' &= R\mathbf{x}R^\dagger \\
R &= \exp\hat{B}\theta/2
\end{aligned} \tag{105.13}$$

The magnitude of the position vector is rotation invariant

$$(\mathbf{x}')^2 = R\mathbf{x}R^\dagger R\mathbf{x}R^\dagger = \mathbf{x}^2, \tag{105.14}$$

as is our the square of the transformed velocity. The transformed velocity is

$$\frac{d\mathbf{x}'}{dt} = \dot{R}\mathbf{x}R + R\dot{\mathbf{x}}R^\dagger + R\mathbf{x}\dot{R}^\dagger \tag{105.15}$$

but with $\dot{\theta} = 0$, $\dot{R} = 0$ its square is just

$$(\mathbf{v}')^2 = R\mathbf{v}R^\dagger R\dot{\mathbf{v}}R^\dagger = \mathbf{v}^2. \tag{105.16}$$

We therefore have a Lagrangian that is invariant under this rotational transformation

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L}, \tag{105.17}$$

and by Noether's theorem (essentially application of the chain rule), we have

$$\begin{aligned}\frac{d\mathcal{L}'}{d\theta} &= \frac{d}{dt}\left(\frac{d\mathbf{x}'}{d\theta} \cdot \nabla_{\mathbf{v}'}\mathcal{L}\right) \\ &= \frac{d}{dt}\left((\hat{B} \cdot \mathbf{x}') \cdot \mu\mathbf{v}'\right).\end{aligned} \tag{105.18}$$

But $d\mathcal{L}'/d\theta = 0$, so we have for any $\hat{B}$

$$(\hat{B} \cdot \mathbf{x}') \cdot (\mu\mathbf{v}') = \hat{B} \cdot (\mathbf{x}' \wedge (\mu\mathbf{v}')) = \text{constant} \tag{105.19}$$

Dropping primes this is

$$L = \mathbf{x} \wedge (\mu\mathbf{v}) = \text{constant}, \tag{105.20}$$

a constant bivector for the conserved center of mass (reduced-mass) angular momentum associated with the Lagrangian of this system.

105.2.4 *Quantized angular momentum for circular solution*

In terms of the circular solution of eq. (105.7) the angular momentum bivector is

$$
\begin{aligned}
L = \mathbf{x} \wedge (\mu \mathbf{v}) &= \left\langle \mathbf{x}_0 e^{i\omega t} \mu \mathbf{x}_0 i \omega e^{i\omega t} \right\rangle_2 \\
&= \left\langle e^{-i\omega t} \mathbf{x}_0 \mu \mathbf{x}_0 \omega e^{i\omega t} i \right\rangle_2 \\
&= (\mathbf{x}_0)^2 \mu \omega i \\
&= ie \sqrt{\frac{\mu |\mathbf{x}_0|}{4\pi\epsilon_0}}
\end{aligned}
\tag{105.21}
$$

Now if this angular momentum is quantized with quantum magnitude l we have we have for the bivector angular momentum the values

$$
L = inl = ie \sqrt{\frac{\mu |\mathbf{x}_0|}{4\pi\epsilon_0}}
\tag{105.22}
$$

Which with $l = \hbar$ (where experiment in the form of the spectral hydrogen line values is required to fix this constant and relate it to Plank's black body constant) is the momentum equation in terms of the Bohr radius $\mathbf{x}_0$ at each energy level. Writing that radius $r_n = |\mathbf{x}_0|$ explicitly as a function of n, we have

$$
r_n = \frac{4\pi\epsilon_0}{\mu} \left(\frac{n\hbar}{e} \right)^2
\tag{105.23}
$$

105.2.4.1 *Velocity*

One of the assumptions of this treatment is a $|\mathbf{v}_e| << c$ requirement so that Coulombs law is valid (ie: slow enough that all the other Maxwell's equations can be neglected). Let us evaluate the velocity numerically at the some of the quantization levels and see how this compares to the speed of light.

First we need an expression for the velocity itself. This is

$$
\begin{aligned}
\mathbf{v}^2 &= (\mathbf{x}_0 i \omega e^{i\omega t})^2 \\
&= \frac{e^2}{4\pi\epsilon_0 \mu r_n} \\
&= \frac{e^4}{(4\pi\epsilon_0)^2 (n\hbar)^2}.
\end{aligned}
\tag{105.24}
$$

For

$$\begin{aligned} v_n &= \frac{e^2}{4\pi\epsilon_0 n\hbar} \\ &= 2.1 \times 10^6 m/s \end{aligned} \tag{105.25}$$

This is the $1/137$ of the speed of light value that one sees googling electron speed in hydrogen, and only decreases with quantum number so the non-relativistic speed approximation holds ($\gamma = 1.00002663$). This speed is still pretty zippy, even if it is not relativistic, so it is not unreasonable to attempt to repeat this treatment trying to incorporate the remainder of Maxwell's equations.

Interestingly the velocity is not a function of the reduced mass at all, but just the charge and quantum numbers. One also gets a good hint at why the Bohr theory breaks down for larger atoms. An electron in circular orbit around an ion of Gold would have a velocity of $79/137$ the speed of light!

106

SCHRÖDINGER EQUATION PROBABILITY CONSERVATION

106.1 MOTIVATION

In [33] is a one dimensional probability conservation derivation from Schrödinger's equation. Do this for the three dimensional case.

106.2

Consider the time rate of change of the probability as expressed in terms of the wave function

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\partial \psi^* \psi}{\partial t} \\
&= \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}
\end{aligned}
\tag{106.1}
$$

This can be calculated from Schrödinger's equation and its complex conjugate

$$
\begin{aligned}
\partial_t \psi &= \left(-\frac{\hbar}{2mi}\boldsymbol{\nabla}^2 + \frac{1}{i\hbar}V\right)\psi \\
\partial_t \psi^* &= \left(\frac{\hbar}{2mi}\boldsymbol{\nabla}^2 - \frac{1}{i\hbar}V\right)\psi^*
\end{aligned}
\tag{106.2}
$$

Multiplying by the conjugate wave functions and adding we have

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \psi^*\left(-\frac{\hbar}{2mi}\boldsymbol{\nabla}^2 + \frac{1}{i\hbar}V\right)\psi + \psi\left(\frac{\hbar}{2mi}\boldsymbol{\nabla}^2 - \frac{1}{i\hbar}V\right)\psi^* \\
&= \frac{\hbar}{2mi}\left(-\psi^*\boldsymbol{\nabla}^2\psi + \psi\boldsymbol{\nabla}^2\psi^*\right)
\end{aligned}
\tag{106.3}
$$

So we have the following conservation law

$$
\frac{\partial \rho}{\partial t} + \frac{\hbar}{2mi}\left(\psi^*\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\psi^*\right) = 0
\tag{106.4}
$$

The text indicates that the second order terms here can be written as a divergence. Somewhat loosely, by treating ψ as a scalar field one can show that this is the case

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*\right) &= \left\langle \boldsymbol{\nabla} \left(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*\right)\right\rangle \\
&= \left\langle (\boldsymbol{\nabla}\psi^*)(\boldsymbol{\nabla}\psi) - (\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\psi^*) + \psi^* \boldsymbol{\nabla}^2\psi - \psi \boldsymbol{\nabla}^2\psi^* \right\rangle \\
&= \left\langle 2(\boldsymbol{\nabla}\psi^*) \wedge (\boldsymbol{\nabla}\psi) + \psi^* \boldsymbol{\nabla}^2\psi - \psi \boldsymbol{\nabla}^2\psi^* \right\rangle \\
&= \psi^* \boldsymbol{\nabla}^2\psi - \psi \boldsymbol{\nabla}^2\psi^*
\end{aligned} \tag{106.5}$$

Assuming that this procedure is justified. Equation (106.4) therefore can be written in terms of a probability current very reminiscent of the current density vector of electrodynamics

$$\begin{aligned}
\mathbf{J} &= \frac{\hbar}{2mi} \left(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*\right) \\
0 &= \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}
\end{aligned} \tag{106.6}$$

Regarding justification, this should be revisited. It appears to give the right answer, despite the fact that ψ is a complex (mixed grade) object, which likely has some additional significance.

106.3

Now, having calculated the probability conservation eq. (106.6), it is interesting to note the similarity to the relativistic spacetime divergence from Maxwell's equation.

We can write

$$0 = \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = \nabla \cdot \left(c\rho\gamma_0 + \mathbf{J}\gamma_0\right) \tag{106.7}$$

and form something that has the appearance of a relativistic four vector, re-writing the conservation equation as

$$\begin{aligned}
J &= c\rho\gamma_0 + \mathbf{J}\gamma_0 \\
0 &= \nabla \cdot J
\end{aligned} \tag{106.8}$$

Expanding this four component vector shows an interesting form:

$$J = c\rho\gamma_0 + \frac{\hbar}{2mi} \left(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*\right)\gamma_0 \tag{106.9}$$

Now, if one assumes the wave function can be represented as a even grade object with the following complex structure

$$\psi = \alpha + \gamma^m \wedge \gamma^n \beta_{mn} \tag{106.10}$$

then γ_0 will commute with ψ. Noting that $\boldsymbol{\nabla}\gamma_0 = \sum_k \gamma_k \partial_k = -\gamma^k \partial_k$, we have

$$mJ = mc\psi^* \psi \gamma_0 + \frac{i\hbar}{2}\left(\psi^* \gamma^k \partial_k \psi - \psi \gamma^k \partial_k \psi^*\right) \tag{106.11}$$

Now, this is an interesting form. In particular compare this to the Dirac Lagrangian, as given in the wikipedia Dirac equation article.

$$L = mc\bar{\psi}\psi - \frac{i\hbar}{2}(\bar{\psi}\gamma^\mu(\partial_\mu \psi) - (\partial_\mu \bar{\psi})\gamma^\mu \psi) \tag{106.12}$$

Although the Schrödinger equation is a non-relativistic equation, it appears that the probability current, when we add the $\gamma^0 \partial_0$ term required to put this into a covariant form, is in fact the Lagrangian density for the Dirac equation (when scaled by mass).

I do not know enough yet about QM to see what exactly the implications of this are, but I suspect that there is something of some interesting significance to this particular observation.

106.4 ON THE GRADES OF THE QM COMPLEX NUMBERS

To get to eq. (106.4), no assumptions about the representation of the field variable ψ were required. However, to make the identification

$$\psi^* \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \psi^* = \boldsymbol{\nabla} \cdot \left(\psi^* \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \psi^*\right) \tag{106.13}$$

we need some knowledge or assumptions about the representation. The assumption made initially was that we could treat ψ as a scalar, but then we later see there is value trying to switch to the Dirac representation (which appears to be the logical way to relativistically extend the probability current).

For example, with a geometric algebra multivector representation we have many ways to construct complex quantities. Assuming a Euclidean basis we can construct a complex number we can factor out one of the basis vectors

$$\sigma_1 x_1 + \sigma_2 x_2 = \sigma_1 (x_1 + \sigma_1 \sigma_2 x_2) \tag{106.14}$$

However, this is not going to commute with vectors (ie: such as the gradient), unless that vector is perpendicular to the plane spanned by this vector. As an example

$$i = \sigma_1 \sigma_2 \tag{106.15}$$

$$\begin{aligned}
i\sigma_1 &= -\sigma_1 i \\
i\sigma_2 &= -\sigma_2 i \\
i\sigma_3 &= \sigma_3 i
\end{aligned} \tag{106.16}$$

What would work is a complex representation using the $\mathbb{R}^3$ pseudoscalar (aka the Dirac pseudoscalar).

$$\psi = \alpha + \sigma_1 \sigma_2 \sigma_3 \beta = \alpha + \gamma_0 \gamma_1 \gamma_2 \gamma_3 \beta \tag{106.17}$$

107

DIRAC LAGRANGIAN

107.1 DIRAC LAGRANGIAN WITH FEYNMAN SLASH NOTATION

Wikipedia's Dirac Lagrangian entry lists the Lagrangian as

$$\mathcal{L} = \bar{\psi}(i\hbar c \not{D} - mc^2)\psi \tag{107.1}$$

"where ψ is a Dirac spinor, $\bar{\psi} = \psi^\dagger \gamma^0$ is its Dirac adjoint, D is the gauge covariant derivative, and $\not{D}$ is Feynman slash notation|Feynman notation for $\gamma^\sigma D_\sigma$."

Let us decode this. First, what is D_σ?

From Gauge theory

$$D_\mu := \partial_\mu - ieA_\mu \tag{107.2}$$

where A_μ is the electromagnetic vector potential.

So, in four-vector notation we have

$$\begin{aligned} \not{D} &= \gamma^\mu \partial_\mu - ie\gamma^\mu A_\mu \\ &= \nabla - ieA \end{aligned} \tag{107.3}$$

So our Lagrangian written out in full is left as

$$\mathcal{L} = \psi^\dagger \gamma^0 (i\hbar c\nabla + \hbar ceA - mc^2)\psi \tag{107.4}$$

How about this $\gamma^0 i\nabla$ term? If we assume that $i = \gamma_0\gamma_1\gamma_2\gamma_3$ is the four space pseudoscalar, then this is

$$\begin{aligned} \gamma^0 i\nabla &= -i\gamma^0(\gamma^0\partial_0 + \gamma^i\partial_i) \\ &= -i(\partial_0 + \sigma_i\partial_i) \end{aligned} \tag{107.5}$$

So, operationally, we have the dual of a quaternion like gradient operator. If ψ is an even grade object, as I had guess can be implied by the term spinor, then there is some sense to requiring a gradient operation that has scalar and spacetime bivector components.

Let us write this

$$\gamma^0 \nabla = \partial_0 + \sigma_i \partial_i = \nabla_{0,2} \tag{107.6}$$

Now, how about the meaning of $\bar{\psi} = \psi^\dagger \gamma^0$? I initially assumed that $\psi^\dagger$ was the reverse operation. However, looking in the quantum treatment of [10] and their earlier relativity content, I see that they explicitly avoid dagger as a reverse in a relativistic context since it is used for "something-else" in a quantum context. It appears that their mapping from matrix algebra to Clifford algebra is

$$\psi^\dagger \equiv \gamma_0 \tilde{\psi} \gamma_0, \tag{107.7}$$

where tilde is used for the reverse operation.

This then implies that

$$\bar{\psi} = \psi^\dagger \gamma^0 = \gamma_0 \tilde{\psi} \tag{107.8}$$

We now have an expression of the Lagrangian in full in terms of geometric objects

$$\mathcal{L} = \gamma_0 \tilde{\psi} (i \hbar c \nabla + \hbar c e A - m c^2) \psi. \tag{107.9}$$

Assuming that this is now the correct geometric interpretation of the Lagrangian, why bother having that first γ_0 factor. It should not change the field equations (just as a constant factor should not make a difference). It seems more natural to instead write the Lagrangian as just

$$\mathcal{L} = \tilde{\psi} \left(i \nabla + e A - \frac{mc}{\hbar} \right) \psi, \tag{107.10}$$

where both the constant vector factor γ_0, the redundant common factor of c have been removed, and we divide throughout by $\hbar$ to tidy up a bit. Perhaps this tidy up should be omitted since it sacrifices the energy dimensionality of the original.

107.1.1 *Dirac adjoint field*

The reverse sandwich operation of $\gamma_0 \tilde{\psi} \gamma_0$ to produce the Dirac adjoint field from ψ can be recognized as very similar to the mechanism used to split the Faraday bivector for the electromagnetic field into electric and magnetic terms. There addition and subtraction of the sandwich'ed fields with the original acted as a spacetime split operation, producing separate electric field spacetime (Pauli) bivectors and pure spatial bivectors (magnetic components) from the total field. Here we have a quaternion like field variable with scalar and bivector terms. Is there a physical (observables) significance only for a subset of the six possible bivectors that make up the spinor field? If so, then this adjoint operation can be used as a filter to select only the desired components.

Recall that the Faraday bivector is

$$\begin{aligned} F &= \mathbf{E} + ic\mathbf{B} \\ &= E^j\sigma_j + icB^j\sigma_j \\ &= E^j\gamma_j\gamma_0 + icB^j\gamma_j\gamma_0 \end{aligned} \tag{107.11}$$

So we have

$$\begin{aligned} \gamma_0 F\gamma_0 &= E^j\gamma_0\gamma_j + \gamma_0 icB^j\gamma_j \\ &= -E^j\sigma_j + icB^j\sigma_j \\ &= -\mathbf{E} + ic\mathbf{B} \end{aligned} \tag{107.12}$$

So we have

$$\begin{aligned} \frac{1}{2}\left(F - \gamma_0 F\gamma_0\right) &= \mathbf{E} \\ \frac{1}{2i}\left(F + \gamma_0 F\gamma_0\right) &= c\mathbf{B} \end{aligned} \tag{107.13}$$

How does this sandwich operation act on other grade objects?

- scalar

$$\gamma_0\alpha\gamma_0 = \alpha \tag{107.14}$$

- vector

$$\begin{aligned} \gamma_0\gamma_\mu\gamma_0 &= \left(2\gamma_0\cdot\gamma_\mu - \gamma_\mu\gamma_0\right)\gamma_0 \\ &= 2(\gamma_0\cdot\gamma_\mu)\gamma_0 - \gamma_\mu \\ &= \begin{cases} \gamma_0 & \text{if } \mu = 0 \\ -\gamma_i & \text{if } \mu = i \neq 0 \end{cases} \end{aligned} \tag{107.15}$$

- trivector

  For the duals of the vectors we have the opposite split, where for the dual of γ_0 we have a sign toggle

$$\gamma_0\gamma_i\gamma_j\gamma_k\gamma_0 = -\gamma_i\gamma_j\gamma_k \tag{107.16}$$

  whereas for the duals of γ_k we have invariant sign under sandwich

$$\gamma_0\gamma_i\gamma_j\gamma_0\gamma_0 = \gamma_i\gamma_j\gamma_0 \tag{107.17}$$

- pseudoscalar

$$\begin{aligned}\gamma_0 i\gamma_0 &= \gamma_0\gamma_0\gamma_1\gamma_2\gamma_3\gamma_0 \\ &= -i\end{aligned} \tag{107.18}$$

Ah ha! Recalling the conjugation results from 109, one sees that this sandwich operation is in fact just the equivalent of the conjugate operation on Dirac matrix algebra elements. So we can write

$$\psi^* \equiv \gamma_0\psi\gamma_0 \tag{107.19}$$

and can thus identify $\gamma_0\tilde{\psi}\gamma_0 = \psi^\dagger$ as the reverse of that conjugate quantity. That is

$$\psi^\dagger = (\psi^*)\tilde{} \tag{107.20}$$

This does not really help identify the significance of this term but this identification may prove useful later.

107.1.2 *Field equations*

Now, how to recover the field equation from eq. (107.10) ? If one assumes that the Euler-Lagrange field equations

$$\frac{\partial\mathcal{L}}{\partial\eta} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\eta)} = 0 \tag{107.21}$$

hold for these even grade field variables ψ, then treating ψ and $\bar{\psi}$ as separate field variables one has for the reversed field variable

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \tilde{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \tilde{\psi})} = 0 \\
\left(i\nabla + eA - \frac{mc}{\hbar}\right)\psi - (0) = 0
\end{aligned} \tag{107.22}$$

Or

$$\hbar(i\nabla + eA)\psi = mc\psi \tag{107.23}$$

Except for the additional eA term here, this is the Dirac equation that we get from taking square roots of the Klein-Gordon equation. Should A be considered a field variable? More likely is that this is a supplied potential as in the V of the non-relativistic Schrödinger's equation.

Being so loose with the math here (ie: taking partials with respect to non-scalar variables) is somewhat disturbing but developing some intuition is worthwhile before getting the details down.

107.1.3 *Conjugate field equation*

Our Lagrangian is not at all symmetric looking, having derivatives of ψ, but not $\bar{\psi}$. Compare this to the Lagrangians for the Schrödinger's equation, and Klein-Gordon equation respectively, which are

$$\begin{aligned}
\mathcal{L} &= \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\psi) \cdot (\boldsymbol{\nabla}\psi^*) + V\psi\psi^* + i\hbar\left(\psi\partial_t\psi^* - \psi^*\partial_t\psi\right) \\
\mathcal{L} &= -\partial^\nu\psi\partial_\nu\psi^* + \frac{m^2c^2}{\hbar^2}\psi\psi^*.
\end{aligned} \tag{107.24}$$

With these Lagrangians one gets the field equation for ψ, differentiating with respect to the conjugate field ψ^*, and the conjugate equation with differentiation with respect to ψ (where ψ and ψ^* are treated as independent field variables).

It is not obvious that evaluating the Euler-Lagrange equations will produce a similarly regular result, so let us compute the derivatives with respect to the ψ field variables to compute the equations for $\bar{\psi}$ or $\tilde{\psi}$ to see what results. Written out in coordinates so that we can apply the Euler-Lagrange equations, our Lagrangian (with A terms omitted) is

$$\mathcal{L} = \tilde{\psi}\left(i\gamma^\mu\partial_\mu + eA - \frac{mc}{\hbar}\right)\psi \tag{107.25}$$

Again abusing the Euler Lagrange equations, ignoring the possible issues with commuting partials taken with respect to spinors (not scalar variables), blinding plugging into the formulas we have

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \psi} &= \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \\ \tilde{\psi}\left(eA - \frac{mc}{\hbar}\right) &= \partial_\mu \left(\tilde{\psi} i \gamma^\mu\right)\end{aligned} \tag{107.26}$$

reversing this entire equation we have

$$\left(eA - \frac{mc}{\hbar}\right)\psi = \gamma^\mu i \partial_\mu \psi = -i\nabla\psi \tag{107.27}$$

Or

$$\hbar\left(i\nabla + eA\right)\psi = mc\psi \tag{107.28}$$

So we do in fact get the same field equation regardless of which of the two field variables one differentiates with. That is not obvious looking at the Lagrangian.

107.2 ALTERNATE DIRAC LAGRANGIAN WITH ANTISYMMETRIC TERMS

Now, the wikipedia article Adjoint equation and Dirac current lists the Lagrangian as

$$\mathcal{L} = mc\bar{\psi}\psi - \frac{1}{2} i\hbar(\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi) \tag{107.29}$$

Computing the Euler Lagrange equations for this potential free Lagrangian we have

$$mc\psi - \frac{1}{2} i\hbar\gamma^\mu\partial_\mu\psi = \partial_\mu\left(\frac{1}{2} i\hbar\gamma^\mu\psi\right) \tag{107.30}$$

Or,

$$mc\psi = i\hbar\nabla\psi \tag{107.31}$$

And the same computation, treating $\bar{\psi}$ as the independent field variable of interest we have

$$mc\psi + \frac{1}{2} i\hbar\partial_\mu\bar{\psi}\gamma^\mu = -\frac{1}{2} i\hbar\partial_\mu\bar{\psi}\gamma^\mu \tag{107.32}$$

which is

$$
\begin{aligned}
mc\bar{\psi} &= -i\hbar\partial_\mu\bar{\psi}\gamma^\mu \\
mc\gamma_0\tilde{\psi} &= -i\hbar\partial_\mu\gamma_0\tilde{\psi}\gamma^\mu \\
mc\tilde{\psi} &= i\hbar\partial_\mu\tilde{\psi}\gamma^\mu \\
mc\psi &= \hbar\nabla\psi i
\end{aligned}
\tag{107.33}
$$

Or,

$$
i\hbar\nabla\psi = -mc\psi \tag{107.34}
$$

FIXME: This differs in sign from the same calculation with the Lagrangian of eq. (107.25). Based on the possibility of both roots in the Klein-Gordon equation, I suspect I have made a sign error in the first calculation.

107.3 APPENDIX

107.3.1 *Pseudoscalar reversal*

The pseudoscalar reverses to itself

$$
\begin{aligned}
\tilde{i} &= \gamma_{3210} \\
&= -\gamma_{2103} \\
&= -\gamma_{1023} \\
&= \gamma_{0123} \\
&= i,
\end{aligned}
\tag{107.35}
$$

107.3.2 *Form of the spinor*

The specific structure of the spinor has not been defined here. It has been assumed to be quaternion like, and contain only even grades, but in the Dirac/Minkowski algebra that gives us two possibilities

$$
\begin{aligned}
\psi &= \alpha + P^a\gamma_a\gamma_0 \\
&= \alpha + P^a\sigma_a
\end{aligned}
\tag{107.36}
$$

Or

$$\begin{aligned}
\psi &= \alpha + P^c \gamma_a \wedge \gamma_b \\
&= \alpha - P^c \sigma_a \wedge \sigma_b \\
&= \alpha - i\epsilon_{abc} P^c \sigma_c
\end{aligned} \tag{107.37}$$

Spinors in Doran/Lasenby appear to use the latter form of dual Pauli vectors (wedge products of the Pauli spatial basis elements). This actually makes sense since one wants a spatial bivector for rotation (ie: "spin"), and not the spacetime bivectors, which provide a Lorentz boost action.

108

PAULI MATRICES

108.1 MOTIVATION

Having learned Geometric (Clifford) Algebra from [10], [19], [11], and other sources before studying any quantum mechanics, trying to work with (and talk to people familiar with) the Pauli and Dirac matrix notation as used in traditional quantum mechanics becomes difficult.

The aim of these notes is to work through equivalents to many Clifford algebra expressions entirely in commutator and anticommutator notations. This will show the mapping between the (generalized) dot product and the wedge product, and also show how the different grade elements of the Clifford algebra $C_{\{3,0\}}$ manifest in their matrix forms.

108.2 PAULI MATRICES

The matrices in question are:

$$
\begin{aligned}
\sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
\sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
\tag{108.1}
$$

These all have positive square as do the traditional Euclidean unit vectors $\mathbf{e}_i$, and so can be used algebraically as a vector basis for $\mathbb{R}^3$. So any vector that we can write in coordinates

$$\mathbf{x} = x^i \mathbf{e}_i, \tag{108.2}$$

we can equivalently write (an isomorphism) in terms of the Pauli matrix's

$$x = x^i \sigma_i. \tag{108.3}$$

108.2.1 *Pauli Vector*

[44] introduces the Pauli vector as a mechanism for mapping between a vector basis and this matrix basis

$$\boldsymbol{\sigma} = \sum \sigma_i \mathbf{e}_i \tag{108.4}$$

This is a curious looking construct with products of $2x2$ matrices and $\mathbb{R}^3$ vectors. Obviously these are not the usual $3x1$ column vector representations. This Pauli vector is thus really a notational construct. If one takes the dot product of a vector expressed using the standard orthonormal Euclidean basis $\{\mathbf{e}_i\}$ basis, and then takes the dot product with the Pauli matrix in a mechanical fashion

$$\begin{aligned}
\mathbf{x} \cdot \boldsymbol{\sigma} &= (x^i \mathbf{e}_i) \cdot \sum \sigma_j \mathbf{e}_j \\
&= \sum_{i,j} x^i \sigma_j \mathbf{e}_i \cdot \mathbf{e}_j \\
&= x^i \sigma_i
\end{aligned} \tag{108.5}$$

one arrives at the matrix representation of the vector in the Pauli basis $\{\sigma_i\}$. Does this construct have any value? That I do not know, but for the rest of these notes the coordinate representation as in equation eq. (108.3) will be used directly.

108.2.2 *Matrix squares*

It was stated that the Pauli matrices have unit square. Direct calculation of this is straightforward, and confirms the assertion

$$\begin{aligned}
{\sigma_1}^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
{\sigma_2}^2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
{\sigma_3}^2 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\end{aligned} \tag{108.6}$$

Note that unlike the vector (Clifford) square the identity matrix and not a scalar.

108.2.3 *Length*

If we are to operate with Pauli matrices how do we express our most basic vector operation, the length?

Examining a vector lying along one direction, say, $\mathbf{a} = \alpha\hat{\mathbf{x}}$ we expect

$$\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a} = \alpha^2\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \alpha^2. \tag{108.7}$$

Lets contrast this to the Pauli square for the same vector $y = \alpha\sigma_1$

$$y^2 = \alpha^2{\sigma_1}^2 = \alpha^2 I \tag{108.8}$$

The wiki article mentions trace, but no application for it. Since $\mathrm{Tr}\left(I\right) = 2$, an observable application is that the trace operator provides a mechanism to convert a diagonal matrix to a scalar. In particular for this scaled unit vector y we have

$$\alpha^2 = \frac{1}{2}\,\mathrm{Tr}\left(y^2\right) \tag{108.9}$$

It is plausible to guess that the squared length will be related to the matrix square in the general case as well

$$|x|^2 = \frac{1}{2}\,\mathrm{Tr}\left(x^2\right) \tag{108.10}$$

Let us see if this works by performing the coordinate expansion

$$\begin{aligned} x^2 &= (x^i\sigma_i)(x^j\sigma_j) \\ &= x^i x^j \sigma_i\sigma_j \end{aligned} \tag{108.11}$$

A split into equal and different indices thus leaves

$$x^2 = \sum_{i<j} x^i x^j(\sigma_i\sigma_j + \sigma_j\sigma_i) + \sum_i (x^i)^2{\sigma_i}^2 \tag{108.12}$$

As an algebra that is isomorphic to the Clifford Algebra $C_{\{3,0\}}$ it is expected that the $\sigma_i \sigma_j$ matrices anticommute for $i \neq j$. Multiplying these out verifies this

$$\begin{aligned}
\sigma_1\sigma_2 &= i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} &&= i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &&= i\sigma_3 \\
\sigma_2\sigma_1 &= i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &&= i\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} &&= -i\sigma_3 \\
\sigma_3\sigma_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &&= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &&= i\sigma_2 \\
\sigma_1\sigma_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} &&= -i\sigma_2 \\
\sigma_2\sigma_3 &= i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &&= i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &&= i\sigma_1 \\
\sigma_3\sigma_2 &= i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} &&= i\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} &&= -i\sigma_3
\end{aligned} . \tag{108.13}$$

Thus in eq. (108.12) the sum over the $\{i < j\} = \{12, 23, 13\}$ indices is zero.

Having computed this, our vector square leaves us with the vector length multiplied by the identity matrix

$$x^2 = \sum_i (x^i)^2 I. \tag{108.14}$$

Invoking the trace operator will therefore extract just the scalar length desired

$$|x|^2 = \frac{1}{2}\operatorname{Tr}\left(x^2\right) = \sum_i (x^i)^2. \tag{108.15}$$

108.2.3.1 *Aside: Summarizing the multiplication table*

It is worth pointing out that the multiplication table above used to confirm the antisymmetric behavior of the Pauli basis can be summarized as

$$\sigma_a \sigma_b = 2i\epsilon_{abc}\sigma_c \tag{108.16}$$

108.2.4 *Scalar product*

Having found the expression for the length of a vector in the Pauli basis, the next logical desirable identity is the dot product. One can guess that this will be the trace of a scaled symmetric product, but can motivate this without guessing in the usual fashion, by calculating the length of an orthonormal sum.

Consider first the length of a general vector sum. To calculate this we first wish to calculate the matrix square of this sum.

$$(x+y)^2 = x^2 + y^2 + xy + yx \tag{108.17}$$

If these vectors are perpendicular this equals $x^2 + y^2$. Thus orthonormality implies that

$$xy + yx = 0 \tag{108.18}$$

or,

$$yx = -xy \tag{108.19}$$

We have already observed this by direct calculation for the Pauli matrices themselves. Now, this is not any different than the usual description of perpendicularity in a Clifford Algebra, and it is notable that there are not any references to matrices in this argument. One only requires that a well defined vector product exists, where the squared vector has a length interpretation.

One matrix dependent observation that can be made is that since the left hand side and the x^2, and y^2 terms are all diagonal, this symmetric sum must also be diagonal. Additionally, for the length of this vector sum we then have

$$|x+y|^2 = |x|^2 + |y|^2 + \frac{1}{2}\operatorname{Tr}\left(xy + yx\right) \tag{108.20}$$

For correspondence with the Euclidean dot product of two vectors we must then have

$$x \bullet y = \frac{1}{4}\operatorname{Tr}\left(xy + yx\right). \tag{108.21}$$

Here $x \bullet y$ has been used to denote this scalar product (ie: a plain old number), since $x \cdot y$ will be used later for a matrix dot product (this times the identity matrix) which is more natural in many ways for this Pauli algebra.

Observe the symmetric product that is found embedded in this scalar selection operation. In physics this is known as the anticommutator, where the commutator is the antisymmetric sum. In the physics notation the anticommutator (symmetric sum) is

$$\{x, y\} = xy + yx \tag{108.22}$$

So this scalar selection can be written

$$x \bullet y = \frac{1}{4} \operatorname{Tr} \{x, y\} \tag{108.23}$$

Similarly, the commutator, an antisymmetric product, is denoted:

$$\left[x, y\right] = xy - yx, \tag{108.24}$$

A close relationship between this commutator and the wedge product of Clifford Algebra is expected.

108.2.5 *Symmetric and antisymmetric split*

As with the Clifford product, the symmetric and antisymmetric split of a vector product is a useful concept. This can be used to write the product of two Pauli basis vectors in terms of the anticommutator and commutator products

$$\begin{aligned} xy &= \frac{1}{2}\{x, y\} + \frac{1}{2}\left[x, y\right] \\ yx &= \frac{1}{2}\{x, y\} - \frac{1}{2}\left[x, y\right] \end{aligned} \tag{108.25}$$

These follows from the definition of the anticommutator eq. (108.22) and commutator eq. (108.24) products above, and are the equivalents of the Clifford symmetric and antisymmetric split into dot and wedge products

$$\begin{aligned} xy &= x \cdot y + x \wedge y \\ yx &= x \cdot y - x \wedge y \end{aligned} \tag{108.26}$$

Where the dot and wedge products are respectively

$$\begin{aligned} x \cdot y &= \frac{1}{2}(xy + yx) \\ x \wedge y &= \frac{1}{2}(xy - yx) \end{aligned} \tag{108.27}$$

Note the factor of two differences in the two algebraic notations. In particular very handy Clifford vector product reversal formula

$$yx = -xy + 2x \cdot y \tag{108.28}$$

has no factor of two in its Pauli anticommutator equivalent

$$yx = -xy + \{x, y\} \tag{108.29}$$

108.2.6 *Vector inverse*

It has been observed that the square of a vector is diagonal in this matrix representation, and can therefore be inverted for any non-zero vector

$$\begin{aligned}
x^2 &= |x|^2 I \\
(x^2)^{-1} &= |x|^{-2} I \\
&\Longrightarrow \\
x^2 (x^2)^{-1} &= I
\end{aligned} \tag{108.30}$$

So it is therefore quite justifiable to define

$$x^{-2} = \frac{1}{x^2} \equiv |x|^{-2} I \tag{108.31}$$

This allows for the construction of a dual sided vector inverse operation.

$$\begin{aligned}
x^{-1} &\equiv \frac{1}{|x|^2} x \\
&= \frac{1}{x^2} x \\
&= x \frac{1}{x^2}
\end{aligned} \tag{108.32}$$

This inverse is a scaled version of the vector itself.

The diagonality of the squared matrix or the inverse of that allows for commutation with x. This diagonality plays the same role as the scalar in a regular Clifford square. In either case the square can commute with the vector, and that commutation allows the inverse to have both left and right sided action.

Note that like the Clifford vector inverse when the vector is multiplied with this inverse, the product resides outside of the proper $\mathbb{R}^3$ Pauli basis since the identity matrix is required.

108.2.7 *Coordinate extraction*

Given a vector in the Pauli basis, we can extract the coordinates using the scalar product

$$x = \sum_i \frac{1}{4} \operatorname{Tr} \{x, \sigma_i\} \sigma_i \tag{108.33}$$

But do not need to convert to strict scalar form if we are multiplying by a Pauli matrix. So in anticommutator notation this takes the form

$$\begin{aligned} x &= x^i \sigma_i = \sum_i \frac{1}{2} \{x, \sigma_i\} \sigma_i \\ x^i &= \frac{1}{2} \{x, \sigma_i\} \end{aligned} \tag{108.34}$$

108.2.8 *Projection and rejection*

The usual Clifford algebra trick for projective and rejective split maps naturally to matrix form. Write

$$\begin{aligned} x &= x a a^{-1} \\ &= (xa) a^{-1} \\ &= \left(\frac{1}{2} \{x, a\} + \frac{1}{2} [x, a] \right) a^{-1} \\ &= \left(\frac{1}{2} (xa + ax) + \frac{1}{2} (xa - ax) \right) a^{-1} \\ &= \frac{1}{2} \left(x + a x a^{-1} \right) + \frac{1}{2} \left(x - a x a^{-1} \right) \end{aligned} \tag{108.35}$$

Since $\{x, a\}$ is diagonal, this first term is proportional to a^{-1}, and thus lines in the direction of a itself. The second term is perpendicular to a.

These are in fact the projection of x in the direction of a and rejection of x from the direction of a respectively.

$$\begin{aligned} x &= x_{\|} + x_{\perp} \\ x_{\|} &= \operatorname{Proj}_a(x) = \frac{1}{2} \{x, a\} a^{-1} = \frac{1}{2} \left(x + a x a^{-1} \right) \\ x_{\perp} &= \operatorname{Rej}_a(x) = \frac{1}{2} [x, a] a^{-1} = \frac{1}{2} \left(x - a x a^{-1} \right) \end{aligned} \tag{108.36}$$

To complete the verification of this note that the perpendicularity of the $x_\perp$ term can be verified by taking dot products

$$
\begin{aligned}
\frac{1}{2}\{a, x_\perp\} &= \frac{1}{4}\left(a\left(x - axa^{-1}\right) + \left(x - axa^{-1}\right)a\right) \\
&= \frac{1}{4}\left(ax - aaxa^{-1} + xa - axa^{-1}a\right) \\
&= \frac{1}{4}\left(ax - xa + xa - ax\right) \\
&= 0
\end{aligned}
\tag{108.37}
$$

108.2.9 *Space of the vector product*

Expansion of the anticommutator and commutator in coordinate form shows that these entities lie in a different space than the vectors itself.

For real coordinate vectors in the Pauli basis, all the commutator values are imaginary multiples and thus not representable

$$
\begin{aligned}
[x, y] &= x^a \sigma_a y^b \sigma_b - y^a \sigma_a x^b \sigma_b \\
&= (x^a y^b - y^a x^b)\sigma_a \sigma_b \\
&= 2i(x^a y^b - y^a x^b)\epsilon_{abc}\sigma_c
\end{aligned}
\tag{108.38}
$$

Similarly, the anticommutator is diagonal, which also falls outside the Pauli vector basis:

$$
\begin{aligned}
\{x, y\} &= x^a \sigma_a y^b \sigma_b + y^a \sigma_a x^b \sigma_b \\
&= (x^a y^b + y^a x^b)\sigma_a \sigma_b \\
&= (x^a y^b + y^a x^b)(I\delta_{ab} + i\epsilon_{abc}\sigma_c) \\
&= \sum_a (x^a y^a + y^a x^a) I + \sum_{a<b} (x^a y^b + y^a x^b) i(\overbrace{\boxed{\epsilon_{abc} + \epsilon_{bac}}}^{=0})\sigma_c \\
&= \sum_a (x^a y^a + y^a x^a) I \\
&= 2 \sum_a x^a y^a I,
\end{aligned}
\tag{108.39}
$$

These correspond to the Clifford dot product being scalar (grade zero), and the wedge defining a grade two space, where grade expresses the minimal degree that a product can be reduced to. By example a Clifford product of normal unit vectors such as

$$\begin{aligned}\mathbf{e}_1\mathbf{e}_3\mathbf{e}_4\mathbf{e}_1\mathbf{e}_3\mathbf{e}_4\mathbf{e}_3 &\propto \mathbf{e}_3 \\ \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_1\mathbf{e}_3\mathbf{e}_4\mathbf{e}_3\mathbf{e}_5 &\propto \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_5\end{aligned} \tag{108.40}$$

are grade one and four respectively. The proportionality constant will be dependent on metric of the underlying vector space and the number of permutations required to group terms in pairs of matching indices.

108.2.10 *Completely antisymmetrized product of three vectors*

In a Clifford algebra no imaginary number is required to express the antisymmetric (commutator) product. However, the bivector space can be enumerated using a dual basis defined by multiplication of the vector basis elements with the unit volume trivector. That is also the case here and gives a geometrical meaning to the imaginaries of the Pauli formulation.

How do we even write the unit volume element in Pauli notation? This would be

$$\begin{aligned}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 &= (\sigma_1 \wedge \sigma_2) \wedge \sigma_3 \\ &= \frac{1}{2}\left[\sigma_1, \sigma_2\right] \wedge \sigma_3 \\ &= \frac{1}{4}\left(\left[\sigma_1, \sigma_2\right]\sigma_3 + \sigma_3\left[\sigma_1, \sigma_2\right]\right)\end{aligned} \tag{108.41}$$

So we have

$$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \frac{1}{8}\left\{\left[\sigma_1, \sigma_2\right], \sigma_3\right\} \tag{108.42}$$

Similar expansion of $\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \sigma_1 \wedge (\sigma_2 \wedge \sigma_3)$, or $\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = (\sigma_3 \wedge \sigma_1) \wedge \sigma_2$ shows that we must also have

$$\left\{\left[\sigma_1, \sigma_2\right], \sigma_3\right\} = \left\{\sigma_1, \left[\sigma_2, \sigma_3\right]\right\} = \left\{\left[\sigma_3, \sigma_1\right], \sigma_2\right\} \tag{108.43}$$

Until now the differences in notation between the anticommutator/commutator and the dot/wedge product of the Pauli algebra and Clifford algebra respectively have only differed by factors of two, which is not much of a big deal. However, having to express the naturally associative wedge product operation in the non-associative looking notation of equation eq. (108.42) is rather unpleasant seeming. Looking at an expression of the form gives no mnemonic hint of the underlying associativity, and actually seems obfuscating. I suppose that one could get used to it though.

We expect to get a three by three determinant out of the trivector product. Let us verify this by expanding this in Pauli notation for three general coordinate vectors

$$
\begin{aligned}
\{[x, y], z\} &= \left\{\left[x^a \sigma_a, y^b \sigma_b\right], z^c \sigma_c\right\} \\
&= 2i\epsilon_{abd} x^a y^b z^c \{\sigma_d, \sigma_c\} \\
&= 4i\epsilon_{abd} x^a y^b z^c \delta_{cd} I \\
&= 4i\epsilon_{abc} x^a y^b z^c I \\
&= 4i \begin{vmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ z_a & z_b & z_c \end{vmatrix} I
\end{aligned}
\tag{108.44}
$$

In particular, our unit volume element is

$$
\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \frac{1}{4}\{[\sigma_1, \sigma_2], \sigma_3\} = iI \tag{108.45}
$$

So one sees that the complex number i in the Pauli algebra can logically be replaced by the unit pseudoscalar iI, and relations involving i, like the commutator expansion of a vector product, is restored to the expected dual form of Clifford algebra

$$
\begin{aligned}
\sigma_a \wedge \sigma_b &= \frac{1}{2}[\sigma_a, \sigma_b] \\
&= i\epsilon_{abc}\sigma_c \\
&= (\sigma_a \wedge \sigma_b \wedge \sigma_c)\sigma_c
\end{aligned}
\tag{108.46}
$$

Or

$$
\sigma_a \wedge \sigma_b = (\sigma_a \wedge \sigma_b \wedge \sigma_c) \cdot \sigma_c \tag{108.47}
$$

108.2.11 *Duality*

We have seen that multiplication by i is a duality operation, which is expected since iI is the matrix equivalent of the unit pseudoscalar. Logically this means that for a vector x, the product $(iI)x$ represents a plane quantity (torque, angular velocity/momentum, ...). Similarly if B is a plane object, then $(iI)B$ will have a vector interpretation.

In particular, for the antisymmetric (commutator) part of the vector product xy

$$\begin{aligned}\frac{1}{2}\left[x, y\right] &= \frac{1}{2} x^a y^b \left[\sigma_a, \sigma_b\right] \\ &= x^a y^b i\epsilon_{abc}\sigma_c\end{aligned} \tag{108.48}$$

a "vector" in the dual space spanned by $\{i\sigma_a\}$ is seen to be more naturally interpreted as a plane quantity (bivector in Clifford algebra).

As in Clifford algebra, we can write the cross product in terms of the antisymmetric product

$$a \times b = \frac{1}{2i}\left[a, b\right]. \tag{108.49}$$

With the factor of 2 in the denominator here (like the exponential form of sine), it is interesting to contrast this to the cross product in its trigonometric form

$$\begin{aligned}a \times b &= |a||b| \sin(\theta)\hat{\mathbf{n}} \\ &= |a||b|\frac{1}{2i}(e^{i\theta} - e^{-i\theta})\hat{\mathbf{n}}\end{aligned} \tag{108.50}$$

This shows we can make the curious identity

$$\left[\hat{a}, \hat{b}\right] = (e^{i\theta} - e^{-i\theta})\hat{\mathbf{n}} \tag{108.51}$$

If one did not already know about the dual sides half angle rotation formulation of Clifford algebra, this is a hint about how one could potentially work towards that. We have the commutator (or wedge product) as a rotation operator that leaves the normal component of a vector untouched (commutes with the normal vector).

108.2.12 *Complete algebraic space*

Pauli equivalents for all the elements in the Clifford algebra have now been determined.

- scalar

$$\alpha \rightarrow \alpha I \tag{108.52}$$

- vector

$$\begin{aligned}u^i\sigma_i &\rightarrow \begin{bmatrix} 0 & u^1 \\ u^1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -iu^2 \\ iu^2 & 0 \end{bmatrix} + \begin{bmatrix} u^3 & 0 \\ 0 & -u^3 \end{bmatrix} \\ &= \begin{bmatrix} u^3 & u^1 - iu^2 \\ u^1 + iu^2 & -u^3 \end{bmatrix}\end{aligned} \tag{108.53}$$

- bivector

$$\begin{aligned}\sigma_1\sigma_2\sigma_3 v^a\sigma_a &\rightarrow i v^a\sigma_a \\ &= \begin{bmatrix} iv^3 & iv^1 + v^2 \\ iv^1 - v^2 & -iv^3 \end{bmatrix}\end{aligned} \tag{108.54}$$

- pseudoscalar

$$\beta\sigma_1\sigma_2\sigma_3 \rightarrow i\beta I \tag{108.55}$$

Summing these we have the mapping from Clifford basis to Pauli matrix as follows

$$\alpha + \beta I + u^i\sigma_i + I v^a\sigma_a \rightarrow \begin{bmatrix} (\alpha + u^3) + i(\beta + v^3) & (u^1 + v^2) + i(-u^2 + v^1) \\ (u^1 - v^2) + i(u^2 - v^1) & (\alpha - u^3) + i(\beta - v^3) \end{bmatrix} \tag{108.56}$$

Thus for any given sum of scalar, vector, bivector, and trivector elements we can completely express this in Pauli form as a general $2x2$ complex matrix.

Provided that one can also extract the coordinates for each of the grades involved, this also provides a complete Clifford algebra characterization of an arbitrary complex $2x2$ matrix.

Computationally this has some nice looking advantages. Given any canned complex matrix software, one should be able to easily cook up with little work a working $\mathbb{R}^3$ Clifford calculator.

As for the coordinate extraction, part of the work can be done by taking real and imaginary components. Let an element of the general algebra be denoted

$$P = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \tag{108.57}$$

We therefore have

$$\begin{aligned}\Re(P) &= \begin{bmatrix} \alpha + u^3 & u^1 + v^2 \\ u^1 - v^2 & \alpha - u^3 \end{bmatrix} \\ \Im(P) &= \begin{bmatrix} \beta + v^3 & -u^2 + v^1 \\ u^2 + v^1 & \beta - v^3 \end{bmatrix}\end{aligned} \tag{108.58}$$

By inspection, symmetric and antisymmetric sums of the real and imaginary parts recovers the coordinates as follows

$$\begin{aligned}
\alpha &= \frac{1}{2}\Re(z_{11} + z_{22}) \\
u^3 &= \frac{1}{2}\Re(z_{11} - z_{22}) \\
u^1 &= \frac{1}{2}\Re(z_{12} + z_{21}) \\
v^2 &= \frac{1}{2}\Re(z_{12} - z_{21}) \\
\beta &= \frac{1}{2}\Im(z_{11} + z_{22}) \\
v^3 &= \frac{1}{2}\Im(z_{11} - z_{22}) \\
v^1 &= \frac{1}{2}\Im(z_{12} + z_{21}) \\
u^2 &= \frac{1}{2}\Im(-z_{12} + z_{21})
\end{aligned} \tag{108.59}$$

In terms of grade selection operations the decomposition by grade

$$P = \langle P \rangle + \langle P \rangle_1 + \langle P \rangle_2 + \langle P \rangle_3, \tag{108.60}$$

is

$$\begin{aligned}
\langle P \rangle &= \frac{1}{2}\Re(z_{11} + z_{22}) = \frac{1}{2}\Re(\mathrm{Tr}\, P) \\
\langle P \rangle_1 &= \frac{1}{2}\left(\Re(z_{12} + z_{21})\sigma_1 + \Im(-z_{12} + z_{21})\sigma_2 + \Re(z_{11} - z_{22})\sigma_3\right) \\
\langle P \rangle_2 &= \frac{1}{2}\left(\Im(z_{12} + z_{21})\sigma_2 \wedge \sigma_3 + \Re(z_{12} - z_{21})\sigma_3 \wedge \sigma_1 + \Im(z_{11} - z_{22})\sigma_1 \wedge \sigma_2\right) \\
\langle P \rangle_3 &= \frac{1}{2}\Im(z_{11} + z_{22})I = \frac{1}{2}\Im(\mathrm{Tr}\, P)\sigma_1 \wedge \sigma_2 \wedge \sigma_3
\end{aligned} \tag{108.61}$$

Employing $\Im(z) = \Re(-iz)$, and $\Re(z) = \Im(iz)$ this can be made slightly more symmetrical, with Real operations selecting the vector coordinates and imaginary operations selecting the bivector coordinates.

$$
\begin{aligned}
\langle P \rangle &= \frac{1}{2}\Re(z_{11} + z_{22}) = \frac{1}{2}\Re(\operatorname{Tr} P) \\
\langle P \rangle_1 &= \frac{1}{2}\left(\Re(z_{12} + z_{21})\sigma_1 + \Re(iz_{12} - iz_{21})\sigma_2 + \Re(z_{11} - z_{22})\sigma_3\right) \\
\langle P \rangle_2 &= \frac{1}{2}\left(\Im(z_{12} + z_{21})\sigma_2 \wedge \sigma_3 + \Im(iz_{12} - iz_{21})\sigma_3 \wedge \sigma_1 + \Im(z_{11} - z_{22})\sigma_1 \wedge \sigma_2\right) \\
\langle P \rangle_3 &= \frac{1}{2}\Im(z_{11} + z_{22})I = \frac{1}{2}\Im(\operatorname{Tr} P)\sigma_1 \wedge \sigma_2 \wedge \sigma_3
\end{aligned}
\tag{108.62}
$$

Finally, returning to the Pauli algebra, this also provides the following split of the Pauli multivector matrix into its geometrically significant components $P = \langle P \rangle + \langle P \rangle_1 + \langle P \rangle_2 + \langle P \rangle_3$,

$$
\begin{aligned}
\langle P \rangle &= \frac{1}{2}\Re(z_{11} + z_{22})I \\
\langle P \rangle_1 &= \frac{1}{2}\left(\Re(z_{12} + z_{21})\sigma_1 + \Re(iz_{12} - iz_{21})\sigma_2 + \Re(z_{11} - z_{22})\sigma_3\right) \\
\langle P \rangle_2 &= \frac{1}{2}\left(\Im(z_{12} + z_{21})i\sigma_1 + \Im(iz_{12} - iz_{21})i\sigma_2 + \Im(z_{11} - z_{22})i\sigma_k\right) \\
\langle P \rangle_3 &= \frac{1}{2}\Im(z_{11} + z_{22})iI
\end{aligned}
\tag{108.63}
$$

108.2.13 *Reverse operation*

The reversal operation switches the order of the product of perpendicular vectors. This will change the sign of grade two and three terms in the Pauli algebra. Since σ_2 is imaginary, conjugation does not have the desired effect, but Hermitian conjugation (conjugate transpose) does the trick.

Since the reverse operation can be written as Hermitian conjugation, one can also define the anticommutator and commutator in terms of reversion in a way that seems particularly natural for complex matrices. That is

$$
\begin{aligned}
\{a, b\} &= ab + (ab)^* \\
[a, b] &= ab - (ab)^*
\end{aligned}
\tag{108.64}
$$

108.2.14 *Rotations*

Rotations take the normal Clifford, dual sided quaterionic form. A rotation about a unit normal n will be

$$R(x) = e^{-in\theta/2} x e^{in\theta/2} \tag{108.65}$$

The Rotor $R = e^{-in\theta/2}$ commutes with any component of the vector x that is parallel to the normal (perpendicular to the plane), whereas it anticommutes with the components in the plane. Writing the vector components perpendicular and parallel to the plane respectively as $x = x_\perp + x_\|$, the essence of the rotation action is this selective commutation or anti-commutation behavior

$$\begin{aligned} R x_\| R^* &= x_\| R^* \\ R x_\perp R^* &= x_\perp R R^* = x_\perp \end{aligned} \tag{108.66}$$

Here the exponential has the obvious meaning in terms of exponential series, so for this bivector case we have

$$\exp(i\hat{\mathbf{n}}\theta/2) = \cos(\theta/2) I + i\hat{\mathbf{n}} \sin(\theta/2) \tag{108.67}$$

The unit bivector $B = i\hat{\mathbf{n}}$ can also be defined explicitly in terms of two vectors a, and b in the plane

$$B = \frac{1}{\left|[a, b]\right|} [a, b] \tag{108.68}$$

Where the bivector length is defined in terms of the conjugate square (bivector times bivector reverse)

$$\left|[a, b]\right|^2 = [a, b] [a, b]^* \tag{108.69}$$

Examples to complete this subsection would make sense. As one of the most powerful and useful operations in the algebra, it is a shame in terms of completeness to skimp on this. However, except for some minor differences like substitution of the Hermitian conjugate operation for reversal, the use of the identity matrix I in place of the scalar in the exponential expansion, the treatment is exactly the same as in the Clifford algebra.

108.2.15 *Grade selection*

Coordinate equations for grade selection were worked out above, but the observation that reversion and Hermitian conjugation are isomorphic operations can partially clean this up. In particular a Hermitian conjugate symmetrization and anti-symmetrization of the general matrix provides a nice split into quaternion and dual quaternion parts (say $P = Q + R$ respectively). That is

$$\begin{aligned} Q &= \langle P\rangle + \langle P\rangle_1 = \frac{1}{2}(P + P^*) \\ R &= \langle P\rangle_2 + \langle P\rangle_3 = \frac{1}{2}(P - P^*) \end{aligned} \tag{108.70}$$

Now, having done that, how to determine $\langle Q\rangle$, $\langle Q\rangle_1$, $\langle R\rangle_2$, and $\langle R\rangle_3$ becomes the next question. Once that is done, the individual coordinates can be picked off easily enough. For the vector parts, a Fourier decomposition as in equation eq. (108.34) will retrieve the desired coordinates.

The dual vector coordinates can be picked off easily enough taking dot products with the dual basis vectors

$$\begin{aligned} B &= B^k i\sigma_k = \sum_k \frac{1}{2}\left\{B, \frac{1}{i\sigma_k}\right\} i\sigma_k \\ B^k &= \frac{1}{2}\left\{B, \frac{1}{i\sigma_k}\right\} \end{aligned} \tag{108.71}$$

For the quaternion part Q the aim is to figure out how to isolate or subtract out the scalar part. This is the only tricky bit because the diagonal bits are all mixed up with the σ_3 term which is also real, and diagonal. Consideration of the sum

$$aI + b\sigma_3 = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}, \tag{108.72}$$

shows that trace will recover the value $2a$, so we have

$$\begin{aligned} \langle Q\rangle &= \frac{1}{2}\operatorname{Tr}(Q)\, I \\ \langle Q\rangle_1 &= Q - \frac{1}{2}\operatorname{Tr}(Q)\, I. \end{aligned} \tag{108.73}$$

Next is isolation of the pseudoscalar part of the dual quaternion R. As with the scalar part, consideration of the sum of the $i\sigma_3$ term and the iI term is required

$$iaI + ib\sigma_3 = \begin{bmatrix} ia + ib & 0 \\ 0 & ia - ib \end{bmatrix}, \tag{108.74}$$

So the trace of the dual quaternion provides the $2a$, leaving the bivector and pseudoscalar grade split

$$\begin{aligned} \langle R \rangle_3 &= \frac{1}{2} \operatorname{Tr}\left(R\right) I \\ \langle R \rangle_2 &= R - \frac{1}{2} \operatorname{Tr}\left(R\right) I. \end{aligned} \tag{108.75}$$

A final assembly of these results provides the following coordinate free grade selection operators

$$\begin{aligned} \langle P \rangle &= \frac{1}{4} \operatorname{Tr}\left(P + P^*\right) I \\ \langle P \rangle_1 &= \frac{1}{2}(P + P^*) - \frac{1}{4} \operatorname{Tr}\left(P + P^*\right) I \\ \langle P \rangle_2 &= \frac{1}{2}(P - P^*) - \frac{1}{4} \operatorname{Tr}\left(P - P^*\right) I \\ \langle P \rangle_3 &= \frac{1}{4} \operatorname{Tr}\left(P - P^*\right) I \end{aligned} \tag{108.76}$$

108.2.16 *Generalized dot products*

Here the equivalent of the generalized Clifford bivector/vector dot product will be computed, as well as the associated distribution equation

$$(a \wedge b) \cdot c = a(b \cdot c) - b(a \cdot c) \tag{108.77}$$

To translate this write

$$(a \wedge b) \cdot c = \frac{1}{2} \left((a \wedge b)c - c(a \wedge b)\right) \tag{108.78}$$

Then with the identifications

$$\begin{aligned} a \cdot b &\equiv \frac{1}{2}\{a, b\} \\ a \wedge b &\equiv \frac{1}{2}\left[a, b\right] \end{aligned} \tag{108.79}$$

we have

$$\begin{aligned} (a \wedge b) \cdot c &\equiv \frac{1}{4}\left[\left[a, b\right], c\right] \\ &= \frac{1}{2}\left(a\{b, c\} - \{b, c\}a\right) \end{aligned} \tag{108.80}$$

From this we also get the strictly Pauli algebra identity

$$\left[\left[a, b\right], c\right] = 2\left(a\{b, c\} - \{b, c\}a\right) \tag{108.81}$$

But the geometric meaning of this is unfortunately somewhat obfuscated by the notation.

108.2.17 *Generalized dot and wedge product*

The fundamental definitions of dot and wedge products are in terms of grade

$$\langle A \rangle_r \cdot \langle B \rangle_s = \langle AB \rangle_{|r-s|} \tag{108.82}$$

$$\langle A \rangle_r \wedge \langle B \rangle_s = \langle AB \rangle_{r+s} \tag{108.83}$$

Use of the trace and Hermitian conjugate split grade selection operations above, we can calculate these for each of the four grades in the Pauli algebra.

108.2.17.1 *Grade zero*

There are three dot products consider, vector/vector, bivector/bivector, and trivector/trivector. In each case we want to compute

$$\begin{aligned} A \cdot B &= \langle A \rangle B \\ &= \frac{1}{4}\operatorname{Tr}\left(AB + (AB)^*\right) I \\ &= \frac{1}{4}\operatorname{Tr}\left(AB + B^* A^*\right) I \end{aligned} \tag{108.84}$$

For vectors we have $a^* = a$, since the Pauli basis is Hermitian, whereas for bivectors and trivectors we have $a^* = -a$. Therefore, in all cases where A, and B have equal grades we have

$$\begin{aligned} A \cdot B &= \langle A \rangle B I \\ &= \frac{1}{4} \operatorname{Tr} \left(AB + BA \right) I \\ &= \frac{1}{4} \operatorname{Tr} \{A, B\} I \end{aligned} \tag{108.85}$$

108.2.17.2 *Grade one*

We have two dot products that produce vectors, bivector/vector, and trivector/bivector, and in each case we need to compute

$$\langle AB \rangle_1 = \frac{1}{2}(AB + (AB)^*) - \frac{1}{4} \operatorname{Tr} \left(AB + (AB)^* \right) \tag{108.86}$$

For the bivector/vector dot product we have

$$(Ba)^* = -aB \tag{108.87}$$

For bivector $B = ib^k \sigma_k$, and vector $a = a^k \sigma_k$ our symmetric Hermitian sum in coordinates is

$$\begin{aligned} Ba + (Ba)^* &= Ba - aB \\ &= ib^k \sigma_k a^m \sigma_m - a^m \sigma_m ib^k \sigma_k \end{aligned} \tag{108.88}$$

Any $m = k$ terms will vanish, leaving only the bivector terms, which are traceless. We therefore have

$$\begin{aligned} B \cdot a &= \langle Ba \rangle_1 \\ &= \frac{1}{2}(Ba - aB) \\ &= \frac{1}{2} \left[B, a \right]. \end{aligned} \tag{108.89}$$

This result was borrowed without motivation from Clifford algebra in equation eq. (108.78), and thus not satisfactory in terms of a logically derived sequence.

For a trivector T dotted with bivector B we have

$$(BT)^* = (-T)(-B) = TB = BT. \tag{108.90}$$

This is also traceless, and the trivector/bivector dot product is therefore reduced to just

$$\begin{aligned} B \cdot T &= \langle BT \rangle_1 \\ &= \frac{1}{2}\{B, T\} \\ &= BT \\ &= TB. \end{aligned} \tag{108.91}$$

This is the duality relationship for bivectors. Multiplication by the unit pseudoscalar (or any multiple of it), produces a vector, the dual of the original bivector.

108.2.17.3 *Grade two*

We have two products that produce a grade two term, the vector wedge product, and the vector/trivector dot product. For either case we must compute

$$\langle AB \rangle_2 = \frac{1}{2}(AB - (AB)^*) - \frac{1}{4}\operatorname{Tr}\left(AB - (AB)^*\right) \tag{108.92}$$

For a vector a, and trivector T we need the antisymmetric Hermitian sum

$$aT - (aT)^* = aT + Ta = 2aT = 2Ta \tag{108.93}$$

This is a pure bivector, and thus traceless, leaving just

$$\begin{aligned} a \cdot T &= \langle aT \rangle_2 \\ &= aT \\ &= Ta \end{aligned} \tag{108.94}$$

Again we have the duality relation, pseudoscalar multiplication with a vector produces a bivector, and is equivalent to the dot product of the two.

Now for the wedge product case, with vector $a = a^m \sigma_m$, and $b = b^k \sigma_k$ we must compute

$$\begin{aligned} ab - (ab)^* &= ab - ba \\ &= a^m \sigma_m b^k \sigma_k - b^k \sigma_k a^m \sigma_m \end{aligned} \tag{108.95}$$

All the $m = n$ terms vanish, leaving a pure bivector which is traceless, so only the first term of eq. (108.92) is relevant, and is in this case a commutator

$$\begin{aligned} a \wedge b &= \langle ab \rangle_2 \\ &= \frac{1}{2}\left[a, b\right] \end{aligned} \tag{108.96}$$

108.2.17.4 *Grade three*

There are two ways we can produce a grade three term in the algebra. One is a wedge of a vector with a bivector, and the other is the wedge product of three vectors. The triple wedge product is the grade three term of the product of the three

$$\begin{aligned} a \wedge b \wedge c &= \langle abc \rangle_3 \\ &= \frac{1}{4} \operatorname{Tr} \left(abc - (abc)^* \right) \\ &= \frac{1}{4} \operatorname{Tr} \left(abc - cba \right) \end{aligned} \tag{108.97}$$

With a split of the bc and cb terms into symmetric and antisymmetric terms we have

$$abc - cba = \frac{1}{2}(a\{b, c\} - \{c, b\}a) + \frac{1}{2}(a\,[b, c] - [c, b]\,a) \tag{108.98}$$

The symmetric term is diagonal so it commutes (equivalent to scalar commutation with a vector in Clifford algebra), and this therefore vanishes. Writing $B = b \wedge c = \frac{1}{2}[b, c]$, and noting that $[b, c] = -[c, b]$ we therefore have

$$\begin{aligned} a \wedge B &= \langle aB \rangle_3 \\ &= \frac{1}{4} \operatorname{Tr} \left(aB + Ba \right) \\ &= \frac{1}{4} \operatorname{Tr} \{a, B\} \end{aligned} \tag{108.99}$$

In terms of the original three vectors this is

$$\begin{aligned} a \wedge b \wedge c &= \langle aB \rangle_3 \\ &= \frac{1}{8} \operatorname{Tr} \{a, [b, c]\}. \end{aligned} \tag{108.100}$$

Since this could have been expanded by grouping ab instead of bc we also have

$$a \wedge b \wedge c = \frac{1}{8} \operatorname{Tr} \{[a, b]\,, c\}. \tag{108.101}$$

108.2.18 *Plane projection and rejection*

Projection of a vector onto a plane follows like the vector projection case. In the Pauli notation this is

$$\begin{aligned} x &= xB\frac{1}{B} \\ &= \frac{1}{2}\{x, B\}\frac{1}{B} + \frac{1}{2}\left[x, B\right]\frac{1}{B} \end{aligned} \tag{108.102}$$

Here the plane is a bivector, so if vectors a, and b are in the plane, the orientation and attitude can be represented by the commutator

So we have

$$x = \frac{1}{2}\left\{x, [a, b]\right\}\frac{1}{[a, b]} + \frac{1}{2}\left[x, [a, b]\right]\frac{1}{[a, b]} \tag{108.103}$$

Of these the second term is our projection onto the plane, while the first is the normal component of the vector.

108.3 EXAMPLES

108.3.1 *Radial decomposition*

108.3.1.1 *Velocity and momentum*

A decomposition of velocity into radial and perpendicular components should be straightforward in the Pauli algebra as it is in the Clifford algebra.

With a radially expressed position vector

$$x = |x|\hat{x}, \tag{108.104}$$

velocity can be written by taking derivatives

$$v = x' = |x|'\hat{x} + |x|\hat{x}' \tag{108.105}$$

or as above in the projection calculation with

$$\begin{aligned} v &= v\frac{1}{x}x \\ &= \frac{1}{2}\left\{v, \frac{1}{x}\right\}x + \frac{1}{2}\left[v, \frac{1}{x}\right]x \\ &= \frac{1}{2}\{v, \hat{x}\}\hat{x} + \frac{1}{2}\left[v, \hat{x}\right]\hat{x} \end{aligned} \tag{108.106}$$

By comparison we have

$$
\begin{aligned}
|x|' &= \frac{1}{2}\{v, \hat{x}\} \\
\hat{x}' &= \frac{1}{2|x|}\left[v, \hat{x}\right]\hat{x}
\end{aligned}
\tag{108.107}
$$

In assembled form we have

$$
v = \frac{1}{2}\{v, \hat{x}\}\hat{x} + x\omega
\tag{108.108}
$$

Here the commutator has been identified with the angular velocity bivector ω

$$
\omega = \frac{1}{2x^2}\left[x, v\right].
\tag{108.109}
$$

Similarly, the linear and angular momentum split of a momentum vector is

$$
\begin{aligned}
p_{\parallel} &= \frac{1}{2}\{p, \hat{x}\}\hat{x} \\
p_{\perp} &= \frac{1}{2}\left[p, \hat{x}\right]\hat{x}
\end{aligned}
\tag{108.110}
$$

and in vector form

$$
p = \frac{1}{2}\{p, \hat{x}\}\hat{x} + mx\omega
\tag{108.111}
$$

Writing $J = mx^2$ for the moment of inertia we have for our commutator

$$
L = \frac{1}{2}\left[x, p\right] = mx^2\omega = J\omega
\tag{108.112}
$$

With the identification of the commutator with the angular momentum bivector L we have the total momentum as

$$
p = \frac{1}{2}\{p, \hat{x}\}\hat{x} + \frac{1}{x}L
\tag{108.113}
$$

108.3.1.2 *Acceleration and force*

Having computed velocity, and its radial split, the next logical thing to try is acceleration.

The acceleration will be

$$a = v' = |x|'' \hat{x} + 2|x|' \hat{x}' + |x| \hat{x}'' \tag{108.114}$$

We need to compute $\hat{x}''$ to continue, which is

$$\begin{aligned}
\hat{x}'' &= \left(\frac{1}{2|x|^3} \left[v, x \right] x \right)' \\
&= \frac{-3}{2|x|^4} |x|' \left[v, x \right] x + \frac{1}{2|x|^3} \left[a, x \right] x + \frac{1}{2|x|^3} \left[v, x \right] v \\
&= \frac{-3}{4|x|^5} \{v, x\} \left[v, x \right] x + \frac{1}{2|x|^3} \left[a, x \right] x + \frac{1}{2|x|^3} \left[v, x \right] v
\end{aligned} \tag{108.115}$$

Putting things back together is a bit messy, but starting so gives

$$\begin{aligned}
a &= |x|'' \hat{x} + 2 \frac{1}{4|x|^4} \{v, x\} \left[v, x \right] x + \frac{-3}{4|x|^4} \{v, x\} \left[v, x \right] x + \frac{1}{2|x|^2} \left[a, x \right] x + \frac{1}{2|x|^2} \left[v, x \right] v \\
&= |x|'' \hat{x} - \frac{1}{4|x|^4} \{v, x\} \left[v, x \right] x + \frac{1}{2|x|^2} \left[a, x \right] x + \frac{1}{2|x|^2} \left[v, x \right] v \\
&= |x|'' \hat{x} + \frac{1}{4|x|^4} \left[v, x \right] \left(-\{v, x\} x + 2x^2 v \right) + \frac{1}{2|x|^2} \left[a, x \right] x
\end{aligned} \tag{108.116}$$

The anticommutator can be eliminated above using

$$\begin{aligned}
vx &= \frac{1}{2} \{v, x\} + \frac{1}{2} \left[v, x \right] \\
&\implies \\
-\{v, x\} x + 2x^2 v &= -(2vx - \left[v, x \right]) x + 2x^2 v \\
&= \left[v, x \right] x
\end{aligned} \tag{108.117}$$

Finally reassembly of the assembly is thus

$$\begin{aligned}
a &= |x|'' \hat{x} + \frac{1}{4|x|^4} \left[v, x \right]^2 x + \frac{1}{2|x|^2} \left[a, x \right] x \\
&= |x|'' \hat{x} + \omega^2 x + \frac{1}{2} \left[a, x \right] \frac{1}{x}
\end{aligned} \tag{108.118}$$

The second term is the inwards facing radially directed acceleration, while the last is the rejective component of the acceleration.

It is usual to express this last term as the rate of change of angular momentum (torque). Because $[v, v] = 0$, we have

$$\frac{d\,[x, v]}{dt} = [x, a] \tag{108.119}$$

So, for constant mass, we can write the torque as

$$\begin{aligned}\tau &= \frac{d}{dt}\left(\frac{1}{2}\,[x, p]\right) \\ &= \frac{dL}{dt}\end{aligned} \tag{108.120}$$

and finally have for the force

$$\begin{aligned}F &= m|x|''\hat{x} + m\omega^2 x + \frac{1}{x}\frac{dL}{dt} \\ &= m\left(|x|'' - \frac{\left|\omega^2\right|}{|x|}\right)\hat{x} + \frac{1}{x}\frac{dL}{dt}\end{aligned} \tag{108.121}$$

108.4 CONCLUSION

Although many of the GA references that can be found downplay the Pauli algebra as unnecessarily employing matrices as a basis, I believe this shows that there are some nice computational and logical niceties in the complete formulation of the $\mathbb{R}^3$ Clifford algebra in this complex matrix formulation. If nothing else it takes some of the abstraction away, which is good for developing intuition. All of the generalized dot and wedge product relationships are easily derived showing specific examples of the general pattern for the dot and blade product equations which are sometimes supplied as definitions instead of consequences.

Also, the matrix concepts (if presented right which I likely have not done) should also be accessible to most anybody out of high school these days since both matrix algebra and complex numbers are covered as basics these days (at least that is how I recall it from fifteen years back;)

Hopefully, having gone through the exercise of examining all the equivalent constructions will be useful in subsequent Quantum physics study to see how the matrix algebra that is used in that subject is tied to the classical geometrical vector constructions.

Expressions that were scary and mysterious looking like

$$[L_x, L_y] = i\hbar L_z \tag{108.122}$$

are no longer so bad since some of the geometric meaning that backs this operator expression is now clear (this is a quantization of angular momentum in a specific plane, and encodes the plane orientation as well as the magnitude). Knowing that $[a, b]$ was an antisymmetric sum, but not realizing the connection between that and the wedge product previously made me wonder "where the hell did the i come from"?

This commutator equation is logically and geometrically a plane operation. It can therefore be expressed with a vector duality relationship employing the $\mathbb{R}^3$ unit pseudoscalar $iI = \sigma_1\sigma_2\sigma_3$. This is a good nice step towards taking some of the mystery out of the math behind the physics of the subject (which has enough intrinsic mystery without the mathematical language adding to it).

It is unfortunate that QM uses this matrix operator formulation and none of classical physics does. By the time one gets to QM learning an entirely new language is required despite the fact that there are many powerful applications of this algebra in the classical domain, not just for rotations which is recognized (in [16] for example where he uses the Pauli algebra to express his rotation quaternions.)

109

GAMMA MATRICES

109.1 DIRAC MATRICES

The Dirac matrices γ^μ can be used as a Minkowski basis. The basic defining relationship is the Minkowski metric, where the dot products satisfy

$$\begin{aligned}\gamma^\mu \bullet \gamma^\nu &= \pm\delta_{\mu\nu} \\ (\gamma^0 \bullet \gamma^0)(\gamma^a \bullet \gamma^a) &= -1 \quad \text{where } a \in \{1, 2, 3\}\end{aligned} \tag{109.1}$$

There is freedom to pick the positive square for either γ^0 or γ^a, and both conventions are common.

One of the matrix representations for these vectors listed in the Dirac matrix wikipedia article is

$$\begin{aligned}\gamma^0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\ \gamma^2 &= \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \quad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\end{aligned} \tag{109.2}$$

For this particular basis we have a $+---$ metric signature. In the matrix form this takes the specific meaning that $(\gamma^0)^2 = I$, and $(\gamma^a)^2 = -I$.

A table of all the possible product variants of eq. (109.2) can be found below in the appendix.

109.1.1 anticommutator product

Noting that the matrices square in the fashion just described and that they reverse sign when multiplication order is reversed allows for summarizing the dot products relationships as follows

$$\begin{aligned}\{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\ &= 2\eta^{\mu\nu} I,\end{aligned} \tag{109.3}$$

where the metric tensor $\eta^{\mu\nu} = \gamma^\mu \bullet \gamma^\nu$ is commonly summarized as coordinates of a matrix as in

$$\left[\eta^{\mu\nu}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{109.4}$$

The relationship eq. (109.3) is taken as the defining relationship for the Dirac matrices, but can be seen to be just a matricized statement of the Clifford vector dot product.

109.1.2 Written as Pauli matrices

Using the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{109.5}$$

one can write the Dirac matrices and all their products (reading from the multiplication table) more concisely as

$$
\begin{aligned}
\gamma^0 &= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \\
\gamma^a &= \begin{bmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{bmatrix} \\
\gamma^0\gamma^a &= \begin{bmatrix} 0 & \sigma_a \\ \sigma_a & 0 \end{bmatrix} \\
\gamma^a\gamma^b &= -i\epsilon_{abc} \begin{bmatrix} \sigma_c & 0 \\ 0 & \sigma_c \end{bmatrix} \\
\gamma^1\gamma^2\gamma^3 &= i \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \\
\gamma^0\gamma^1\gamma^2 &= i \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \\
\gamma^3\gamma^0\gamma^1 &= i \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix} \\
\gamma^0\gamma^1\gamma^2 &= i \begin{bmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}
\end{aligned}
\tag{109.6}
$$

109.1.3 *Deriving properties using the Pauli matrices*

From the multiplication table a number of properties can be observed. Using the Pauli matrices one can arrive at these more directly using the multiplication identity for those matrices

$$
\sigma_a\sigma_b = 2i\epsilon_{abc}\sigma_c \tag{109.7}
$$

Actually taking the time to type this out in full does not seem worthwhile and is a fairly straightforward exercise.

109.1.4 Conjugation behavior

Unlike the Pauli matrices, the Dirac matrices do not split nicely via conjugation. Instead we have the time basis vector and its dual are Hermitian

$$\begin{aligned}(\gamma^0)^* &= \gamma^0 \\ (\gamma^1\gamma^2\gamma^3)^* &= \gamma^1\gamma^2\gamma^3\end{aligned} \tag{109.8}$$

whereas the spacelike basis vectors and their duals are all anti-Hermitian

$$\begin{aligned}(\gamma^a)^* &= -\gamma^a \\ (\gamma^a\gamma^b\gamma^c)^* &= -\gamma^a\gamma^b\gamma^c.\end{aligned} \tag{109.9}$$

For the scalar and the pseudoscalar parts we have a Hermitian split

$$\begin{aligned}I^* &= I \\ (\gamma^0\gamma^1\gamma^2\gamma^3)^* &= -(\gamma^0\gamma^1\gamma^2\gamma^3)^*\end{aligned} \tag{109.10}$$

and finally, also have a Hermitian split of the bivector parts into spacetime (relative vectors), and the purely spatial bivectors

$$\begin{aligned}(\gamma^0\gamma^a)^* &= \gamma^0\gamma^a \\ (\gamma^a\gamma^b)^* &= -\gamma^a\gamma^b\end{aligned} \tag{109.11}$$

Is there a logical and simple set of matrix operations that splits things nicely into scalar, vector, bivector, trivector, and pseudoscalar parts as there was with the Pauli matrices?

109.2 APPENDIX. TABLE OF ALL GENERATED PRODUCTS

A small C++ program using boost::numeric::ublas and std::complex, plus some perl to generate part of that, was written to generate the multiplication table for the gamma matrix products for this particular basis. The metric tensor and the antisymmetry of the wedge products can be seen from these.

$$\gamma^0\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \gamma^1\gamma^1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{109.12}$$

$$\gamma^2\gamma^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \gamma^3\gamma^3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{109.13}$$

$$\gamma^0\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \gamma^1\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{109.14}$$

$$\gamma^0\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \quad \gamma^2\gamma^0 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{109.15}$$

$$\gamma^0\gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \gamma^3\gamma^0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{109.16}$$

$$\gamma^1\gamma^2 = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \quad \gamma^2\gamma^1 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \tag{109.17}$$

$$\gamma^1\gamma^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \gamma^3\gamma^1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{109.18}$$

$$\gamma^2\gamma^3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} \quad \gamma^3\gamma^2 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \tag{109.19}$$

$$\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \quad \gamma^2\gamma^3\gamma^0 = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \tag{109.20}$$

$$\gamma^3\gamma^0\gamma^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \gamma^0\gamma^1\gamma^2 = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \tag{109.21}$$

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \tag{109.22}$$

110

BIVECTOR FORM OF QUANTUM ANGULAR MOMENTUM OPERATOR

110.1 SPATIAL BIVECTOR REPRESENTATION OF THE ANGULAR MOMENTUM OPERATOR

Reading [2] on the angular momentum operator, the form of the operator is suggested by analogy where components of $\mathbf{x} \times \mathbf{p}$ with the position representation $\mathbf{p} \sim -i\hbar\boldsymbol{\nabla}$ used to expand the coordinate representation of the operator.

The result is the following coordinate representation of the operator

$$
\begin{aligned}
L_1 &= -i\hbar(x_2\partial_3 - x_3\partial_2) \\
L_2 &= -i\hbar(x_3\partial_1 - x_1\partial_3) \\
L_3 &= -i\hbar(x_1\partial_2 - x_2\partial_1)
\end{aligned}
\tag{110.1}
$$

It is interesting to put these in vector form, and then employ the freedom to use for $i = \sigma_1\sigma_2\sigma_3$ the spatial pseudoscalar.

$$
\begin{aligned}
\mathbf{L} &= -\sigma_1(\sigma_1\sigma_2\sigma_3)\hbar(x_2\partial_3 - x_3\partial_2) - \sigma_2(\sigma_2\sigma_3\sigma_1)\hbar(x_3\partial_1 - x_1\partial_3) - \sigma_3(\sigma_3\sigma_1\sigma_2)\hbar(x_1\partial_2 - x_2\partial_1) \\
&= -\sigma_2\sigma_3\hbar(x_2\partial_3 - x_3\partial_2) - \sigma_3\sigma_1\hbar(x_3\partial_1 - x_1\partial_3) - \sigma_1\sigma_2\hbar(x_1\partial_2 - x_2\partial_1) \\
&= -\hbar(\sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3) \wedge (\sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3)
\end{aligned}
\tag{110.2}
$$

The choice to use the pseudoscalar for this imaginary seems a logical one and the end result is a pure bivector representation of angular momentum operator

$$
\mathbf{L} = -\hbar\mathbf{x} \wedge \boldsymbol{\nabla}
\tag{110.3}
$$

The choice to represent angular momentum as a bivector $\mathbf{x} \wedge \mathbf{p}$ is also natural in classical mechanics (encoding the orientation of the plane and the magnitude of the momentum in the bivector), although its dual form the axial vector $\mathbf{x} \times \mathbf{p}$ is more common, at least in introductory mechanics. Observe that there is no longer any explicit imaginary in eq. (110.3), since the bivector itself has an implicit complex structure.

110.2 FACTORING THE GRADIENT AND LAPLACIAN

The form of eq. (110.3) suggests a more direct way to extract the angular momentum operator from the Hamiltonian (i.e. from the Laplacian). Bohm uses the spherical polar representation of the Laplacian as the starting point. Instead let us project the gradient itself in a specific constant direction **a**, much as we can do to find the polar form angular velocity and acceleration components.

Write

$$\begin{aligned}\boldsymbol{\nabla} &= \frac{1}{\mathbf{a}}\mathbf{a}\boldsymbol{\nabla} \\ &= \frac{1}{\mathbf{a}}(\mathbf{a}\cdot\boldsymbol{\nabla} + \mathbf{a}\wedge\boldsymbol{\nabla})\end{aligned} \tag{110.4}$$

Or

$$\begin{aligned}\boldsymbol{\nabla} &= \boldsymbol{\nabla}\mathbf{a}\frac{1}{\mathbf{a}} \\ &= (\boldsymbol{\nabla}\cdot\mathbf{a} + \boldsymbol{\nabla}\wedge\mathbf{a})\frac{1}{\mathbf{a}} \\ &= (\mathbf{a}\cdot\boldsymbol{\nabla} - \mathbf{a}\wedge\boldsymbol{\nabla})\frac{1}{\mathbf{a}}\end{aligned} \tag{110.5}$$

The Laplacian is therefore

$$\begin{aligned}\boldsymbol{\nabla}^2 &= \left\langle \boldsymbol{\nabla}^2 \right\rangle \\ &= \left\langle (\mathbf{a}\cdot\boldsymbol{\nabla} - \mathbf{a}\wedge\boldsymbol{\nabla})\frac{1}{\mathbf{a}}\frac{1}{\mathbf{a}}(\mathbf{a}\cdot\boldsymbol{\nabla} + \mathbf{a}\wedge\boldsymbol{\nabla}) \right\rangle \\ &= \frac{1}{\mathbf{a}^2}\langle(\mathbf{a}\cdot\boldsymbol{\nabla} - \mathbf{a}\wedge\boldsymbol{\nabla})(\mathbf{a}\cdot\boldsymbol{\nabla} + \mathbf{a}\wedge\boldsymbol{\nabla})\rangle \\ &= \frac{1}{\mathbf{a}^2}((\mathbf{a}\cdot\boldsymbol{\nabla})^2 - (\mathbf{a}\wedge\boldsymbol{\nabla})^2)\end{aligned} \tag{110.6}$$

So we have for the Laplacian a representation in terms of projection and rejection components

$$\begin{aligned}\boldsymbol{\nabla}^2 &= (\hat{\mathbf{a}}\cdot\boldsymbol{\nabla})^2 - \frac{1}{\mathbf{a}^2}(\mathbf{a}\wedge\boldsymbol{\nabla})^2 \\ &= (\hat{\mathbf{a}}\cdot\boldsymbol{\nabla})^2 - (\hat{\mathbf{a}}\wedge\boldsymbol{\nabla})^2\end{aligned} \tag{110.7}$$

The vector **a** was arbitrary, and just needed to be constant with respect to the factorization operations. Setting $\mathbf{a} = \mathbf{x}$, the radial position from the origin one may guess that we have

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{\mathbf{x}^2}(\mathbf{x}\wedge\boldsymbol{\nabla})^2 \tag{110.8}$$

however, with the switch to a non-constant position vector **x**, this cannot possibly be right.

110.3 THE CORIOLIS TERM

The radial factorization of the gradient relied on the direction vector **a** being constant. If we evaluate eq. (110.8), then there should be a non-zero remainder compared to the Laplacian. Evaluation by coordinate expansion is one way to verify this, and should produce the difference. Let us do this in two parts, starting with the scalar part of $(x \wedge \nabla)^2$. Summation will be implied by mixed indices, and for generality a general basis and associated reciprocal frame will be used.

$$
\begin{aligned}
\left\langle (x \wedge \nabla)^2 \right\rangle f &= ((x^\mu \gamma_\mu) \wedge (\gamma_\nu \partial^\nu)) \cdot ((x_\alpha \gamma^\alpha) \wedge (\gamma^\beta \partial_\beta)) f \\
&= (\gamma_\mu \wedge \gamma_\nu) \cdot (\gamma^\alpha \wedge \gamma^\beta) x^\mu \partial^\nu (x_\alpha \partial_\beta) f \\
&= ({\delta_\mu}^\beta {\delta_\nu}^\alpha - {\delta_\mu}^\alpha {\delta_\nu}^\beta) x^\mu \partial^\nu (x_\alpha \partial_\beta) f \\
&= x^\mu \partial^\nu ((x_\nu \partial_\mu) - x_\mu \partial_\nu) f \\
&= x^\mu (\partial^\nu x_\nu) \partial_\mu f - x^\mu (\partial^\nu x_\mu) \partial_\nu f \\
&+ x^\mu x_\nu \partial^\nu \partial_\mu f - x^\mu x_\mu \partial^\nu \partial_\nu f \\
&= (n-1) x \cdot \nabla f + x^\mu x_\nu \partial^\nu \partial_\mu f - x^2 \nabla^2 f
\end{aligned}
\tag{110.9}
$$

For the dot product we have

$$
\begin{aligned}
\left\langle (x \cdot \nabla)^2 \right\rangle f &= x^\mu \partial_\mu (x^\nu \partial_\nu) f \\
&= x^\mu (\partial_\mu x^\nu) \partial_\nu f + x^\mu x^\nu \partial_\mu \partial_\nu f \\
&= x^\mu \partial_\mu f + x^\mu x_\nu \partial^\nu \partial_\mu f \\
&= x \cdot \nabla f + x^\mu x_\nu \partial^\nu \partial_\mu f
\end{aligned}
\tag{110.10}
$$

So, forming the difference we have

$$
(x \cdot \nabla)^2 f - \left\langle (x \wedge \nabla)^2 \right\rangle f = -(n-2) x \cdot \nabla f + x^2 \nabla^2 f
\tag{110.11}
$$

Or

$$
\nabla^2 = \frac{1}{x^2} (x \cdot \nabla)^2 - \frac{1}{x^2} \left\langle (x \wedge \nabla)^2 \right\rangle + (n-2) \frac{1}{x} \cdot \nabla
\tag{110.12}
$$

110.4 ON THE BIVECTOR AND QUADVECTOR COMPONENTS OF THE SQUARED ANGULAR MOMENTUM OPERATOR

The requirement for a scalar selection on all the $(x \wedge \nabla)^2$ terms is a bit ugly, but omitting it would be incorrect for two reasons. One reason is that this is a bivector operator and not a bivector (where the squaring operates on itself). The other is that we derived a result for arbitrary dimension, and the product of two bivectors in a general space has grade 2 and grade 4 terms in addition to the scalar terms. Without taking only the scalar parts, lets expand this product a bit more carefully, starting with

$$(x \wedge \nabla)^2 = (\gamma_\mu \wedge \gamma_\nu)(\gamma^\alpha \wedge \gamma^\beta) x^\mu \partial^\nu x_\alpha \partial_\beta \tag{110.13}$$

Just expanding the multivector factor for now, we have

$$\begin{aligned}
&2(\gamma_\mu \wedge \gamma_\nu)(\gamma^\alpha \wedge \gamma^\beta) \\
&= \gamma_\mu \gamma_\nu (\gamma^\alpha \wedge \gamma^\beta) - \gamma_\nu \gamma_\mu (\gamma^\alpha \wedge \gamma^\beta) \\
&= \gamma_\mu \left({\delta_\nu}^\alpha \gamma^\beta - {\delta_\nu}^\beta \gamma^\alpha + \gamma_\nu \wedge \gamma^\alpha \wedge \gamma^\beta \right) - \gamma_\nu \left({\delta_\mu}^\alpha \gamma^\beta - {\delta_\mu}^\beta \gamma^\alpha + \gamma_\mu \wedge \gamma^\alpha \wedge \gamma^\beta \right) \\
&= {\delta_\nu}^\alpha {\delta_\mu}^\beta - {\delta_\nu}^\beta {\delta_\mu}^\alpha - {\delta_\mu}^\alpha {\delta_\nu}^\beta + {\delta_\mu}^\beta {\delta_\nu}^\alpha \\
&+ \gamma_\mu \wedge \gamma_\nu \wedge \gamma^\alpha \wedge \gamma^\beta - \gamma_\nu \wedge \gamma_\mu \wedge \gamma^\alpha \wedge \gamma^\beta \\
&+ \gamma_\mu \cdot (\gamma_\nu \wedge \gamma^\alpha \wedge \gamma^\beta) - \gamma_\nu \cdot (\gamma_\mu \wedge \gamma^\alpha \wedge \gamma^\beta)
\end{aligned} \tag{110.14}$$

Our split into grades for this operator is then, the scalar

$$\begin{aligned}
\left\langle (x \wedge \nabla)^2 \right\rangle &= (x \wedge \nabla) \cdot (x \wedge \nabla) \\
&= \left({\delta_\nu}^\alpha {\delta_\mu}^\beta - {\delta_\nu}^\beta {\delta_\mu}^\alpha \right) x^\mu \partial^\nu x_\alpha \partial_\beta
\end{aligned} \tag{110.15}$$

the pseudoscalar (or grade 4 term in higher than 4D spaces).

$$\begin{aligned}
\left\langle (x \wedge \nabla)^2 \right\rangle_4 &= (x \wedge \nabla) \wedge (x \wedge \nabla) \\
&= \left(\gamma_\mu \wedge \gamma_\nu \wedge \gamma^\alpha \wedge \gamma^\beta \right) x^\mu \partial^\nu x_\alpha \partial_\beta
\end{aligned} \tag{110.16}$$

If we work in dimensions less than or equal to three, we will have no grade four term since this wedge product is zero (irrespective of the operator action), so in 3D we have only a bivector term in excess of the scalar part of this operator.

The bivector term deserves some reduction, but is messy to do so. This has been done separately in (112)

We can now write for the squared operator

$$(x \wedge \nabla)^2 = (n-2)(x \wedge \nabla) + (x \wedge \nabla) \wedge (x \wedge \nabla) + (x \wedge \nabla) \cdot (x \wedge \nabla) \tag{110.17}$$

and then eliminate the scalar selection from the eq. (110.12)

$$\nabla^2 = \frac{1}{x^2}(x \cdot \nabla)^2 + (n-2)\frac{1}{x} \cdot \nabla - \frac{1}{x^2}\left((x \wedge \nabla)^2 - (n-2)(x \wedge \nabla) - (x \wedge \nabla) \wedge (x \wedge \nabla)\right) \tag{110.18}$$

In 3D this is

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2}(\mathbf{x} \cdot \boldsymbol{\nabla})^2 + \frac{1}{\mathbf{x}} \cdot \boldsymbol{\nabla} - \frac{1}{\mathbf{x}^2}\left(\mathbf{x} \wedge \boldsymbol{\nabla} - 1\right)(\mathbf{x} \wedge \boldsymbol{\nabla}) \tag{110.19}$$

Wow, that was an ugly mess of algebra. The worst of it for the bivector grades was initially incorrect and the correct handling omitted. There is likely a more clever coordinate free way to do the same expansion. We will see later that at least a partial verification of eq. (110.19) can be obtained by considering of the Quantum eigenvalue problem, examining simultaneous eigenvalues of $\mathbf{x} \wedge \boldsymbol{\nabla}$, and $\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle$. However, lets revisit this after examining the radial terms in more detail, and also after verifying that at least in the scalar selection form, this factorized Laplacian form has the same structure as the Laplacian in scalar r, θ, and ϕ operator form.

110.5 CORRESPONDENCE WITH EXPLICIT RADIAL FORM

We have seen above that we can factor the 3D Laplacian as

$$\boldsymbol{\nabla}^2 \psi = \frac{1}{\mathbf{x}^2}((\mathbf{x} \cdot \boldsymbol{\nabla})^2 + \mathbf{x} \cdot \boldsymbol{\nabla} - \left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle)\psi \tag{110.20}$$

Contrast this to the explicit r, θ, ϕ form as given in (Bohm's [2], 14.2)

$$\boldsymbol{\nabla}^2 \psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta + \frac{1}{\sin^2\theta} + \partial_{\phi\phi}\right)\psi \tag{110.21}$$

Let us expand out the non-angular momentum operator terms explicitly as a partial verification of this factorization. The radial term in Bohm's Laplacian formula expands out to

$$\begin{aligned}
\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) &= \frac{1}{r}\partial_r(\partial_r r\psi) \\
&= \frac{1}{r}\partial_r(\psi + r\partial_r\psi) \\
&= \frac{1}{r}\partial_r\psi + \frac{1}{r}(\partial_r\psi + r\partial_{rr}\psi) \\
&= \frac{2}{r}\partial_r\psi + \partial_{rr}\psi
\end{aligned} \tag{110.22}$$

On the other hand, with $\mathbf{x} = r\hat{\mathbf{r}}$, what we expect to correspond to the radial term in the vector factorization is

$$
\begin{aligned}
\frac{1}{\mathbf{x}^2}((\mathbf{x}\cdot\boldsymbol{\nabla})^2 + \mathbf{x}\cdot\boldsymbol{\nabla})\psi &= \frac{1}{r^2}((r\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})^2 + r\hat{\mathbf{r}}\cdot\boldsymbol{\nabla})\psi \\
&= \frac{1}{r^2}((r\partial_r)^2 + r\partial_r)\psi \\
&= \frac{1}{r^2}(r\partial_r\psi + r^2\partial_{rr}\psi + r\partial_r\psi) \\
&= \frac{2}{r}\partial_r\psi + \partial_{rr}\psi
\end{aligned}
\tag{110.23}
$$

Okay, good. It is a brute force way to verify things, but it works. With $\mathbf{x}\wedge\boldsymbol{\nabla} = I(\mathbf{x}\times\boldsymbol{\nabla})$ we can eliminate the wedge product from the factorization expression eq. (110.20) and express things completely in quantities that can be understood without any resort to Geometric Algebra. That is

$$
\boldsymbol{\nabla}^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}\left\langle(\mathbf{x}\times\boldsymbol{\nabla})^2\right\rangle\psi
\tag{110.24}
$$

Bohm resorts to analogy and an operatorization of $L_c = \epsilon_{abc}(x_a p_b - x_b p_a)$, then later a spherical polar change of coordinates to match exactly the L^2 expression with eq. (110.21). With the GA formalism we see this a bit more directly, although it is not the least bit obvious that the operator $\mathbf{x}\times\boldsymbol{\nabla}$ has no radial dependence. Without resorting to a comparison with the explicit r, θ, ϕ form that would not be so easy to see.

110.6 RAISING AND LOWERING OPERATORS IN GA FORM

Having seen in (111) that we have a natural GA form for the $l = 1$ spherical harmonic eigenfunctions ψ_1^m, and that we have the vector angular momentum operator $\mathbf{x}\times\boldsymbol{\nabla}$ showing up directly in a sort-of-radial factorization of the Laplacian, it is natural to wonder what the GA form of the raising and lowering operators are. At least for the $l = 1$ harmonics use of $i = I\mathbf{e}_3$ (unit bivector for the $x - y$ plane) for the imaginary ended up providing a nice geometric interpretation.

Let us see what that provides for the raising and lowering operators. First we need to express L_x and L_y in terms of our bivector angular momentum operator. Let us switch notations and drop the $-i\hbar$ factor from eq. (110.3) writing just

$$
\mathbf{L} = \mathbf{x}\wedge\boldsymbol{\nabla}
\tag{110.25}
$$

We can now write this in terms of components with respect to the basis bivectors $I\mathbf{e}_k$. That is

$$\mathbf{L} = \sum_k \left((\mathbf{x} \wedge \boldsymbol{\nabla}) \cdot \frac{1}{I\mathbf{e}_k} \right) I\mathbf{e}_k \tag{110.26}$$

These scalar product results are expected to match the L_x, L_y, and L_z components at least up to a sign. Let us check, picking L_z as representative

$$\begin{aligned}
(\mathbf{x} \wedge \boldsymbol{\nabla}) \cdot \frac{1}{I\mathbf{e}_3} &= (\sigma_m \wedge \sigma^k) \cdot -\sigma_1\sigma_2\sigma_3\sigma_3 x^m \partial_k \\
&= (\sigma_m \wedge \sigma^k) \cdot -\sigma_1\sigma_2 x^m \partial_k \\
&= -(x^2\partial_1 - x^1\partial_2)
\end{aligned} \tag{110.27}$$

With the $-i\hbar$ factors dropped this is $L_z = L_3 = x^1\partial_2 - x^2\partial_1$, the projection of $\mathbf{L}$ onto the $x - y$ plane $I\mathbf{e}_k$. So, now how about the raising and lowering operators

$$\begin{aligned}
L_x \pm iL_y &= L_x \pm I\mathbf{e}_3 L_y \\
&= \mathbf{L} \cdot \frac{1}{I\mathbf{e}_1} \pm I\mathbf{e}_3 \mathbf{L} \cdot \frac{1}{I\mathbf{e}_2} \\
&= -\mathbf{e}_1 I \left(I\mathbf{e}_1 \mathbf{L} \cdot \frac{1}{I\mathbf{e}_1} \pm I\mathbf{e}_2 \mathbf{L} \cdot \frac{1}{I\mathbf{e}_2} \right)
\end{aligned} \tag{110.28}$$

Or

$$(I\mathbf{e}_1) L_x \pm iL_y = I\mathbf{e}_1 \mathbf{L} \cdot \frac{1}{I\mathbf{e}_1} \pm I\mathbf{e}_2 \mathbf{L} \cdot \frac{1}{I\mathbf{e}_2} \tag{110.29}$$

Compare this to the projective split of $\mathbf{L}$ eq. (110.26). We have projections of the bivector angular momentum operator onto the bivector directions $I\mathbf{e}_1$ and $I\mathbf{e}_2$ (really the bivectors for the planes perpendicular to the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions).

We have the Laplacian in explicit vector form and have a clue how to vectorize (really bivectorize) the raising and lowering operators. We have also seen how to geometrize the first spherical harmonics. The next logical step is to try to apply this vector form of the raising and lowering operators to the vector form of the spherical harmonics.

110.7 EXPLICIT EXPANSION OF THE ANGULAR MOMENTUM OPERATOR

There is a couple of things to explore before going forward. One is an explicit verification that $\mathbf{x} \wedge \boldsymbol{\nabla}$ has no radial dependence (something not obvious). Another is that we should be able to compare the $\mathbf{x}^{-2}(\mathbf{x} \wedge \boldsymbol{\nabla})^2$ (as done for the $\mathbf{x} \cdot \boldsymbol{\nabla}$ terms) the explicit r, θ, ϕ expression for the Laplacian to verify consistency and correctness.

For the spherical polar rotation we use the rotor

$$R = e^{\mathbf{e}_{31}\theta/2} e^{\mathbf{e}_{12}\phi/2} \tag{110.30}$$

Our position vector and gradient in spherical polar coordinates are

$$\mathbf{x} = r\tilde{R}\mathbf{e}_3 R \tag{110.31}$$

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\partial_r + \hat{\boldsymbol{\theta}}\frac{1}{r}\partial_\theta + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\partial_\phi \tag{110.32}$$

with the unit vectors translate from the standard basis as

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \tilde{R} \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} R \tag{110.33}$$

This last mapping can be used to express the gradient unit vectors in terms of the standard basis, as we did for the position vector $\mathbf{x}$. That is

$$\boldsymbol{\nabla} = \tilde{R}\left(\mathbf{e}_3 R\partial_r + \mathbf{e}_1 R\frac{1}{r}\partial_\theta + \mathbf{e}_2 R\frac{1}{r\sin\theta}\partial_\phi\right) \tag{110.34}$$

Okay, we have now got all the pieces collected, ready to evaluate $\mathbf{x} \wedge \boldsymbol{\nabla}$

$$\begin{aligned} \mathbf{x} \wedge \boldsymbol{\nabla} &= r\left\langle \tilde{R}\mathbf{e}_3 R\tilde{R}\left(\mathbf{e}_3 R\partial_r + \mathbf{e}_1 R\frac{1}{r}\partial_\theta + \mathbf{e}_2 R\frac{1}{r\sin\theta}\partial_\phi\right)\right\rangle_2 \\ &= r\left\langle \tilde{R}\left(R\partial_r + \mathbf{e}_3\mathbf{e}_1 R\frac{1}{r}\partial_\theta + \mathbf{e}_3\mathbf{e}_2 R\frac{1}{r\sin\theta}\partial_\phi\right)\right\rangle_2 \end{aligned} \tag{110.35}$$

Observe that the ${\mathbf{e}_3}^2$ contribution is only a scalar, so bivector selection of that is zero. In the remainder we have cancellation of r/r factors, leaving just

$$\mathbf{x} \wedge \boldsymbol{\nabla} = \tilde{R}\left(\mathbf{e}_3\mathbf{e}_1 R\partial_\theta + \mathbf{e}_3\mathbf{e}_2 R\frac{1}{\sin\theta}\partial_\phi\right) \tag{110.36}$$

Using eq. (110.33) this is

$$\mathbf{x} \wedge \boldsymbol{\nabla} = \hat{\mathbf{r}} \left(\hat{\boldsymbol{\theta}} \partial_\theta + \hat{\boldsymbol{\phi}} \frac{1}{\sin\theta} \partial_\phi \right) \tag{110.37}$$

As hoped, there is no explicit radial dependence here, taking care of the first of the desired verifications.

Next we want to square this operator. It should be noted that in the original derivation where we "factored" the gradient operator with respect to the reference vector $\mathbf{x}$ our Laplacian really followed by considering $(\mathbf{x} \wedge \boldsymbol{\nabla})^2 \equiv \left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle$. That is worth noting since a regular bivector would square to a negative constant, whereas the operator factors of the vectors in this expression do not intrinsically commute.

An additional complication for evaluating the square of $\mathbf{x} \wedge \boldsymbol{\nabla}$ using the result of eq. (110.37) is that $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{r}}$ are functions of θ and ϕ, so we would have to operate on those too. Without that operator subtlety we get the wrong answer

$$\begin{aligned} -\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle &= \left\langle \tilde{R} \left(\mathbf{e}_1 R \partial_\theta + \frac{\mathbf{e}_2 R}{\sin\theta} \partial_\phi \right) \tilde{R} \left(\mathbf{e}_1 R \partial_\theta + \frac{\mathbf{e}_2 R}{\sin\theta} \partial_\phi \right) \right\rangle \\ &\neq \partial_{\theta\theta} + \frac{1}{\sin^2\theta} \partial_{\phi\phi} \end{aligned} \tag{110.38}$$

Equality above would only be if the unit vectors were fixed. By comparison we also see that this is missing a $\cot\theta\partial_\theta$ term. That must come from the variation of the unit vectors with position in the second application of $\mathbf{x} \wedge \boldsymbol{\nabla}$.

110.8 DERIVATIVES OF THE UNIT VECTORS

To properly evaluate the angular momentum square we will need to examine the ∂_θ and ∂_ϕ variation of the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. Some part of this question can be evaluated without reference to the specific vector or even which derivative is being evaluated. Writing e for one of $\mathbf{e}_1$, $\mathbf{e}_2$, or $\mathbf{e}_k$, and $\sigma = \tilde{R}eR$ for the mapping of this vector under rotation, and ∂ for the desired θ or ϕ partial derivative, we have

$$\partial(\tilde{R}eR) = (\partial\tilde{R})eR + \tilde{R}e(\partial R) \tag{110.39}$$

Since $\tilde{R}R = 1$, we have

$$\begin{aligned} 0 &= \partial(\tilde{R}R) \\ &= (\partial\tilde{R})R + \tilde{R}(\partial R) \end{aligned} \tag{110.40}$$

So substitution of $(\partial \tilde{R}) = -\tilde{R}(\partial R)\tilde{R}$, back into eq. (110.39) supplies

$$
\begin{aligned}
\partial(\tilde{R}eR) &= -\tilde{R}(\partial R)\tilde{R}eR + \tilde{R}e(\partial R) \\
&= -\tilde{R}(\partial R)(\tilde{R}eR) + (\tilde{R}eR)\tilde{R}(\partial R) \\
&= -\tilde{R}(\partial R)\sigma + \sigma\tilde{R}(\partial R)
\end{aligned}
\tag{110.41}
$$

Writing the bivector term as

$$
\Omega = \tilde{R}(\partial R) \tag{110.42}
$$

The change in the rotated vector is seen to be entirely described by the commutator of that vectors image under rotation with Ω. That is

$$
\partial\sigma = [\sigma, \Omega] \tag{110.43}
$$

Our spherical polar rotor was given by

$$
R = e^{\mathbf{e}_{31}\theta/2} e^{\mathbf{e}_{12}\phi/2} \tag{110.44}
$$

Lets calculate the Ω bivector for each of the θ and ϕ partials. For θ we have

$$
\begin{aligned}
\Omega_\theta &= \tilde{R}\partial_\theta R \\
&= \frac{1}{2} e^{-\mathbf{e}_{12}\phi/2} e^{-\mathbf{e}_{31}\theta/2} \mathbf{e}_{31} e^{\mathbf{e}_{31}\theta/2} e^{\mathbf{e}_{12}\phi/2} \\
&= \frac{1}{2} e^{-\mathbf{e}_{12}\phi/2} \mathbf{e}_{31} e^{\mathbf{e}_{12}\phi/2} \\
&= \frac{1}{2} \mathbf{e}_3 e^{-\mathbf{e}_{12}\phi/2} \mathbf{e}_1 e^{\mathbf{e}_{12}\phi/2} \\
&= \frac{1}{2} \mathbf{e}_{31} e^{\mathbf{e}_{12}\phi}
\end{aligned}
\tag{110.45}
$$

Explicitly, this is the bivector $\Omega_\theta = (\mathbf{e}_{31}\cos\theta + \mathbf{e}_{32}\sin\theta)/2$, a wedge product of a vectors in $\hat{\mathbf{z}}$ direction with one in the perpendicular $x - y$ plane (curiously a vector in the $x - y$ plane rotated by polar angle θ, not the equatorial angle ϕ).

FIXME: picture. Draw this plane cutting through the sphere.

For the ϕ partial variation of any of our unit vectors our bivector rotation generator is

$$\begin{aligned}\Omega_\phi &= \tilde{R}\partial_\phi R \\ &= \frac{1}{2}e^{-\mathbf{e}_{12}\phi/2}e^{-\mathbf{e}_{31}\theta/2}e^{\mathbf{e}_{31}\theta/2}\mathbf{e}_{12}e^{\mathbf{e}_{12}\phi/2} \\ &= \frac{1}{2}\mathbf{e}_{12}\end{aligned} \tag{110.46}$$

This one has no variation at all with angle whatsoever. If this is all correct so far perhaps that is not surprising given the fact that we expect an extra $\cot\theta$ in the angular momentum operator square, so a lack of ϕ dependence in the rotation generator likely means that any additional ϕ dependence will cancel out. Next step is to take these rotation generator bivectors, apply them via commutator products to the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ vectors, and see what we get.

110.9 APPLYING THE VECTOR DERIVATIVE COMMUTATOR (OR NOT)

Let us express the $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ unit vectors explicitly in terms of the standard basis. Starting with $\hat{\boldsymbol{\theta}}$ we have

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \tilde{R}\mathbf{e}_1 R \\ &= e^{-\mathbf{e}_{12}\phi/2}e^{-\mathbf{e}_{31}\theta/2}\mathbf{e}_1 e^{\mathbf{e}_{31}\theta/2}e^{\mathbf{e}_{12}\phi/2} \\ &= e^{-\mathbf{e}_{12}\phi/2}\mathbf{e}_1 e^{\mathbf{e}_{31}\theta}e^{\mathbf{e}_{12}\phi/2} \\ &= e^{-\mathbf{e}_{12}\phi/2}(\mathbf{e}_1\cos\theta - \mathbf{e}_3\sin\theta)e^{\mathbf{e}_{12}\phi/2} \\ &= \mathbf{e}_1\cos\theta e^{\mathbf{e}_{12}\phi} - \mathbf{e}_3\sin\theta\end{aligned} \tag{110.47}$$

Explicitly in vector form, eliminating the exponential, this is $\hat{\boldsymbol{\theta}} = \mathbf{e}_1\cos\theta\cos\phi + \mathbf{e}_2\cos\theta\sin\phi - \mathbf{e}_3\sin\theta$, but it is more convenient to keep the exponential as is.

For $\hat{\boldsymbol{\phi}}$ we have

$$\begin{aligned}\hat{\boldsymbol{\phi}} &= \tilde{R}\mathbf{e}_2 R \\ &= e^{-\mathbf{e}_{12}\phi/2}e^{-\mathbf{e}_{31}\theta/2}\mathbf{e}_2 e^{\mathbf{e}_{31}\theta/2}e^{\mathbf{e}_{12}\phi/2} \\ &= e^{-\mathbf{e}_{12}\phi/2}\mathbf{e}_2 e^{\mathbf{e}_{12}\phi/2} \\ &= \mathbf{e}_2 e^{\mathbf{e}_{12}\phi}\end{aligned} \tag{110.48}$$

Again, explicitly this is $\hat{\boldsymbol{\phi}} = \mathbf{e}_2 \cos\phi - \mathbf{e}_1 \sin\phi$, but we will use the exponential form above. Last we want $\hat{\mathbf{r}}$

$$\begin{aligned}
\hat{\mathbf{r}} &= \tilde{R}\mathbf{e}_3 R \\
&= e^{-\mathbf{e}_{12}\phi/2} e^{-\mathbf{e}_{31}\theta/2} \mathbf{e}_3 e^{\mathbf{e}_{31}\theta/2} e^{\mathbf{e}_{12}\phi/2} \\
&= e^{-\mathbf{e}_{12}\phi/2} \mathbf{e}_3 e^{\mathbf{e}_{31}\theta} e^{\mathbf{e}_{12}\phi/2} \\
&= e^{-\mathbf{e}_{12}\phi/2} (\mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta) e^{\mathbf{e}_{12}\phi/2} \\
&= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{\mathbf{e}_{12}\phi}
\end{aligned} \tag{110.49}$$

Summarizing we have

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= \mathbf{e}_1 \cos\theta e^{\mathbf{e}_{12}\phi} - \mathbf{e}_3 \sin\theta \\
\hat{\boldsymbol{\phi}} &= \mathbf{e}_2 e^{\mathbf{e}_{12}\phi} \\
\hat{\mathbf{r}} &= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{\mathbf{e}_{12}\phi}
\end{aligned} \tag{110.50}$$

Or without exponentials

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= \mathbf{e}_1 \cos\theta \cos\phi + \mathbf{e}_2 \cos\theta \sin\phi - \mathbf{e}_3 \sin\theta \\
\hat{\boldsymbol{\phi}} &= \mathbf{e}_2 \cos\phi - \mathbf{e}_1 \sin\phi \\
\hat{\mathbf{r}} &= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta \cos\phi + \mathbf{e}_2 \sin\theta \sin\phi
\end{aligned} \tag{110.51}$$

Now, having worked out the cool commutator result, it appears that it will actually be harder to use it, then to just calculate the derivatives directly (at least for the $\hat{\boldsymbol{\phi}}$ derivatives). For those we have

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\phi}} &= \partial_\theta \mathbf{e}_2 e^{\mathbf{e}_{12}\phi} \\
&= 0
\end{aligned} \tag{110.52}$$

and

$$\begin{aligned}
\partial_\phi \hat{\boldsymbol{\phi}} &= \partial_\phi \mathbf{e}_2 e^{\mathbf{e}_{12}\phi} \\
&= \mathbf{e}_2 \mathbf{e}_{12} e^{\mathbf{e}_{12}\phi} \\
&= -\mathbf{e}_{12} \hat{\boldsymbol{\phi}}
\end{aligned} \tag{110.53}$$

This multiplication takes $\hat{\boldsymbol{\phi}}$ a vector in the x, y plane and rotates it 90 degrees, leaving an inwards facing radial unit vector in the x,y plane.

Now, having worked out the commutator method, lets at least verify that it works.

$$
\begin{aligned}
\partial_\theta \hat{\boldsymbol{\phi}} &= \left[\hat{\boldsymbol{\phi}}, \Omega_\theta\right] \\
&= \hat{\boldsymbol{\phi}}\Omega_\theta - \Omega_\theta\hat{\boldsymbol{\phi}} \\
&= \frac{1}{2}(\hat{\boldsymbol{\phi}}\mathbf{e}_{31}e^{\mathbf{e}_{12}\phi} - \mathbf{e}_{31}e^{\mathbf{e}_{12}\phi}\hat{\boldsymbol{\phi}}) \\
&= \frac{1}{2}(\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1 e^{-\mathbf{e}_{12}\phi}e^{\mathbf{e}_{12}\phi} - \mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 e^{-\mathbf{e}_{12}\phi}e^{\mathbf{e}_{12}\phi}) \\
&= \frac{1}{2}(-\mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_3\mathbf{e}_1\mathbf{e}_2) \\
&= 0
\end{aligned}
\tag{110.54}
$$

Much harder this way compared to taking the derivative directly, but we at least get the right answer. For the ϕ derivative using the commutator we have

$$
\begin{aligned}
\partial_\phi \hat{\boldsymbol{\phi}} &= \left[\hat{\boldsymbol{\phi}}, \Omega_\phi\right] \\
&= \hat{\boldsymbol{\phi}}\Omega_\phi - \Omega_\phi\hat{\boldsymbol{\phi}} \\
&= \frac{1}{2}(\hat{\boldsymbol{\phi}}\mathbf{e}_{12} - \mathbf{e}_{12}\hat{\boldsymbol{\phi}}) \\
&= \frac{1}{2}(\mathbf{e}_2 e^{\mathbf{e}_{12}\phi}\mathbf{e}_{12} - \mathbf{e}_{12}\mathbf{e}_2 e^{\mathbf{e}_{12}\phi}) \\
&= \frac{1}{2}(-\mathbf{e}_{12}\mathbf{e}_2 e^{\mathbf{e}_{12}\phi} - \mathbf{e}_{12}\mathbf{e}_2 e^{\mathbf{e}_{12}\phi}) \\
&= -\mathbf{e}_{12}\hat{\boldsymbol{\phi}}
\end{aligned}
\tag{110.55}
$$

Good, also consistent with direct calculation. How about our $\hat{\boldsymbol{\theta}}$ derivatives? Lets just calculate these directly without bothering at all with the commutator. This is

$$
\begin{aligned}
\partial_\phi \hat{\boldsymbol{\theta}} &= \mathbf{e}_1 \cos\theta \mathbf{e}_1 2 e^{\mathbf{e}_{12}\phi} \\
&= \mathbf{e}_2 \cos\theta e^{\mathbf{e}_{12}\phi} \\
&= \cos\theta \hat{\boldsymbol{\phi}}
\end{aligned}
\tag{110.56}
$$

and

$$
\begin{aligned}
\partial_\theta \hat{\boldsymbol{\theta}} &= -\mathbf{e}_1 \sin\theta e^{\mathbf{e}_{12}\phi} - \mathbf{e}_3 \cos\theta \\
&= -\mathbf{e}_{12} \sin\theta \hat{\boldsymbol{\phi}} - \mathbf{e}_3 \cos\theta
\end{aligned}
\tag{110.57}
$$

Finally, last we have the derivatives of $\hat{\mathbf{r}}$. Those are

$$
\begin{aligned}
\partial_\phi \hat{\mathbf{r}} &= \mathbf{e}_2 \sin\theta e^{\mathbf{e}_{12}\phi} \\
&= \sin\theta \hat{\boldsymbol{\phi}}
\end{aligned}
\tag{110.58}
$$

and

$$\begin{aligned}
\partial_\theta \hat{\mathbf{r}} &= -\mathbf{e}_3 \sin\theta + \mathbf{e}_1 \cos\theta e^{\mathbf{e}_{12}\phi} \\
&= -\mathbf{e}_3 \sin\theta + \mathbf{e}_{12} \cos\theta \hat{\boldsymbol{\phi}}
\end{aligned} \tag{110.59}$$

Summarizing, all the derivatives we need to evaluate the square of the angular momentum operator are

$$\begin{aligned}
\partial_\theta \hat{\boldsymbol{\phi}} &= 0 \\
\partial_\phi \hat{\boldsymbol{\phi}} &= -\mathbf{e}_{12} \hat{\boldsymbol{\phi}} \\
\partial_\theta \hat{\boldsymbol{\theta}} &= -\mathbf{e}_{12} \sin\theta \hat{\boldsymbol{\phi}} - \mathbf{e}_3 \cos\theta \\
\partial_\phi \hat{\boldsymbol{\theta}} &= \cos\theta \hat{\boldsymbol{\phi}} \\
\partial_\theta \hat{\mathbf{r}} &= -\mathbf{e}_3 \sin\theta + \mathbf{e}_{12} \cos\theta \hat{\boldsymbol{\phi}} \\
\partial_\phi \hat{\mathbf{r}} &= \sin\theta \hat{\boldsymbol{\phi}}
\end{aligned} \tag{110.60}$$

Bugger. We actually want the derivatives of the bivectors $\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}$ so we are not ready to evaluate the squared angular momentum. There is three choices, one is to use these results and apply the chain rule, or start over and directly take the derivatives of these bivectors, or use the commutator result (which did not actually assume vectors and we can apply it to bivectors too if we really wanted to).

An attempt to use the chain rule get messy, but it looks like the bivectors reduce nicely, making it pointless to even think about the commutator method. Introducing some notational conveniences, first write $i = \mathbf{e}_{12}$. We will have to be a bit careful with this since it commutes with $\mathbf{e}_3$, but anticommutes with $\mathbf{e}_1$ or $\mathbf{e}_2$ (and therefore $\hat{\boldsymbol{\phi}}$). As usual we also write $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ for the Euclidean pseudoscalar (which commutes with all vectors and bivectors).

$$\begin{aligned}
\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} &= (\mathbf{e}_3 \cos\theta + i \sin\theta \hat{\boldsymbol{\phi}})(\cos\theta i \hat{\boldsymbol{\phi}} - \mathbf{e}_3 \sin\theta) \\
&= \mathbf{e}_3 \cos^2\theta i \hat{\boldsymbol{\phi}} - i \sin^2\theta \hat{\boldsymbol{\phi}} \mathbf{e}_3 + (i\hat{\boldsymbol{\phi}} i \hat{\boldsymbol{\phi}} - \mathbf{e}_3\mathbf{e}_3) \cos\theta \sin\theta \\
&= i\mathbf{e}_3 (\cos^2\theta + \sin^2\theta) \hat{\boldsymbol{\phi}} + (-\hat{\boldsymbol{\phi}} i^2 \hat{\boldsymbol{\phi}} - 1) \cos\theta \sin\theta
\end{aligned} \tag{110.61}$$

This gives us just

$$\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} = I\hat{\boldsymbol{\phi}} \tag{110.62}$$

and calculation of the bivector partials will follow exclusively from the $\hat{\boldsymbol{\phi}}$ partials tabulated above.

Our other bivector does not reduce quite as cleanly. We have

$$\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} = (\mathbf{e}_3 \cos\theta + i \sin\theta \hat{\boldsymbol{\phi}})\hat{\boldsymbol{\phi}} \tag{110.63}$$

So for this one we have

$$\hat{\mathbf{r}}\hat{\boldsymbol{\phi}} = \mathbf{e}_3\hat{\boldsymbol{\phi}}\cos\theta + i\sin\theta \tag{110.64}$$

Tabulating all the bivector derivatives (details omitted) we have

$$\begin{aligned}
\partial_\theta(\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}) &= 0 \\
\partial_\phi(\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}) &= \mathbf{e}_3\hat{\boldsymbol{\phi}} \\
\partial_\theta(\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}) &= -\mathbf{e}_3\hat{\boldsymbol{\phi}}\sin\theta + i\cos\theta = ie^{I\hat{\boldsymbol{\phi}}\theta} \\
\partial_\phi(\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}) &= -I\hat{\boldsymbol{\phi}}\cos\theta
\end{aligned} \tag{110.65}$$

Okay, we should now be armed to do the squaring of the angular momentum.

110.10 SQUARING THE ANGULAR MOMENTUM OPERATOR

It is expected that we have the equivalence of the squared bivector form of angular momentum with the classical scalar form in terms of spherical angles ϕ, and θ. Specifically, if no math errors have been made playing around with this GA representation, we should have the following identity for the scalar part of the squared angular momentum operator

$$-\left\langle(\mathbf{x}\wedge\boldsymbol{\nabla})^2\right\rangle = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{110.66}$$

To finally attempt to verify this we write the angular momentum operator in polar form, using $i = \mathbf{e}_1\mathbf{e}_2$ as

$$\mathbf{x}\wedge\boldsymbol{\nabla} = \hat{\mathbf{r}}\left(\hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_\phi\right) \tag{110.67}$$

Expressing the unit vectors in terms of $\hat{\boldsymbol{\phi}}$ and after some rearranging we have

$$\mathbf{x}\wedge\boldsymbol{\nabla} = I\hat{\boldsymbol{\phi}}\left(\partial_\theta + ie^{I\hat{\boldsymbol{\phi}}\theta}\frac{1}{\sin\theta}\partial_\phi\right) \tag{110.68}$$

Using this lets now compute the partials. First for the θ partials we have

$$\begin{aligned}
\partial_\theta(\mathbf{x} \wedge \boldsymbol{\nabla}) &= I\hat{\boldsymbol{\phi}}\left(\partial_{\theta\theta} + iI\hat{\boldsymbol{\phi}}e^{I\hat{\boldsymbol{\phi}}\theta}\frac{1}{\sin\theta}\partial_\phi + ie^{I\hat{\boldsymbol{\phi}}\theta}\frac{\cos\theta}{\sin^2\theta}\partial_\phi + ie^{I\hat{\boldsymbol{\phi}}\theta}\frac{1}{\sin\theta}\partial_{\theta\phi}\right) \\
&= I\hat{\boldsymbol{\phi}}\left(\partial_{\theta\theta} + i(I\hat{\boldsymbol{\phi}}e^{I\hat{\boldsymbol{\phi}}\theta}\sin\theta + e^{I\hat{\boldsymbol{\phi}}\theta}\cos\theta)\frac{1}{\sin^2\theta}\partial_\phi + ie^{I\hat{\boldsymbol{\phi}}\theta}\frac{1}{\sin\theta}\partial_{\theta\phi}\right) \\
&= I\hat{\boldsymbol{\phi}}\left(\partial_{\theta\theta} + ie^{2I\hat{\boldsymbol{\phi}}\theta}\frac{1}{\sin^2\theta}\partial_\phi + ie^{I\hat{\boldsymbol{\phi}}\theta}\frac{1}{\sin\theta}\partial_{\theta\phi}\right)
\end{aligned} \tag{110.69}$$

Premultiplying by $I\hat{\boldsymbol{\phi}}$ and taking scalar parts we have the first part of the application of eq. (110.68) on itself,

$$\left\langle I\hat{\boldsymbol{\phi}}\partial_\theta(\mathbf{x} \wedge \boldsymbol{\nabla})\right\rangle = -\partial_{\theta\theta} \tag{110.70}$$

For the ϕ partials it looks like the simplest option is using the computed bivector ϕ partials $\partial_\phi(\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}) = \mathbf{e}_3\hat{\boldsymbol{\phi}}$, $\partial_\phi(\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}) = -I\hat{\boldsymbol{\phi}}\cos\theta$. Doing so we have

$$\begin{aligned}
\partial_\phi(\mathbf{x} \wedge \boldsymbol{\nabla}) &= \partial_\phi\left(\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\partial_\theta + \hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_\phi\right) \\
&= \mathbf{e}_3\hat{\boldsymbol{\phi}}\partial_\theta + +\hat{\mathbf{r}}\hat{\boldsymbol{\theta}}\partial_{\phi\theta} - I\hat{\boldsymbol{\phi}}\cot\theta\partial_\phi + \hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_{\phi\phi}
\end{aligned} \tag{110.71}$$

So the remaining terms of the squared angular momentum operator follow by premultiplying by $\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}/\sin\theta$, and taking scalar parts. This is

$$\left\langle \hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_\phi(\mathbf{x} \wedge \boldsymbol{\nabla})\right\rangle = \frac{1}{\sin\theta}\left\langle -\hat{\mathbf{r}}\mathbf{e}_3\partial_\theta + -\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\theta}}\partial_{\phi\theta} - \hat{\mathbf{r}}I\cot\theta\partial_\phi\right\rangle - \frac{1}{\sin^2\theta}\partial_{\phi\phi} \tag{110.72}$$

The second and third terms in the scalar selection have only bivector parts, but since $\hat{\mathbf{r}} = \mathbf{e}_3\cos\theta + \mathbf{e}_1\sin\theta e^{\mathbf{e}_{12}\phi}$ has component in the $\mathbf{e}_3$ direction, we have

$$\left\langle \hat{\mathbf{r}}\hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_\phi(\mathbf{x} \wedge \boldsymbol{\nabla})\right\rangle = -\cot\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial_{\phi\phi} \tag{110.73}$$

Adding results from eq. (110.70), and eq. (110.73) we have

$$-\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2\right\rangle = \partial_{\theta\theta} + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_{\phi\phi} \tag{110.74}$$

A final verification of eq. (110.66) now only requires a simple calculus expansion

$$\begin{aligned}
\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\psi &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\partial_\theta\psi \\
&= \frac{1}{\sin\theta}(\cos\theta\partial_\theta\psi + \sin\theta\partial_{\theta\theta}\psi) \\
&= \cot\theta\partial_\theta\psi + \partial_{\theta\theta}\psi
\end{aligned} \tag{110.75}$$

Voila. This exercise demonstrating that what was known to have to be true, is in fact explicitly true, is now done. There is no new or interesting results in this in and of itself, but we get some additional confidence in the new methods being experimented with.

110.11 3D QUANTUM HAMILTONIAN

Going back to the quantum Hamiltonian we do still have the angular momentum operator as one of the distinct factors of the Laplacian. As operators we have something akin to the projection of the gradient onto the radial direction, as well as terms that project the gradient onto the tangential plane to the sphere at the radial point

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V = -\frac{\hbar^2}{2m}\left(\frac{1}{\mathbf{x}^2}(\mathbf{x}\cdot\boldsymbol{\nabla})^2 - \frac{1}{\mathbf{x}^2}\left\langle(\mathbf{x}\wedge\boldsymbol{\nabla})^2\right\rangle + \frac{1}{\mathbf{x}}\cdot\boldsymbol{\nabla}\right) + V \tag{110.76}$$

Using the result of eq. (110.19) and the radial formulation for the rest, we can write this

$$\begin{aligned}
0 &= \left(\boldsymbol{\nabla}^2 - \frac{2m}{\hbar^2}(V - E)\right)\psi \\
&= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\left(\mathbf{x}\wedge\boldsymbol{\nabla} - 1\right)(\mathbf{x}\wedge\boldsymbol{\nabla})\psi - \frac{2m}{\hbar^2}(V - E)\psi
\end{aligned} \tag{110.77}$$

If $V = V(r)$, then a radial split by separation of variables is possible. Writing $\psi = R(r)Y$, we get

$$\frac{r}{R}\frac{\partial}{\partial r}r\frac{\partial R}{\partial r} - \frac{2mr^2}{\hbar^2}(V(r) - E) = \frac{1}{Y}\left(\mathbf{x}\wedge\boldsymbol{\nabla} - 1\right)(\mathbf{x}\wedge\boldsymbol{\nabla})Y = \text{constant} \tag{110.78}$$

For the constant, lets use c, and split this into a pair of equations

$$r\frac{\partial}{\partial r}r\frac{\partial R}{\partial r} - \frac{2mr^2R}{\hbar^2}(V(r) - E) = cR \tag{110.79}$$

$$\left(\mathbf{x} \wedge \boldsymbol{\nabla} - 1\right)(\mathbf{x} \wedge \boldsymbol{\nabla})Y = cY \tag{110.80}$$

In this last we can examine simultaneous eigenvalues of $\mathbf{x} \wedge \boldsymbol{\nabla}$, and $\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle$. Suppose that Y_λ is an eigenfunction of $\mathbf{x} \wedge \boldsymbol{\nabla}$ with eigenvalue λ. We then have

$$\begin{aligned}
\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle Y_\lambda &= \left(\mathbf{x} \wedge \boldsymbol{\nabla} - 1\right)(\mathbf{x} \wedge \boldsymbol{\nabla})Y_\lambda \\
&= \left(\mathbf{x} \wedge \boldsymbol{\nabla} - 1\right) \lambda Y_\lambda \\
&= \lambda \left(\lambda - 1\right) Y_\lambda
\end{aligned} \tag{110.81}$$

We see immediately that Y_λ is then also an eigenfunction of $\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle$, with eigenvalue

$$\lambda \left(\lambda - 1\right) \tag{110.82}$$

Bohm gives results for simultaneous eigenfunctions of L_x, L_y, or L_z with L^2, in which case the eigenvalues match. He also shows that eigenfunctions of raising and lowering operators, $L_x \pm iL_y$ are also simultaneous eigenfunctions of L^2, but having $m(m \pm 1)$ eigenvalues. This is something slightly different since we are not considering any specific components, but we still see that eigenfunctions of the bivector angular momentum operator $\mathbf{x} \wedge \boldsymbol{\nabla}$ are simultaneous eigenfunctions of the scalar squared angular momentum operator $\langle \mathbf{x} \wedge \boldsymbol{\nabla} \rangle$ (Q: is that identical to the scalar operator L^2).

Moving on, the next order of business is figuring out how to solve the multivector eigenvalue problem

$$(\mathbf{x} \wedge \boldsymbol{\nabla})Y_\lambda = \lambda Y_\lambda \tag{110.83}$$

110.12 ANGULAR MOMENTUM POLAR FORM, FACTORING OUT THE RAISING AND LOWERING OPERATORS, AND SIMULTANEOUS EIGENVALUES

After a bit more manipulation we find that the angular momentum operator polar form representation, again using $i = \mathbf{e}_1\mathbf{e}_2$, is

$$\mathbf{x} \wedge \boldsymbol{\nabla} = I\hat{\boldsymbol{\phi}}(\partial_\theta + i \cot\theta \partial_\phi + \mathbf{e}_{23} e^{i\phi} \partial_\phi) \tag{110.84}$$

Observe how similar the exponential free terms within the braces are to the raising operator as given in Bohm's equation (14.40)

$$
\begin{aligned}
L_x + iL_y &= e^{i\phi}(\partial_\theta + i\cot\theta\partial_\phi) \\
L_z &= \frac{1}{i}\partial_\phi
\end{aligned}
\tag{110.85}
$$

In fact since $\mathbf{e}_{23}e^{i\phi} = e^{-i\phi}\mathbf{e}_{23}$, the match can be made even closer

$$
\mathbf{x}\wedge\boldsymbol{\nabla} = I\hat{\boldsymbol{\phi}}e^{-i\phi}\Big(\overbrace{\boxed{e^{i\phi}(\partial_\theta + i\cot\theta\partial_\phi)}}^{=L_x+iL_y} + \mathbf{e}_{13}\overbrace{\boxed{\frac{1}{i}\partial_\phi}}^{=L_z}\Big)
\tag{110.86}
$$

This is a surprising factorization, but noting that $\hat{\boldsymbol{\phi}} = \mathbf{e}_2 e^{i\phi}$ we have

$$
\mathbf{x}\wedge\boldsymbol{\nabla} = \mathbf{e}_{31}\left(e^{i\phi}(\partial_\theta + i\cot\theta\partial_\phi) + \mathbf{e}_{13}\frac{1}{i}\partial_\phi\right)
\tag{110.87}
$$

It appears that the factoring out from the left of a unit bivector (in this case $\mathbf{e}_{31}$) from the bivector angular momentum operator, leaves as one of the remainders the raising operator.

Similarly, noting that $\mathbf{e}_{13}$ anticommutes with $i = \mathbf{e}_{12}$, we have the right factorization

$$
\mathbf{x}\wedge\boldsymbol{\nabla} = \left(e^{-i\phi}(\partial_\theta - i\cot\theta\partial_\phi) - \mathbf{e}_{13}\frac{1}{i}\partial_\phi\right)\mathbf{e}_{31}
\tag{110.88}
$$

Now in the remainder, we see the polar form representation of the lowering operator $L_x - iL_y = e^{-i\phi}(\partial_\theta - i\cot\theta\partial_\phi)$.

I was not expecting the raising and lowering operators "to fall out" as they did by simply expressing the complete bivector operator in polar form. This is actually fortuitous since it shows why this peculiar combination is of interest.

If we find a zero solution to the raising or lowering operator, that is also a solution of the eigenproblem $(\partial_\phi - \lambda)\psi = 0$, then this is necessarily also an eigensolution of $\mathbf{x}\wedge\boldsymbol{\nabla}$. A secondary implication is that this is then also an eigensolution of $\left\langle(\mathbf{x}\wedge\boldsymbol{\nabla})^2\right\rangle\psi = \lambda'\psi$. This was the starting point in Bohm's quest for the spherical harmonics, but why he started there was not clear to me.

Saying this without the words, let us look for eigenfunctions for the non-raising portion of eq. (110.87). That is

$$
\mathbf{e}_{31}\mathbf{e}_{13}\frac{1}{i}\partial_\phi f = \lambda f
\tag{110.89}
$$

Since $\mathbf{e}_{31}\mathbf{e}_{13} = 1$ we want solutions of

$$\partial_\phi f = i\lambda f \tag{110.90}$$

Solutions are

$$f = \kappa(\theta)e^{i\lambda\phi} \tag{110.91}$$

A demand that this is a zero eigenfunction for the raising operator, means we are looking for solutions of

$$\mathbf{e}_{31}e^{i\phi}(\partial_\theta + i\cot\theta\partial_\phi)\kappa(\theta)e^{i\lambda\phi} = 0 \tag{110.92}$$

It is sufficient to find zero eigenfunctions of

$$(\partial_\theta + i\cot\theta\partial_\phi)\kappa(\theta)e^{i\lambda\phi} = 0 \tag{110.93}$$

Evaluation of the ϕ partials and rearrangement leaves us with an equation in θ only

$$\frac{\partial\kappa}{\partial\theta} = \lambda\cot\theta\kappa \tag{110.94}$$

This has solutions $\kappa = A(\phi)(\sin\theta)^\lambda$, where because of the partial derivatives in eq. (110.94) we are free to make the integration constant a function of ϕ. Since this is the functional dependence that is a zero of the raising operator, including this at the θ dependence of eq. (110.91) means that we have a simultaneous zero of the raising operator, and an eigenfunction of eigenvalue λ for the remainder of the angular momentum operator.

$$f(\theta, \phi) = (\sin\theta)^\lambda e^{i\lambda\phi} \tag{110.95}$$

This is very similar seeming to the process of adding homogeneous solutions to specific ones, since we augment the specific eigenvalued solutions for one part of the operator by ones that produce zeros for the rest.

As a check lets apply the angular momentum operator to this as a test and see if the results match our expectations.

$$\begin{aligned}
(\mathbf{x} \wedge \boldsymbol{\nabla})(\sin\theta)^\lambda e^{i\lambda\phi} &= \hat{\mathbf{r}}\left(\hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_\phi\right)(\sin\theta)^\lambda e^{i\lambda\phi} \\
&= \hat{\mathbf{r}}\left(\hat{\boldsymbol{\theta}}\lambda(\sin\theta)^{\lambda-1}\cos\theta + \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}(\sin\theta)^\lambda(i\lambda)\right)e^{i\lambda\phi} \\
&= \lambda\hat{\mathbf{r}}\left(\hat{\boldsymbol{\theta}}\cos\theta + \hat{\boldsymbol{\phi}}i\right)e^{i\lambda\phi}(\sin\theta)^{\lambda-1}
\end{aligned} \tag{110.96}$$

From eq. (110.64) we have

$$\begin{aligned}
\hat{\mathbf{r}}\hat{\boldsymbol{\phi}}i &= \mathbf{e}_3\hat{\boldsymbol{\phi}}i\cos\theta - \sin\theta \\
&= \mathbf{e}_{32}ie^{i\phi}\cos\theta - \sin\theta \\
&= \mathbf{e}_{13}e^{i\phi}\cos\theta - \sin\theta
\end{aligned} \tag{110.97}$$

and from eq. (110.62) we have

$$\begin{aligned}
\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} &= I\hat{\boldsymbol{\phi}} \\
&= \mathbf{e}_{31}e^{i\phi}
\end{aligned} \tag{110.98}$$

Putting these together shows that $(\sin\theta)^\lambda e^{i\lambda\phi}$ is an eigenfunction of $\mathbf{x}\wedge\boldsymbol{\nabla}$,

$$(\mathbf{x}\wedge\boldsymbol{\nabla})(\sin\theta)^\lambda e^{i\lambda\phi} = -\lambda(\sin\theta)^\lambda e^{i\lambda\phi} \tag{110.99}$$

This negation surprised me at first, but I do not see any errors here in the arithmetic. Observe that this provides a verification of messy algebra that led to eq. (110.19). That was

$$\left\langle(\mathbf{x}\wedge\boldsymbol{\nabla})^2\right\rangle \stackrel{?}{=} \left(\mathbf{x}\wedge\boldsymbol{\nabla} - 1\right)(\mathbf{x}\wedge\boldsymbol{\nabla}) \tag{110.100}$$

Using this and eq. (110.99) the operator effect of $\left\langle(\mathbf{x}\wedge\boldsymbol{\nabla})^2\right\rangle$ for the eigenvalue we have is

$$\begin{aligned}
\left\langle(\mathbf{x}\wedge\boldsymbol{\nabla})^2\right\rangle(\sin\theta)^\lambda e^{i\lambda\phi} &= \left(\mathbf{x}\wedge\boldsymbol{\nabla} - 1\right)(\mathbf{x}\wedge\boldsymbol{\nabla})(\sin\theta)^\lambda e^{i\lambda\phi} \\
&= ((-\lambda)^2 - (-\lambda))(\sin\theta)^\lambda e^{i\lambda\phi}
\end{aligned} \tag{110.101}$$

So the eigenvalue is $\lambda(\lambda+1)$, consistent with results obtained with coordinate and scalar polar form tools.

110.13 SUMMARY

Having covered a fairly wide range in the preceding Geometric Algebra exploration of the angular momentum operator, it seems worthwhile to attempt to summarize what was learned.

The exploration started with a simple observation that the use of the spatial pseudoscalar for the imaginary of the angular momentum operator in its coordinate form

$$\begin{aligned}
L_1 &= -i\hbar(x_2\partial_3 - x_3\partial_2) \\
L_2 &= -i\hbar(x_3\partial_1 - x_1\partial_3) \\
L_3 &= -i\hbar(x_1\partial_2 - x_2\partial_1)
\end{aligned} \tag{110.102}$$

allowed for expressing the angular momentum operator in its entirety as a bivector valued operator

$$\mathbf{L} = -\hbar \mathbf{x} \wedge \boldsymbol{\nabla} \tag{110.103}$$

The bivector representation has an intrinsic complex behavior, eliminating the requirement for an explicitly imaginary i as is the case in the coordinate representation.

It was then assumed that the Laplacian can be expressed directly in terms of $\mathbf{x} \wedge \boldsymbol{\nabla}$. This is not an unreasonable thought since we can factor the gradient with components projected onto and perpendicular to a constant reference vector $\hat{\mathbf{a}}$ as

$$\boldsymbol{\nabla} = \hat{\mathbf{a}}(\hat{\mathbf{a}} \cdot \boldsymbol{\nabla}) + \hat{\mathbf{a}}(\hat{\mathbf{a}} \wedge \boldsymbol{\nabla}) \tag{110.104}$$

and this squares to a Laplacian expressed in terms of these constant reference directions

$$\boldsymbol{\nabla}^2 = (\hat{\mathbf{a}} \cdot \boldsymbol{\nabla})^2 - (\hat{\mathbf{a}} \cdot \boldsymbol{\nabla})^2 \tag{110.105}$$

a quantity that has an angular momentum like operator with respect to a constant direction. It was then assumed that we could find an operator representation of the form

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2}\left((\mathbf{x} \cdot \boldsymbol{\nabla})^2 - \left\langle (\mathbf{x} \cdot \boldsymbol{\nabla})^2 \right\rangle + f(\mathbf{x}, \boldsymbol{\nabla})\right) \tag{110.106}$$

Where $f(\mathbf{x}, \boldsymbol{\nabla})$ was to be determined, and was found by subtraction. Thinking ahead to relativistic applications this result was obtained for the n-dimensional Laplacian and was found to be

$$\nabla^2 = \frac{1}{x^2}\left((n - 2 + x \cdot \nabla)(x \cdot \nabla) - \left\langle (x \wedge \nabla)^2 \right\rangle\right) \tag{110.107}$$

For the 3D case specifically this is

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2}\left((1 + \mathbf{x} \cdot \boldsymbol{\nabla})(\mathbf{x} \cdot \boldsymbol{\nabla}) - \left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle\right) \tag{110.108}$$

While the scalar selection above is good for some purposes, it interferes with observations about simultaneous eigenfunctions for the angular momentum operator and the scalar part of its square as seen in the Laplacian. With some difficulty and tedium, by subtracting the bivector and quadvector grades from the squared angular momentum operator $(x \wedge \nabla)^2$ it was eventually found that eq. (110.107) can be written as

$$\nabla^2 = \frac{1}{x^2} \left((n-2+x\cdot\nabla)(x\cdot\nabla) + (n-2-x\wedge\nabla)(x\wedge\nabla) + (x\wedge\nabla)\wedge(x\wedge\nabla) \right) \tag{110.109}$$

In the 3D case the quadvector vanishes and eq. (110.108) with the scalar selection removed is reduced to

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2} \left((1+\mathbf{x}\cdot\boldsymbol{\nabla})(\mathbf{x}\cdot\boldsymbol{\nabla}) + (1-\mathbf{x}\wedge\boldsymbol{\nabla})(\mathbf{x}\wedge\boldsymbol{\nabla}) \right) \tag{110.110}$$

In 3D we also have the option of using the duality relation between the cross and the wedge $\mathbf{a}\wedge\mathbf{b} = i(\mathbf{a}\times\mathbf{b})$ to express the Laplacian

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2} \left((1+\mathbf{x}\cdot\boldsymbol{\nabla})(\mathbf{x}\cdot\boldsymbol{\nabla}) + (1-i(\mathbf{x}\times\boldsymbol{\nabla}))i(\mathbf{x}\times\boldsymbol{\nabla}) \right) \tag{110.111}$$

Since it is customary to express angular momentum as $\mathbf{L} = -i\hbar(\mathbf{x}\times\boldsymbol{\nabla})$, we see here that the imaginary in this context should perhaps necessarily be viewed as the spatial pseudoscalar. It was that guess that led down this path, and we come full circle back to this considering how to factor the Laplacian in vector quantities. Curiously this factorization is in no way specific to Quantum Theory.

A few verifications of the Laplacian in eq. (110.111) were made. First it was shown that the directional derivative terms containing $\mathbf{x}\cdot\boldsymbol{\nabla}$, are equivalent to the radial terms of the Laplacian in spherical polar coordinates. That is

$$\frac{1}{\mathbf{x}^2}(1+\mathbf{x}\cdot\boldsymbol{\nabla})(\mathbf{x}\cdot\boldsymbol{\nabla})\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) \tag{110.112}$$

Employing the quaternion operator for the spherical polar rotation

$$\begin{aligned} R &= e^{\mathbf{e}_{31}\theta/2} e^{\mathbf{e}_{12}\phi/2} \\ \mathbf{x} &= r\tilde{R}\mathbf{e}_3 R \end{aligned} \tag{110.113}$$

it was also shown that there was explicitly no radial dependence in the angular momentum operator which takes the form

$$\begin{aligned} \mathbf{x}\wedge\boldsymbol{\nabla} &= \tilde{R}\left(\mathbf{e}_3\mathbf{e}_1 R\partial_\theta + \mathbf{e}_3\mathbf{e}_2 R\frac{1}{\sin\theta}\partial_\phi \right) \\ &= \hat{\mathbf{r}}\left(\hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\partial_\phi \right) \end{aligned} \tag{110.114}$$

Because there is a θ, and ϕ dependence in the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$, squaring the angular momentum operator in this form means that the unit vectors are also operated on. Those vectors were given by the triplet

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \tilde{R} \begin{pmatrix} \mathbf{e}_3 \\ \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} R \tag{110.115}$$

Using $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ for the spatial pseudoscalar, and $i = \mathbf{e}_1\mathbf{e}_2$ (a possibly confusing switch of notation) for the bivector of the x-y plane we can write the spherical polar unit vectors in exponential form as

$$\begin{pmatrix} \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_2 e^{i\phi} \\ \mathbf{e}_3 e^{I\hat{\boldsymbol{\phi}}\theta} \\ i\hat{\boldsymbol{\phi}} e^{I\hat{\boldsymbol{\phi}}\theta} \end{pmatrix} \tag{110.116}$$

These or related expansions were used to verify (with some difficulty) that the scalar squared bivector operator is identical to the expected scalar spherical polar coordinates parts of the Laplacian

$$-\left\langle (\mathbf{x} \wedge \boldsymbol{\nabla})^2 \right\rangle = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \tag{110.117}$$

Additionally, by left or right dividing a unit bivector from the angular momentum operator, we are able to find that the raising and lowering operators are left as one of the factors

$$\begin{aligned} \mathbf{x} \wedge \boldsymbol{\nabla} &= \mathbf{e}_{31} \left(e^{i\phi} (\partial_\theta + i \cot\theta \partial_\phi) + \mathbf{e}_{13} \frac{1}{i} \partial_\phi \right) \\ \mathbf{x} \wedge \boldsymbol{\nabla} &= \left(e^{-i\phi} (\partial_\theta - i \cot\theta \partial_\phi) - \mathbf{e}_{13} \frac{1}{i} \partial_\phi \right) \mathbf{e}_{31} \end{aligned} \tag{110.118}$$

Both of these use $i = \mathbf{e}_1\mathbf{e}_2$, the bivector for the plane, and not the spatial pseudoscalar. We are then able to see that in the context of the raising and lowering operator for the radial equation the interpretation of the imaginary should be one of a plane.

Using the raising operator factorization, it was calculated that $(\sin\theta)^\lambda e^{i\lambda\phi}$ was an eigenfunction of the bivector operator $\mathbf{x} \wedge \boldsymbol{\nabla}$ with eigenvalue $-\lambda$. This results in the simultaneous eigenvalue of $\lambda(\lambda + 1)$ for this eigenfunction with the scalar squared angular momentum operator.

There are a few things here that have not been explored to their logical conclusion.

The bivector Fourier projections $I\mathbf{e}_k(\mathbf{x} \wedge \boldsymbol{\nabla}) \cdot (-I\mathbf{e}_k)$ do not obey the commutation relations of the scalar angular momentum components, so an attempt to directly use these to construct raising and lowering operators does not produce anything useful. The raising and lowering operators in a form that could be used to find eigensolutions were found by factoring out $\mathbf{e}_{13}$ from the bivector operator. Making this particular factorization was a fluke and only because it was desirable to express the bivector operator entirely in spherical polar form. It is curious that this results in raising and lowering operators for the x,y plane, and understanding this further would be nice.

In the eigen solutions for the bivector operator, no quantization condition was imposed. I do not understand the argument that Bohm used to do so in the traditional treatment, and revisiting this once that is done is in order.

I am also unsure exactly how Bohm knows that the inner product for the eigenfunctions should be a surface integral. This choice works, but what drives it. Can that be related to anything here?

111

GRAPHICAL REPRESENTATION OF SPHERICAL HARMONICS FOR $l = 1$

111.1 FIRST OBSERVATIONS

In Bohm's QT [2], 14.17), the properties of $l = 1$ associated Legendre polynomials are examined under rotation. Wikipedia ([46] calls these eigen functions the spherical harmonics.

The unnormalized eigenfunctions are given (eqn (14.47) in Bohm) for $s \in [0, l]$, with $\cos\theta = \zeta$ by

$$\psi_l^{l-s} = \frac{e^{i(l-s)\phi}}{(1-\zeta^2)^{(l-s)/2}} \frac{\partial^s}{\partial\zeta^s}(1-\zeta^2)^l \tag{111.1}$$

The normalization is provided by a surface area inner product

$$(u, v) = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} uv^* \sin\theta d\theta d\phi \tag{111.2}$$

Computing these for $l = 1$, and disregarding any normalization these eigenfunctions can be found to be

$$\begin{aligned} \psi_1^1 &= \sin\theta e^{i\phi} \\ \psi_1^0 &= \cos\theta \\ \psi_1^{-1} &= \sin\theta e^{-i\phi} \end{aligned} \tag{111.3}$$

There is a direct relationship between these eigenfunctions with a triple of vectors associated with a point on the unit sphere. Referring to fig. 111.1, observe the three doubled arrow vectors, all associated with a point on the unit sphere $\mathbf{x} = (x, y, z) = (\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta)$.

The normal to the x, y plane from $\mathbf{x}$, designated $\mathbf{n}$ has the vectorial value

$$\mathbf{n} = \cos\theta\mathbf{e}_3 \tag{111.4}$$

From the origin to the point of of the x, y plane intersection to the normal we have

$$\boldsymbol{\rho} = \sin\theta(\cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2) = \mathbf{e}_1\sin\theta e^{\mathbf{e}_1\mathbf{e}_2\phi} \tag{111.5}$$

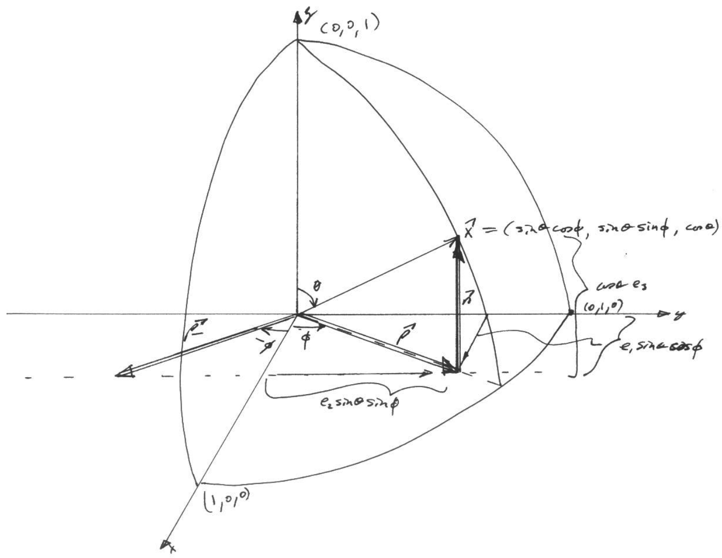

Figure 111.1: Vectoring the $l = 1$ associated Legendre polynomials

and finally in the opposite direction also in the plane and mirroring $\boldsymbol{\rho}$ we have the last of this triplet of vectors

$$\boldsymbol{\rho}_- = \sin\theta(\cos\phi \mathbf{e}_1 - \sin\phi \mathbf{e}_2) = \mathbf{e}_1 \sin\theta e^{-\mathbf{e}_1\mathbf{e}_2\phi} \tag{111.6}$$

So, if we choose to use $i = \mathbf{e}_1\mathbf{e}_2$ (the bivector for the plane normal to the z-axis), then we can in fact vectorize these eigenfunctions. The vectors $\boldsymbol{\rho}$ (i.e. ψ_1^1), and $\boldsymbol{\rho}_-$ (i.e. ψ_1^{-1}) are both normal to $\mathbf{n}$ (i.e. ψ_1^0), but while the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_-$ are both in the plane one is produced with a counterclockwise rotation of $\mathbf{e}_1$ by ϕ in the plane and the other with an opposing rotation.

Summarizing, we can write the unnormalized vectors the relations

$$\begin{aligned}
\psi_1^1 &= \mathbf{e}_1\boldsymbol{\rho} &&= \sin\theta e^{\mathbf{e}_1\mathbf{e}_2\phi} \\
\psi_1^0 &= \mathbf{e}_3\mathbf{n} &&= \cos\theta \\
\psi_1^{-1} &= \mathbf{e}_1\boldsymbol{\rho}_- &&= \sin\theta e^{-\mathbf{e}_1\mathbf{e}_2\phi}
\end{aligned}$$

I have no familiarity yet with the $l = 2$ or higher Legendre eigenfunctions. Do they also admit a geometric representation?

111.2 EXPRESSING LEGENDRE EIGENFUNCTIONS USING ROTATIONS

We can express a point on a sphere with a pair of rotation operators. First rotating $\mathbf{e}_3$ towards $\mathbf{e}_1$ in the z, x plane by θ, then in the x, y plane by ϕ we have the point $\mathbf{x}$ in fig. 111.1

Writing the result of the first rotation as $\mathbf{e}_3'$ we have

$$\mathbf{e}_3' = \mathbf{e}_3 e^{\mathbf{e}_{31}\theta} = e^{-\mathbf{e}_{31}\theta/2}\mathbf{e}_3 e^{\mathbf{e}_{31}\theta/2} \tag{111.7}$$

One more rotation takes $\mathbf{e}_3'$ to $\mathbf{x}$. That is

$$\mathbf{x} = e^{-\mathbf{e}_{12}\phi/2}\mathbf{e}_3' e^{\mathbf{e}_{12}\phi/2} \tag{111.8}$$

All together, writing $R_\theta = e^{\mathbf{e}_{31}\theta/2}$, and $R_\phi = e^{\mathbf{e}_{12}\phi/2}$, we have

$$\mathbf{x} = \tilde{R}_\phi \tilde{R}_\theta \mathbf{e}_3 R_\theta R_\phi \tag{111.9}$$

It is worth a quick verification that this produces the desired result.

$$\begin{aligned}
\tilde{R}_\phi \tilde{R}_\theta \mathbf{e}_3 R_\theta R_\phi &= \tilde{R}_\phi \mathbf{e}_3 e^{\mathbf{e}_{31}\theta} R_\phi \\
&= e^{-\mathbf{e}_{12}\phi/2}(\mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta) e^{\mathbf{e}_{12}\phi/2} \\
&= \mathbf{e}_3 \cos\theta + \mathbf{e}_1 \sin\theta e^{\mathbf{e}_{12}\phi}
\end{aligned} \tag{111.10}$$

This is the expected result

$$\mathbf{x} = \mathbf{e}_3 \cos\theta + \sin\theta(\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta) \tag{111.11}$$

The projections onto the $\mathbf{e}_3$ and the x, y plane are then, respectively,

$$\begin{aligned}
\mathbf{x}_z &= \mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{x}) = \mathbf{e}_3 \cos\theta \\
\mathbf{x}_{x,y} &= \mathbf{e}_3(\mathbf{e}_3 \wedge \mathbf{x}) = \sin\theta(\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta)
\end{aligned} \tag{111.12}$$

So if $\mathbf{x}_\pm$ is the point on the unit sphere associated with the rotation angles $\theta, \pm\phi$, then we have for the $l = 1$ associated Legendre polynomials

$$\begin{aligned}
\psi_1^0 &= \mathbf{e}_3 \cdot \mathbf{x} \\
\psi_1^{\pm 1} &= \mathbf{e}_1 \mathbf{e}_3(\mathbf{e}_3 \wedge \mathbf{x}_\pm)
\end{aligned} \tag{111.13}$$

Note that the $\pm$ was omitted from $\mathbf{x}$ for ψ_1^0 since either produces the same $\mathbf{e}_3$ component. This gives us a nice geometric interpretation of these eigenfunctions. We see that ψ_1^0 is the biggest when $\mathbf{x}$ is close to straight up, and when this occurs $\psi_1^{\pm 1}$ are correspondingly reduced, but when $\mathbf{x}$ is close to the x, y plane where $\psi_1^{\pm 1}$ will be greatest the z-axis component is reduced.

112

BIVECTOR GRADES OF THE SQUARED ANGULAR MOMENTUM OPERATOR

112.1 MOTIVATION

The aim here is to extract the bivector grades of the squared angular momentum operator

$$\left\langle (x \wedge \nabla)^2 \right\rangle_2 \stackrel{?}{=} \cdots \tag{112.1}$$

I had tried this before and believe gotten it wrong. Take it super slow and dumb and careful.

112.2 NON-OPERATOR EXPANSION

Suppose P is a bivector, $P = (\gamma^k \wedge \gamma^m) P_{km}$, the grade two product with a different unit bivector is

$$\begin{aligned}
&\left\langle (\gamma_a \wedge \gamma_b)(\gamma^k \wedge \gamma^m) \right\rangle_2 P_{km} \\
&= \left\langle (\gamma_a \gamma_b - \gamma_a \cdot \gamma_b)(\gamma^k \wedge \gamma^m) \right\rangle_2 P_{km} \\
&= \left\langle \gamma_a (\gamma_b \cdot (\gamma^k \wedge \gamma^m)) \right\rangle_2 P_{km} + \left\langle \gamma_a (\gamma_b \wedge (\gamma^k \wedge \gamma^m)) \right\rangle_2 P_{km} - (\gamma_a \cdot \gamma_b)(\gamma^k \wedge \gamma^m) P_{km} \\
&= (\gamma_a \wedge \gamma^m) P_{bm} - (\gamma_a \wedge \gamma^k) P_{kb} - (\gamma_a \cdot \gamma_b)(\gamma^k \wedge \gamma^m) P_{km} \\
&+ (\gamma_a \cdot \gamma_b)(\gamma^k \wedge \gamma^m) P_{km} - (\gamma_b \wedge \gamma^m) P_{am} + (\gamma_b \wedge \gamma^k) P_{ka} \\
&= (\gamma_a \wedge \gamma^c)(P_{bc} - P_{cb}) + (\gamma_b \wedge \gamma^c)(P_{ca} - P_{ac})
\end{aligned} \tag{112.2}$$

This same procedure will be used for the operator square, but we have the complexity of having the second angular momentum operator change the first bivector result.

112.3 OPERATOR EXPANSION

In the first few lines of the bivector product expansion above, a blind replacement $\gamma_a \rightarrow x$, and $\gamma_b \rightarrow \nabla$ gives us

$$\begin{aligned}
&\left\langle (x \wedge \nabla)(\gamma^k \wedge \gamma^m) \right\rangle_2 P_{km} \\
&= \left\langle (x\nabla - x \cdot \nabla)(\gamma^k \wedge \gamma^m) \right\rangle_2 P_{km} \\
&= \left\langle x(\nabla \cdot (\gamma^k \wedge \gamma^m)) \right\rangle_2 P_{km} + \left\langle x(\nabla \wedge (\gamma^k \wedge \gamma^m)) \right\rangle_2 P_{km} - (x \cdot \nabla)(\gamma^k \wedge \gamma^m) P_{km}
\end{aligned} \tag{112.3}$$

Using $P_{km} = x_k \partial_m$, eliminating the coordinate expansion we have an intermediate result that gets us partway to the desired result

$$\left\langle (x \wedge \nabla)^2 \right\rangle_2 = \langle x(\nabla \cdot (x \wedge \nabla)) \rangle_2 + \langle x(\nabla \wedge (x \wedge \nabla)) \rangle_2 - (x \cdot \nabla)(x \wedge \nabla) \tag{112.4}$$

An expansion of the first term should be easier than the second. Dropping back to coordinates we have

$$\begin{aligned}
\langle x(\nabla \cdot (x \wedge \nabla)) \rangle_2 &= \left\langle x(\nabla \cdot (\gamma^k \wedge \gamma^m)) \right\rangle_2 x_k \partial_m \\
&= \left\langle x(\gamma_a \partial^a \cdot (\gamma^k \wedge \gamma^m)) \right\rangle_2 x_k \partial_m \\
&= \left\langle x \gamma^m \partial^k \right\rangle_2 x_k \partial_m - \left\langle x \gamma^k \partial^m \right\rangle_2 x_k \partial_m \\
&= x \wedge (\partial^k x_k \gamma^m \partial_m) - x \wedge (\partial^m \gamma^k x_k \partial_m)
\end{aligned} \tag{112.5}$$

Okay, a bit closer. Backpedaling with the reinsertion of the complete vector quantities we have

$$\langle x(\nabla \cdot (x \wedge \nabla)) \rangle_2 = x \wedge (\partial^k x_k \nabla) - x \wedge (\partial^m x \partial_m) \tag{112.6}$$

Expanding out these two will be conceptually easier if the functional operation is made explicit. For the first

$$\begin{aligned}
x \wedge (\partial^k x_k \nabla)\phi &= x \wedge x_k \partial^k (\nabla \phi) + x \wedge ((\partial^k x_k)\nabla)\phi \\
&= x \wedge ((x \cdot \nabla)(\nabla \phi)) + n(x \wedge \nabla)\phi
\end{aligned} \tag{112.7}$$

In operator form this is

$$x \wedge (\partial^k x_k \nabla) = n(x \wedge \nabla) + x \wedge ((x \cdot \nabla)\nabla) \tag{112.8}$$

Now consider the second half of eq. (112.6). For that we expand

$$x \wedge (\partial^m x \partial_m)\phi = x \wedge (x \partial_m \partial^m \phi) + x \wedge ((\partial^m x)\partial_m \phi) \tag{112.9}$$

Since $x \wedge x = 0$, and $\partial^m x = \partial^m x_k \gamma^k = \gamma^m$, we have

$$\begin{aligned} x \wedge (\partial^m x \partial_m)\phi &= x \wedge (\gamma^m \partial_m)\phi \\ &= (x \wedge \nabla)\phi \end{aligned} \tag{112.10}$$

Putting things back together we have for eq. (112.6)

$$\langle x(\nabla \cdot (x \wedge \nabla))\rangle_2 = (n-1)(x \wedge \nabla) + x \wedge ((x \cdot \nabla)\nabla) \tag{112.11}$$

This now completes a fair amount of the bivector selection, and a substitution back into eq. (112.4) yields

$$\left\langle (x \wedge \nabla)^2 \right\rangle_2 = (n - 1 - x \cdot \nabla)(x \wedge \nabla) + x \wedge ((x \cdot \nabla)\nabla) + x \cdot (\nabla \wedge (x \wedge \nabla)) \tag{112.12}$$

The remaining task is to explicitly expand the last vector-trivector dot product. To do that we use the basic alternation expansion identity

$$a \cdot (b \wedge c \wedge d) = (a \cdot b)(c \wedge d) - (a \cdot c)(b \wedge d) + (a \cdot d)(b \wedge c) \tag{112.13}$$

To see how to apply this to the operator case lets write that explicitly but temporarily in coordinates

$$\begin{aligned} x \cdot ((\nabla \wedge (x \wedge \nabla))\phi &= (x^\mu \gamma_\mu) \cdot ((\gamma^\nu \partial_\nu) \wedge (x_\alpha \gamma^\alpha \wedge (\gamma^\beta \partial_\beta)))\phi \\ &= x \cdot \nabla (x \wedge \nabla)\phi - x \cdot \gamma^\alpha \nabla \wedge x_\alpha \nabla \phi + x^\mu \nabla \wedge x \gamma_\mu \cdot \gamma^\beta \partial_\beta \phi \\ &= x \cdot \nabla (x \wedge \nabla)\phi - x^\alpha \nabla \wedge x_\alpha \nabla \phi + x^\mu \nabla \wedge x \partial_\mu \phi \end{aligned} \tag{112.14}$$

Considering this term by term starting with the second one we have

$$\begin{aligned} x^\alpha \nabla \wedge x_\alpha \nabla \phi &= x_\alpha (\gamma^\mu \partial_\mu) \wedge x^\alpha \nabla \phi \\ &= x_\alpha \gamma^\mu \wedge (\partial_\mu x^\alpha) \nabla \phi + x_\alpha \gamma^\mu \wedge x^\alpha \partial_\mu \nabla \phi \\ &= x_\mu \gamma^\mu \wedge \nabla \phi + x_\alpha x^\alpha \gamma^\mu \wedge \partial_\mu \nabla \phi \\ &= x \wedge \nabla \phi + x^2 \nabla \wedge \nabla \phi \end{aligned} \tag{112.15}$$

The curl of a gradient is zero, since summing over an product of antisymmetric and symmetric indices $\gamma^\mu \wedge \gamma^\nu \partial_{\mu\nu}$ is zero. Only one term remains to evaluate in the vector-trivector dot product now

$$x \cdot (\nabla \wedge x \wedge \nabla) = (-1 + x \cdot \nabla)(x \wedge \nabla) + x^\mu \nabla \wedge x \partial_\mu \tag{112.16}$$

Again, a completely dumb and brute force expansion of this is

$$\begin{aligned} x^\mu \nabla \wedge x \partial_\mu \phi &= x^\mu (\gamma^\nu \partial_\nu) \wedge (x^\alpha \gamma_\alpha) \partial_\mu \phi \\ &= x^\mu \gamma^\nu \wedge (\partial_\nu (x^\alpha \gamma_\alpha)) \partial_\mu \phi + x^\mu \gamma^\nu \wedge (x^\alpha \gamma_\alpha) \partial_\nu \partial_\mu \phi \\ &= x^\mu (\gamma^\alpha \wedge \gamma_\alpha) \partial_\mu \phi + x^\mu \gamma^\nu \wedge x \partial_\nu \partial_\mu \phi \end{aligned} \tag{112.17}$$

With $\gamma^\mu = \pm\gamma_\mu$, the wedge in the first term is zero, leaving

$$\begin{aligned} x^\mu \nabla \wedge x \partial_\mu \phi &= -x^\mu x \wedge \gamma^\nu \partial_\nu \partial_\mu \phi \\ &= -x^\mu x \wedge \gamma^\nu \partial_\mu \partial_\nu \phi \\ &= -x \wedge x^\mu \partial_\mu \gamma^\nu \partial_\nu \phi \end{aligned} \tag{112.18}$$

In vector form we have finally

$$x^\mu \nabla \wedge x \partial_\mu \phi = -x \wedge (x \cdot \nabla) \nabla \phi \tag{112.19}$$

The final expansion of the vector-trivector dot product is now

$$x \cdot (\nabla \wedge x \wedge \nabla) = (-1 + x \cdot \nabla)(x \wedge \nabla) - x \wedge (x \cdot \nabla) \nabla \phi \tag{112.20}$$

This was the last piece we needed for the bivector grade selection. Incorporating this into eq. (112.12), both the $x \cdot \nabla x \wedge \nabla$, and the $x \wedge (x \cdot \nabla) \nabla$ terms cancel leaving the surprising simple result

$$\left\langle (x \wedge \nabla)^2 \right\rangle_2 = (n-2)(x \wedge \nabla) \tag{112.21}$$

The power of this result is that it allows us to write the scalar angular momentum operator from the Laplacian as

$$\begin{aligned} \left\langle (x \wedge \nabla)^2 \right\rangle &= (x \wedge \nabla)^2 - \left\langle (x \wedge \nabla)^2 \right\rangle_2 - (x \wedge \nabla) \wedge (x \wedge \nabla) \\ &= (x \wedge \nabla)^2 - (n-2)(x \wedge \nabla) - (x \wedge \nabla) \wedge (x \wedge \nabla) \\ &= (-(n-2) + (x \wedge \nabla) - (x \wedge \nabla) \wedge)(x \wedge \nabla) \end{aligned} \tag{112.22}$$

The complete Laplacian is

$$\nabla^2 = \frac{1}{x^2}(x \cdot \nabla)^2 + (n-2)\frac{1}{x} \cdot \nabla - \frac{1}{x^2}\Big((x \wedge \nabla)^2 - (n-2)(x \wedge \nabla) - (x \wedge \nabla) \wedge (x \wedge \nabla)\Big) \tag{112.23}$$

In particular in less than four dimensions the quad-vector term is necessarily zero. The 3D Laplacian becomes

$$\boldsymbol{\nabla}^2 = \frac{1}{\mathbf{x}^2}(1 + \mathbf{x} \cdot \boldsymbol{\nabla})(\mathbf{x} \cdot \boldsymbol{\nabla}) + \frac{1}{\mathbf{x}^2}(1 - \mathbf{x} \wedge \boldsymbol{\nabla})(\mathbf{x} \wedge \boldsymbol{\nabla}) \tag{112.24}$$

So any eigenfunction of the bivector angular momentum operator $\mathbf{x} \wedge \boldsymbol{\nabla}$ is necessarily a simultaneous eigenfunction of the scalar operator.

Part XI

FOURIER TREATMENTS

113 FOURIER SOLUTIONS TO HEAT AND WAVE EQUATIONS

113.1 MOTIVATION

Stanford iTunesU has some Fourier transform lectures by Prof. Brad Osgood. He starts with Fourier series and by Lecture 5 has covered this and the solution of the Heat equation on a ring as an example.

Now, for these lectures I get only sound on my ipod. I can listen along and pick up most of the lectures since this is review material, but here is some notes to firm things up.

Since this heat equation

$$\nabla^2 u = \kappa \partial_t u \tag{113.1}$$

is also the Schrödinger equation for a free particle in one dimension (once the constant is fixed appropriately), we can also apply the Fourier technique to a particle constrained to a circle. It would be interesting afterwards to contrast this with Susskind's solution of the same problem (where he used the Fourier transform and algebraic techniques instead).

113.2 PRELIMINARIES

113.2.1 *Laplacian*

Osgood wrote the heat equation for the ring as

$$\frac{1}{2} u_{xx} = u_t \tag{113.2}$$

where x represented an angular position on the ring, and where he set the heat diffusion constant to $1/2$ for convenience. To apply this to the Schrödinger equation retaining all the desired units we want to be a bit more careful, so let us start with the Laplacian in polar coordinates.

In polar coordinates our gradient is

$$\nabla = \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{r}} \frac{\partial}{\partial r} \tag{113.3}$$

squaring this we have

$$
\begin{aligned}
\nabla^2 = \nabla \cdot \nabla &= \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} \cdot \left(\hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \hat{\mathbf{r}} \frac{\partial}{\partial r} \cdot \left(\hat{\mathbf{r}} \frac{\partial}{\partial r} \right) \\
&= \frac{-1}{r^3} \frac{\partial r}{\partial \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2} \\
&= \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r^2}
\end{aligned}
\tag{113.4}
$$

So for the circularly constrained where r is constant case we have simply

$$
\nabla^2 = \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \tag{113.5}
$$

and our heat equation to solve becomes

$$
\frac{\partial^2 u(\theta, t)}{\partial \theta^2} = (r^2 \kappa) \frac{\partial u(\theta, t)}{\partial t} \tag{113.6}
$$

113.2.2 *Fourier series*

Now we also want Fourier series for a given period. Assuming the absence of the "Rigor Police" as Osgood puts it we write for a periodic function $f(x)$ known on the interval $I = [a, a + T]$

$$
f(x) = \sum c_k e^{2\pi i k x/T} \tag{113.7}
$$

$$
\begin{aligned}
\int_{\partial I} f(x) e^{-2\pi i n x/T} &= \sum c_k \int_{\partial I} e^{2\pi i (k-n) x/T} \\
&= c_n T
\end{aligned}
\tag{113.8}
$$

So our Fourier coefficient is

$$
\hat{f}(n) = c_n = \frac{1}{T} \int_{\partial I} f(x) e^{-2\pi i n x/T} \tag{113.9}
$$

113.3 SOLUTION OF HEAT EQUATION

113.3.1 *Basic solution*

Now we are ready to solve the radial heat equation

$$u_{\theta\theta} = r^2 \kappa u_t, \tag{113.10}$$

by assuming a Fourier series solution.
Suppose

$$\begin{aligned} u(\theta, t) &= \sum c_n(t) e^{2\pi i n\theta/T} \\ &= \sum c_n(t) e^{in\theta} \end{aligned} \tag{113.11}$$

Taking derivatives of this assumed solution we have

$$\begin{aligned} u_{\theta\theta} &= \sum (in)^2 c_n e^{in\theta} \\ u_t &= \sum c'_n e^{in\theta} \end{aligned} \tag{113.12}$$

Substituting this back into eq. (113.10) we have

$$\sum -n^2 c_n e^{in\theta} = \sum c'_n r^2 \kappa e^{in\theta} \tag{113.13}$$

equating components we have

$$c'_n = -\frac{n^2}{r^2\kappa} c_n \tag{113.14}$$

which is also just an exponential.

$$c_n = A_n \exp\left(-\frac{n^2}{r^2\kappa} t\right) \tag{113.15}$$

Reassembling we have the time variation of the solution now fixed and can write

$$u(\theta, t) = \sum A_n \exp\left(-\frac{n^2}{r^2\kappa} t + in\theta\right) \tag{113.16}$$

113.3.2 *As initial value problem*

For the heat equation case, we can assume a known initial heat distribution $f(\theta)$. For an initial time $t = 0$ we can then write

$$u(\theta, 0) = \sum A_n e^{in\theta} = f(\theta) \tag{113.17}$$

This is just another Fourier series, with Fourier coefficients

$$A_n = \frac{1}{2\pi} \int_{\partial I} f(v) e^{-inv} dv \tag{113.18}$$

Final reassembly of the results gives us

$$u(\theta, t) = \sum \exp\left(-\frac{n^2}{r^2\kappa} t + in\theta\right) \frac{1}{2\pi} \int_{\partial I} f(v) e^{-inv} dv \tag{113.19}$$

113.3.3 *Convolution*

Osgood's next step, also with the rigor police in hiding, was to exchange orders of integration and summation, to write

$$u(\theta, t) = \int_{\partial I} f(v) dv \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2}{r^2\kappa} t - in(v - \theta)\right) \tag{113.20}$$

Introducing a Green's function $g(v, t)$, we then have the complete solution in terms of convolution

$$\begin{aligned} g(v, t) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2}{r^2\kappa} t - inv\right) \\ u(\theta, t) &= \int_{\partial I} f(v) g(v - \theta, t) dv \end{aligned} \tag{113.21}$$

Now, this Green's function is fairly interesting. By summing over paired negative and positive indices, we have a set of weighted Gaussians.

$$g(v, t) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \exp\left(-\frac{n^2}{r^2\kappa} t\right) \frac{\cos(nv)}{\pi} \tag{113.22}$$

Recalling that the delta function can be expressed as a limit of a sinc function, seeing something similar in this Green's function is not entirely unsurprising seeming.

113.4 WAVE EQUATION

The QM equation for a free particle is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\partial_t\psi \tag{113.23}$$

This has the same form of the heat equation, so for the free particle on a circle our wave equation is

$$\psi_{\theta\theta} = -\frac{2mir^2}{\hbar}\partial_t\psi \qquad \text{ie: } \kappa = -2mi/\hbar \tag{113.24}$$

So, if the wave equation was known at an initial time $\psi(\theta, 0) = \phi(\theta)$, we therefore have by comparison the time evolution of the particle's wave function is

$$\begin{aligned}
g(w, t) &= \frac{1}{2\pi} + \sum_{n=1}^{\infty} \exp\left(-\frac{i\hbar n^2 t}{2mr^2}\right)\frac{\cos(nw)}{\pi} \\
\psi(\theta, t) &= \int_{\partial I} \phi(v)g(v-\theta, t)dv
\end{aligned} \tag{113.25}$$

113.5 FOURIER TRANSFORM SOLUTION

Now, lets try this one dimensional heat problem with a Fourier transform instead to compare. Here we do not try to start with an assumed solution, but instead take the Fourier transform of both sides of the equation directly.

$$\mathcal{F}(u_{xx}) = \kappa\mathcal{F}(u_t) \tag{113.26}$$

Let us start with the left hand side, where we can evaluate by integrating by parts

$$\begin{aligned}
\mathcal{F}(u_{xx}) &= \int_{-\infty}^{\infty} u_{xx}(x, t)e^{-2\pi isx}dx \\
&= \int_{-\infty}^{\infty} \frac{\partial u_x(x, t)}{\partial x}e^{-2\pi isx}dx \\
&= \left(u_x(x, t)e^{-2\pi isx}\Big|_{x=-\infty}^{\infty} - (-2\pi is)\int_{-\infty}^{\infty} u_x(x, t)e^{-2\pi isx}dx\right)
\end{aligned} \tag{113.27}$$

So if we assume (or require) that the derivative of our unknown function u is zero at infinity, and then similarly require the function itself to be zero there, we have

$$\begin{aligned}\mathcal{F}(u_{xx}) &= (2\pi i s)\int_{-\infty}^{\infty} \frac{\partial u_x(x,t)}{\partial x} e^{-2\pi i s x} dx \\ &= (2\pi i s)^2 \int_{-\infty}^{\infty} u(x,t) e^{-2\pi i s x} dx \\ &= (2\pi i s)^2 \mathcal{F}(u)\end{aligned} \tag{113.28}$$

Now, for the time derivative. We want

$$\mathcal{F}(u_t) = \int_{-\infty}^{\infty} u_t(x,t) e^{-2\pi i s x} dx \tag{113.29}$$

But can pull the derivative out of the integral for

$$\begin{aligned}\mathcal{F}(u_t) &= \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} u(x,t) e^{-2\pi i s x} dx\right) \\ &= \frac{\partial \mathcal{F}(u)}{\partial t}\end{aligned} \tag{113.30}$$

So, now we have an equation relating time derivatives only of the Fourier transformed solution.

Writing $\mathcal{F}(u) = \hat{u}$ this is

$$(2\pi i s)^2 \hat{u} = \kappa \frac{\partial \hat{u}}{\partial t} \tag{113.31}$$

With a solution of

$$\hat{u} = A(s) e^{-4\pi^2 s^2 t/\kappa} \tag{113.32}$$

Here $A(s)$ is an arbitrary constant in time integration constant, which may depend on s since it is a solution of our simpler frequency domain partial differential equation eq. (113.31).

Performing an inverse transform to recover $u(x,t)$ we thus have

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty} \hat{u} e^{2\pi i x s} ds \\ &= \int_{-\infty}^{\infty} A(s) e^{-4\pi^2 s^2 t/\kappa} e^{2\pi i x s} ds\end{aligned} \tag{113.33}$$

Now, how about initial conditions. Suppose we have $u(x,0) = f(x)$, then

$$f(x) = \int_{-\infty}^{\infty} A(s) e^{2\pi i x s} ds \tag{113.34}$$

Which is just an inverse Fourier transform in terms of the integration "constant" $A(s)$. We can therefore write the $A(s)$ in terms of the initial time domain conditions.

$$\begin{aligned} A(s) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} dx \\ &= \hat{f}(s) \end{aligned} \tag{113.35}$$

and finally have a complete solution of the one dimensional Heat equation. That is

$$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{-4\pi^2 s^2 t/\kappa} e^{2\pi i x s} ds \tag{113.36}$$

113.5.1 *With Green's function?*

If we put in the integral for $\hat{f}(s)$ explicitly and switch the order as was done with the Fourier series will we get a similar result? Let us try

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) e^{-2\pi i s u} du \right) e^{-4\pi^2 s^2 t/\kappa} e^{2\pi i x s} ds \\ &= \int_{-\infty}^{\infty} du f(u) \int_{-\infty}^{\infty} e^{-4\pi^2 s^2 t/\kappa} e^{2\pi i (x-u) s} ds \end{aligned} \tag{113.37}$$

Cool. So, with the introduction of a Green's function $g(w,t)$ for the fundamental solution of the heat equation, we therefore have our solution in terms of convolution with the initial conditions. It does not get any more general than this!

$$\begin{aligned} g(w,t) &= \int_{-\infty}^{\infty} \exp\left(-\frac{4\pi^2 s^2 t}{\kappa} + 2\pi i w s \right) ds \\ u(x,t) &= \int_{-\infty}^{\infty} f(u) g(x-u,t) du \end{aligned} \tag{113.38}$$

Compare this to eq. (113.21), the solution in terms of Fourier series. The form is almost identical, but the requirement for periodicity has been removed by switch to the continuous frequency domain!

113.5.2 *Wave equation*

With only a change of variables, setting $\kappa = -2mi/\hbar$ we have the general solution to the one dimensional zero potential wave equation eq. (113.23) in terms of an initial wave function. However, we have a form of the Fourier transform that obscures the physics has been picked here unfortunately. Let us start over in super speed mode directly from the wave equation, using the form of the Fourier transform that substitutes $2\pi s \rightarrow k$ for wave number.

We want to solve

$$-\frac{\hbar^2}{2m}\psi_{xx} = i\hbar\psi_t \tag{113.39}$$

Now calculate

$$\begin{aligned}
\mathcal{F}(\psi_{xx}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_{xx}(x,t)e^{-ikx}dx \\
&= \frac{1}{2\pi}\psi_x(x,t)e^{-ikx}\Big|_{-\infty}^{\infty} - (-ik)\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_x(x,t)e^{-ikx}dx \\
&= \cdots \\
&= \frac{1}{2\pi}(ik)^2\hat{\psi}(k)
\end{aligned} \tag{113.40}$$

So we have

$$-\frac{\hbar^2}{2m}(ik)^2\hat{\psi}(k,t) = i\hbar\frac{\partial\hat{\psi}(k,t)}{\partial t} \tag{113.41}$$

This provides us the fundamental solutions to the wave function in the wave number domain

$$\begin{aligned}
\hat{\psi}(k,t) &= A(k)\exp\left(-\frac{i\hbar k^2}{2m}t\right) \\
\psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)\exp\left(-\frac{i\hbar k^2}{2m}t\right)\exp(ikx)dk
\end{aligned} \tag{113.42}$$

In particular, as before, with an initial time wave function $\psi(x,0) = \phi(x)$ we have

$$\begin{aligned}
\phi(x) = \psi(x,0) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)\exp(ikx)dk \\
&= \mathcal{F}^{-1}(A(k))
\end{aligned} \tag{113.43}$$

So, $A(k) = \hat{\phi}$, and we have

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}(k) \exp\left(-\frac{i\hbar k^2}{2m}t\right) \exp(ikx) dk \tag{113.44}$$

So, ending the story we have finally, the general solution for the time evolution of our one dimensional wave function given initial conditions

$$\psi(x,t) = \mathcal{F}^{-1}\left(\hat{\phi}(k) \exp\left(-\frac{i\hbar k^2}{2m}t\right)\right) \tag{113.45}$$

or, alternatively, in terms of momentum via $k = p/\hbar$ we have

$$\psi(x,t) = \mathcal{F}^{-1}\left(\hat{\phi}(p) \exp\left(-\frac{ip^2}{2m\hbar}t\right)\right) \tag{113.46}$$

Pretty cool! Observe that in the wave number or momentum domain the time evolution of the wave function is just a continual phase shift relative to the initial conditions.

113.5.3 *Wave function solutions by Fourier transform for a particle on a circle*

Now, thinking about how to translate this Fourier transform method to the wave equation for a particle on a circle (as done by Susskind in his online lectures) makes me realize that one is free to use any sort of integral transform method appropriate for the problem (Fourier, Laplace, ...). It does not have to be the Fourier transform. Now, if we happen to pick an integral transform with $\theta \in [0, \pi]$ bounds, what do we have? This is nothing more than the inner product for the Fourier series, and we come full circle!

Now, the next thing to work out in detail is how to translate from the transform methods to the algebraic bra ket notation. This looks like it will follow immediately if one calls out the inner product in use explicitly, but that is an exploration for a different day.

114

POISSON AND RETARDED POTENTIAL GREEN'S FUNCTIONS FROM FOURIER KERNELS

114.1 MOTIVATION

Having recently attempted a number of Fourier solutions to the Heat, Schrödinger, Maxwell vacuum, and inhomogeneous Maxwell equation, a reading of [36] inspired me to have another go. In particular, he writes the Poisson equation solution explicitly in terms of a Green's function.

The Green's function for the Poisson equation

$$G(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \tag{114.1}$$

is not really derived, rather is just pointed out. However, it is a nice closed form that does not have any integrals. Contrast this to the Fourier transform method, where one ends up with a messy threefold integral that is not particularly obvious how to integrate.

In the PF thread Fourier transform solution to electrostatics Poisson equation? I asked if anybody knew how to reduce this integral to the potential kernel of electrostatics. Before getting any answer from PF I found it in [5], a book recently purchased, but not yet read.

Go through this calculation here myself in full detail to get more comfort with the ideas. Some of these ideas can probably also be applied to previous incomplete Fourier solution attempts. In particular, the retarded time potential solutions likely follow. Can these same ideas be applied to the STA form of the Maxwell equation, explicitly inverting it, as [10] indicate is possible (but do not spell out).

114.2 POISSON EQUATION

114.2.1 *Setup*

As often illustrated with the Heat equation, we seek a Fourier transform solution of the electrostatics Poisson equation

$$\nabla^2 \phi = -\rho/\epsilon_0 \tag{114.2}$$

Our 3D Fourier transform pairs are defined as

$$\begin{aligned}
\hat{f}(\mathbf{k}) &= \frac{1}{(\sqrt{2\pi})^3} \iiint f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3 x \\
f(\mathbf{x}) &= \frac{1}{(\sqrt{2\pi})^3} \iiint \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3 k
\end{aligned} \tag{114.3}$$

Applying the transform we get

$$\begin{aligned}
\phi(\mathbf{x}) &= \frac{1}{\epsilon_0} \iiint \rho(\mathbf{x}') G(\mathbf{x} - \mathbf{x}') d^3 x' \\
G(\mathbf{x}) &= \frac{1}{(2\pi)^3} \iiint \frac{1}{\mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{x}} d^3 k
\end{aligned} \tag{114.4}$$

Green's functions are usually defined by their delta function operational properties. Doing so, as defined above we have

$$\nabla^2 G(\mathbf{x}) = -4\pi\delta^3(\mathbf{x}) \tag{114.5}$$

(note that there are different sign conventions for this delta function identification.) Application to the Poisson equation eq. (114.2) gives

$$\int \nabla^2 G(\mathbf{x} - \mathbf{x}')\phi(\mathbf{x}') = \int (-4\pi\delta^3(\mathbf{x} - \mathbf{x}'))\phi(\mathbf{x}') = -4\pi\phi(\mathbf{x}) \tag{114.6}$$

and with expansion in the alternate sequence

$$\int \nabla^2 G(\mathbf{x} - \mathbf{x}')\phi(\mathbf{x}') = \int G(\mathbf{x} - \mathbf{x}')(\nabla'^2 \phi(\mathbf{x}')) = -\frac{1}{\epsilon_0} \int G(\mathbf{x} - \mathbf{x}')\rho(\mathbf{x}') \tag{114.7}$$

With prior knowledge of electrostatics we should therefore find

$$G(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}. \tag{114.8}$$

Our task is to actually compute this from the Fourier integral.

114.2.2 *Evaluating the convolution kernel integral*

114.2.2.1 *Some initial thoughts*

Now it seems to me that this integral G only has to be evaluated around a small neighborhood of the origin. For example if one evaluates one of the integrals

$$\int_{-\infty}^{\infty} \frac{1}{{k_1}^2 + {k_2}^2 + {k_3}^3} e^{ik_1 x_1} dk_1 \tag{114.9}$$

using a an upper half plane contour the result is zero unless $k_2 = k_3 = 0$. So one is left with something loosely like

$$G(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{1}{(2\pi)^3} \int_{k_1=-\epsilon}^{\epsilon} dk_1 \int_{k_2=-\epsilon}^{\epsilon} dk_2 \int_{k_3=-\epsilon}^{\epsilon} dk_3 \frac{1}{\mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{114.10}$$

How to reduce this? Somehow it must be possible to take this Fourier convolution kernel and somehow evaluate the integral to produce the electrostatics potential.

114.2.2.2 *An attempt*

The answer of how to do so, as pointed out above, was found in [5]. Instead of trying to evaluate this integral which has a pole at the origin, they cleverly evaluate a variant of it

$$I = \iiint \frac{1}{\mathbf{k}^2 + a^2} e^{i\mathbf{k}\cdot\mathbf{x}} d^3 k \tag{114.11}$$

which splits and shifts the repeated pole into two first order poles away from the origin. After a change to spherical polar coordinates, the new integral can be evaluated, and the Poisson Green's function in potential form follows by letting a tend to zero.

Very cool. It seems worthwhile to go through the motions of this myself, omitting no details I would find valuable.

First we want the volume element in spherical polar form, and our vector. That is

$$\begin{aligned}
\rho &= k \cos\phi \\
dA &= (\rho d\theta)(k d\phi) \\
d^3 k &= dk dA = k^2 \cos\phi d\theta d\phi dk \\
\mathbf{k} &= (\rho\cos\theta, \rho\sin\theta, k\sin\theta) \\
&= k(\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)
\end{aligned} \tag{114.12}$$

FIXME: scan picture to show angle conventions picked.

This produces

$$\begin{aligned}
I = &\int_{\theta=0}^{2\pi} \int_{\phi=-\pi/2}^{\pi/2} k^2 \int_{k=0}^{\infty} \cos\phi d\theta d\phi dk \\
&\frac{1}{k^2+a^2} \exp\left(ik(\cos\phi\cos\theta x_1 + \cos\phi\sin\theta + x_2 + \sin\phi x_3)\right)
\end{aligned} \tag{114.13}$$

Now, this is a lot less tractable than the Byron/Fuller treatment. In particular they were able to make a $t = \cos\phi$ substitution, and if I try this I get

$$I = -\int_{\theta=0}^{2\pi} \int_{t=-1}^{1} \int_{k=0}^{\infty} \frac{1}{k^2+a^2} \exp\left(ik(t\cos\theta x_1 + t\sin\theta x_2 + \sqrt{1-t^2}x_3)\right) k^2 dt d\theta dk \tag{114.14}$$

Now, this is still a whole lot different, and in particular it has $ik(t\sin\theta x_2 + \sqrt{1-t^2}x_3)$ in the exponential. I puzzled over this for a while, but it becomes clear on writing. Freedom to orient the axis along a preferable direction has been used, and some basis for which $\mathbf{x} = x_j\mathbf{e}^j + = x\mathbf{e}^1$ has been used! We are now left with

$$\begin{aligned}
I &= -\int_{\theta=0}^{2\pi} \int_{t=-1}^{1} \int_{k=0}^{\infty} \frac{1}{k^2+a^2} \exp\left(ikt\cos\theta x\right) k^2 dt d\theta dk \\
&= -\int_{\theta=0}^{2\pi} \int_{k=0}^{\infty} \frac{2}{(k^2+a^2)\cos\theta} \sin\left(kt\cos\theta x\right) k d\theta dk \\
&= -\int_{\theta=0}^{2\pi} \int_{k=-\infty}^{\infty} \frac{1}{(k^2+a^2)\cos\theta} \sin\left(kt\cos\theta x\right) k d\theta dk
\end{aligned} \tag{114.15}$$

Here the fact that our integral kernel is even in k has been used to double the range and half the kernel.

However, looking at this, one can see that there is trouble. In particular, we have $\cos\theta$ in the denominator, with a range that allows zeros. How did the text avoid this trouble?

114.2.3 *Take II*

After mulling it over for a bit, it appears that aligning $\mathbf{x}$ with the x-axis is causing the trouble. Aligning with the z-axis will work out much better, and leave only one trig term in the exponential. Essentially we need to use a volume of rotation about the z-axis, integrating along all sets of constant $\mathbf{k}\cdot\mathbf{x}$. This is a set of integrals over concentric ring volume elements (FIXME: picture).

Our volume element, measuring $\theta \in [0, \pi]$ from the z-axis, and ϕ as our position on the ring

$$
\begin{aligned}
\mathbf{k} \cdot \mathbf{x} &= kx \cos\theta \\
\rho &= k \sin\theta \\
dA &= (\rho d\phi)(k d\theta) \\
d^3 k &= dk dA = k^2 \sin\theta d\theta d\phi dk
\end{aligned}
\tag{114.16}
$$

This gives us

$$
I = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{k=0}^{\infty} \frac{1}{k^2 + a^2} \exp\left(ikx \cos\theta\right) k^2 \sin\theta d\theta d\phi dk
\tag{114.17}
$$

Now we can integrate immediately over ϕ, and make a $t = \cos\theta$ substitution ($dt = -\sin\theta d\theta$)

$$
\begin{aligned}
I &= -2\pi \int_{t=1}^{-1} \int_{k=0}^{\infty} \frac{1}{k^2 + a^2} \exp\left(ikxt\right) k^2 dt dk \\
&= -\frac{2\pi}{ix} \int_{k=0}^{\infty} \frac{1}{k^2 + a^2} \left(e^{-ikx} - e^{ikx}\right) k dk \\
&= \frac{2\pi}{ix} \int_{k=0}^{\infty} \frac{1}{k^2 + a^2} e^{ikx} k dk - \frac{2\pi}{ix} \int_{k=-0}^{-\infty} \frac{1}{k^2 + a^2} e^{ikx} (-k)(-dk) \\
&= \frac{2\pi}{ix} \int_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} e^{ikx} k dk \\
&= \frac{2\pi}{ix} \int_{k=-\infty}^{\infty} \frac{1}{k - ia} \frac{k e^{ikx}}{(k + ia)} dk
\end{aligned}
\tag{114.18}
$$

Now we have something that is in form for contour integration. In the upper half plane we have a pole at $k = ia$. Assuming that the integral over the big semicircular arc vanishes, we can just pick up the residue at that pole contributing. The assumption that this vanishes is actually non-trivial looking since the $k/(k + ia)$ term at a big radius R tends to 1. This is probably where Jordan's lemma comes in, so some study to understand that looks well justified.

$$
\begin{aligned}
0 &= I - 2\pi i \frac{2\pi}{ix} \left.\frac{k e^{ikx}}{(k + ia)}\right|_{k=ia} \\
&= I - 2\pi i \frac{2\pi}{ix} \frac{e^{-ax}}{2}
\end{aligned}
\tag{114.19}
$$

So we have

$$
I = \frac{2\pi^2}{x} e^{-ax}
\tag{114.20}
$$

Now that we have this, the Green's function of eq. (114.4) is

$$\begin{aligned} G(\mathbf{x}) &= \lim_{a \to 0} \frac{1}{(2\pi)^3} \frac{2\pi^2}{x} e^{-ax} \\ &= \frac{1}{4\pi|\mathbf{x}|} \end{aligned} \tag{114.21}$$

Which gives

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV' \tag{114.22}$$

Awesome! All following from the choice to set $\mathbf{E} = -\boldsymbol{\nabla}\phi$, we have a solution for ϕ following directly from the divergence equation $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ via Fourier transformation of this equation.

114.3 RETARDED TIME POTENTIALS FOR THE 3D WAVE EQUATION

114.3.1 *Setup*

If we look at the general inhomogeneous Maxwell equation

$$\nabla F = J/\epsilon_0 c \tag{114.23}$$

In terms of potential $F = \nabla \wedge A$ and employing in the Lorentz gauge $\nabla \cdot A = 0$, we have

$$\nabla^2 A = \left(\frac{1}{c^2}\partial_{tt} - \sum \partial_{jj}\right) A = J/\epsilon_0 c \tag{114.24}$$

As scalar equations with $A = A^\mu \gamma_\mu$, $J = J^\nu \gamma_\nu$ we have four equations all of the same form. A Green's function form for such wave equations was previously calculated in 116. That was

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial x^{j2}}\right)\psi = g \tag{114.25}$$

$$\begin{aligned} \psi(\mathbf{x}, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}', t') G(\mathbf{x} - \mathbf{x}', t - t') d^3x' dt' \\ G(\mathbf{x}, t) &= \theta(t) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{c}{(2\pi)^3|\mathbf{k}|} \sin(|\mathbf{k}|ct) \exp\left(i\mathbf{k} \cdot \mathbf{x}\right) d^3k \end{aligned} \tag{114.26}$$

Here $\theta(t)$ is the unit step function, which meant we only sum the time contributions of the charge density for $t - t' > 0$, or $t' < t$. That is the causal variant of the solution, which was arbitrary mathematically ($t > t'$ would have also worked).

114.3.2 *Reducing the Green's function integral*

Let us see if the spherical polar method works to reduce this equation too. In particular we want to evaluate

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|\mathbf{k}|} \sin(|\mathbf{k}|ct) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) d^3k \tag{114.27}$$

Will we have a requirement to introduce a pole off the origin as above? Perhaps like

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|\mathbf{k}|+\alpha} \sin(|\mathbf{k}|ct) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) d^3k \tag{114.28}$$

Let us omit it for now, but make the same spherical polar substitution used successfully above, writing

$$\begin{aligned} I &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{k=0}^{\infty} \frac{1}{k} \sin\left(kct\right) \exp\left(ikx\cos\theta\right) k^2 \sin\theta d\theta d\phi dk \\ &= 2\pi \int_{\theta=0}^{\pi} \int_{k=0}^{\infty} \sin\left(kct\right) \exp\left(ikx\cos\theta\right) k \sin\theta d\theta dk \end{aligned} \tag{114.29}$$

Let $\tau = \cos\theta$, $-d\tau = \sin\theta d\theta$, for

$$\begin{aligned} I &= 2\pi \int_{\tau=1}^{-1} \int_{k=0}^{\infty} \sin\left(kct\right) \exp\left(ikx\tau\right) k(-d\tau)dk \\ &= -2\pi \int_{k=0}^{\infty} \sin\left(kct\right) \frac{2}{2ikx} \left(\exp\left(-ikx\right) - \exp\left(ikx\right)\right) kdk \\ &= \frac{4\pi}{x} \int_{k=0}^{\infty} \sin\left(kct\right) \sin\left(kx\right) dk \\ &= \frac{2\pi}{x} \int_{k=0}^{\infty} \left(\cos\left(k(x-ct)\right) - \cos\left(k(x+ct)\right)\right) dk \end{aligned} \tag{114.30}$$

Okay, this is much simpler, but still not in a form that is immediately obvious how to apply contour integration to, since it has no poles. The integral kernel here is however an even function, so we can use the trick of doubling the integral range.

$$I = \frac{\pi}{x} \int_{k=-\infty}^{\infty} \left(\cos\left(k(x-ct)\right) - \cos\left(k(x+ct)\right)\right) dk \tag{114.31}$$

Having done this, this integral is not really any more well defined. With the Rigor police on holiday, let us assume we want the principle value of this integral

$$
\begin{aligned}
I &= \lim_{R\to\infty} \frac{\pi}{x} \int_{k=-R}^{R} \left(\cos\left(k(x-ct)\right) - \cos\left(k(x+ct)\right)\right) dk \\
&= \lim_{R\to\infty} \frac{\pi}{x} \int_{k=-R}^{R} d\left(\frac{\sin\left(k(x-ct)\right)}{x-ct} - \frac{\sin\left(k(x+ct)\right)}{x+ct}\right) \\
&= \lim_{R\to\infty} \frac{2\pi^2}{x} \left(\frac{\sin\left(R(x-ct)\right)}{\pi(x-ct)} - \frac{\sin\left(R(x+ct)\right)}{\pi(x+ct)}\right)
\end{aligned}
\tag{114.32}
$$

This sinc limit has been seen before being functionally identified with the delta function (the wikipedia article calls these "nascent delta function"), so we can write

$$
I = \frac{2\pi^2}{x} \left(\delta(x-ct) - \delta(x+ct)\right) \tag{114.33}
$$

For our Green's function we now have

$$
\begin{aligned}
G(\mathbf{x}, t) &= \theta(t)\frac{c}{(2\pi)^3}\frac{2\pi^2}{|\mathbf{x}|} \left(\delta(x-ct) - \delta(x+ct)\right) \\
&= \theta(t)\frac{c}{4\pi|\mathbf{x}|} \left(\delta(x-ct) - \delta(x+ct)\right)
\end{aligned}
\tag{114.34}
$$

And finally, our wave function (switching variables to convolve with the charge density) instead of the Green's function

$$
\begin{aligned}
\psi(\mathbf{x}, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}-\mathbf{x}', t-t')\theta(t')\frac{c}{4\pi|\mathbf{x}'|}\delta(\left|\mathbf{x}'\right| - ct') d^3x' dt' \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}-\mathbf{x}', t-t')\theta(t')\frac{c}{4\pi|\mathbf{x}'|}\delta(\left|\mathbf{x}'\right| + ct') d^3x' dt'
\end{aligned}
\tag{114.35}
$$

Let us break these into two parts

$$
\psi(\mathbf{x}, t) = \psi_-(\mathbf{x}, t) + \psi_+(\mathbf{x}, t) \tag{114.36}
$$

Where the first part, ψ_- is for the $-ct'$ delta function and one ψ_- for the $+ct'$. Making a $\tau = t - t'$ change of variables, this first portion is

$$
\begin{aligned}
\psi_-(\mathbf{x}, t) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}-\mathbf{x}', \tau)\theta(t-\tau)\frac{c}{4\pi|\mathbf{x}'|}\delta(\left|\mathbf{x}'\right| - ct + c\tau) d^3x' d\tau \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}-\mathbf{x}'', t - \left|\mathbf{x}''\right|/c)\frac{c}{4\pi|\mathbf{x}''|} d^3x''
\end{aligned}
\tag{114.37}
$$

One more change of variables, $\mathbf{x}' = \mathbf{x} - \mathbf{x}''$, $d^3x'' = -d^3x$, gives the final desired retarded potential result. The ψ_+ result is similar (see below), and assembling all we have

$$\begin{aligned}
\psi_-(\mathbf{x}, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right|/c)\frac{c}{4\pi|\mathbf{x} - \mathbf{x}'|}d^3x' \\
\psi_+(\mathbf{x}, t) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}, t + \left|\mathbf{x} - \mathbf{x}'\right|/c)\frac{c}{4\pi|\mathbf{x} - \mathbf{x}'|}d^3x'
\end{aligned} \tag{114.38}$$

It looks like my initial interpretation of the causal nature of the unit step in the original functional form was not really right. It is not until the Green's function is "integrated" do we get this causal and non-causal split into two specific solutions. In the first of these solutions is only charge contributions at the position in space offset by the wave propagation speed effects the potential (this is the causal case). On the other hand we have a second specific solution to the wave equation summing the charge contributions at all the future positions, this time offset by the time it takes a wave to propagate backwards from that future spacetime

The final mathematical result is consistent with statements seen elsewhere, such as in [12], although it is likely that the path taken by others to get this result was less ad-hoc than mine. It is been a couple years since seeing this for the first time in Feynman's text. It was not clear to me how somebody could possibly come up with those starting with Maxwell's equations. Here by essentially applying undergrad Engineering Fourier methods, we get the result in an admittedly ad-hoc fashion, but at least the result is not pulled out of a magic hat.

114.3.3 *Omitted Details. Advanced time solution*

Similar to the above for ψ_+ we have

$$\begin{aligned}
\psi_+(\mathbf{x}, t) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x} - \mathbf{x}', t - t')\theta(t')\frac{c}{4\pi|\mathbf{x}'|}\delta(\left|\mathbf{x}'\right| + ct')d^3x'dt' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x} - \mathbf{x}', \tau)\theta(t - \tau)\frac{c}{4\pi|\mathbf{x}'|}\delta(\left|\mathbf{x}'\right| + c(t - \tau))d^3x'd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x} - \mathbf{x}', \tau)\theta(t - \tau)\frac{c}{4\pi|\mathbf{x}'|}\delta(\left|\mathbf{x}'\right| + ct - c\tau)d^3x'd\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x} - \mathbf{x}', t + \left|\mathbf{x}'\right|/c)\frac{c}{4\pi|\mathbf{x}'|}d^3x' \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\mathbf{x}, t + \left|\mathbf{x} - \mathbf{x}'\right|/c)\frac{c}{4\pi|\mathbf{x} - \mathbf{x}'|}d^3x'
\end{aligned} \tag{114.39}$$

Is there an extra factor of -1 here?

114.4 1D WAVE EQUATION

It is somewhat irregular seeming to treat the 3D case before what should be the simpler 1D case, so let us try evaluating the Green's function for the 1D wave equation too.

We have found that Fourier transforms applied to the forced wave equation

$$\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)\psi = g(x,t) \tag{114.40}$$

result in the following integral solution.

$$\begin{aligned}
\psi(x,t) &= \int_{x'=-\infty}^{\infty}\int_{t'=0}^{\infty} g(x-x', t-t')G(x',t')dx'dt' \\
G(x,t) &= \int_{k=-\infty}^{\infty} \frac{v}{2\pi k}\sin(kvt)\exp(ikx)dk
\end{aligned} \tag{114.41}$$

As in the 3D case above can this reduced to something that does not involve such an unpalatable integral. Given the 3D result, it would be reasonable to get a result involving $g(x \pm vt)$ terms.

First let us get rid of the sine term, and express G entirely in exponential form. That is

$$\begin{aligned}
G(x,t) &= \int_{k=-\infty}^{\infty} \frac{v}{4\pi ki}\left(\exp(kvt) - \exp(-kvt)\right)\exp(ikx)dk \\
&= \int_{k=-\infty}^{\infty} \frac{v}{4\pi ki}\left(e^{k(x+vt)} - e^{k(x-vt)}\right)dk
\end{aligned} \tag{114.42}$$

Using the unit step function identification from eq. (114.53), we have

$$G(x,t) = \frac{v}{2}\left(\theta(x+vt) - \theta(x-vt)\right) \tag{114.43}$$

If this identification works our solution then becomes

$$\begin{aligned}
\psi(x,t) &= \int_{x'=-\infty}^{\infty}\int_{t'=0}^{\infty} g(x-x', t-t')\frac{v}{2}\left(\theta(x'+vt') - \theta(x'-vt')\right)dx'dt' \\
&= \int_{x'=-\infty}^{\infty}\int_{s=0}^{\infty} g(x-x', t-s/v)\frac{1}{2}\left(\theta(x'+s) - \theta(x'-s)\right)dx'ds
\end{aligned} \tag{114.44}$$

This is already much simpler than the original, but additional reduction should be possible by breaking this down into specific intervals. An alternative, perhaps is to use integration by parts and the delta function as the derivative of the unit step identification.

Let us try a pair of variable changes

$$\begin{aligned}\psi(x,t) &= \int_{u=-\infty}^{\infty}\int_{s=0}^{\infty} g(x-u+s, t-s/v)\frac{1}{2}\theta(u)duds \\ &- \int_{u=-\infty}^{\infty}\int_{s=0}^{\infty} g(x-u-s, t-s/v)\frac{1}{2}\theta(u)duds\end{aligned} \tag{114.45}$$

Like the retarded time potential solution to the 3D wave equation, we now have the wave function solution entirely specified by a weighted sum of the driving function

$$\psi(x,t) = \frac{1}{2}\int_{u=0}^{\infty}\int_{s=0}^{\infty} \left(g(x-u+s, t-s/v) - g(x-u-s, t-s/v)\right) duds \tag{114.46}$$

Can this be tidied at all? Let us do a change of variables here, writing $-\tau = t - s/v$.

$$\begin{aligned}\psi(x,t) &= \frac{1}{2}\int_{u=0}^{\infty}\int_{\tau=-t}^{\infty} \left(g(x+vt-(u-v\tau), \tau) - g(x-vt-(u+v\tau), \tau)\right) dud\tau \\ &= \frac{1}{2}\int_{u=0}^{\infty}\int_{\tau=-\infty}^{t} \left(g(x+vt-(u+v\tau), -\tau) - g(x-vt-(u-v\tau), -\tau)\right) dud\tau\end{aligned} \tag{114.47}$$

Is that any better? I am not so sure, and intuition says there is a way to reduce this to a single integral summing only over spatial variation.

114.4.1 *Followup to verify*

There has been a lot of guessing and loose mathematics here. However, if this is a valid solution despite all that, we should be able to apply the wave function operator $\frac{1}{v^2}\partial_{tt} + \partial_{xx}$ as a consistency check and get back $g(x,t)$ by differentiating under the integral sign.

FIXME: First have to think about how exactly to do this differentiation.

114.5 APPENDIX

114.5.1 *Integral form of unit step function*

The wiki article on the Heaviside unit step function lists an integral form

$$\begin{aligned}I_\epsilon &= \frac{1}{2\pi i}\,\mathrm{PV}\int_{-\infty}^{\infty}\frac{e^{ix\tau}}{\tau - i\epsilon}d\tau \\ \theta(x) &= \lim_{\epsilon\to 0} I_\epsilon\end{aligned} \tag{114.48}$$

How does this make sense? For $x > 0$ we can evaluate this with an upper half plane semicircular contour (FIXME: picture). Along the arc $z = Re^{i\phi}$ we have

$$
\begin{aligned}
|I_\epsilon| &= \left| \frac{1}{2\pi i} \int_{\phi=0}^{\pi} \frac{e^{iR(\cos\phi + i \sin\phi)}}{Re^{i\phi} - i\epsilon} Rie^{i\phi} d\phi \right| \\
&\approx \left| \frac{1}{2\pi} \int_{\phi=0}^{\pi} e^{iR\cos\phi} e^{-R\sin\phi} d\phi \right| \\
&\le \frac{1}{2\pi} \int_{\phi=0}^{\pi} e^{-R\sin\phi} d\phi \\
&\le \frac{1}{2\pi} \int_{\phi=0}^{\pi} e^{-R} d\phi \\
&= \frac{1}{2} e^{-R}
\end{aligned}
\tag{114.49}
$$

This tends to zero as $R \to \infty$, so evaluating the residue, we have for $x > 0$

$$
\begin{aligned}
I_\epsilon &= -(-2\pi i) \frac{1}{2\pi i} e^{ix\tau}\Big|_{\tau=i\epsilon} \\
&= e^{-x\epsilon}
\end{aligned}
\tag{114.50}
$$

Now for $x < 0$ an upper half plane contour will diverge, but the lower half plane can be used. This gives us $I_\epsilon = 0$ in that region. All that remains is the $x = 0$ case. There we have

$$
\begin{aligned}
I_\epsilon(0) &= \frac{1}{2\pi i} \,\mathrm{PV} \int_{-\infty}^{\infty} \frac{1}{\tau - i\epsilon} d\tau \\
&= \frac{1}{2\pi i} \lim_{R\to\infty} \ln\left(\frac{R - i\epsilon}{-R - i\epsilon} \right) \\
&\to \frac{1}{2\pi i} \ln(-1) \\
&= \frac{1}{2\pi i} i\pi
\end{aligned}
\tag{114.51}
$$

Summarizing we have

$$
I_\epsilon(x) = \begin{cases} e^{-x\epsilon} & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}
\tag{114.52}
$$

So in the limit this does work as an integral formulation of the unit step. This will be used to (very loosely) identify

$$\theta(x) \sim \frac{1}{2\pi i} \,\mathrm{PV} \int_{-\infty}^{\infty} \frac{e^{ix\tau}}{\tau} d\tau \tag{114.53}$$

115

FOURIER TRANSFORM SOLUTIONS TO THE WAVE EQUATION

115.1 MECHANICAL WAVE EQUATION SOLUTION

We want to solve

$$\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)\psi = 0 \tag{115.1}$$

A separation of variables treatment of this has been done in 67, and some logical followup for that done in 69 in the context of Maxwell's equation for the vacuum field.

Here the Fourier transform will be used as a tool.

115.2 ONE DIMENSIONAL CASE

Following the heat equation treatment in 113, we take Fourier transforms of both parts of eq. (115.1).

$$\mathcal{F}\left(\frac{1}{v^2}\partial_{tt}\psi\right) = \mathcal{F}\left(\partial_{xx}\psi\right) \tag{115.2}$$

For the x derivatives we can integrate by parts twice

$$\begin{aligned}
\mathcal{F}\left(\partial_{xx}\psi\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\partial_{xx}\psi\right)\exp\left(-ikx\right)dx \\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\partial_{x}\psi\right)\partial_x\left(\exp\left(-ikx\right)\right)dx \\
&= -\frac{-ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\partial_{x}\psi\right)\exp\left(-ikx\right)dx \\
&= \frac{(-ik)^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi\exp\left(-ikx\right)dx
\end{aligned} \tag{115.3}$$

Note that this integration by parts requires that $\partial_x\psi = \psi = 0$ at $\pm\infty$. We are left with

$$\mathcal{F}\left(\partial_{xx}\psi\right) = -k^2\hat{\psi}(k,t) \tag{115.4}$$

Now, for the left hand side, for the Fourier transform of the time partials we can pull the derivative operation out of the integral

$$\begin{aligned}\mathcal{F}\left(\frac{1}{v^2}\partial_{tt}\psi\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{v^2}\partial_{tt}\psi\right)\exp\left(-ikx\right)dx \\ &= \frac{1}{v^2}\partial_{tt}\hat{\psi}(k,t)\end{aligned} \tag{115.5}$$

We are left with our harmonic oscillator differential equation for the transformed wave function

$$\frac{1}{v^2}\partial_{tt}\hat{\psi}(k,t) = -k^2\hat{\psi}(k,t). \tag{115.6}$$

Since we have a partial differential equation, for the integration constant we are free to pick any function of k. The solutions of this are therefore of the form

$$\hat{\psi}(k,t) = A(k)\exp\left(\pm ivkt\right) \tag{115.7}$$

Performing an inverse Fourier transform we now have the wave equation expressed in terms of this unknown (so far) frequency domain function $A(k)$. That is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)\exp\left(\pm ivkt + ikx\right)dk \tag{115.8}$$

Now, suppose we fix the boundary value conditions by employing a known value of the wave function at $t = 0$, say $\psi(x,0) = \phi(x)$. We then have

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}A(k)\exp\left(ikx\right)dk \tag{115.9}$$

From which we have $A(k)$ in terms of ϕ by inverse transform

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\exp\left(-ikx\right)dx \tag{115.10}$$

One could consider the problem fully solved at this point, but it can be carried further. Let us substitute eq. (115.10) back into eq. (115.8). This is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(u)\exp\left(-iku\right)du\right)\exp\left(\pm ivkt + ikx\right)dk \tag{115.11}$$

With the Rigor police on holiday, exchange the order of integration

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty} \phi(u) du \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-iku \pm ivkt + ikx\right) dk \\ &= \int_{-\infty}^{\infty} \phi(u) du \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(ik(x - u \pm vt)\right) dk\end{aligned} \tag{115.12}$$

The principle value of this inner integral is

$$\begin{aligned}\text{PV}\,\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(ik(x - u \pm vt)\right) dk &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \exp\left(ik(x - u \pm vt)\right) dk \\ &= \lim_{R\to\infty} \frac{\sin\left(R(x - u \pm vt)\right)}{\pi(x - u \pm vt)}\end{aligned} \tag{115.13}$$

And here we make the usual identification with the delta function $\delta(x - u \pm vt)$. We are left with

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty} \phi(u)\delta(x - u \pm vt) du \\ &= \phi(x \pm vt)\end{aligned} \tag{115.14}$$

We find, amazingly enough, just by application of the Fourier transform, that the time evolution of the wave function follows propagation of the initial wave packet down the x-axis in one of the two directions with velocity v.

This is a statement well known to any first year student taking a vibrations and waves course, but it is nice to see it follow from the straightforward application of transform techniques straight out of the Engineer's toolbox.

115.3 two dimensional case

Next, using a two dimensional Fourier transform

$$\begin{aligned}\hat{f}(k,m) &= \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} f(x,y) \exp\left(-ikx - imy\right) dxdy \\ f(x,y) &= \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} \hat{f}(k,m) \exp\left(ikx + imy\right) dkdm,\end{aligned} \tag{115.15}$$

let us examine the two dimensional wave equation

$$\mathcal{F}\left(\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx} - \partial_{yy}\right)\psi = 0\right) \tag{115.16}$$

Applying the same technique as above we have

$$\frac{1}{v^2}\partial_{tt}\hat{\psi}(k,m,t) = \left((-ik)^2 + (-im)^2\right)\hat{\psi}(k,m,t) \tag{115.17}$$

With a solution

$$\hat{\psi}(k,m,t) = A(k,m)\exp\left(\pm i\sqrt{k^2+m^2}vt\right). \tag{115.18}$$

Inverse transforming we have our spatial domain function

$$\psi(x,y,t) = \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty} A(k,m)\exp\left(ikx + imy \pm i\sqrt{k^2+m^2}vt\right)dkdm \tag{115.19}$$

Again introducing an initial value function $\psi(x,y,0) = \phi(x,y)$ we have

$$\begin{aligned} A(k,m) &= \hat{\phi}(k,m) \\ &= \frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\phi(u,w)\exp\left(-iku - imw\right)dudw \end{aligned} \tag{115.20}$$

From which we can produce a final solution for the time evolution of an initial wave function, in terms of a Green's function for the wave equation.

$$\begin{aligned} \psi(x,y,t) &= \int_{-\infty}^{\infty}\phi(u,w)G(x-u,y-w,t)dudw \\ G(x,y,t) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\exp\left(ikx + imy \pm i\sqrt{k^2+m^2}vt\right)dkdm \end{aligned} \tag{115.21}$$

Pretty cool even if it is incomplete.

115.3.1 *A (busted) attempt to reduce this Green's function to deltas*

Now, for this inner integral kernel in eq. (115.21), our Green's function, or fundamental solution for the wave equation, we expect to have the action of a delta function. If it weare not for that root term we could make that identification easily since it could be factored into independent bits:

$$\begin{aligned} &\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\exp\left(ik(x-u) + im(y-w)\right)dkdm \\ &\quad= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(ik(x-u)\right)dk\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(im(y-w)\right)dm\right) \\ &\quad\sim \delta(x-u)\delta(y-w) \end{aligned} \tag{115.22}$$

Having seen previously that functions of the form $f(\hat{\mathbf{k}} \cdot \mathbf{x} - vt)$ are general solutions to the wave equation in higher dimensions suggests rewriting the integral kernel of the wave function in the following form

$$\begin{aligned}
&\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp\left(ik(x-u) + im(y-w) \pm i\sqrt{k^2+m^2} vt \right) dkdm \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp\left(ik(x-u \pm vt) \right) \\
&\quad \times \frac{1}{2\pi} \int_{-\infty}^{\infty} dm \exp\left(im(y-w \pm vt) \right) \\
&\quad \times \exp\left(\pm ivt(\sqrt{k^2+m^2} - k - m) \right)
\end{aligned} \tag{115.23}$$

Now, the first two integrals have the form that we associate with one dimensional delta functions, and one can see that when either k or m separately large (and positive) relative to the other than the third factor is approximately zero. In a loose fashion one can guesstimate that this combined integral has the following delta action

$$\begin{aligned}
&\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp\left(ik(x-u) + im(y-w) \pm i\sqrt{k^2+m^2} vt \right) dkdm \\
&\quad \sim \delta(x-u \pm vt)\delta(y-w \pm vt)
\end{aligned} \tag{115.24}$$

If that is the case then our final solution becomes

$$\begin{aligned}
\psi(x,y,t) &= \int_{-\infty}^{\infty} \phi(u,w)\delta(x-u \pm vt)\delta(y-w \pm vt) dudw \\
&= \phi(x \pm vt, y \pm vt)
\end{aligned} \tag{115.25}$$

This is a bit different seeming than the unit wave number dot product form, but lets see if it works. We want to expand

$$\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx} - \partial_{yy} \right)\psi \tag{115.26}$$

Let us start with the time partials

$$\begin{aligned}
\partial_{tt}\phi(x \pm vt, y \pm vt) &= \partial_t\partial_t\phi(x \pm vt, y \pm vt) \\
&= \partial_t(\partial_x\phi(\pm v) + \partial_y\phi(\pm v)) \\
&= (\pm v)(\partial_x\partial_t\phi + \partial_y\partial_t\phi) \\
&= (\pm v)^2(\partial_x(\partial_x\phi + \partial_y\phi) + \partial_y(\partial_x\phi + \partial_y\phi)) \\
&= (\pm v)^2(\partial_{xx}\phi + \partial_{yy}\phi + \partial_{yx}\phi + \partial_{xy}\phi)
\end{aligned} \tag{115.27}$$

So application of this test solution to the original wave equation is not zero, since these cross partials are not necessarily zero

$$\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx} - \partial_{yy}\right)\psi = \partial_{yx}\phi + \partial_{xy}\phi \tag{115.28}$$

This indicates that an incorrect guess was made about the delta function action of the integral kernel found via this Fourier transform technique. The remainder of that root term does not in fact cancel out, which appeared may occur, but was just too convenient. Oh well.

115.4 THREE DIMENSIONAL WAVE FUNCTION

It is pretty clear that a three dimensional Fourier transform

$$\begin{aligned}\hat{f}(k,m,n) &= \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty} f(x,y,z)\exp\left(-ikx - imy - inz\right)dxdydz \\ f(x,y,z) &= \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty} \hat{f}(k,m,n)\exp\left(ikx + imy + inz\right)dkdmdn,\end{aligned} \tag{115.29}$$

applied to a three dimensional wave equation

$$\mathcal{F}\left(\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx} - \partial_{yy} - \partial_{zz}\right)\psi = 0\right) \tag{115.30}$$

will lead to the similar results, but since this result did not work, it is not worth perusing this more general case just yet.

Despite the failure in the hopeful attempt to reduce the Green's function to a product of delta functions, one still gets a general solution from this approach for the three dimensional case.

$$\begin{aligned}\psi(x,y,z,t) &= \int_{-\infty}^{\infty} \phi(u,w,r)G(x-u,y-w,z-r,t)dudwdr \\ G(x,y,z,t) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \exp\left(ikx + imy + inz \pm i\sqrt{k^2+m^2+n^2}vt\right)dkdmdn\end{aligned} \tag{115.31}$$

So, utilizing this or reducing it to the familiar $f(\hat{\mathbf{k}}\cdot\mathbf{x} \pm vt)$ solutions becomes the next step. Intuition says that we need to pick a different inner product to get that solution. For the two dimensional case that likely has to be an inner product with a circular contour, and for the three dimensional case a spherical surface inner product of some sort.

Now, also interestingly, one can see hints here of the non-vacuum Maxwell retarded time potential wave solutions. This inspires an attempt to try to tackle that too.

116

FOURIER TRANSFORM SOLUTIONS TO MAXWELL'S EQUATION

116.1 MOTIVATION

In 115 a Green's function solution to the homogeneous wave equation

$$\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx} - \partial_{yy} - \partial_{zz}\right)\psi = 0 \tag{116.1}$$

was found to be

$$\begin{aligned}
\psi(x, y, z, t) &= \int_{-\infty}^{\infty} \phi(u, w, r) G(x - u, y - w, z - r, t) du d\tau dr \\
G(x, y, z, t) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp\left(ikx + imy + inz \pm i\sqrt{k^2 + m^2 + n^2} vt\right) dkdmdn
\end{aligned} \tag{116.2}$$

The aim of this set of notes is to explore the same ideas to the forced wave equations for the four vector potentials of the Lorentz gauge Maxwell equation.

Such solutions can be used to find the Faraday bivector or its associated tensor components.

Note that the specific form of the Fourier transform used in these notes continues to be

$$\begin{aligned}
\hat{f}(\mathbf{k}) &= \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} f(\mathbf{x}) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) d^n x \\
f(\mathbf{x}) &= \frac{1}{(\sqrt{2\pi})^n} \int_{-\infty}^{\infty} \hat{f}(\mathbf{k}) \exp\left(i\mathbf{k} \cdot \mathbf{x}\right) d^n k
\end{aligned} \tag{116.3}$$

116.2 FORCED WAVE EQUATION

116.2.1 *One dimensional case*

A good starting point is the reduced complexity one dimensional forced wave equation

$$\left(\frac{1}{v^2}\partial_{tt} - \partial_{xx}\right)\psi = g \tag{116.4}$$

Fourier transforming to to the wave number domain, with application of integration by parts twice (each toggling the sign of the spatial derivative term) we have

$$\frac{1}{v^2}\hat{\psi}_{tt} - (-ik)^2\hat{\psi} = \hat{g} \tag{116.5}$$

This leaves us with a linear differential equation of the following form to solve

$$f'' + \alpha^2 f = h \tag{116.6}$$

Out of line solution of this can be found below in eq. (116.48), where we have $f = \hat{\psi}$, $\alpha = kv$, and $h = \hat{g}v^2$. Our solution for the wave function in the wave number domain is now completely specified

$$\hat{\psi}(k,t) = \left|\frac{v}{k}\right| \int_{u=t_0(k)}^{t} \hat{g}(u) \sin(|kv|(t-u))du \tag{116.7}$$

Here because of the partial differentiation we have the flexibility to make the initial time a function of the wave number k, but it is probably more natural to just set $t_0 = -\infty$. Also let us explicitly pick $v > 0$ so that absolutes are only required on the factors of k

$$\hat{\psi}(k,t) = \frac{v}{|k|} \int_{u=-\infty}^{t} \hat{g}(k,u) \sin(|k|v(t-u))du \tag{116.8}$$

But seeing the integral in this form suggests a change of variables $\tau = t - u$, which gives us our final wave function in the wave number domain with all the time dependency removed from the integration limits

$$\hat{\psi}(k,t) = \frac{v}{|k|} \int_{\tau=0}^{\infty} \hat{g}(k, t-\tau) \sin(|k|v\tau)d\tau \tag{116.9}$$

With this our wave function is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{v}{k} \int_{\tau=0}^{\infty} \hat{g}(k, t-\tau) \sin(|k|v\tau)d\tau \right) \exp(ikx)dk \tag{116.10}$$

But we also have

$$\hat{g}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x,t) \exp(-ikx)dx \tag{116.11}$$

Reassembling we have

$$\psi(x,t) = \int_{k=-\infty}^{\infty}\int_{\tau=0}^{\infty}\int_{y=-\infty}^{\infty}\frac{v}{2\pi|k|}g(y,t-\tau)\sin(|k|v\tau)\exp(ik(x-y))dyd\tau dk \tag{116.12}$$

Rearranging a bit, and noting that $\text{sinc}(|k|x) = \text{sinc}(kx)$ we have

$$\begin{aligned}\psi(x,t) &= \int_{x'=-\infty}^{\infty}\int_{t'=0}^{\infty} g(x-x',t-t')G(x',t')dx'dt' \\ G(x,t) &= \int_{k=-\infty}^{\infty}\frac{v}{2\pi k}\sin(kvt)\exp(ikx)dk\end{aligned} \tag{116.13}$$

We see that our charge density summed over all space contributes to the wave function, but it is the charge density at that spatial location as it existed at a specific previous time.

The Green's function that we convolve with in eq. (116.13) is a rather complex looking function. As seen later in 114 it was possible to evaluate a 3D variant of such an integral in ad-hoc methods to produce a form in terms of retarded time and advanced time delta functions. A similar reduction, also in 114, of the Green's function above yields a unit step function identification

$$G(x,t) = \frac{v}{2}\left(\theta(x+vt) - \theta(x-vt)\right) \tag{116.14}$$

(This has to be verified more closely to see if it works).

116.2.2 *Three dimensional case*

Now, lets move on to the 3D case that is of particular interest for electrodynamics. Our wave equation is now of the form

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial {x^j}^2}\right)\psi = g \tag{116.15}$$

and our Fourier transformation produces almost the same result, but we have a wave number contribution from each of the three dimensions

$$\frac{1}{v^2}\hat{\psi}_{tt} + \mathbf{k}^2\hat{\psi} = \hat{g} \tag{116.16}$$

Our wave number domain solution is therefore

$$\hat{\psi}(\mathbf{k},t) = \frac{v}{|\mathbf{k}|}\int_{\tau=0}^{\infty}\hat{g}(\mathbf{k},t-\tau)\sin(|\mathbf{k}|v\tau)d\tau \tag{116.17}$$

But our wave number domain charge density is

$$\hat{g}(\mathbf{k}, t) = \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} g(\mathbf{x}, t) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) d^3x \tag{116.18}$$

Our wave number domain result in terms of the charge density is therefore

$$\hat{\psi}(\mathbf{k}, t) = \frac{v}{|\mathbf{k}|} \int_{\tau=0}^{\infty} \left(\frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} g(\mathbf{r}, t - \tau) \exp\left(-i\mathbf{k} \cdot \mathbf{r}\right) d^3r \right) \sin(|\mathbf{k}| v\tau) d\tau \tag{116.19}$$

And finally inverse transforming back to the spatial domain we have a complete solution for the inhomogeneous wave equation in terms of the spatial and temporal charge density distribution

$$\begin{aligned} \psi(\mathbf{x}, t) &= \int_{-\infty}^{\infty} \int_{t'=0}^{\infty} g(\mathbf{x} - \mathbf{x}', t - t') G(\mathbf{x}', t') d^3x' dt' \\ G(\mathbf{x}, t) &= \int_{-\infty}^{\infty} \frac{v}{(2\pi)^3 |\mathbf{k}|} \sin(|\mathbf{k}| vt) \exp\left(i\mathbf{k} \cdot \mathbf{x}\right) d^3k \end{aligned} \tag{116.20}$$

For computational purposes we are probably much better off using eq. (116.17), however, from an abstract point of form this expression is much prettier.

One can also see the elements of the traditional retarded time expressions for the potential hiding in there. See 114 for an evaluation of this integral (in an ad-hoc non-rigorous fashion) eventually producing the retarded time solution.

116.2.2.1 *Tweak this a bit to put into proper Green's function form*

Now, it makes sense to redefine $G(\mathbf{x}, t)$ above so that we can integrate uniformly over all space and time. To do so we can add a unit step function into the definition, so that $G(\mathbf{x}, t < 0) = 0$. Additionally, if we express this convolution it is slightly tidier (and consistent with the normal Green's function notation) to put the parameter differences in the kernel term. Such a change of variables will alter the sign of the integral limits by a factor of $(-1)^4$, but we also have a $(-1)^4$ term from the differentials. After making these final adjustments we have a final variation of our integral solution

$$\begin{aligned} \psi(\mathbf{x}, t) &= \int_{-\infty}^{\infty} g(\mathbf{x}', t') G(\mathbf{x} - \mathbf{x}', t - t') d^3x' dt' \\ G(\mathbf{x}, t) &= \theta(t) \int_{-\infty}^{\infty} \frac{v}{(2\pi)^3 |\mathbf{k}|} \sin(|\mathbf{k}| vt) \exp\left(i\mathbf{k} \cdot \mathbf{x}\right) d^3k \end{aligned} \tag{116.21}$$

Now our inhomogeneous solution is expressed nicely as the convolution of our current density over all space and time with an integral kernel. That integral kernel is precisely the Green's function for this forced wave equation.

This solution comes with a large number of assumptions. Along the way we have the assumption that both our wave function and the charge density was Fourier transformable, and that the wave number domain products were inverse transformable. We also had an assumption that the wave function is sufficiently small at the limits of integration that the intermediate contributions from the integration by parts vanished, and finally the big assumption that we were perfectly free to interchange integration order in an extremely ad-hoc and non-rigorous fashion!

116.3 MAXWELL EQUATION SOLUTION

Having now found Green's function form for the forced wave equation, we can now move to Maxwell's equation

$$\nabla F = J/\epsilon_0 c \tag{116.22}$$

In terms of potentials we have $F = \nabla \wedge A$, and may also impose the Lorentz gauge $\nabla \cdot A = 0$, to give us our four charge/current forced wave equations

$$\nabla^2 A = J/\epsilon_0 c \tag{116.23}$$

As scalar equations these are

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \sum_j \frac{\partial^2}{\partial {x^j}^2}\right) A^\mu = \frac{J^\mu}{\epsilon_0 c} \tag{116.24}$$

So, from above, also writing $x^0 = ct$, we have

$$\begin{aligned} A^\mu(x) &= \frac{1}{\epsilon_0 c}\int J^\mu(x')G(x-x')d^4x' \\ G(x) &= \theta(x^0)\int \frac{1}{(2\pi)^3|\mathbf{k}|}\sin(|\mathbf{k}|x^0)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right)d^3k \end{aligned} \tag{116.25}$$

116.3.1 *Four vector form for the Green's function*

Can we put the sine and exponential product in a more pleasing form? It would be nice to merge the $\mathbf{x}$ and ct terms into a single four vector form. One possibility is merging the two

$$
\begin{aligned}
&\sin(|\mathbf{k}|x^0)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \\
&\qquad = \frac{1}{2i}\left(\exp\left(i\left(\mathbf{k}\cdot\mathbf{x}+|\mathbf{k}|x^0\right)\right)-\exp\left(i\left(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|x^0\right)\right)\right) \\
&\qquad = \frac{1}{2i}\left(\exp\left(i|\mathbf{k}|\left(\hat{\mathbf{k}}\cdot\mathbf{x}+x^0\right)\right)-\exp\left(i|\mathbf{k}|\left(\hat{\mathbf{k}}\cdot\mathbf{x}-x^0\right)\right)\right)
\end{aligned}
\tag{116.26}
$$

Here we have a sort of sine like conjugation in the two exponentials. Can we tidy this up? Let us write the unit wave number vector in terms of direction cosines

$$
\begin{aligned}
\hat{\mathbf{k}} &= \sum_m \sigma_m \alpha_m \\
&= \sum_m \gamma_m \gamma_0 \alpha_m
\end{aligned}
\tag{116.27}
$$

Allowing us to write

$$
\sum_m \gamma^m \alpha_m = -\hat{\mathbf{k}}\gamma_0 \tag{116.28}
$$

This gives us

$$
\begin{aligned}
\hat{\mathbf{k}}\cdot\mathbf{x}+x^0 &= \alpha_m x^m + x^0 \\
&= (\alpha_m\gamma^m)\cdot(\gamma_j x^j) + \gamma^0\cdot\gamma_0 x^0 \\
&= (-\hat{\mathbf{k}}\gamma_0+\gamma_0)\cdot\gamma_\mu x^\mu \\
&= (-\hat{\mathbf{k}}\gamma_0+\gamma_0)\cdot x
\end{aligned}
\tag{116.29}
$$

Similarly we have

$$
\hat{\mathbf{k}}\cdot\mathbf{x}-x^0 = (-\hat{\mathbf{k}}\gamma_0-\gamma_0)\cdot x \tag{116.30}
$$

and can now put G in explicit four vector form

$$
G(x) = \frac{\theta(x\cdot\gamma_0)}{(2\pi)^3 2i}\int\left(\exp\left(i((|\mathbf{k}|-\mathbf{k})\gamma_0)\cdot x\right)-\exp\left(-i((|\mathbf{k}|+\mathbf{k})\gamma_0)\cdot x\right)\right)\frac{d^3k}{|\mathbf{k}|} \tag{116.31}
$$

Hmm, is that really any better? Intuition says that this whole thing can be written as sine with some sort of geometric product conjugate terms.

I get as far as writing

$$i(\mathbf{k} \cdot \mathbf{x} \pm |\mathbf{k}| x^0) = (i\gamma_0) \wedge (\mathbf{k} \pm |\mathbf{k}|) \cdot x \tag{116.32}$$

But that does not quite have the conjugate form I was looking for (or does it)? Have to go back and look at Hestenes's multivector conjugation operation. Think it had something to do with reversion, but do not recall.

Failing that tidy up the following

$$G(x) = \frac{\theta(x \cdot \gamma_0)}{(2\pi)^3} \int \sin(|\mathbf{k}| x \cdot \gamma_0) \exp\left(-i(\mathbf{k}\gamma_0) \cdot x\right) \frac{d^3 k}{|\mathbf{k}|} \tag{116.33}$$

is probably about as good as it gets for now. Note the interesting feature that we end up essentially integrating over a unit ball in our wave number space. This suggests the possibility of simplification using the divergence theorem.

116.3.2 *Faraday tensor*

Attempting to find a tidy four vector form for the four vector potentials was in preparation for taking derivatives. Specifically, applied to eq. (116.25) we have

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{116.34}$$

subject to the Lorentz gauge constraint

$$0 = \partial_\mu A^\mu \tag{116.35}$$

If we switch the convolution indices for our potentials

$$A^\mu(\mathbf{x}, t) = \frac{1}{\epsilon_0 c} \int J^\mu(x - x') G(x') d^4 x' \tag{116.36}$$

Then the Lorentz gauge condition, after differentiation under the integral sign, is

$$0 = \partial_\mu A^\mu = \frac{1}{\epsilon_0 c} \int \left(\partial_\mu J^\mu(x - x')\right) G(x') d^4 x' \tag{116.37}$$

So we see that the Lorentz gauge seems to actually imply the continuity equation

$$\partial_\mu J^\mu(x) = 0 \tag{116.38}$$

Similarly, it appears that we can write our tensor components in terms of current density derivatives

$$F^{\mu\nu} = \frac{1}{\epsilon_0 c} \int \left(\partial^\mu J^\nu(x - x') - \partial^\nu J^\mu(x - x')\right) G(x') d^4 x' \tag{116.39}$$

Logically, I suppose that one can consider the entire problem solved here, pending the completion of this calculus exercise.

In terms of tidiness, it would be nicer seeming use the original convolution, and take derivative differences of the Green's function. However, how to do this is not clear to me since this function has no defined derivative at the $t = 0$ points due to the unit step.

116.4 APPENDIX. MECHANICAL DETAILS

116.4.1 *Solving the wave number domain differential equation*

We wish to solve equation the inhomogeneous eq. (116.6). Writing this in terms of a linear operator equation this is

$$\begin{aligned} L(y) &= y'' + \alpha^2 y \\ L(y) &= h \end{aligned} \tag{116.40}$$

The solutions of this equation will be formed from linear combinations of the homogeneous problem plus a specific solution of the inhomogeneous problem

By inspection the homogeneous problem has solutions in $\text{span}\{e^{i\alpha x}, e^{-i\alpha x}\}$. We can find a solution to the inhomogeneous problem using the variation of parameters method, assuming a solution of the form

$$y = u e^{i\alpha x} + v e^{-i\alpha x} \tag{116.41}$$

Taking derivatives we have

$$y' = u' e^{i\alpha x} + v' e^{-i\alpha x} + i\alpha(u e^{i\alpha x} - v e^{-i\alpha x}) \tag{116.42}$$

The trick to solving this is to employ the freedom to set the u', and v' terms above to zero

$$u' e^{i\alpha x} + v' e^{-i\alpha x} = 0 \tag{116.43}$$

Given this choice we then have

$$\begin{aligned} y' &= i\alpha(u e^{i\alpha x} - v e^{-i\alpha x}) \\ y'' &= (i\alpha)^2 (u e^{i\alpha x} + v e^{-i\alpha x}) i\alpha(u' e^{i\alpha x} - v' e^{-i\alpha x}) \end{aligned} \tag{116.44}$$

So we have

$$\begin{aligned} L(y) &= (i\alpha)^2 (u e^{i\alpha x} + v e^{-i\alpha x}) \\ &\quad + i\alpha(u' e^{i\alpha x} - v' e^{-i\alpha x}) + (\alpha)^2 (u e^{i\alpha x} + v e^{-i\alpha x}) \\ &= i\alpha(u' e^{i\alpha x} - v' e^{-i\alpha x}) \end{aligned} \tag{116.45}$$

With this and eq. (116.43) we have a set of simultaneous first order linear differential equations to solve

$$\begin{aligned} \begin{bmatrix} u' \\ v' \end{bmatrix} &= \begin{bmatrix} e^{i\alpha x} & -e^{-i\alpha x} \\ e^{i\alpha x} & e^{-i\alpha x} \end{bmatrix}^{-1} \begin{bmatrix} h/i\alpha \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-i\alpha x} & e^{-i\alpha x} \\ -e^{i\alpha x} & e^{i\alpha x} \end{bmatrix} \begin{bmatrix} h/i\alpha \\ 0 \end{bmatrix} \\ &= \frac{h}{2i\alpha} \begin{bmatrix} e^{-i\alpha x} \\ -e^{i\alpha x} \end{bmatrix} \end{aligned} \tag{116.46}$$

Substituting back into the assumed solution we have

$$\begin{aligned} y &= \frac{1}{2i\alpha} \left(e^{i\alpha x} \int h e^{-i\alpha x} - e^{-i\alpha x} \int h e^{i\alpha x} \right) \\ &= \frac{1}{2i\alpha} \int_{u=x_0}^{x} h(u) \left(e^{-i\alpha(u-x)} - e^{i\alpha(u-x)} \right) du \end{aligned} \tag{116.47}$$

So our solution appears to be

$$y = \frac{1}{\alpha} \int_{u=x_0}^{x} h(u) \sin(\alpha(x-u)) du \tag{116.48}$$

A check to see if this is correct is in order to verify this. Differentiating using eq. (116.57) we have

$$
\begin{aligned}
y' &= \frac{1}{\alpha} h(u) \sin(\alpha(x-u))\Big|_{u=x} + \frac{1}{\alpha} \int_{u=x_0}^{x} \frac{\partial}{\partial x} h(u) \sin(\alpha(x-u)) du \\
&= \int_{u=x_0}^{x} h(u) \cos(\alpha(x-u)) du
\end{aligned}
\tag{116.49}
$$

and for the second derivative we have

$$
\begin{aligned}
y'' &= h(u) \cos(\alpha(x-u))|_{u=x} - \alpha \int_{u=x_0}^{x} h(u) \sin(\alpha(x-u)) du \\
&= h(x) - \alpha^2 y(x)
\end{aligned}
\tag{116.50}
$$

Excellent, we have $y'' + \alpha^2 y = h$ as desired.

116.4.2 *Differentiation under the integral sign*

Given an function that is both a function of the integral limits and the integrals kernel

$$
f(x) = \int_{u=a(x)}^{b(x)} G(x,u) du,
\tag{116.51}
$$

lets recall how to differentiate the beastie. First let $G(x,u) = \partial F(x,u)/\partial u$ so we have

$$
f(x) = F(x, b(x)) - F(x, a(x))
\tag{116.52}
$$

and our derivative is

$$
\begin{aligned}
f'(x) &= \frac{\partial F}{\partial x}(x, b(x)) \frac{\partial F}{\partial u}(x, b(x)) b' - \frac{\partial F}{\partial x}(x, a(x)) - \frac{\partial F}{\partial u}(x, a(x)) a' \\
&= G(x, b(x)) b' - G(x, a(x)) a' + \frac{\partial F}{\partial x}(x, b(x)) - \frac{\partial F}{\partial x}(x, a(x))
\end{aligned}
\tag{116.53}
$$

Now, we want $\partial F/\partial x$ in terms of G, and to get there, assuming sufficient continuity, we have from the definition

$$
\begin{aligned}
\frac{\partial}{\partial x} G(x,u) &= \frac{\partial}{\partial x} \frac{\partial F(x,u)}{\partial u} \\
&= \frac{\partial}{\partial u} \frac{\partial F(x,u)}{\partial x}
\end{aligned}
\tag{116.54}
$$

Integrating both sides with respect to u we have

$$\begin{aligned}\int \frac{\partial G}{\partial x} du &= \int \frac{\partial}{\partial u} \frac{\partial F(x,u)}{\partial x} du \\ &= \frac{\partial F(x,u)}{\partial x}\end{aligned} \tag{116.55}$$

This allows us to write

$$\frac{\partial F}{\partial x}(x, b(x)) - \frac{\partial F}{\partial x}(x, a(x)) = \int_a^b \frac{\partial G}{\partial x}(x,u) du \tag{116.56}$$

and finally

$$\frac{d}{dx}\int_{u=a(x)}^{b(x)} G(x,u) du = G(x, b(x))b' - G(x, a(x))a' + \int_{a(x)}^{b(x)} \frac{\partial G}{\partial x}(x,u) du \tag{116.57}$$

116.4.2.1 *Argument logic error above to understand*

Is the following not also true

$$\begin{aligned}\int \frac{\partial G}{\partial x} du &= \int \frac{\partial}{\partial u} \frac{\partial F(x,u)}{\partial x} du \\ &= \int \frac{\partial}{\partial u}\left(\frac{\partial F(x,u)}{\partial x} + A(x)\right) du \\ &= \frac{\partial F(x,u)}{\partial x} + A(x)u + B\end{aligned} \tag{116.58}$$

In this case we have

$$\frac{\partial F}{\partial x}(x, b(x)) - \frac{\partial F}{\partial x}(x, a(x)) = \int_a^b \frac{\partial G}{\partial x}(x,u) du - A(x)(b(x) - a(x)) \tag{116.59}$$

How to reconcile this with the answer I expect (and having gotten it, I believe matches my recollection)?

117

FIRST ORDER FOURIER TRANSFORM SOLUTION OF MAXWELL'S EQUATION

117.1 MOTIVATION

In 116 solutions of Maxwell's equation via Fourier transformation of the four potential forced wave equations were explored.

Here a first order solution is attempted, by directly Fourier transforming the Maxwell's equation in bivector form.

117.2 SETUP

Again using a 3D spatial Fourier transform, we want to put Maxwell's equation into an explicit time dependent form, and can do so by premultiplying by our observer's time basis vector γ_0

$$\gamma_0 \nabla F = \gamma_0 \frac{J}{\epsilon_0 c} \tag{117.1}$$

On the left hand side we have

$$\begin{aligned}
\gamma_0 \nabla &= \gamma_0 \left(\gamma^0 \partial_0 + \gamma^k \partial_k \right) \\
&= \partial_0 - \gamma^k \gamma_0 \partial_k \\
&= \partial_0 + \sigma^k \partial_k \\
&= \partial_0 + \boldsymbol{\nabla}
\end{aligned} \tag{117.2}$$

and on the right hand side we have

$$\begin{aligned}
\gamma_0 \frac{J}{\epsilon_0 c} &= \gamma_0 \frac{c\rho \gamma_0 + J^k \gamma_k}{\epsilon_0 c} \\
&= \frac{c\rho - J^k \sigma_k}{\epsilon_0 c} \\
&= \frac{\rho}{\epsilon_0} - \frac{\mathbf{j}}{\epsilon_0 c}
\end{aligned} \tag{117.3}$$

Both the spacetime gradient and the current density four vector have been put in a quaternionic form with scalar and bivector grades in the STA basis. This leaves us with the time centric formulation of Maxwell's equation

$$\left(\partial_0 + \boldsymbol{\nabla}\right) F = \frac{\rho}{\epsilon_0} - \frac{\mathbf{j}}{\epsilon_0 c} \tag{117.4}$$

Except for the fact that we have objects of various grades here, and that this is a first instead of second order equation, these equations have the same form as in the previous Fourier transform attacks. Those were Fourier transform application for the homogeneous and inhomogeneous wave equations, and the heat and Schrödinger equation.

117.3 FOURIER TRANSFORMING A MIXED GRADE OBJECT

Now, here we make the assumption that we can apply 3D Fourier transform pairs to mixed grade objects, as in

$$\begin{aligned}
\hat{\psi}(\mathbf{k}, t) &= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \psi(\mathbf{x}, t) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) d^3 x \\
\psi(\mathbf{x}, t) &= \mathrm{PV} \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \hat{\psi}(\mathbf{k}, t) \exp\left(i\mathbf{k} \cdot \mathbf{x}\right) d^3 k
\end{aligned} \tag{117.5}$$

Now, because of linearity, is it clear enough that this will work, provided this is a valid transform pair for any specific grade. We do however want to be careful of the order of the factors since we want the flexibility to use any particular convenient representation of i, in particular $i = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3$.

Let us repeat our an ad-hoc verification that this transform pair works as desired, being careful with the order of products and specifically allowing for ψ to be a non-scalar function. Writing $\mathbf{k} = k_m\sigma^m$, $\mathbf{r} = \sigma_m r^m$, $\mathbf{x} = \sigma_m x^m$, that is an expansion of

$$\begin{aligned}
&\mathrm{PV}\frac{1}{(\sqrt{2\pi})^3}\int\left(\frac{1}{(\sqrt{2\pi})^3}\int\psi(\mathbf{r},t)\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right)d^3r\right)\exp\left(i\mathbf{k}\cdot\mathbf{x}\right)d^3k \\
&= \int\psi(\mathbf{r},t)d^3r\,\mathrm{PV}\,\frac{1}{(2\pi)^3}\int\exp\left(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{r})\right)d^3k \\
&= \int\psi(\mathbf{r},t)d^3r\prod_{m=1}^{3}\mathrm{PV}\,\frac{1}{2\pi}\int\exp\left(ik_m(x^m-r^m)\right)dk_m \\
&= \int\psi(\mathbf{r},t)d^3r\prod_{m=1}^{3}\lim_{R\to\infty}\frac{\sin\left(R(x^m-r^m)\right)}{\pi(x^m-r^m)} \\
&\sim \int\psi(\mathbf{r},t)\delta(x^1-r^1)\delta(x^2-r^2)\delta(x^3-r^3)d^3r \\
&= \psi(\mathbf{x},t)
\end{aligned} \tag{117.6}$$

In the second last step above we make the ad-hoc identification of that sinc limit with the Dirac delta function, and recover our original function as desired (the Rigor police are on holiday again).

117.3.1 *Rotor form of the Fourier transform?*

Although the formulation picked above appears to work, it is not the only choice to potentially make for the Fourier transform of multivector. Would it be more natural to pick an explicit Rotor formulation? This perhaps makes more sense since it is then automatically grade preserving.

$$\begin{aligned}
\hat{\psi}(\mathbf{k},t) &= \frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty}\exp\left(\frac{1}{2}i\mathbf{k}\cdot\mathbf{x}\right)\psi(\mathbf{x},t)\exp\left(-\frac{1}{2}i\mathbf{k}\cdot\mathbf{x}\right)d^nx \\
\psi(\mathbf{x},t) &= \mathrm{PV}\,\frac{1}{(\sqrt{2\pi})^n}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}i\mathbf{k}\cdot\mathbf{x}\right)\hat{\psi}(\mathbf{k},t)\exp\left(\frac{1}{2}i\mathbf{k}\cdot\mathbf{x}\right)d^nk
\end{aligned} \tag{117.7}$$

This is not a moot question since I later tried to make an assumption that the grade of a transformed object equals the original grade. That does not work with the Fourier transform definition that has been picked in eq. (117.5). It may be necessary to revamp the complete treatment, but for now at least an observation that the grades of transform pairs do not necessarily match is required.

Does the transform pair work? For the $n = 1$ case this is

$$
\begin{aligned}
\mathcal{F}(f) = \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{1}{2}ikx\right) f(x) \exp\left(-\frac{1}{2}ikx\right) dx \\
\mathcal{F}^{-1}(\hat{f}) = f(x) &= \mathrm{PV}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}ikx\right) \hat{f}(k) \exp\left(\frac{1}{2}ikx\right) dk
\end{aligned}
\tag{117.8}
$$

Will the computation of $\mathcal{F}^{-1}(\mathcal{F}(f(x)))$ produce $f(x)$? Let us try

$$
\begin{aligned}
&\mathcal{F}^{-1}(\mathcal{F}(f(x))) \\
&= \mathrm{PV}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}ikx\right) \left(\int_{-\infty}^{\infty} \exp\left(\frac{1}{2}iku\right) f(u) \exp\left(-\frac{1}{2}iku\right) du \right) \exp\left(\frac{1}{2}ikx\right) dk \\
&= \mathrm{PV}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}ik(x-u)\right) f(u) \exp\left(\frac{1}{2}ik(x-u)\right) dudk
\end{aligned}
\tag{117.9}
$$

Now, this expression can not obviously be identified with the delta function as in the single sided transformation. Suppose we decompose f into grades that commute and anticommute with i. That is

$$
\begin{aligned}
f &= f_{\parallel} + f_{\perp} \\
f_{\parallel} i &= i f_{\parallel} \\
f_{\perp} i &= -i f_{\perp}
\end{aligned}
\tag{117.10}
$$

This is also sufficient to determine how these components of f behave with respect to the exponentials. We have

$$
\begin{aligned}
e^{i\theta} &= \sum_m \frac{(i\theta)^m}{m!} \\
&= \cos(\theta) + i\sin(\theta)
\end{aligned}
\tag{117.11}
$$

So we also have

$$
\begin{aligned}
f_{\parallel} e^{i\theta} &= e^{i\theta} f_{\parallel} \\
f_{\perp} e^{i\theta} &= e^{-i\theta} f_{\perp}
\end{aligned}
\tag{117.12}
$$

This gives us

$$
\begin{aligned}
\mathcal{F}^{-1}(\mathcal{F}(f(x))) &= \mathrm{PV}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{\parallel}(u) dudk + \mathrm{PV}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{\perp}(u) \exp\left(ik(x-u)\right) dudk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f_{\parallel}(u) du + \int_{-\infty}^{\infty} f_{\perp}(u) \delta(x-u) du
\end{aligned}
\tag{117.13}
$$

So, we see two things. First is that any $f_{\|} \neq 0$ produces an unpleasant infinite result. One could, in a vague sense, allow for odd valued $f_{\|}$, however, if we were to apply this inversion transformation pair to a function time varying multivector function $f(x, t)$, this would then require that the function is odd for all times. Such a function must be zero valued.

The second thing that we see is that if f entirely anticommutes with i, we do recover it with this transform pair, obtaining $f_{\perp}(x)$.

With respect to Maxwell's equation this immediately means that this double sided transform pair is of no use, since our pseudoscalar $i = \gamma_0\gamma_1\gamma_2\gamma_3$ commutes with our grade two field bivector F.

117.4 FOURIER TRANSFORMING THE SPACETIME SPLIT GRADIENT EQUATION

Now, suppose we have a Maxwell like equation of the form

$$\left(\partial_0 + \boldsymbol{\nabla}\right) \psi = g \tag{117.14}$$

Let us take the Fourier transform of this equation. This gives us

$$\partial_0 \hat{\psi} + \sigma^m \mathcal{F}(\partial_m \psi) = \hat{g} \tag{117.15}$$

Now, we need to look at the middle term in a bit more detail. For the wave, and heat equations this was evaluated with just an integration by parts. Was there any commutation assumption in that previous treatment? Let us write this out in full to make sure we are cool.

$$\mathcal{F}(\partial_m \psi) = \frac{1}{(\sqrt{2\pi})^3} \int \left(\partial_m \psi(\mathbf{x}, t)\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) d^3 x \tag{117.16}$$

Let us also expand the integral completely, employing a permutation of indices $\pi(1, 2, 3) = (m, n, p)$.

$$\mathcal{F}(\partial_m \psi) = \frac{1}{(\sqrt{2\pi})^3} \int_{\partial x^p} dx^p \int_{\partial x^n} dx^n \int_{\partial x^m} dx^m \left(\partial_m \psi(\mathbf{x}, t)\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) \tag{117.17}$$

Okay, now we are ready for the integration by parts. We want a derivative substitution, based on

$$\begin{aligned}
&\partial_m \left(\psi(\mathbf{x}, t) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right)\right) \\
&\quad = \left(\partial_m \psi(\mathbf{x}, t)\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) + \psi(\mathbf{x}, t) \partial_m \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) \\
&\quad = \left(\partial_m \psi(\mathbf{x}, t)\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) + \psi(\mathbf{x}, t)(-ik_m) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right)
\end{aligned} \tag{117.18}$$

Observe that we do not wish to assume that the pseudoscalar i commutes with anything except the exponential term, so we have to leave it sandwiched or on the far right. We also must take care to not necessarily commute the exponential itself with ψ or its derivative. Having noted this we can rearrange as desired for the integration by parts

$$(\partial_m \psi(\mathbf{x}, t)) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) = \partial_m \left(\psi(\mathbf{x}, t) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right)\right) - \psi(\mathbf{x}, t)(-ik_m) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) \tag{117.19}$$

and substitute back into the integral

$$\begin{aligned}\sigma^m \mathcal{F}(\partial_m \psi) = &\frac{1}{(\sqrt{2\pi})^3} \int_{\partial x^p} dx^p \int_{\partial x^n} dx^n \left(\sigma^m \psi(\mathbf{x}, t) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right)\right)\Big|_{\partial x^m} \\ &- \frac{1}{(\sqrt{2\pi})^3} \int_{\partial x^p} dx^p \int_{\partial x^n} dx^n \int_{\partial x^m} dx^m \sigma^m \psi(\mathbf{x}, t)(-ik_m) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right)\end{aligned} \tag{117.20}$$

So, we find that the Fourier transform of our spatial gradient is

$$\mathcal{F}(\nabla\psi) = \mathbf{k}\hat{\psi} i \tag{117.21}$$

This has the specific ordering of the vector products for our possibility of non-commutative factors.

From this, without making any assumptions about grade, we have the wave number domain equivalent for the spacetime split of the gradient eq. (117.14)

$$\partial_0 \hat{\psi} + \mathbf{k}\hat{\psi} i = \hat{g} \tag{117.22}$$

117.5 BACK TO SPECIFICS. MAXWELL'S EQUATION IN WAVE NUMBER DOMAIN

For Maxwell's equation our field variable F is grade two in the STA basis, and our specific transform pair is:

$$\begin{aligned}\left(\partial_0 + \boldsymbol{\nabla}\right) F &= \gamma_0 J/\epsilon_0 c \\ \partial_0 \hat{F} + \mathbf{k}\hat{F} i &= \gamma_0 \hat{J}/\epsilon_0 c\end{aligned} \tag{117.23}$$

Now, $\exp(i\theta)$ and i commute, and i also commutes with both F and $\mathbf{k}$. This is true since our field F as well as the spatial vector $\mathbf{k}$ are grade two in the STA basis. Two sign interchanges occur as we commute with each vector factor of these bivectors.

This allows us to write our transformed equation in the slightly tidier form

$$\partial_0 \hat{F} + (i\mathbf{k})\hat{F} = \gamma_0 \hat{J}/\epsilon_0 c \tag{117.24}$$

We want to find a solution to this equation. If the objects in question were all scalars this would be simple enough, and is a problem of the form

$$B' + AB = Q \tag{117.25}$$

For our electromagnetic field our transform is a summation of the following form

$$(\mathbf{E} + ic\mathbf{B})(\cos\theta + i\sin\theta) = (\mathbf{E}\cos\theta - c\mathbf{B}\sin\theta) + i(\mathbf{E}\sin\theta + c\mathbf{B}\cos\theta) \tag{117.26}$$

The summation of the integral itself will not change the grades, so $\hat{F}$ is also a grade two multivector. The dual of our spatial wave number vector $i\mathbf{k}$ is also grade two with basis bivectors $\gamma_m\gamma_n$ very much like the magnetic field portions of our field vector $ic\mathbf{B}$.

Having figured out the grades of all the terms in eq. (117.24), what does a grade split of this equation yield? For the equation to be true do we not need it to be true for all grades? Our grade zero, four, and two terms respectively are

$$\begin{aligned}(i\mathbf{k}) \cdot \hat{F} &= \hat{\rho}/\epsilon_0 \\ (i\mathbf{k}) \wedge \hat{F} &= 0 \\ \partial_0 \hat{F} + (i\mathbf{k}) \times \hat{F} &= -\hat{\mathbf{j}}/\epsilon_0 c\end{aligned} \tag{117.27}$$

Here the (antisymmetric) commutator product $\langle ab \rangle_2 = a \times b = (ab - ba)/2$ has been used in the last equation for this bivector product.

It is kind of interesting that an unmoving charge density contributes nothing to the time variation of the field in the wave number domain, instead only the current density (spatial) vectors contribute to our differential equation.

117.5.1 *Solving this first order inhomogeneous problem*

We want to solve the inhomogeneous scalar equation eq. (117.25) but do so in a fashion that is also valid for the grades for the Maxwell equation problem.

Application of variation of parameters produces the desired result. Let us write this equation in operator form

$$L(B) = B' + AB \tag{117.28}$$

and start with the solution of the homogeneous problem

$$L(B) = 0 \tag{117.29}$$

This is

$$B' = -AB \tag{117.30}$$

so we expect exponential solutions will do the trick, but have to get the ordering right due to the possibility of non-commutative factors. How about one of

$$\begin{aligned} B &= Ce^{-At} \\ B &= e^{-At}C \end{aligned} \tag{117.31}$$

Where C is constant, but not necessarily a scalar, and does not have to commute with A. Taking derivatives of the first we have

$$B' = C(-A)e^{-At} \tag{117.32}$$

This does not have the desired form unless C and A commute. How about the second possibility? That one has the derivative

$$\begin{aligned} B' &= (-A)e^{-At}C \\ &= -AB \end{aligned} \tag{117.33}$$

which is what we want. Now, for the inhomogeneous problem we want to use a test solution replacing C with an function to be determined. That is

$$B = e^{-At}U \tag{117.34}$$

For this we have

$$\begin{aligned} L(B) &= (-A)e^{-At}U + e^{-At}U' + AB \\ &= e^{-At}U' \end{aligned} \tag{117.35}$$

Our inhomogeneous problem $L(B) = Q$ is therefore reduced to

$$e^{-At}U' = Q \tag{117.36}$$

Or

$$U = \int e^{At} Q(t) dt \tag{117.37}$$

As an indefinite integral this gives us

$$B(t) = e^{-At} \int e^{At} Q(t) dt \tag{117.38}$$

And finally in definite integral form, if all has gone well, we have a specific solution to the forced problem

$$B(t) = \int_{t_0}^{t} e^{-A(t-\tau)} Q(\tau) d\tau \tag{117.39}$$

117.5.1.1 *Verify*

With differentiation under the integral sign we have

$$\begin{aligned} \frac{dB}{dt} &= e^{-A(t-\tau)} Q(\tau) \Big|_{\tau=t} + \int_{t_0}^{t} -A e^{-A(t-\tau)} Q(\tau) d\tau \\ &= Q(t) - AB \end{aligned} \tag{117.40}$$

Great!

117.5.2 *Back to Maxwell's*

Switching to explicit time derivatives we have

$$\partial_t \hat{F} + (ic\mathbf{k}) \hat{F} = \gamma_0 \hat{J} / \epsilon_0 \tag{117.41}$$

By eq. (117.39), this has, respectively, homogeneous and inhomogeneous solutions

$$\begin{aligned} \hat{F}(\mathbf{k}, t) &= e^{-ic\mathbf{k}t} C(\mathbf{k}) \\ \hat{F}(\mathbf{k}, t) &= \frac{1}{\epsilon_0} \int_{t_0(\mathbf{k})}^{t} e^{-(ic\mathbf{k})(t-\tau)} \gamma_0 \hat{J}(\mathbf{k}, \tau) d\tau \end{aligned} \tag{117.42}$$

For the homogeneous term at $t = 0$ we have

$$\hat{F}(\mathbf{k}, 0) = C(\mathbf{k}) \tag{117.43}$$

So, $C(\mathbf{k})$ is just the Fourier transform of an initial wave packet. Reassembling all the bits in terms of fully specified Fourier and inverse Fourier transforms we have

$$\begin{aligned} F(\mathbf{x}, t) &= \frac{1}{(\sqrt{2\pi})^3} \int \left(\frac{1}{(\sqrt{2\pi})^3} \int e^{-ic\mathbf{k}t} F(\mathbf{u}, 0) e^{-i\mathbf{k}\cdot\mathbf{u}} d^3u \right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k \\ &= \frac{1}{(2\pi)^3} \int e^{-ic\mathbf{k}t} F(\mathbf{u}, 0) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{u})} d^3u d^3k \end{aligned} \tag{117.44}$$

We have something like a double sided Green's function, with which we do a spatial convolution over all space with to produce a function of wave number. One more integration over all wave numbers gives us our inverse Fourier transform. The final result is a beautiful closed form solution for the time evolution of an arbitrary wave packet for the field specified at some specific initial time.

Now, how about that forced term? We want to inverse Fourier transform our $\hat{J}$ based equation in eq. (117.42). Picking our $t_0 = -\infty$ this is

$$\begin{aligned} F(\mathbf{x}, t) &= \frac{1}{(\sqrt{2\pi})^3} \int \left(\frac{1}{\epsilon_0} \int_{\tau=-\infty}^{t} e^{-(ic\mathbf{k})(t-\tau)} \gamma_0 \hat{J}(\mathbf{k}, \tau) d\tau \right) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k \\ &= \frac{1}{\epsilon_0 (2\pi)^3} \int \int_{\tau=-\infty}^{t} e^{-(ic\mathbf{k})(t-\tau)} \gamma_0 J(\mathbf{u}, \tau) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{u})} d\tau d^3u d^3k \end{aligned} \tag{117.45}$$

Again we have a double sided Green's function. We require a convolution summing the four vector current density contributions over all space and for all times less than t.

Now we can combine the vacuum and charge present solutions for a complete solution to Maxwell's equation. This is

$$F(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int e^{-ic\mathbf{k}t} \left(F(\mathbf{u}, 0) + \frac{1}{\epsilon_0} \int_{\tau=-\infty}^{t} e^{ic\mathbf{k}\tau} \gamma_0 J(\mathbf{u}, \tau) d\tau \right) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{u})} d^3u d^3k \tag{117.46}$$

Now, this may not be any good for actually computing with, but it sure is pretty!

There is a lot of verification required to see if all this math actually works out, and also a fair amount of followup required to play with this and see what other goodies fall out if this is used. I had expect that this result ought to be usable to show familiar results like the Biot-Savart law.

How do our energy density and Poynting energy momentum density conservation relations, and the stress energy tensor terms, look given a closed form expression for F?

It is also kind of interesting to see the time phase term coupled to the current density here in the forcing term. That looks awfully similar to some QM expressions, although it could be coincidental.

118

4D FOURIER TRANSFORMS APPLIED TO MAXWELL'S EQUATION

118.1 NOTATION

Please see A for a summary of much of the notation used here.

118.2 MOTIVATION

In 117, a solution of the first order Maxwell equation

$$\nabla F = \frac{J}{\epsilon_0 c} \tag{118.1}$$

was found to be

$$F(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int e^{-ic\mathbf{k}t} \left(F(\mathbf{u}, 0) + \frac{1}{\epsilon_0} \int_{\tau=-\infty}^{t} e^{ic\mathbf{k}\tau} \gamma_0 J(\mathbf{u}, \tau) d\tau \right) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{u})} d^3u d^3k \tag{118.2}$$

This does not have the spacetime uniformity that is expected for a solution of a Lorentz invariant equation.

Similarly, in 116 solutions of the second order Maxwell equation in the Lorentz gauge $\nabla \cdot A = 0$

$$\begin{aligned} F &= \nabla \wedge A \\ \nabla^2 A &= J/\epsilon_0 c \end{aligned} \tag{118.3}$$

were found to be

$$\begin{aligned} A^\mu(x) &= \frac{1}{\epsilon_0 c} \int J^\mu(x') G(x - x') d^4x' \\ G(x) &= \frac{u(x \cdot \gamma_0)}{(2\pi)^3} \int \sin(|\mathbf{k}| x \cdot \gamma_0) \exp\left(-i(\mathbf{k}\gamma_0) \cdot x\right) \frac{d^3k}{|\mathbf{k}|} \end{aligned} \tag{118.4}$$

Here our convolution kernel G also does not exhibit a uniform four vector form that one could logically expect.

In these notes an attempt to rework these problems using a 4D spacetime Fourier transform will be made.

118.3 4D FOURIER TRANSFORM

As before we want a multivector friendly Fourier transform pair, and choose the following

$$
\begin{aligned}
\hat{\psi}(k) &= \frac{1}{(\sqrt{2\pi})^4} \int_{-\infty}^{\infty} \psi(x) \exp\left(-ik \cdot x\right) d^4 x \\
\psi(x) &= \text{PV} \frac{1}{(\sqrt{2\pi})^4} \int_{-\infty}^{\infty} \hat{\psi}(k) \exp\left(ik \cdot x\right) d^4 k
\end{aligned}
\tag{118.5}
$$

Here we use $i = \gamma_0\gamma_1\gamma_2\gamma_3$ as our pseudoscalar, and have to therefore be careful of order of operations since this does not necessarily commute with multivector ψ or $\hat{\psi}$ functions.

For our dot product and vectors, with summation over matched upstairs downstairs indices implied, we write

$$
\begin{aligned}
x &= x^\mu \gamma_\mu = x_\mu \gamma^\mu \\
k &= k^\mu \gamma_\mu = k_\mu \gamma^\mu \\
x \cdot k &= x^\mu k_\mu = x_\mu k^\mu
\end{aligned}
\tag{118.6}
$$

Finally our differential volume elements are defined to be

$$
\begin{aligned}
d^4 x &= dx^0 dx^1 dx^2 dx^3 \\
d^4 k &= dk_0 dk_1 dk_2 dk_3
\end{aligned}
\tag{118.7}
$$

Note the opposite pairing of upstairs and downstairs indices in the coordinates.

118.4 POTENTIAL EQUATIONS

118.4.1 *Inhomogeneous case*

First for the attack is the Maxwell potential equations. As well as using a 4D transform, having learned how to do Fourier transformations of multivectors, we will attack this one in vector form as well. Our equation to invert is

$$
\nabla^2 A = J/\epsilon_0 c
\tag{118.8}
$$

There is nothing special to do for the transformation of the current term, but the left hand side will require two integration parts

$$\begin{aligned}\mathcal{F}(\nabla^2 A) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left(\left(\partial_{00} - \sum_m \partial_{mm}\right)A\right)e^{-ik_\mu x^\mu}d^4x \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}A\left((-ik_0)^2 - \sum_m(-ik_m)^2\right)e^{-ik_\mu x^\mu}d^4x\end{aligned} \tag{118.9}$$

As usual it is required that A and $\partial_\mu A$ vanish at infinity. Now for the scalar in the interior we have

$$(-ik_0)^2 - \sum_m(-ik_m)^2 = -(k_0)^2 + \sum_m(k_m)^2 \tag{118.10}$$

But this is just the (negation) of the square of our wave number vector

$$\begin{aligned}k^2 &= k_\mu\gamma^\mu \cdot k_\nu\gamma^\nu \\ &= k_\mu k_\nu \gamma^\mu \cdot \gamma^\nu \\ &= k_0 k_0 \gamma_0 \cdot \gamma^0 - \sum_{a,b} k_a k_b \gamma_a \cdot \gamma^b \\ &= (k_0)^2 - \sum_a (k_a)^2\end{aligned} \tag{118.11}$$

Putting things back together we have for our potential vector in the wave number domain

$$\hat{A} = \frac{\hat{J}}{-k^2\epsilon_0 c} \tag{118.12}$$

Inverting, and substitution for $\hat{J}$ gives us our spacetime domain potential vector in one fell swoop

$$\begin{aligned}A(x) &= \frac{1}{(\sqrt{2\pi})^4}\int_{-\infty}^{\infty}\left(\frac{1}{-k^2\epsilon_0 c}\frac{1}{(\sqrt{2\pi})^4}\int_{-\infty}^{\infty}J(x')e^{-ik\cdot x'}d^4x'\right)e^{ik\cdot x}d^4k \\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}J(x')\frac{1}{-k^2\epsilon_0 c}e^{ik\cdot(x-x')}d^4k d^4x'\end{aligned} \tag{118.13}$$

This allows us to write this entire specific solution to the forced wave equation problem as a convolution integral

$$\begin{aligned} A(x) &= \frac{1}{\epsilon_0 c} \int_{-\infty}^{\infty} J(x')G(x-x')d^4x' \\ G(x) &= \frac{-1}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{e^{ik\cdot x}}{k^2} d^4k \end{aligned} \tag{118.14}$$

Pretty slick looking, but actually also problematic if one thinks about it. Since k^2 is null in some cases $G(x)$ may blow up in some conditions. My assumption however, is that a well defined meaning can be associated with this integral, I just do not know what it is yet. A way to define this more exactly may require picking a more specific orthonormal basis once the exact character of J is known.

FIXME: In 114 I worked through how to evaluate such an integral (expanding on a too brief treatment found in [5]). To apply such a technique here, where our Green's function has precisely the same form as the Green's function for the Poisson's equation, a way to do the equivalent of a spherical polar parametrization will be required. How would that be done in 4D? Have seen such treatments in [13] for hypervolume and surface integration, but they did not make much sense then. Perhaps they would now?

118.4.2 *The homogeneous case*

The missing element here is the addition of any allowed homogeneous solutions to the wave equation. The form of such solutions cannot be obtained with the 4D transform since that produces

$$-k^2\hat{A} = 0 \tag{118.15}$$

and no meaningful inversion of that is possible.

For the homogeneous problem we are forced to re-express the spacetime Laplacian with an explicit bias towards either time or a specific direction in space, and attack with a Fourier transform on the remaining coordinates. This has been done previously, but we can revisit this using our new vector transform.

Now we switch to a spatial Fourier transform

$$\begin{aligned} \hat{\psi}(\mathbf{k}, t) &= \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \psi(\mathbf{x}, t) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) d^3x \\ \psi(\mathbf{x}, t) &= \mathrm{PV} \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \hat{\psi}(\mathbf{k}, t) \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) d^3k \end{aligned} \tag{118.16}$$

Using a spatial transform we have

$$\mathcal{F}((\partial_{00} - \sum_m \partial_{mm})A) = \partial_{00}\hat{A} - \sum_m \hat{A}(-ik_m)^2 \tag{118.17}$$

Carefully keeping the pseudoscalar factors all on the right of our vector as the integration by parts was performed does not make a difference since we just end up with a scalar in the end. Our equation in the wave number domain is then just

$$\partial_{tt}\hat{A}(\mathbf{k}, t) + (c^2\mathbf{k}^2)\hat{A}(\mathbf{k}, t) = 0 \tag{118.18}$$

with exponential solutions

$$\hat{A}(\mathbf{k}, t) = C(\mathbf{k}) \exp(\pm ic|\mathbf{k}|t) \tag{118.19}$$

In particular, for $t = 0$ we have

$$\hat{A}(\mathbf{k}, 0) = C(\mathbf{k}) \tag{118.20}$$

Reassembling then gives us our homogeneous solution

$$A(\mathbf{x}, t) = \frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} \left(\frac{1}{(\sqrt{2\pi})^3} \int_{-\infty}^{\infty} A(\mathbf{x}', 0) e^{-i\mathbf{k}\cdot\mathbf{x}'} d^3x' \right) e^{\pm ic|\mathbf{k}|t} e^{i\mathbf{k}\cdot\mathbf{x}} d^3k \tag{118.21}$$

This is

$$\begin{aligned} A(\mathbf{x}, t) &= \int_{-\infty}^{\infty} A(\mathbf{x}', 0) G(\mathbf{x} - \mathbf{x}') d^3x' \\ G(\mathbf{x}) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp\left(i\mathbf{k} \cdot \mathbf{x} \pm ic|\mathbf{k}|t\right) d^3k \end{aligned} \tag{118.22}$$

Here also we have to be careful to keep the Green's function on the right hand side of A since they will not generally commute.

118.4.3 *Summarizing*

Assembling both the homogeneous and inhomogeneous parts for a complete solution we have for the Maxwell four vector potential

$$
\begin{aligned}
A(x) &= \int_{-\infty}^{\infty} \left(A(\mathbf{x}', 0) H(\mathbf{x} - \mathbf{x}') + \frac{1}{\epsilon_0 c} \int_{-\infty}^{\infty} J(x') G(x - x') dx^0 \right) dx^1 dx^2 dx^3 \\
H(\mathbf{x}) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \exp\left(i\mathbf{k} \cdot \mathbf{x} \pm ic|\mathbf{k}|t \right) d^3 k \\
G(x) &= \frac{-1}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{e^{ik\cdot x}}{k^2} d^4 k
\end{aligned}
\tag{118.23}
$$

Here for convenience both four vectors and spatial vectors were used with

$$
\begin{aligned}
x &= x^\mu \gamma_\mu \\
\mathbf{x} &= x^m \sigma_m = x \wedge \gamma_0
\end{aligned}
\tag{118.24}
$$

As expected, operating where possible in a Four vector context does produce a simpler convolution kernel for the vector potential.

118.5 FIRST ORDER MAXWELL EQUATION TREATMENT

Now we want to Fourier transform Maxwell's equation directly. That is

$$
\mathcal{F}(\nabla F = J/\epsilon_0 c)
\tag{118.25}
$$

For the LHS we have

$$
\begin{aligned}
\mathcal{F}(\nabla F) &= \mathcal{F}(\gamma^\mu \partial_\mu F) \\
&= \gamma^\mu \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (\partial_\mu F) e^{-ik\cdot x} d^4 x \\
&= -\gamma^\mu \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F \partial_\mu (e^{-ik_\sigma x^\sigma}) d^4 x \\
&= -\gamma^\mu \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F(-ik_\mu) e^{-ik\cdot x} d^4 x \\
&= -i\gamma^\mu k_\mu \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} F e^{-ik\cdot x} d^4 x \\
&= -ik\hat{F}
\end{aligned}
\tag{118.26}
$$

This gives us

$$-ik\hat{F} = \hat{J}/\epsilon_0 c \tag{118.27}$$

So to solve the forced Maxwell equation we have only to inverse transform the following

$$\hat{F} = \frac{1}{-ik\epsilon_0 c}\hat{J} \tag{118.28}$$

This is

$$F = \frac{1}{(\sqrt{2\pi})^4}\int_{-\infty}^{\infty}\frac{1}{-ik\epsilon_0 c}\left(\frac{1}{(\sqrt{2\pi})^4}\int_{-\infty}^{\infty}J(x')e^{-ik\cdot x'}d^4x'\right)e^{ik\cdot x}d^4k \tag{118.29}$$

Adding to this a solution to the homogeneous equation we now have a complete solution in terms of the given four current density and an initial field wave packet

$$F = \frac{1}{(2\pi)^3}\int e^{-ic\mathbf{k}t}F(\mathbf{x}', 0)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}d^3x'd^3k + \frac{1}{(2\pi)^4\epsilon_0 c}\int \frac{i}{k}J(x')e^{ik\cdot(x-x')}d^4kd^4x' \tag{118.30}$$

Observe that we can not make a single sided Green's function to convolve J with since the vectors k and J may not commute.

As expected working in a relativistic context for our inherently relativistic equation turns out to be much simpler and produce a simpler result. As before trying to actually evaluate these integrals is a different story.

119 FOURIER SERIES VACUUM MAXWELL'S EQUATIONS

119.1 MOTIVATION

In [2], after finding a formulation of Maxwell's equations that he likes, his next step is to assume the electric and magnetic fields can be expressed in a 3D Fourier series form, with periodicity in some repeated volume of space, and then proceeds to evaluate the energy of the field.

119.1.1 *Notation*

See the notational table A for much of the notation assumed here.

119.2 SETUP

Let us try this. Instead of using the sine and cosine Fourier series which looks more complex than it ought to be, use of a complex exponential ought to be cleaner.

119.2.1 *3D Fourier series in complex exponential form*

For a multivector function $f(\mathbf{x}, t)$, periodic in some rectangular spatial volume, let us assume that we have a 3D Fourier series representation.

Define the element of volume for our fundamental wavelengths to be the region bounded by three intervals in the x^1, x^2, x^3 directions respectively

$$\begin{aligned} I_1 &= [a^1, a^1 + \lambda_1] \\ I_2 &= [a^2, a^2 + \lambda_2] \\ I_3 &= [a^3, a^3 + \lambda_3] \end{aligned} \tag{119.1}$$

Our assumed Fourier representation is then

$$f(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}}(t) \exp\left(- \sum_j \frac{2\pi i k_j x^j}{\lambda_j} \right) \tag{119.2}$$

Here $\hat{f}_{\mathbf{k}} = \hat{f}_{\{k_1,k_2,k_3\}}$ is indexed over a triplet of integer values, and the k_1, k_2, k_3 indices take on all integer values in the $[-\infty, \infty]$ range.

Note that we also wish to allow i to not just be a generic complex number, but allow for the use of either the Euclidean or Minkowski pseudoscalar

$$i = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3 \tag{119.3}$$

Because of this we should not assume that we can commute i, or our exponentials with the functions $f(\mathbf{x}, t)$, or $\hat{f}_{\mathbf{k}}(t)$.

$$\begin{aligned} &\int_{x^1=\partial I_1}\int_{x^2=\partial I_2}\int_{x^3=\partial I_3} f(\mathbf{x}, t) e^{2\pi i m_j x^j/\lambda_j} dx^1 dx^2 dx^3 \\ &\qquad = \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}}(t) \int_{x^1=\partial I_1}\int_{x^2=\partial I_2}\int_{x^3=\partial I_3} dx^1 dx^2 dx^3 e^{2\pi i (m_j - k_j) x^j/\lambda_j} dx^1 dx^2 dx^3 \end{aligned} \tag{119.4}$$

But each of these integrals is just $\delta_{\mathbf{k},\mathbf{m}}\lambda_1\lambda_2\lambda_3$, giving us

$$\hat{f}_{\mathbf{k}}(t) = \frac{1}{\lambda_1\lambda_2\lambda_3}\int_{x^1=\partial I_1}\int_{x^2=\partial I_2}\int_{x^3=\partial I_3} f(\mathbf{x}, t)\exp\left(\sum_j \frac{2\pi i k_j x^j}{\lambda_j}\right) dx^1 dx^2 dx^3 \tag{119.5}$$

To tidy things up lets invent (or perhaps abuse) some notation to tidy things up. As a subscript on our Fourier coefficients we have used $\mathbf{k}$ as an index. Let us also use it as a vector, and define

$$\mathbf{k} \equiv 2\pi \sum_m \frac{\sigma^m k_m}{\lambda_m} \tag{119.6}$$

With our spatial vector $\mathbf{x}$ written

$$\mathbf{x} = \sum_m \sigma_m x^m \tag{119.7}$$

We now have a $\mathbf{k} \cdot \mathbf{x}$ term in the exponential, and can remove when desirable the coordinate summation. If we write $V = \lambda_1\lambda_2\lambda_3$ it leaves a nice tidy notation for the 3D Fourier series over the volume

$$\begin{aligned} f(\mathbf{x}, t) &= \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \hat{f}_{\mathbf{k}}(t) &= \frac{1}{V}\int f(\mathbf{x}, t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \end{aligned} \tag{119.8}$$

This allows us to proceed without caring about the specifics of the lengths of the sides of the rectangular prism that defines the periodicity of the signal in question.

119.2.2 *Vacuum equation*

Now that we have a desirable seeming Fourier series representation, we want to apply this to Maxwell's equation for the vacuum. We will use the STA formulation of Maxwell's equation, but use the unit convention of Bohm's book.

In 101 the STA equivalent to Bohm's notation for Maxwell's equations was found to be

$$\begin{aligned} F &= \mathcal{E} + i\mathcal{H} \\ J &= (\rho + \mathbf{j})\gamma_0 \\ \nabla F &= 4\pi J \end{aligned} \tag{119.9}$$

This is the CGS form of Maxwell's equation, but with the old style $\mathcal{H}$ for $c\mathbf{B}$, and $\mathcal{E}$ for $\mathbf{E}$. In more recent texts $\mathcal{E}$ (as a non-vector) is reserved for electromotive flux. In this set of notes I use Bohm's notation, since the aim is to clarify for myself aspects of his treatment.

For the vacuum equation, we make an explicit spacetime split by premultiplying with γ_0

$$\begin{aligned} \gamma_0 \nabla &= \gamma_0 \left(\gamma^0 \partial_0 + \gamma^k \partial_k\right) \\ &= \partial_0 - \gamma^k \gamma_0 \partial_k \\ &= \partial_0 + \gamma_k \gamma_0 \partial_k \\ &= \partial_0 + \sigma_k \partial_k \\ &= \partial_0 + \boldsymbol{\nabla} \end{aligned} \tag{119.10}$$

So our vacuum equation is just

$$(\partial_0 + \boldsymbol{\nabla})F = 0 \tag{119.11}$$

119.3 FIRST ORDER VACUUM SOLUTION WITH FOURIER SERIES

119.3.1 *Basic solution in terms of undetermined coefficients*

Now that a notation for the 3D Fourier series has been established, we can assume a series solution for our field of the form

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{F}_{\mathbf{k}}(t) e^{-2\pi i k_j x^j/\lambda_j} \tag{119.12}$$

can now apply this to the vacuum Maxwell equation eq. (119.11). This gives us

$$\begin{aligned}\sum_{\mathbf{k}} \left(\partial_t \hat{F}_{\mathbf{k}}(t)\right) e^{-2\pi i k_j x^j/\lambda_j} &= -c \sum_{\mathbf{k},m} \sigma^m \hat{F}_{\mathbf{k}}(t) \frac{\partial}{\partial x^m} e^{-2\pi i k_j x^j/\lambda_j} \\ &= -c \sum_{\mathbf{k},m} \sigma^m \hat{F}_{\mathbf{k}}(t) \left(-2\pi \frac{k_m}{\lambda_m}\right) e^{-2\pi i k_j x^j/\lambda_j} \\ &= 2\pi c \sum_{\mathbf{k}} \sum_m \frac{\sigma^m k_m}{\lambda_m} \hat{F}_{\mathbf{k}}(t) i e^{-2\pi i k_j x^j/\lambda_j}\end{aligned} \tag{119.13}$$

Note that i commutes with $\mathbf{k}$ and since F is also an STA bivector i commutes with F. Putting all this together we have

$$\sum_{\mathbf{k}} \left(\partial_t \hat{F}_{\mathbf{k}}(t)\right) e^{-i\mathbf{k}\cdot\mathbf{x}} = ic \sum_{\mathbf{k}} \mathbf{k} \hat{F}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{119.14}$$

Term by term we now have a (big ass, triple infinite) set of very simple first order differential equations, one for each $\mathbf{k}$ triplet of indices. Specifically this is

$$\hat{F}'_{\mathbf{k}} = ic\mathbf{k}\hat{F}_{\mathbf{k}} \tag{119.15}$$

With solutions

$$\begin{aligned}\hat{F}_0 &= C_0 \\ \hat{F}_{\mathbf{k}} &= \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}}\end{aligned} \tag{119.16}$$

Here $C_{\mathbf{k}}$ is an undetermined STA bivector. For now we keep this undetermined coefficient on the right hand side of the exponential since no demonstration that it commutes with a factor of the form $\exp(i\mathbf{k}\phi)$. Substitution back into our assumed solution sum we have a solution to Maxwell's equation, in terms of a set of as yet undetermined (bivector) coefficients

$$F(\mathbf{x}, t) = C_0 + \sum_{\mathbf{k}\neq 0} \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x}) \tag{119.17}$$

The special case of $\mathbf{k} = 0$ is now seen to be not so special and can be brought into the sum.

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x}) \tag{119.18}$$

We can also take advantage of the bivector nature of $C_{\mathbf{k}}$, which implies the complex exponential can commute to the left, since the two fold commutation with the pseudoscalar with change sign twice.

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) C_{\mathbf{k}} \tag{119.19}$$

119.3.2 *Solution as time evolution of initial field*

Now, observe the form of this sum for $t = 0$. This is

$$F(\mathbf{x}, 0) = \sum_{\mathbf{k}} C_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{x}) \tag{119.20}$$

So, the C_k coefficients are precisely the Fourier coefficients of $F(\mathbf{x}, 0)$. This is to be expected having repeatedly seen similar results in the Fourier transform treatments of 116, 117, and 118. We then have an equation for the complete time evolution of any spatially periodic electrodynamic field in terms of the field value at all points in the region at some initial time. Summarizing so far this is

$$\begin{aligned} F(\mathbf{x}, t) &= \sum_{\mathbf{k}} \exp\left(ic\mathbf{k}t\right) C_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{x}) \\ C_{\mathbf{k}} &= \frac{1}{V} \int F(\mathbf{x}', 0) \exp\left(i\mathbf{k} \cdot \mathbf{x}'\right) d^3x' \end{aligned} \tag{119.21}$$

Regrouping slightly we can write this as a convolution with a Fourier kernel (a Green's function). That is

$$F(\mathbf{x}, t) = \frac{1}{V} \int \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})\right) F(\mathbf{x}', 0) d^3x' \tag{119.22}$$

Or

$$\begin{aligned} F(\mathbf{x}, t) &= \int G(\mathbf{x} - \mathbf{x}', t) F(\mathbf{x}', 0) d^3x' \\ G(\mathbf{x}, t) &= \frac{1}{V} \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) \end{aligned} \tag{119.23}$$

Okay, that is cool. We have now got the basic periodicity result directly from Maxwell's equation in one shot. No need to drop down to potentials, or even the separate electric or magnetic components of our field $F = \boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}$.

119.3.3 *Prettying it up? Questions of commutation*

Now, it is tempting here to write eq. (119.19) as a single exponential

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct - i\mathbf{k}\cdot\mathbf{x}\right) C_{\mathbf{k}} \qquad \text{VALID?} \tag{119.24}$$

This would probably allow for a prettier four vector form in terms of $x = x^\mu \gamma_\mu$ replacing the separate $\mathbf{x}$ and $x^0 = ct$ terms. However, such a grouping is not allowable unless one first demonstrates that $e^{i\mathbf{u}}$, and $e^{i\alpha}$, for spatial vector $\mathbf{u}$ and scalar α commute!

To demonstrate that this is in fact the case note that exponential of this dual spatial vector can be written

$$\exp(i\mathbf{u}) = \cos(\mathbf{u}) + i\sin(\mathbf{u}) \tag{119.25}$$

This spatial vector cosine, $\cos(\mathbf{u})$, is a scalar (even powers only), and our sine, $\sin(\mathbf{u}) \propto \mathbf{u}$, is a spatial vector in the direction of $\mathbf{u}$ (odd powers leaves a vector times a scalar). Spatial vectors commute with i (toggles sign twice percolating its way through), therefore pseudoscalar exponentials also commute with i.

This will simplify a lot, and it shows that eq. (119.24) is in fact a valid representation.

Now, there is one more question of commutation here. Namely, does a dual spatial vector exponential commute with the field itself (or equivalently, one of the Fourier coefficients).

Expanding such a product and attempting term by term commutation should show

$$\begin{aligned}
e^{i\mathbf{u}}F &= (\cos\mathbf{u} + i\sin\mathbf{u})(\mathcal{E} + i\mathcal{H}) \\
&= i\sin\mathbf{u}(\mathcal{E} + i\mathcal{H}) + (\mathcal{E} + i\mathcal{H})\cos\mathbf{u} \\
&= i(\sin\mathbf{u})\mathcal{E} - (\sin\mathbf{u})\mathcal{H} + F\cos\mathbf{u} \\
&= i(-\mathcal{E}\sin\mathbf{u} + 2\mathcal{E}\cdot\sin\mathbf{u}) + (\mathcal{H}\sin\mathbf{u} - 2\mathcal{H}\cdot\sin\mathbf{u}) + F\cos\mathbf{u} \\
&= 2\sin\mathbf{u}\cdot(\mathcal{E} - \mathcal{H}) + F(\cos\mathbf{u} - i\sin\mathbf{u})
\end{aligned} \tag{119.26}$$

That is

$$e^{i\mathbf{u}}F = 2\sin\mathbf{u}\cdot(\mathcal{E} - \mathcal{H}) + Fe^{-i\mathbf{u}} \tag{119.27}$$

This exponential has one anticommuting term, but also has a scalar component introduced by the portions of the electric and magnetic fields that are colinear with the spatial vector $\mathbf{u}$.

119.4 FIELD ENERGY AND MOMENTUM

Given that we have the same structure for our four vector potential solutions as the complete bivector field, it does not appear that there is much reason to work in the second order quantities. Following Bohm we should now be prepared to express the field energy density and momentum density in terms of the Fourier coefficients, however unlike Bohm, let us try this using the first order solutions found above.

In CGS units (see 101 for verification) these field energy and momentum densities (Poynting vector $\mathbf{P}$) are, respectively

$$\begin{aligned} E &= \frac{1}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2\right) \\ \mathbf{P} &= \frac{1}{4\pi}(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) \end{aligned} \tag{119.28}$$

Given that we have a complete field equation without an explicit separation of electric and magnetic components, perhaps this is easier to calculate from the stress energy four vector for energy/momentum. In CGS units this must be

$$T(\gamma_0) = \frac{1}{8\pi} F \gamma_0 \tilde{F} \tag{119.29}$$

An expansion of this to verify the CGS conversion seems worthwhile.

$$\begin{aligned} T(\gamma_0) &= \frac{1}{8\pi} F \gamma_0 \tilde{F} \\ &= \frac{-1}{8\pi}(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}})\gamma_0(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}}) \\ &= \frac{1}{8\pi}(\boldsymbol{\mathcal{E}} + i\boldsymbol{\mathcal{H}})(\boldsymbol{\mathcal{E}} - i\boldsymbol{\mathcal{H}})\gamma_0 \\ &= \frac{1}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 - (i\boldsymbol{\mathcal{H}})^2 + i(\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{H}})\right)\gamma_0 \\ &= \frac{1}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2 + 2i^2\boldsymbol{\mathcal{H}} \times \boldsymbol{\mathcal{E}}\right)\gamma_0 \\ &= \frac{1}{8\pi}\left(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2\right)\gamma_0 + \frac{1}{4\pi}\left(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}\right)\gamma_0 \end{aligned} \tag{119.30}$$

Good, as expected we have

$$\begin{aligned} E &= T(\gamma_0) \cdot \gamma_0 \\ \mathbf{P} &= T(\gamma_0) \wedge \gamma_0 \end{aligned} \tag{119.31}$$

FIXME: units here for $\mathbf{P}$ are off by a factor of c. This does not matter so much in four vector form $T(\gamma_0)$ where the units naturally take care of themselves.

Okay, let us apply this to our field eq. (119.22), and try to percolate the γ_0 through all the terms of $\tilde{F}(\mathbf{x}, t)$

$$\begin{aligned}\gamma_0 \tilde{F}(\mathbf{x}, t) &= -\gamma_0 F(\mathbf{x}, t) \\ &= -\gamma_0 \frac{1}{V} \int \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})\right) F(\mathbf{x}', 0) d^3 x'\end{aligned} \tag{119.32}$$

Taking one factor at a time

$$\begin{aligned}\gamma_0 \exp\left(i\mathbf{k}ct\right) &= \gamma_0 (\cos\left(\mathbf{k}ct\right) + i \sin\left(\mathbf{k}ct\right)) \\ &= \cos\left(\mathbf{k}ct\right) \gamma_0 - i\gamma_0 \sin\left(\mathbf{k}ct\right)) \\ &= \cos\left(\mathbf{k}ct\right) \gamma_0 - i \sin\left(\mathbf{k}ct\right))\gamma_0 \\ &= \exp\left(-i\mathbf{k}ct\right) \gamma_0\end{aligned} \tag{119.33}$$

Next, percolate γ_0 through the pseudoscalar exponential.

$$\begin{aligned}\gamma_0 e^{i\phi} &= \gamma_0 (\cos\phi + i \sin\phi) \\ &= \cos\phi \gamma_0 - i\gamma_0 \sin\phi \\ &= e^{-i\phi} \gamma_0\end{aligned} \tag{119.34}$$

Again, the percolation produces a conjugate effect. Lastly, as noted previously F commutes with i. We have therefore

$$\begin{aligned}\tilde{F}(\mathbf{x}, t)\gamma_0 F(\mathbf{x}, t)\gamma_0 &= \frac{1}{V^2} \int \sum_{\mathbf{k},\mathbf{m}} F(\mathbf{a}, 0) e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x})} e^{i\mathbf{k}ct} e^{-i\mathbf{m}ct} e^{-i\mathbf{m}\cdot(\mathbf{b}-\mathbf{x})} F(\mathbf{b}, 0) d^3 a d^3 b \\ &= \frac{1}{V^2} \int \sum_{\mathbf{k},\mathbf{m}} F(\mathbf{a}, 0) e^{i\mathbf{k}\cdot\mathbf{a} - i\mathbf{m}\cdot\mathbf{b} + i(\mathbf{k}-\mathbf{m})ct - i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}} F(\mathbf{b}, 0) d^3 a d^3 b \\ &= \frac{1}{V^2} \int \sum_{\mathbf{k}} F(\mathbf{a}, 0) F(\mathbf{b}, 0) e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})} d^3 a d^3 b \\ &\quad + \frac{1}{V^2} \int \sum_{\mathbf{k}\neq\mathbf{m}} F(\mathbf{a}, 0) e^{i\mathbf{k}\cdot\mathbf{a} - i\mathbf{m}\cdot\mathbf{b} + i(\mathbf{k}-\mathbf{m})ct - i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}} F(\mathbf{b}, 0) d^3 a d^3 b \\ &= \frac{1}{V^2} \int \sum_{\mathbf{k}} F(\mathbf{a}, 0) F(\mathbf{b}, 0) e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})} d^3 a d^3 b \\ &\quad + \frac{1}{V^2} \int \sum_{\mathbf{m},\mathbf{k}\neq 0} F(\mathbf{a}, 0) e^{i\mathbf{m}\cdot(\mathbf{a}-\mathbf{b}) + i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x}) + i\mathbf{k}ct} F(\mathbf{b}, 0) d^3 a d^3 b\end{aligned} \tag{119.35}$$

Hmm. Messy. The scalar bits of the above are our energy. We have a F^2 like term in the first integral (like the Lagrangian density), but it is at different points, and we have to integrate those with a sort of vector convolution. Given the reciprocal relationships between convolution and multiplication moving between the frequency and time domains in Fourier transforms I had expect that this first integral can somehow be turned into the sum of the squares of all the Fourier coefficients

$$\sum_{\mathbf{k}} C_{\mathbf{k}}^2 \tag{119.36}$$

which is very much like a discrete version of the Rayleigh energy theorem as derived in C, and is in this case a constant (not a function of time or space) and is dependent on only the initial field. That would mean that the remainder is the Poynting vector, which looks reasonable since it has the appearance of being somewhat antisymmetric.

Hmm, having mostly figured it out without doing the math in this case, the answer pops out. This first integral can be separated cleanly since the pseudoscalar exponentials commute with the bivector field. We then have

$$\begin{aligned}
&\frac{1}{V^2}\int \sum_{\mathbf{k}} F(\mathbf{a},0)F(\mathbf{b},0)e^{i\mathbf{k}\cdot(\mathbf{a}-\mathbf{b})}d^3a d^3b \\
&\quad= \frac{1}{V}\int \sum_{\mathbf{k}} F(\mathbf{a},0)e^{i\mathbf{k}\cdot\mathbf{a}}d^3a \int F(\mathbf{b},0)e^{-i\mathbf{k}\cdot\mathbf{b}}d^3b \\
&\quad= \sum_{\mathbf{k}} \hat{F}_{-\mathbf{k}}\hat{F}_{\mathbf{k}}
\end{aligned} \tag{119.37}$$

A side note on subtle notational sneakiness here. In the assumed series solution of eq. (119.12) $\hat{F}_{\mathbf{k}}(t)$ was the $\mathbf{k}$ Fourier coefficient of $F(\mathbf{x},t)$, whereas here the use of $\hat{F}_{\mathbf{k}}$ has been used to denote the $\mathbf{k}$ Fourier coefficient of $F(\mathbf{x},0)$. An alternative considered and rejected was something messier like $F\widehat{(t=0)}_{\mathbf{k}}$, or the use of the original, less physically significant, $C_{\mathbf{k}}$ coefficients.

The second term could also use a simplification, and it looks like we can separate these **a** and **b** integrals too

$$\begin{aligned}
&\frac{1}{V^2}\int \sum_{\mathbf{m},\mathbf{k}\neq 0} F(\mathbf{a},0)e^{i\mathbf{m}\cdot(\mathbf{a}-\mathbf{b})+i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x})+i\mathbf{k}ct}F(\mathbf{b},0)d^3a d^3b \\
&\quad= \frac{1}{V}\int \sum_{\mathbf{m},\mathbf{k}\neq 0} F(\mathbf{a},0)e^{i(\mathbf{m}+\mathbf{k})\cdot\mathbf{a}}d^3a e^{i\mathbf{k}ct-i\mathbf{k}\cdot\mathbf{x}}\frac{1}{V}\int F(\mathbf{b},0)e^{-i\mathbf{m}\cdot\mathbf{b}}d^3b \\
&\quad= \sum_{\mathbf{m}}\sum_{\mathbf{k}\neq 0} \hat{F}_{-\mathbf{m}-\mathbf{k}}e^{i\mathbf{k}ct-i\mathbf{k}\cdot\mathbf{x}}\hat{F}_{\mathbf{m}}
\end{aligned} \tag{119.38}$$

Making an informed guess that the first integral is a scalar, and the second is a spatial vector, our energy and momentum densities (Poynting vector) respectively are

$$\begin{aligned} U &\stackrel{?}{=} \frac{1}{8\pi} \sum_{\mathbf{k}} \hat{F}_{-\mathbf{k}} \hat{F}_{\mathbf{k}} \\ \mathbf{P} &\stackrel{?}{=} \frac{1}{8\pi} \sum_{\mathbf{m}} \sum_{\mathbf{k} \neq 0} \hat{F}_{-\mathbf{m}-\mathbf{k}} e^{i\mathbf{k}ct - i\mathbf{k}\cdot\mathbf{x}} \hat{F}_{\mathbf{m}} \end{aligned} \tag{119.39}$$

Now that much of the math is taken care of, more consideration about the physics implications is required. In particular, relating these abstract quantities to the frequencies and the harmonic oscillator model as Bohm did is desirable (that was the whole point of the exercise).

On the validity of eq. (119.39), it is not unreasonable to expect that $\partial U/\partial t = 0$, and $\boldsymbol{\nabla} \cdot \mathbf{P} = 0$ separately in these current free conditions from the energy momentum conservation relation

$$\frac{\partial}{\partial t} \frac{1}{8\pi} \left(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2 \right) + \frac{1}{4\pi} \boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}) = -\boldsymbol{\mathcal{E}} \cdot \mathbf{j} \tag{119.40}$$

Note that an SI derivation of this relation can be found in 94. So it therefore makes some sense that all the time dependence ends up in what has been labeled as the Poynting vector. A proof that the spatial divergence of this quantity is zero would help validate the guess made (or perhaps invalidate it).

Hmm. Again on the validity of identifying the first sum with the energy. It does not appear to work for the $\mathbf{k} = 0$ case, since that gives you

$$\frac{1}{8\pi V^2} \int F(\mathbf{a}, 0) F(\mathbf{b}, 0) d^3a d^3b \tag{119.41}$$

That is only a scalar if the somehow all the non-scalar parts of that product somehow magically cancel out. Perhaps it is true that the second sum has no scalar part, and if that is the case one would have

$$U \stackrel{?}{=} \frac{1}{8\pi} \sum_{\mathbf{k}} \left\langle \hat{F}_{-\mathbf{k}} \hat{F}_{\mathbf{k}} \right\rangle \tag{119.42}$$

An explicit calculation of $T(\gamma_0) \cdot \gamma_0$ is probably justified to discarding all other grades, and get just the energy.

So, instead of optimistically hoping that the scalar and spatial vector terms will automatically fall out, it appears that we have to explicitly calculate the dot and wedge products, as in

$$\begin{aligned}
U &= -\frac{1}{16\pi}(F\gamma_0 F\gamma_0 + \gamma_0 F\gamma_0 F) \\
\mathbf{P} &= -\frac{1}{16\pi}(F\gamma_0 F\gamma_0 - \gamma_0 F\gamma_0 F)
\end{aligned} \tag{119.43}$$

and then substitute our Fourier series solution for F to get the desired result. This appears to be getting more complex instead of less so unfortunately, but hopefully following this to a logical conclusion will show in retrospect a faster way to the desired result. A first attempt to do so shows that we have to return to our assumed Fourier solution and revisit some of the assumptions made.

119.5 RETURN TO THE ASSUMED SOLUTIONS TO MAXWELL'S EQUATION

An initial attempt to expand eq. (119.39) properly given the Fourier specification of the Maxwell solution gets into trouble. Consideration of some special cases for specific values of $\mathbf{k}$ shows that there is a problem with the grades of the solution.

Let us reexamine the assumed solution of eq. (119.22) with respect to grade

$$F(\mathbf{x}, t) = \frac{1}{V}\int \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right)\exp\left(i\mathbf{k}\cdot(\mathbf{x}' - \mathbf{x})\right) F(\mathbf{x}', 0) d^3 x' \tag{119.44}$$

For scalar Fourier approximations we are used to the ability to select a subset of the Fourier terms to approximate the field, but except for the $\mathbf{k} = 0$ term it appears that a term by term approximation actually introduces noise in the form of non-bivector grades.

Consider first the $\mathbf{k} = 0$ term. This gives us a first order approximation of the field which is

$$F(\mathbf{x}, t) \approx \frac{1}{V}\int F(\mathbf{x}', 0) d^3 x' \tag{119.45}$$

As summation is grade preserving this spatial average of the initial field conditions does have the required grade as desired. Next consider a non-zero Fourier term such as $\mathbf{k} = \{1, 0, 0\}$. For this single term approximation of the field let us write out the field term as

$$F_{\mathbf{k}}(\mathbf{x}, t) = \frac{1}{V}\int e^{i\hat{\mathbf{k}}|\mathbf{k}|ct + i\mathbf{k}\cdot(\mathbf{x}' - \mathbf{x})}(\boldsymbol{\mathcal{E}}(\mathbf{x}', 0) + i\boldsymbol{\mathcal{H}}(\mathbf{x}', 0)) d^3 x' \tag{119.46}$$

Now, let us expand the exponential. This was shorthand for the product of the exponentials, which seemed to be a reasonable shorthand since we showed they commute. Expanded out this is

$$\begin{aligned}
&\exp(i\hat{\mathbf{k}}|\mathbf{k}|ct + i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})) \\
&\quad = (\cos(\mathbf{k}ct) + i\hat{\mathbf{k}}\sin(|\mathbf{k}|ct))(\cos(\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})) + i\sin(\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})))
\end{aligned} \tag{119.47}$$

For ease of manipulation write $\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) = k\Delta x$, and $\mathbf{k}ct = \omega t$, we have

$$\begin{aligned}
\exp(i\omega t + ik\Delta x) = \cos(\omega t)\cos(k\Delta x) + i\cos(\omega t)\sin(k\Delta x) \\
+ i\sin(\omega t)\cos(k\Delta x) - \sin(\omega t)\sin(k\Delta x)
\end{aligned} \tag{119.48}$$

Note that $\cos(\omega t)$ is a scalar, whereas $\sin(\omega t)$ is a (spatial) vector in the direction of $\mathbf{k}$. Multiplying this out with the initial time field $F(\mathbf{x}', 0) = \boldsymbol{\mathcal{E}}(\mathbf{x}', 0) + i\boldsymbol{\mathcal{H}}(\mathbf{x}', 0) = \boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}'$ we can separate into grades.

$$\begin{aligned}
&\exp(i\omega t + ik\Delta x)(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}') \\
&\quad = \cos(\omega t)(\boldsymbol{\mathcal{E}}'\cos(k\Delta x) - \boldsymbol{\mathcal{H}}'\sin(k\Delta x)) + \sin(\omega t)\times(\boldsymbol{\mathcal{H}}'\sin(k\Delta x) - \boldsymbol{\mathcal{E}}'\cos(k\Delta x)) \\
&\quad + i\cos(\omega t)(\boldsymbol{\mathcal{E}}'\sin(k\Delta x) + \boldsymbol{\mathcal{H}}'\cos(k\Delta x)) - i\sin(\omega t)\times(\boldsymbol{\mathcal{E}}'\sin(k\Delta x) + \boldsymbol{\mathcal{H}}'\cos(k\Delta x)) \\
&\quad - \sin(\omega t)\cdot(\boldsymbol{\mathcal{E}}'\sin(k\Delta x) + \boldsymbol{\mathcal{H}}'\cos(k\Delta x)) \\
&\quad + i(\sin(\omega t)\cdot(\boldsymbol{\mathcal{E}}'\cos(k\Delta x) - \boldsymbol{\mathcal{H}}'\sin(k\Delta x))
\end{aligned} \tag{119.49}$$

The first two lines, once integrated, produce the electric and magnetic fields, but the last two are rogue scalar and pseudoscalar terms. These are allowed in so far as they are still solutions to the differential equation, but do not have the desired physical meaning.

If one explicitly sums over pairs of $\{\mathbf{k}, -\mathbf{k}\}$ of index triplets then some cancellation occurs. The cosine cosine products and sine sine products double and the sine cosine terms cancel. We therefore have

$$\begin{aligned}
&\frac{1}{2}\exp(i\omega t + ik\Delta x)(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}') \\
&= \cos(\omega t)\boldsymbol{\mathcal{E}}'\cos(k\Delta x) + \sin(\omega t)\times\boldsymbol{\mathcal{H}}'\sin(k\Delta x) \\
&+ i\cos(\omega t)\boldsymbol{\mathcal{H}}'\cos(k\Delta x) - i\sin(\omega t)\times\boldsymbol{\mathcal{E}}'\sin(k\Delta x) \\
&- \sin(\omega t)\cdot\boldsymbol{\mathcal{E}}'\sin(k\Delta x) \\
&- i\sin(\omega t)\cdot\boldsymbol{\mathcal{H}}'\sin(k\Delta x) \\
&= (\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\cos(\omega t)\cos(k\Delta x) - i\sin(\omega t)\times(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}}')\sin(k\Delta x) \\
&- \sin(\omega t)\cdot(\boldsymbol{\mathcal{E}}' + i\boldsymbol{\mathcal{H}})\sin(k\Delta x)
\end{aligned} \tag{119.50}$$

Here for grouping purposes i is treated as a scalar, which should be justifiable in this specific case. A final grouping produces

$$\begin{aligned}\frac{1}{2}\exp(i\omega t + ik\Delta x)(\mathcal{E}' + i\mathcal{H}') = (\mathcal{E}' + i\mathcal{H}')\cos(\omega t)\cos(k\Delta x) \\ - i\hat{\mathbf{k}} \times (\mathcal{E}' + i\mathcal{H}')\sin(|\omega| t)\sin(k\Delta x) \\ - \sin(\omega t)\cdot(\mathcal{E}' + i\mathcal{H}')\sin(k\Delta x)\end{aligned} \tag{119.51}$$

Observe that despite the grouping of the summation over the pairs of complementary sign index triplets we still have a pure scalar and pure pseudoscalar term above. Allowable by the math since the differential equation had no way of encoding the grade of the desired solution. That only came from the initial time specification of $F(\mathbf{x}', 0)$, but that is not enough.

Now, from above, we can see that one way to reconcile this grade requirement is to require both $\hat{\mathbf{k}} \cdot \mathcal{E}' = 0$, and $\hat{\mathbf{k}} \cdot \mathcal{H}' = 0$. How can such a requirement make sense given that $\mathbf{k}$ ranges over all directions in space, and that both $\mathcal{E}'$ and $\mathcal{H}'$ could conceivably range over many different directions in the volume of periodicity.

With no other way out, it seems that we have to impose two requirements, one on the allowable wavenumber vector directions (which in turn means we can only pick specific orientations of the Fourier volume), and another on the field directions themselves. The electric and magnetic fields must therefore be directed only perpendicular to the wave number vector direction. Wow, that is a pretty severe implication following strictly from a grade requirement!

Thinking back to eq. (119.27), it appears that an implication of this is that we have

$$e^{i\omega t} F(\mathbf{x}', 0) = F(\mathbf{x}', 0) e^{-i\omega t} \tag{119.52}$$

Knowing this is a required condition should considerably simplify the energy and momentum questions.

120

PLANE WAVE FOURIER SERIES SOLUTIONS TO THE MAXWELL VACUUM EQUATION

120.1 MOTIVATION

In 119 an exploration of spatially periodic solutions to the electrodynamic vacuum equation was performed using a multivector formulation of a 3D Fourier series. Here a summary of the results obtained will be presented in a more coherent fashion, followed by an attempt to build on them. In particular a complete description of the field energy and momentum is desired.

A conclusion from the first analysis was that the orientation of both the electric and magnetic field components must be perpendicular to the angular velocity and wave number vectors within the entire spatial volume. This was a requirement for the field solutions to retain a bivector grade (STA/Dirac basis).

Here a specific orientation of the Fourier volume so that two of the axis lie in the direction of the initial time electric and magnetic fields will be used. This is expected to simplify the treatment.

Also note that having obtained some results in a first attempt hindsight now allows a few choices of variables that will be seen to be appropriate. The natural motivation for any such choices can be found in the initial treatment.

120.1.1 *Notation*

Conventions, definitions, and notation used here will largely follow 119. Also of possible aid in that document is a a table of symbols and their definitions.

120.2 A CONCISE REVIEW OF RESULTS

120.2.1 *Fourier series and coefficients*

A notation for a 3D Fourier series for a spatially periodic function and its Fourier coefficients was developed

$$\begin{aligned} f(\mathbf{x}) &= \sum_{\mathbf{k}} \hat{f}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \hat{f}_{\mathbf{k}} &= \frac{1}{V}\int f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3 x \end{aligned} \tag{120.1}$$

In the vector context $\mathbf{k}$ is

$$\mathbf{k} = 2\pi \sum_m \sigma^m \frac{k_m}{\lambda_m} \tag{120.2}$$

Where λ_m are the dimensions of the volume of integration, $V = \lambda_1\lambda_2\lambda_3$ is the volume, and in an index context $\mathbf{k} = \{k_1, k_2, k_3\}$ is a triplet of integers, positive, negative or zero.

120.2.2 *Vacuum solution and constraints*

We want to find (STA) bivector solutions F to the vacuum Maxwell equation

$$\nabla F = \gamma_0(\partial_0 + \boldsymbol{\nabla})F = 0 \tag{120.3}$$

We start by assuming a Fourier series solution of the form

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{F}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{120.4}$$

For a solution term by term identity is required

$$\begin{aligned} \frac{\partial}{\partial t}\hat{F}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} &= -c\sigma^m \hat{F}_{\mathbf{k}}(t) \frac{\partial}{\partial x^m} \exp\left(-i2\pi\frac{k_j x^j}{\lambda_j}\right) \\ &= ic\mathbf{k}\hat{F}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \end{aligned} \tag{120.5}$$

With $\boldsymbol{\omega} = c\mathbf{k}$, we have a simple first order single variable differential equation

$$\hat{F}'_{\mathbf{k}}(t) = i\omega \hat{F}_{\mathbf{k}}(t) \tag{120.6}$$

with solution

$$\hat{F}_{\mathbf{k}}(t) = e^{i\omega t}\hat{F}_{\mathbf{k}} \tag{120.7}$$

Here, the constant was written as $\hat{F}_{\mathbf{k}}$ given prior knowledge that this is will be the Fourier coefficient of the initial time field. Our assumed solution is now

$$F(\mathbf{x}, t) = \sum_{\mathbf{k}} e^{i\omega t}\hat{F}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{120.8}$$

Observe that for $t = 0$, we have

$$F(\mathbf{x}, 0) = \sum_{\mathbf{k}} \hat{F}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{120.9}$$

which is confirmation of the Fourier coefficient role of $\hat{F}_{\mathbf{k}}$

$$\hat{F}_{\mathbf{k}} = \frac{1}{V}\int F(\mathbf{x}', 0)e^{i\mathbf{k}\cdot\mathbf{x}'}d^3x' \tag{120.10}$$

$$F(\mathbf{x}, t) = \frac{1}{V}\sum_{\mathbf{k}}\int e^{i\omega t}F(\mathbf{x}', 0)e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}d^3x' \tag{120.11}$$

It is straightforward to show that $F(\mathbf{x}, 0)$, and pseudoscalar exponentials commute. Specifically we have

$$Fe^{i\mathbf{k}\cdot\mathbf{x}} = e^{i\mathbf{k}\cdot\mathbf{x}}F \tag{120.12}$$

This follows from the (STA) bivector nature of F.

Another commutativity relation of note is between our time phase exponential and the pseudoscalar exponentials. This one is also straightforward to show and will not be done again here

$$e^{i\omega t}e^{i\mathbf{k}\cdot\mathbf{x}} = e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\omega t} \tag{120.13}$$

Lastly, and most importantly of the commutativity relations, it was also found that the initial field $F(\mathbf{x}, 0)$ must have both electric and magnetic field components perpendicular to all $\boldsymbol{\omega} \propto \mathbf{k}$ at all points $\mathbf{x}$ in the integration volume. This was because the vacuum Maxwell equation eq. (120.3) by itself does not impose any grade requirement on the solution in isolation. An additional requirement that the solution have bivector only values imposes this inherent planar nature in a charge free region, at least for solutions with spatial periodicity. Some revisiting of previous Fourier transform solutions attempts at the vacuum equation is required since similar constraints are expected there too.

The planar constraint can be expressed in terms of dot products of the field components, but an alternate way of expressing the same thing was seen to be a statement of conjugate commutativity between this dual spatial vector exponential and the complete field

$$e^{i\omega t} F = F e^{-i\omega t} \tag{120.14}$$

The set of Fourier coefficients considered in the sum must be restricted to those values that eq. (120.14) holds. An effective way to achieve this is to pick a specific orientation of the coordinate system so the angular velocity bivector is quantized in the same plane as the field. This means that the angular velocity takes on integer multiples k of this value

$$i\omega_k = 2\pi i c k \frac{\boldsymbol{\sigma}}{\lambda} \tag{120.15}$$

Here $\boldsymbol{\sigma}$ is a unit vector describing the perpendicular to the plane of the field, or equivalently via a duality relationship $i\boldsymbol{\sigma}$ is a unit bivector with the same orientation as the field.

120.2.3 *Conjugate operations*

In order to tackle expansion of energy and momentum in terms of Fourier coefficients, some conjugation operations will be required.

Such a conjugation is found when computing electric and magnetic field components and also in the $T(\gamma_0) \propto F\gamma_0 F$ energy momentum four vector. In both cases it involves products with γ_0.

120.2.4 *Electric and magnetic fields*

From the total field one can obtain the electric and magnetic fields via coordinates as in

$$\begin{aligned} \boldsymbol{\mathcal{E}} &= \sigma^m (F \cdot \sigma_m) \\ \boldsymbol{\mathcal{H}} &= \sigma^m ((-iF) \cdot \sigma_m) \end{aligned} \tag{120.16}$$

However, due to the conjugation effect of γ_0 (a particular observer's time basis vector) on F, we can compute the electric and magnetic field components without resorting to coordinates

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= \frac{1}{2}(F - \gamma_0 F \gamma_0) \\ \boldsymbol{\mathcal{H}} &= \frac{1}{2i}(F + \gamma_0 F \gamma_0)\end{aligned} \tag{120.17}$$

Such a split is expected to show up when examining the energy and momentum of our Fourier expressed field in detail.

120.2.5 *Conjugate effects on the exponentials*

Now, since γ_0 anticommutes with i we have a conjugation operation on percolation of γ_0 through the products of an exponential

$$\gamma_0 e^{i\mathbf{k}\cdot\mathbf{x}} = e^{-i\mathbf{k}\cdot\mathbf{x}}\gamma_0 \tag{120.18}$$

However, since γ_0 also anticommutes with any spatial basis vector $\sigma_k = \gamma_k\gamma_0$, we have for a dual spatial vector exponential

$$\gamma_0 e^{i\omega t} = e^{i\omega t}\gamma_0 \tag{120.19}$$

We should now be armed to consider the energy momentum questions that were the desired goal of the initial treatment.

120.3 PLANE WAVE ENERGY AND MOMENTUM IN TERMS OF FOURIER COEFFICIENTS

120.3.1 *Energy momentum four vector*

To obtain the energy component U of the energy momentum four vector (given here in CGS units)

$$T(\gamma_0) = \frac{1}{8\pi} F\gamma_0\tilde{F} = \frac{-1}{8\pi}(F\gamma_0 F) \tag{120.20}$$

we want a calculation of the field energy for the plane wave solutions of Maxwell's equation

$$\begin{aligned}U &= T(\gamma_0)\cdot\gamma_0 \\ &= -\frac{1}{16\pi}(F\gamma_0 F\gamma_0 + \gamma_0 F\gamma_0 F)\end{aligned} \tag{120.21}$$

Given the observed commutativity relationships, at least some parts of this calculation can be performed by direct multiplication of eq. (120.11) summed over two sets of wave number vector indices as in.

$$\begin{aligned} F(\mathbf{x}, t) &= \frac{1}{V} \sum_{\mathbf{k}} \int e^{i\omega_k t + i\mathbf{k}\cdot(\mathbf{a}-\mathbf{x})} F(\mathbf{a}, 0) d^3 a \\ &= \frac{1}{V} \sum_{\mathbf{m}} \int e^{i\omega_m t + i\mathbf{m}\cdot(\mathbf{b}-\mathbf{x})} F(\mathbf{b}, 0) d^3 b \end{aligned} \tag{120.22}$$

However, this gets messy fast. Looking for an alternate approach requires some mechanism for encoding the effect of the γ_0 sandwich on the Fourier coefficients of the field bivector. It has been observed that this operation has a conjugate effect. The form of the stress energy four vector suggests that a natural conjugate definition will be

$$F^\dagger = \gamma_0 \tilde{F} \gamma_0 \tag{120.23}$$

where $\tilde{F}$ is the multivector reverse operation.

This notation for conjugation is in fact what , for Quantum Mechanics, [10] calls the Hermitian adjoint.

In this form our stress energy vector is

$$T(\gamma_0) = \frac{1}{8\pi} F F^\dagger \gamma_0 \tag{120.24}$$

While the trailing γ_0 term here may look a bit out of place, the energy density and the Poynting vector end up with a very complementary structure

$$\begin{aligned} U &= \frac{1}{16\pi} \left(F F^\dagger + (F F^\dagger)\tilde{} \right) \\ \mathbf{P} &= \frac{1}{16\pi c} \left(F F^\dagger - (F F^\dagger)\tilde{} \right) \end{aligned} \tag{120.25}$$

Having this conjugate operation defined it can also be applied to the spacetime split of the electric and the magnetic fields. That can also now be written in a form that calls out the inherent complex nature of the fields

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}(F + F^\dagger) \\ \mathcal{H} &= \frac{1}{2i}(F - F^\dagger) \end{aligned} \tag{120.26}$$

120.3.2 Aside. Applications for the conjugate in non-QM contexts

Despite the existence of the QM notation, it does not appear used in the text or ptIII notes outside of that context. For example, in addition to the stress energy tensor and the spacetime split of the fields, an additional non-QM example where the conjugate operation could be used, is in the ptIII hout8 where Rotors that satisfy

$$v \cdot \gamma_0 = \left\langle \gamma_0 R \gamma_0 \tilde{R} \right\rangle = \left\langle R^\dagger R \right\rangle > 0 \tag{120.27}$$

are called proper orthochronous. There are likely other places involving a time centric projections where this conjugation operator would have a natural fit.

120.3.3 Energy density. Take II

For the Fourier coefficient energy calculation we now take eq. (120.8) as the starting point.

We will need the conjugate of the field

$$\begin{aligned}
F^\dagger &= \gamma_0 \left(\sum_{\mathbf{k}} e^{i\omega t} \hat{F}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right)^{\tilde{}} \gamma_0 \\
&= \gamma_0 \sum_{\mathbf{k}} (e^{-i\mathbf{k}\cdot\mathbf{x}})^{\tilde{}} (-\hat{F}_{\mathbf{k}}) (e^{i\omega t})^{\tilde{}} \gamma_0 \\
&= -\gamma_0 \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{F}_{\mathbf{k}} e^{-i\omega t} \gamma_0 \\
&= -\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \gamma_0 \hat{F}_{\mathbf{k}} \gamma_0 e^{-i\omega t}
\end{aligned} \tag{120.28}$$

This is

$$F^\dagger = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} (\hat{F}_{\mathbf{k}})^\dagger e^{-i\omega t} \tag{120.29}$$

So for the energy we have

$$
\begin{aligned}
FF^\dagger + F^\dagger F &= \sum_{\mathbf{m},\mathbf{k}} e^{i\omega_m t}\hat{F}_\mathbf{m} e^{i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}}(\hat{F}_\mathbf{k})^\dagger e^{-i\omega_k t} + e^{i\mathbf{k}\cdot\mathbf{x}}(\hat{F}_\mathbf{k})^\dagger e^{i(\omega_m-\omega_k)t}\hat{F}_\mathbf{m} e^{-i\mathbf{m}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{m},\mathbf{k}} e^{i\omega_m t}\hat{F}_\mathbf{m}(\hat{F}_\mathbf{k})^\dagger e^{i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}-i\omega_k t} + e^{i\mathbf{k}\cdot\mathbf{x}}(\hat{F}_\mathbf{k})^\dagger\hat{F}_\mathbf{m} e^{-i(\omega_m-\omega_k)t-i\mathbf{m}\cdot\mathbf{x}} \\
&= \sum_{\mathbf{m},\mathbf{k}} \hat{F}_\mathbf{m}(\hat{F}_\mathbf{k})^\dagger e^{i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}-i(\omega_k-\omega_m)t} + (\hat{F}_\mathbf{k})^\dagger\hat{F}_\mathbf{m} e^{i(\omega_k-\omega_m)t+i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}} \\
&= \sum_{\mathbf{k}} \hat{F}_\mathbf{k}(\hat{F}_\mathbf{k})^\dagger + (\hat{F}_\mathbf{k})^\dagger\hat{F}_\mathbf{k} \\
&\quad + \sum_{\mathbf{m}\neq\mathbf{k}} \hat{F}_\mathbf{m}(\hat{F}_\mathbf{k})^\dagger e^{i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}-i(\omega_k-\omega_m)t} + (\hat{F}_\mathbf{k})^\dagger\hat{F}_\mathbf{m} e^{i(\omega_k-\omega_m)t+i(\mathbf{k}-\mathbf{m})\cdot\mathbf{x}}
\end{aligned}
\tag{120.30}
$$

In the first sum all the time dependence and all the spatial dependence that is not embedded in the Fourier coefficients themselves has been eliminated. What is left is something that looks like it is a real quantity (to be verified) Assuming (also to be verified) that $\hat{F}_\mathbf{k}$ commutes with its conjugate we have something that looks like a discrete version of what [18] calls the Rayleigh energy theorem

$$
\int_{-\infty}^{\infty} f(x)f^*(x)dx = \int_{-\infty}^{\infty} \hat{f}(k)\hat{f}^*(k)dk \tag{120.31}
$$

Here $\hat{f}(k)$ is the Fourier transform of $f(x)$.

Before going on it is expected that the $\mathbf{k} \neq \mathbf{m}$ terms all cancel. Having restricted the orientations of the allowed angular velocity bivectors to scalar multiples of the plane formed by the (wedge of) the electric and magnetic fields, we have only a single set of indices to sum over (ie: $\mathbf{k} = 2\pi\sigma k/\lambda$). In particular we can sum over $k < m$, and $k > m$ cases separately and add these with expectation of cancellation. Let us see if this works out.

Write $\boldsymbol{\omega} = 2\pi\sigma/\lambda$, $\omega_k = k\omega$, and $\mathbf{k} = \boldsymbol{\omega}/c$ then we have for these terms

$$
\sum_{\mathbf{m}\neq\mathbf{k}} e^{i(k-m)\omega\cdot\mathbf{x}/c}\left(\hat{F}_\mathbf{m}(\hat{F}_\mathbf{k})^\dagger e^{-i(k-m)\omega t} + (\hat{F}_\mathbf{k})^\dagger\hat{F}_\mathbf{m} e^{i(k-m)\omega t}\right) \tag{120.32}
$$

120.3.3.1 *Hermitian conjugate identities*

To get comfortable with the required manipulations, let us find the Hermitian conjugate equivalents to some of the familiar complex number relationships.

Not all of these will be the same as in "normal" complex numbers. For instance, while for complex numbers, the identities

$$\begin{aligned} z + \bar{z} &= 2\Re(z) \\ \frac{1}{i}(z - \bar{z}) &= 2\Im(z) \end{aligned} \tag{120.33}$$

are both real numbers, we have seen for the electric and magnetic fields that we do not get scalars from the Hermitian conjugates, instead get a spatial vector where we would get a real number in complex arithmetic. Similarly we get a (bi)vector in the dual space for the field minus its conjugate.

Some properties:

- Hermitian conjugate of a product

$$\begin{aligned} (ab)^\dagger &= \gamma_0 \tilde{(ab)} \gamma_0 \\ &= \gamma_0 \tilde{(b)} \tilde{(a)} \gamma_0 \\ &= \left(\gamma_0 \tilde{(b)} \gamma 0\right)\left(\gamma_0 \tilde{(a)} \gamma_0\right) \end{aligned} \tag{120.34}$$

  This is our familiar conjugate of a product is the inverted order product of conjugates.

$$(ab)^\dagger = b^\dagger a^\dagger \tag{120.35}$$

- conjugate of a pure pseudoscalar exponential

$$\begin{aligned} \left(e^{i\alpha}\right)^\dagger &= \gamma_0 \tilde{\left(\cos(\alpha) + i \sin(\alpha)\right)} \gamma_0 \\ &= \cos(\alpha) - i\gamma_0 \sin(\alpha)\gamma_0 \end{aligned} \tag{120.36}$$

  But that is just

$$\left(e^{i\alpha}\right)^\dagger = e^{-i\alpha} \tag{120.37}$$

  Again in sync with complex analysis. Good.

- conjugate of a dual spatial vector exponential

$$\begin{aligned} \left(e^{i\mathbf{k}}\right)^\dagger &= \gamma_0 \tilde{\left(\cos(\mathbf{k}) + i \sin(\mathbf{k})\right)} \gamma_0 \\ &= \gamma_0 \left(\cos(\mathbf{k}) - \sin(\mathbf{k}) i\right) \gamma_0 \\ &= \cos(\mathbf{k}) - i \sin(\mathbf{k}) \end{aligned} \tag{120.38}$$

So, we have

$$\left(e^{i\mathbf{k}}\right)^\dagger = e^{-i\mathbf{k}} \tag{120.39}$$

Again, consistent with complex numbers for this type of multivector object.

- dual spatial vector exponential product with a conjugate.

$$\begin{aligned}
F^\dagger e^{i\mathbf{k}} &= \gamma_0 \tilde{F} \gamma_0 e^{i\mathbf{k}} \\
&= \gamma_0 \tilde{F} e^{i\mathbf{k}} \gamma_0 \\
&= \gamma_0 e^{-i\mathbf{k}} \tilde{F} \gamma_0 \\
&= e^{i\mathbf{k}} \gamma_0 \tilde{F} \gamma_0
\end{aligned} \tag{120.40}$$

So we have conjugate commutation for both the field and its conjugate

$$\begin{aligned}
F^\dagger e^{i\mathbf{k}} &= e^{-i\mathbf{k}} F^\dagger \\
F e^{i\mathbf{k}} &= e^{-i\mathbf{k}} F
\end{aligned} \tag{120.41}$$

- pseudoscalar exponential product with a conjugate.

  For scalar α

$$\begin{aligned}
F^\dagger e^{i\alpha} &= \gamma_0 \tilde{F} \gamma_0 e^{i\alpha} \\
&= \gamma_0 \tilde{F} e^{-i\alpha} \gamma_0 \\
&= \gamma_0 e^{-i\alpha} \tilde{F} \gamma_0 \\
&= e^{i\alpha} \gamma_0 \tilde{F} \gamma_0
\end{aligned} \tag{120.42}$$

In opposition to the dual spatial vector exponential, the plain old pseudoscalar exponentials commute with both the field and its conjugate.

$$\begin{aligned}
F^\dagger e^{i\alpha} &= e^{i\alpha} F^\dagger \\
F e^{i\alpha} &= e^{i\alpha} F
\end{aligned} \tag{120.43}$$

- Pauli vector conjugate.

$$(\sigma_k)^\dagger = \gamma_0 \gamma_0 \gamma_k \gamma_0 = \sigma_k \tag{120.44}$$

Jives with the fact that these in matrix form are called Hermitian.

- pseudoscalar conjugate.

$$i^\dagger = \gamma_0 i \gamma_0 = -i \tag{120.45}$$

- Field Fourier coefficient conjugate.

$$(\hat{F}_\mathbf{k})^\dagger = \frac{1}{V}\int e^{-i\mathbf{k}\cdot\mathbf{x}} F^\dagger(\mathbf{x}, 0) d^3x = \widehat{F^\dagger}_{-\mathbf{k}} \tag{120.46}$$

  The conjugate of the $\mathbf{k}$ Fourier coefficient is the $-\mathbf{k}$ Fourier coefficient of the conjugate field.

Observe that the first three of these properties would have allowed for calculation of eq. (120.29) by inspection.

120.3.4 *Products of Fourier coefficient with another conjugate coefficient*

To progress a relationship between the conjugate products of Fourier coefficients may be required.

120.4 FIXME: FINISH THIS

I am getting tired of trying to show (using Latex as a tool and also on paper) that the $\mathbf{k} \neq \mathbf{m}$ terms vanish and am going to take a break, and move on for a bit. Come back to this later, but start with a electric field and magnetic field expansion of the $(\hat{F}_\mathbf{k})^\dagger\hat{F}_\mathbf{k} + \hat{F}_\mathbf{k}(\hat{F}_\mathbf{k})^\dagger$ term to verify that this ends up being a scalar as desired and expected (this is perhaps an easier first step than showing the cross terms are zero).

121

LORENTZ GAUGE FOURIER VACUUM POTENTIAL SOLUTIONS

121.1 MOTIVATION

In 119 a first order Fourier solution of the Vacuum Maxwell equation was performed. Here a comparative potential solution is obtained.

121.1.1 *Notation*

The 3D Fourier series notation developed for this treatment can be found in the original notes 119. Also included there is a table of notation, much of which is also used here.

121.2 SECOND ORDER TREATMENT WITH POTENTIALS

121.2.1 *With the Lorentz gauge*

Now, it appears that Bohm's use of potentials allows a nice comparison with the harmonic oscillator. Let us also try a Fourier solution of the potential equations. Again, use STA instead of the traditional vector equations, writing $A = (\phi + \mathbf{a})\gamma_0$, and employing the Lorentz gauge $\nabla \cdot A = 0$ we have for $F = \nabla \wedge A$ in CGS units

FIXME: Add $\mathbf{a}$, and ψ to notational table below with definitions in terms of $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{H}}$ (or the other way around).

$$\nabla^2 A = 4\pi J \tag{121.1}$$

Again with a spacetime split of the gradient

$$\nabla = \gamma^0(\partial_0 + \boldsymbol{\nabla}) = (\partial_0 - \boldsymbol{\nabla})\gamma_0 \tag{121.2}$$

our four Laplacian can be written

$$\begin{aligned}(\partial_0 - \boldsymbol{\nabla})\gamma_0\gamma^0(\partial_0 + \boldsymbol{\nabla}) &= (\partial_0 - \boldsymbol{\nabla})(\partial_0 + \boldsymbol{\nabla}) \\ &= \partial_{00} - \boldsymbol{\nabla}^2\end{aligned} \tag{121.3}$$

Our vacuum field equation for the potential is thus

$$\partial_{tt} A = c^2 \boldsymbol{\nabla}^2 A \tag{121.4}$$

Now, as before assume a Fourier solution and see what follows. That is

$$A(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{A}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{121.5}$$

Applied to each component this gives us

$$\begin{aligned}
\hat{A}''_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} &= c^2 \hat{A}_{\mathbf{k}}(t) \sum_m \frac{\partial^2}{(\partial x^m)^2} e^{-2\pi i \sum_j k_j x^j/\lambda_j} \\
&= c^2 \hat{A}_{\mathbf{k}}(t) \sum_m (-2\pi i k_m/\lambda_m)^2 e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&= -c^2 \mathbf{k}^2 \hat{A}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}
\end{aligned} \tag{121.6}$$

So we are left with another big ass set of simplest equations to solve

$$\hat{A}''_{\mathbf{k}} = -c^2 \mathbf{k}^2 \hat{A}_{\mathbf{k}} \tag{121.7}$$

Note that again the origin point $\mathbf{k} = (0, 0, 0)$ is a special case. Also of note this time is that $\hat{A}_{\mathbf{k}}$ has vector and trivector parts, unlike $\hat{F}_{\mathbf{k}}$ which being derived from dual and non-dual components of a bivector was still a bivector.

It appears that solutions can be found with either left or right handed vector valued integration constants

$$\begin{aligned}
\hat{A}_{\mathbf{k}}(t) &= \exp(\pm i c \mathbf{k} t) C_{\mathbf{k}} \\
&= D_{\mathbf{k}} \exp(\pm i c \mathbf{k} t)
\end{aligned} \tag{121.8}$$

Since these are equal at $t = 0$, it appears to imply that these commute with the complex exponentials as was the case for the bivector field.

For the $\mathbf{k} = 0$ special case we have solutions

$$\hat{A}_{\mathbf{k}}(t) = D_0 t + C_0 \tag{121.9}$$

It does not seem unreasonable to require $D_0 = 0$. Otherwise this time dependent DC Fourier component will blow up at large and small values, while periodic solutions are sought.

Putting things back together we have

$$A(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp(\pm ic\mathbf{k}t) C_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{x}) \tag{121.10}$$

Here again for $t = 0$, our integration constants are found to be determined completely by the initial conditions

$$A(\mathbf{x}, 0) = \sum_{\mathbf{k}} C_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{121.11}$$

So we can write

$$C_{\mathbf{k}} = \frac{1}{V} \int A(\mathbf{x}', 0) e^{i\mathbf{k}\cdot\mathbf{x}'} d^3 x' \tag{121.12}$$

In integral form this is

$$A(\mathbf{x}, t) = \int \sum_{\mathbf{k}} \exp(\pm i\mathbf{k}ct) A(\mathbf{x}', 0) \exp(i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')) \tag{121.13}$$

This, somewhat surprisingly, is strikingly similar to what we had for the bivector field. That was:

$$\begin{aligned} F(\mathbf{x}, t) &= \int G(\mathbf{x} - \mathbf{x}', t) F(\mathbf{x}', 0) d^3 x' \\ G(\mathbf{x}, t) &= \frac{1}{V} \sum_{\mathbf{k}} \exp\left(i\mathbf{k}ct\right) \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) \end{aligned} \tag{121.14}$$

We cannot however commute the time phase term to construct a one sided Green's function for this potential solution (or perhaps we can but if so shown or attempted to show that this is possible). We also have a plus or minus variation in the phase term due to the second order nature of the harmonic oscillator equations for our Fourier coefficients.

121.2.2 *Comparing the first and second order solutions*

A consequence of working in the Lorentz gauge ($\nabla \cdot A = 0$) is that our field solution should be a gradient

$$\begin{aligned} F &= \nabla \wedge A \\ &= \nabla A \end{aligned} \tag{121.15}$$

FIXME: expand this out using eq. (121.13) to compare to the first order solution.

Part XII

APPENDIX

A

NOTATION AND DEFINITIONS

Here is a summary of the notation and definitions that will be used.

The following tables summarize a lot of the notation used in these notes. This largely follows conventions from [10].

A.1 COORDINATES AND BASIS VECTORS

Greek letters range over all indices and English indices range over $1, 2, 3$.

Bold vectors are spatial entities and non-bold is used for four vectors and scalars.

Summation convention is often used (less so in earlier notes). This is summation over all sets of matched upper and lower indices is implied.

While many things could be formulated in a metric signature independent fashion, a time positive $(+, -, -, -)$ metric signature should be assumed in most cases. Specifically, that is $(\gamma_0)^2 = 1$, and $(\gamma_k)^2 = -1$.

γ_μ	$\gamma_\mu \cdot \gamma_\nu = \pm\delta^\mu{}_\nu$	Four vector basis vector ($\gamma_\mu \cdot \gamma_\nu = \pm\delta^\mu{}_\nu$)
$(\gamma_0)^2(\gamma_k)^2$	$= -1$	Minkowski metric
$\sigma_k = \sigma^k$	$= \gamma_k \wedge \gamma_0$ $= \gamma_k\gamma_0$	Spatial basis bivector. ($\sigma_k \cdot \sigma_j = \delta_{kj}$)
I	$= \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$ $= \gamma_0\gamma_1\gamma_2\gamma_3$ $= \gamma_{0123}$	Four-vector pseudoscalar
$\gamma^\mu \cdot \gamma_\nu$	$= \delta^\mu{}_\nu$	Reciprocal basis vectors
x^μ	$= x \cdot \gamma^\mu$	Vector coordinate
x_μ	$= x \cdot \gamma_\mu$	Coordinate for reciprocal basis
x	$= \gamma_\mu x^\mu$ $= \gamma^\mu x_\mu$	Four vector in terms of coordinates
x^0	$= x \cdot \gamma^0$ $= ct$	Time coordinate (length dim.)
$\mathbf{x}$	$= x \wedge \gamma_0$ $= x^k\sigma_k$	Spatial vector
x^2	$= x \cdot x$ $= x^\mu x_\mu$	Four vector square.
$\mathbf{x}^2$	$= \mathbf{x} \cdot \mathbf{x}$ $= \sum_{k=1}^3 (x^k)^2$ $= \lvert\mathbf{x}\rvert^2$	Spatial vector square.

If convient sometimes i will be used for the pseudoscalar.

A.2 ELECTROMAGNETISM

SI units are used in most places, but occasionally natural units are used. In some cases, when working with material such as [2], CGS modifications of the notation are employed.

$\mathbf{E}$	$= \sum E^k \sigma_k$	Electric field spatial vector
$\mathbf{B}$	$= \sum B^k \sigma_k$	Magnetic field spatial vector
$\mathcal{E}$	$= E^k \sigma_k$	(CGS)Electric field spatial vector
$\mathcal{H}$	$= H^k \sigma_k$	(CGS)Magnetic field spatial vector
J	$= \gamma_\mu J^\mu$	Current density four vector.
	$= \gamma^\mu J_\mu$	
F	$= \mathbf{E} + Ic\mathbf{B}$	Electromagnetic (Faraday) bivector
	$= F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$	in terms of Faraday tensor
	$= \mathcal{E} + I\mathcal{H}$	(CGS)
J^0	$= J \cdot \gamma^0$	Charge density.
	$= c\rho$	(current density dimensions.)
	$= \rho$	(CGS) (current density dimensions.)
$\mathbf{J}$	$= J \wedge \gamma_0$	Current density spatial vector
	$= J^k \sigma_k$	

A.3 DIFFERENTIAL OPERATORS

∂_μ	$= \partial/\partial x^\mu$	Index up partial.
∂^μ	$= \partial/\partial x_\mu$	Index down partial.
$\partial_{\mu\nu}$	$= \partial/\partial x^\mu \partial/\partial x^\nu$	Index up partial.
∇	$= \sum \gamma^\mu \partial/\partial x^\mu$	Spacetime gradient
	$= \gamma^\mu \partial_\mu$	
	$= \sum \gamma_\mu \partial/\partial x_\mu$	
	$= \gamma_\mu \partial^\mu$	
$\boldsymbol{\nabla}$	$= \sigma^k \partial_k$	Spatial gradient
$\hat{A}_{\mathbf{k}}$	$= \hat{A}_{k_1,k_2,k_3}$	Fourier coefficient, integer indices.
$\nabla^2 A$	$= (\nabla \cdot \nabla) A$	Four Laplacian.
	$= (\partial_{00} - \sum_k \partial_{kk}) A$	
$d^3 x$	$= dx^1 dx^2 dx^3$	Spatial volume element.
$d^4 x$	$= dx^0 dx^1 dx^2 dx^3$	Four volume element.
$\int_{\partial I}$	$= \int_a^b$	Integration range $I = [a, b]$
STA		Space Time Algebra
$(xyz)^\tilde{}$	$= \widetilde{xyz} = zyx$	Reverse of a vector product.

A.4 MISC

The PV notation is taken from [29] where the author uses it in his Riemann integral proof of the inverse Fourier integral.

$\text{PV} \int_{-\infty}^{\infty}$	$= \lim_{R\to\infty} \int_R^R$	Integral Principle value
$\hat{A}(k)$	$= \mathcal{F}(A(x))$	Fourier transform of A
$A(x)$	$= \mathcal{F}^{-1}(A(k))$	Inverse Fourier transform
$\exp(i\mathbf{k}\phi)$	$= \cos(\lvert\mathbf{k}\rvert\phi) + \frac{i\mathbf{k}}{\lvert i\mathbf{k}\rvert} \sin(\lvert\mathbf{k}\rvert\phi)$	bivector exponential.

B

SOME HELPFUL IDENTITIES

Theorem B.1: Distribution of inner products

Given two blades A_s, B_r with grades subject to $s > r > 0$, and a vector b, the inner product distributes according to

$$A_s \cdot (b \wedge B_r) = (A_s \cdot b) \cdot B_r.$$

This will allow us, for example, to expand a general inner product of the form $d^k\mathbf{x} \cdot (\boldsymbol{\partial} \wedge F)$.

The proof is straightforward, but also mechanical. Start by expanding the wedge and dot products within a grade selection operator

$$\begin{aligned} A_s \cdot (b \wedge B_r) &= \langle A_s(b \wedge B_r)\rangle_{s-(r+1)} \\ &= \frac{1}{2}\langle A_s \left(bB_r + (-1)^r B_r b\right)\rangle_{s-(r+1)} \end{aligned} \tag{B.1}$$

Solving for $B_r b$ in

$$2b \cdot B_r = bB_r - (-1)^r B_r b, \tag{B.2}$$

we have

$$\begin{aligned} A_s \cdot (b \wedge B_r) &= \frac{1}{2}\langle A_s b B_r + A_s \left(bB_r - 2b \cdot B_r\right)\rangle_{s-(r+1)} \\ &= \langle A_s b B_r\rangle_{s-(r+1)} - \cancel{\langle A_s \left(b \cdot B_r\right)\rangle_{s-(r+1)}}. \end{aligned} \tag{B.3}$$

The last term above is zero since we are selecting the $s - r - 1$ grade element of a multivector with grades $s - r + 1$ and $s + r - 1$, which has no terms for $r > 0$. Now we can expand the $A_s b$ multivector product, for

$$A_s \cdot (b \wedge B_r) = \langle (A_s \cdot b + A_s \wedge b)\, B_r\rangle_{s-(r+1)}. \tag{B.4}$$

The latter multivector (with the wedge product factor) above has grades $s + 1 - r$ and $s + 1 + r$, so this selection operator finds nothing. This leaves

$$A_s \cdot (b \wedge B_r) = \langle (A_s \cdot b) \cdot B_r + (A_s \cdot b) \wedge B_r\rangle_{s-(r+1)}. \tag{B.5}$$

The first dot products term has grade $s - 1 - r$ and is selected, whereas the wedge term has grade $s - 1 + r \neq s - r - 1$ (for $r > 0$). $\square$

Theorem B.2: Distribution of two bivectors

For vectors **a**, **b**, and bivector B, we have

$$(\mathbf{a} \wedge \mathbf{b}) \cdot B = \frac{1}{2}\left(\mathbf{a} \cdot (\mathbf{b} \cdot B) - \mathbf{b} \cdot (\mathbf{a} \cdot B)\right). \tag{B.6}$$

Proof follows by applying the scalar selection operator, expanding the wedge product within it, and eliminating any of the terms that cannot contribute grade zero values

$$\begin{aligned}
(\mathbf{a} \wedge \mathbf{b}) \cdot B &= \left\langle \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) B \right\rangle \\
&= \frac{1}{2}\langle \mathbf{a}(\mathbf{b} \cdot B + \cancel{\mathbf{b} \wedge B}) - \mathbf{b}(\mathbf{a} \cdot B + \cancel{\mathbf{a} \wedge B})\rangle \\
&= \frac{1}{2}\left\langle \mathbf{a} \cdot (\mathbf{b} \cdot B) + \cancel{\mathbf{a} \wedge (\mathbf{b} \cdot B)} - \mathbf{b} \cdot (\mathbf{a} \cdot B) - \cancel{\mathbf{b} \wedge (\mathbf{a} \cdot B)}\right\rangle \\
&= \frac{1}{2}(\mathbf{a} \cdot (\mathbf{b} \cdot B) - \mathbf{b} \cdot (\mathbf{a} \cdot B)) \qquad \square
\end{aligned} \tag{B.7}$$

Theorem B.3: Inner product of trivector with bivector

Given a bivector B, and trivector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ where **a**, **b** and **c** are vectors, the inner product is

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot B = \mathbf{a}\big((\mathbf{b} \wedge \mathbf{c}) \cdot B\big) + \mathbf{b}\big((\mathbf{c} \wedge \mathbf{a}) \cdot B\big) + \mathbf{c}\big((\mathbf{a} \wedge \mathbf{b}) \cdot B\big). \tag{B.8}$$

This is also problem 1.1(c) from Exercises 2.1 in [19], and submits to a dumb expansion in successive dot products with a final regrouping. With $B = \mathbf{u} \wedge \mathbf{v}$

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot B &= \langle (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) (\mathbf{u} \wedge \mathbf{v}) \rangle_1 \\
&= \langle (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) (\mathbf{u}\mathbf{v} - \mathbf{u} \cdot \mathbf{v}) \rangle_1 \\
&= ((\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{u}) \cdot \mathbf{v} \\
&= (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{v} (\mathbf{c} \cdot \mathbf{u}) + (\mathbf{c} \wedge \mathbf{a}) \cdot \mathbf{v} (\mathbf{b} \cdot \mathbf{u}) + (\mathbf{b} \wedge \mathbf{c}) \cdot \mathbf{v} (\mathbf{a} \cdot \mathbf{u}) \\
&= \mathbf{a} (\mathbf{b} \cdot \mathbf{v}) (\mathbf{c} \cdot \mathbf{u}) - \mathbf{b} (\mathbf{a} \cdot \mathbf{v}) (\mathbf{c} \cdot \mathbf{u}) \\
&\quad + \mathbf{c} (\mathbf{a} \cdot \mathbf{v}) (\mathbf{b} \cdot \mathbf{u}) - \mathbf{a} (\mathbf{c} \cdot \mathbf{v}) (\mathbf{b} \cdot \mathbf{u}) \\
&\quad + \mathbf{b} (\mathbf{c} \cdot \mathbf{v}) (\mathbf{a} \cdot \mathbf{u}) - \mathbf{c} (\mathbf{b} \cdot \mathbf{v}) (\mathbf{a} \cdot \mathbf{u}) \\
&= \mathbf{a} ((\mathbf{b} \cdot \mathbf{v}) (\mathbf{c} \cdot \mathbf{u}) - (\mathbf{c} \cdot \mathbf{v}) (\mathbf{b} \cdot \mathbf{u})) \\
&\quad + \mathbf{b} ((\mathbf{c} \cdot \mathbf{v}) (\mathbf{a} \cdot \mathbf{u}) - (\mathbf{a} \cdot \mathbf{v}) (\mathbf{c} \cdot \mathbf{u})) \\
&\quad + \mathbf{c} ((\mathbf{a} \cdot \mathbf{v}) (\mathbf{b} \cdot \mathbf{u}) - (\mathbf{b} \cdot \mathbf{v}) (\mathbf{a} \cdot \mathbf{u})) \\
&= \mathbf{a} (\mathbf{b} \wedge \mathbf{c}) \cdot (\mathbf{u} \wedge \mathbf{v}) \\
&\quad + \mathbf{b} (\mathbf{c} \wedge \mathbf{a}) \cdot (\mathbf{u} \wedge \mathbf{v}) \\
&\quad + \mathbf{c} (\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{u} \wedge \mathbf{v}) \\
&= \mathbf{a} (\mathbf{b} \wedge \mathbf{c}) \cdot B + \mathbf{b} (\mathbf{c} \wedge \mathbf{a}) \cdot B + \mathbf{c} (\mathbf{a} \wedge \mathbf{b}) \cdot B. \qquad \square
\end{aligned}
\tag{B.9}
$$

Theorem B.4: Distribution of two trivectors

Given a trivector T and three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the entire inner product can be expanded in terms of any successive set inner products, subject to change of sign with interchange of any two adjacent vectors within the dot product sequence

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot T &= \mathbf{a} \cdot (\mathbf{b} \cdot (\mathbf{c} \cdot T)) \\
&= -\mathbf{a} \cdot (\mathbf{c} \cdot (\mathbf{b} \cdot T)) \\
&= \mathbf{b} \cdot (\mathbf{c} \cdot (\mathbf{a} \cdot T)) \\
&= -\mathbf{b} \cdot (\mathbf{a} \cdot (\mathbf{c} \cdot T)) \\
&= \mathbf{c} \cdot (\mathbf{a} \cdot (\mathbf{b} \cdot T)) \\
&= -\mathbf{c} \cdot (\mathbf{b} \cdot (\mathbf{a} \cdot T)) .
\end{aligned}
\tag{B.10}
$$

To show this, we first expand within a scalar selection operator

$$
\begin{aligned}
(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \cdot T &= \langle (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) T \rangle \\
&= \frac{1}{6} \langle \mathbf{abc}T - \mathbf{acb}T + \mathbf{bca}T - \mathbf{bab}T + \mathbf{cab}T - \mathbf{cba}T \rangle
\end{aligned}
\tag{B.11}
$$

Now consider any single term from the scalar selection, such as the first. This can be re-ordered using the vector dot product identity

$$\begin{aligned}\langle \mathbf{abc}T \rangle &= \langle \mathbf{a}\,(-\mathbf{cb} + 2\mathbf{b}\cdot\mathbf{c})\,T \rangle \\ &= -\langle \mathbf{acb}T \rangle + 2\mathbf{b}\cdot\mathbf{c}\cancel{\langle \mathbf{a}T \rangle}.\end{aligned} \tag{B.12}$$

The vector-trivector product in the latter grade selection operation above contributes only bivector and quadvector terms, thus contributing nothing. This can be repeated, showing that

$$\begin{aligned}\langle \mathbf{abc}T \rangle &= -\langle \mathbf{acb}T \rangle \\ &= +\langle \mathbf{bca}T \rangle \\ &= -\langle \mathbf{bac}T \rangle \\ &= +\langle \mathbf{cab}T \rangle \\ &= -\langle \mathbf{cba}T \rangle.\end{aligned} \tag{B.13}$$

Substituting this back into eq. (B.11) proves theorem B.4.

Theorem B.5: Permutation of two successive dot products with trivector

Given a trivector T and two vectors $\mathbf{a}$ and $\mathbf{b}$, alternating the order of the dot products changes the sign

$$\mathbf{a}\cdot(\mathbf{b}\cdot T) = -\mathbf{b}\cdot(\mathbf{a}\cdot T)\,. \tag{B.14}$$

This and theorem B.4 are clearly examples of a more general identity, but I'll not try to prove that here. To show this one, we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\cdot T) &= \langle \mathbf{a}\,(\mathbf{b}\cdot T)\rangle_1 \\ &= \frac{1}{2}\langle \mathbf{ab}T + \mathbf{a}T\mathbf{b}\rangle_1 \\ &= \frac{1}{2}\langle(-\mathbf{ba} + \cancel{2\mathbf{a}\cdot\mathbf{b}})\,T + (\mathbf{a}\cdot T)\,\mathbf{b} + \cancel{\mathbf{a}\wedge T}\mathbf{b}\rangle_1 \\ &= \frac{1}{2}\left(-\mathbf{b}\cdot(\mathbf{a}\cdot T) + (\mathbf{a}\cdot T)\cdot\mathbf{b}\right) \\ &= -\mathbf{b}\cdot(\mathbf{a}\cdot T)\,. \qquad \square\end{aligned} \tag{B.15}$$

Cancellation of terms above was because they could not contribute to a grade one selection. We also employed the relation $\mathbf{x}\cdot B = -B\cdot\mathbf{x}$ for bivector B and vector $\mathbf{x}$.

Theorem B.6: Duality in a plane

For a vector $\mathbf{a}$, and a plane containing $\mathbf{a}$ and $\mathbf{b}$, the dual $\mathbf{a}^*$ of this vector with respect to this plane is

$$\mathbf{a}^* = \frac{\mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b})}{(\mathbf{a} \wedge \mathbf{b})^2}, \tag{B.16}$$

Satisfying

$$\mathbf{a}^* \cdot \mathbf{a} = 1, \tag{B.17}$$

and

$$\mathbf{a}^* \cdot \mathbf{b} = 0. \tag{B.18}$$

To demonstrate, we start with the expansion of

$$\mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b}) = (\mathbf{b} \cdot \mathbf{a})\, \mathbf{b} - \mathbf{b}^2 \mathbf{a}. \tag{B.19}$$

Dotting with **a** we have

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b})) &= \mathbf{a} \cdot \left((\mathbf{b} \cdot \mathbf{a})\, \mathbf{b} - \mathbf{b}^2 \mathbf{a} \right) \\ &= (\mathbf{b} \cdot \mathbf{a})^2 - \mathbf{b}^2 \mathbf{a}^2, \end{aligned} \tag{B.20}$$

but dotting with **b** yields zero

$$\begin{aligned} \mathbf{b} \cdot (\mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b})) &= \mathbf{b} \cdot \left((\mathbf{b} \cdot \mathbf{a})\, \mathbf{b} - \mathbf{b}^2 \mathbf{a} \right) \\ &= (\mathbf{b} \cdot \mathbf{a})\, \mathbf{b}^2 - \mathbf{b}^2 (\mathbf{a} \cdot \mathbf{b}) \\ &= 0. \end{aligned} \tag{B.21}$$

To complete the proof, we note that the product in eq. (B.20) is just the wedge squared

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b})^2 &= \left\langle (\mathbf{a} \wedge \mathbf{b})^2 \right\rangle \\ &= \langle (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})\, (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b}) \rangle \\ &= \langle \mathbf{abab} - 2\, (\mathbf{a} \cdot \mathbf{b})\, \mathbf{ab} \rangle + (\mathbf{a} \cdot \mathbf{b})^2 \\ &= \langle \mathbf{ab}\, (-\mathbf{ba} + 2\mathbf{a} \cdot \mathbf{b}) \rangle - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= (\mathbf{a} \cdot \mathbf{b})^2 - \mathbf{a}^2 \mathbf{b}^2. \end{aligned} \tag{B.22}$$

This duality relation can be recast with a linear denominator

$$\begin{aligned} \mathbf{a}^* &= \frac{\mathbf{b} \cdot (\mathbf{a} \wedge \mathbf{b})}{(\mathbf{a} \wedge \mathbf{b})^2} \\ &= \mathbf{b} \frac{\mathbf{a} \wedge \mathbf{b}}{(\mathbf{a} \wedge \mathbf{b})^2} \\ &= \mathbf{b} \frac{\mathbf{a} \wedge \mathbf{b}}{|\mathbf{a} \wedge \mathbf{b}|} \frac{|\mathbf{a} \wedge \mathbf{b}|}{\mathbf{a} \wedge \mathbf{b}} \frac{1}{(\mathbf{a} \wedge \mathbf{b})}, \end{aligned} \tag{B.23}$$

or

$$\boxed{\mathbf{a}^* = \mathbf{b}\frac{1}{(\mathbf{a} \wedge \mathbf{b})}.} \tag{B.24}$$

We can use this form after scaling it appropriately to express duality in terms of the pseudoscalar.

Theorem B.7: Dual vector in a three vector subspace

In the subspace spanned by $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, the dual of $\mathbf{a}$ is

$$\mathbf{a}^* = \mathbf{b} \wedge \mathbf{c}\frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}},$$

Consider the dot product of $\hat{\mathbf{a}}^*$ with $\mathbf{u} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{a}^* &= \left\langle \mathbf{u}\mathbf{b} \wedge \mathbf{c}\frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} \right\rangle \\
&= \left\langle \mathbf{u} \cdot (\mathbf{b} \wedge \mathbf{c}) \frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} \right\rangle + \left\langle \mathbf{u} \wedge \mathbf{b} \wedge \mathbf{c}\frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} \right\rangle \\
&= \cancel{\left\langle \left((\mathbf{u} \cdot \mathbf{b}) \, \mathbf{c} - (\mathbf{u} \cdot \mathbf{c}) \, \mathbf{b} \right) \frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} \right\rangle} + \left\langle \mathbf{u} \wedge \mathbf{b} \wedge \mathbf{c}\frac{1}{\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}} \right\rangle .
\end{aligned} \tag{B.25}$$

The canceled term is eliminated since it is the product of a vector and trivector producing no scalar term. Substituting $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and noting that $\mathbf{u} \wedge \mathbf{u} = 0$, we have

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{a}^* &= 1 \\
\mathbf{b} \cdot \mathbf{a}^* &= 0 \\
\mathbf{c} \cdot \mathbf{a}^* &= 0.
\end{aligned} \tag{B.26}$$

Theorem B.8: Pseudoscalar selection

For grade k blade $K \in \bigwedge^k$ (i.e. a pseudoscalar), and vectors $\mathbf{a}, \mathbf{b}$, the grade k selection of this blade sandwiched between the vectors is

$$\langle \mathbf{a}K\mathbf{b} \rangle_k = (-1)^{k+1} \langle Kab \rangle_k = (-1)^{k+1} K \left(\mathbf{a} \cdot \mathbf{b}\right).$$

To show this, we have to consider even and odd grades separately. First for even k we have

$$
\begin{aligned}
\langle \mathbf{a}K\mathbf{b} \rangle_k &= \langle (\mathbf{a} \cdot K + \cancel{\mathbf{a} \wedge K})\,\mathbf{b} \rangle_k \\
&= \frac{1}{2} \langle (\mathbf{a}K - K\mathbf{a})\,\mathbf{b} \rangle_k \\
&= \frac{1}{2} \langle \mathbf{a}K\mathbf{b} \rangle_k - \frac{1}{2} \langle K\mathbf{a}\mathbf{b} \rangle_k,
\end{aligned}
\tag{B.27}
$$

or

$$
\begin{aligned}
\langle \mathbf{a}K\mathbf{b} \rangle_k &= -\langle K\mathbf{a}\mathbf{b} \rangle_k \\
&= -K\,(\mathbf{a} \cdot \mathbf{b})\,.
\end{aligned}
\tag{B.28}
$$

Similarly for odd k, we have

$$
\begin{aligned}
\langle \mathbf{a}K\mathbf{b} \rangle_k &= \langle (\mathbf{a} \cdot K + \cancel{\mathbf{a} \wedge K})\,\mathbf{b} \rangle_k \\
&= \frac{1}{2} \langle (\mathbf{a}K + K\mathbf{a})\,\mathbf{b} \rangle_k \\
&= \frac{1}{2} \langle \mathbf{a}K\mathbf{b} \rangle_k + \frac{1}{2} \langle K\mathbf{a}\mathbf{b} \rangle_k,
\end{aligned}
\tag{B.29}
$$

or

$$
\begin{aligned}
\langle \mathbf{a}K\mathbf{b} \rangle_k &= \langle K\mathbf{a}\mathbf{b} \rangle_k \\
&= K\,(\mathbf{a} \cdot \mathbf{b})\,.
\end{aligned}
\tag{B.30}
$$

Adjusting for the signs completes the proof.

C

SOME FOURIER TRANSFORM NOTES

C.1 MOTIVATION

In [33] the Fourier transform pairs are written in a somewhat non-orthodox seeming way.

$$\begin{aligned}\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx \\ \psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{ipx/\hbar} dp\end{aligned} \tag{C.1}$$

The aim here is to do verify this form and do a couple associated calculations (like the Rayleigh energy theorem).

C.2 VERIFY TRANSFORM PAIR

As an exercise to verify, in a not particularly rigorous fashion, that we get back our original function applying the forward and reverse transformations in sequence. Specifically, let us compute

$$\mathcal{F}^{-1}(\mathcal{F}(\psi(x))) = \text{PV} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(u) e^{-ipu/\hbar} du \right) e^{ipx/\hbar} dp \tag{C.2}$$

Here PV is the principle value of the integral, which is the specifically symmetric integration

$$\text{PV} \int_{-\infty}^{\infty} = \lim_{R\to\infty} \int_{-R}^{R} \tag{C.3}$$

We have for the integration

$$\mathcal{F}^{-1}(\mathcal{F}(\psi(x))) = \text{PV} \frac{1}{2\pi\hbar} \int du \psi(u) \int e^{ip(x-u)/\hbar} dp \tag{C.4}$$

Now, let $v = (x-u)/\hbar$, or $u = x - v\hbar$ for

$$\begin{aligned}\frac{1}{2\pi}\int dv \psi(x - v\hbar)\int_{-R}^{R} e^{ipv} dp &= \frac{1}{2\pi}\int dv \psi(x - v\hbar)\,\frac{1}{iv} e^{ipv}\Big|_{p=-R}^{R} \\ &= \int dv \psi(x - v\hbar)\frac{\sin(Rv)}{\pi v}\end{aligned} \tag{C.5}$$

In a hand-waving (aka. Engineering) fashion, one can identify the limit of $\sin(Rv)/\pi v$ as the Dirac delta function and then declare that this does in fact recover the value of $\psi(x)$ by a Dirac delta filtering around the point $v = 0$.

This does in fact work out, but as a strict integration exercise one ought to be able to do better. Observe that the integral performed here was not really valid for $v = 0$ in which case the exponential takes the value of one, so it would be better to treat the neighborhood of $v = 0$ more carefully. Doing so

$$\begin{aligned}\frac{1}{2\pi}\int dv \psi(x - v\hbar)\int_{-R}^{R} e^{ipv} dp &= \int_{v=-\infty}^{-\epsilon} dv \psi(x - v\hbar)\frac{\sin(Rv)}{\pi v} \\ &+ \int_{v=\epsilon}^{\infty} dv \psi(x - v\hbar)\frac{\sin(Rv)}{\pi v} \\ &+ \frac{1}{2\pi}\int_{v=-\epsilon}^{\epsilon} dv \psi(x - v\hbar)\int_{-R}^{R} e^{ipv} dp \\ &= \int_{v=\epsilon}^{\infty} dv \left(\psi(x - v\hbar) + \psi(x + v\hbar)\right)\frac{\sin(Rv)}{\pi v} \\ &+ \frac{1}{2\pi}\int_{v=-\epsilon}^{\epsilon} dv \psi(x - v\hbar)\int_{-R}^{R} e^{ipv} dp\end{aligned} \tag{C.6}$$

Now, evaluating this with ϵ allowing to tend to zero and R tending to infinity simultaneously is troublesome seeming. I seem to recall that one can do something to the effect of setting $\epsilon = 1/R$, and then carefully take the limit, but it is not obvious to me how exactly to do this without pulling out an old text. While some kind of ad-hoc limit process can likely be done and justified in some fashion, one can see why the hard core mathematicians had to invent an alternate stricter mathematics to deal with this stuff rigorously.

That said, from an intuitive point of view, it is fairly clear that the filtering involved here will recover the average of $\psi(x)$ in the neighborhood assuming that it is piecewise continuous:

$$\mathcal{F}^{-1}(\mathcal{F}(\psi(x))) = \frac{1}{2}\left(\psi(x - \epsilon) + \psi(x + \epsilon)\right) \tag{C.7}$$

After digging through my old texts I found a treatment of the Fourier integral very similar to what I have done above in [29], but the important details are not omitted (like integrability conditions). I had read that and some of my treatment is obviously was based on that. That text treats this still with Riemann (and not Lebesgue) integration, but very carefully.

C.3 PARSEVAL'S THEOREM

In [33] he notes that Parseval's theorem tells us

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)g(x)dx &= \int_{-\infty}^{\infty} F(k)G^*(k)dk \\
\int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} |F(k)|^2 dk
\end{aligned}
\tag{C.8}
$$

The last of these in [18] is called Rayleigh's energy theorem. As a refresher in Fourier manipulation, and to translate to the QM Fourier transform notation, let us go through the arguments required to prove these.

C.3.1 *Convolution*

We will need convolution in the QM notation as a first step to express the transform of a product.

Suppose we have two functions $\phi_i(x)$, and their transform pairs $\tilde{\phi}_i(x) = \mathcal{F}(\phi_i)$, then the transform of the product is

$$
\tilde{\Phi}_{12}(p) = \mathcal{F}(\phi_1(x)\phi_2(x)) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)e^{-ipx/\hbar}dx
\tag{C.9}
$$

Now write $\phi_2(x)$ in terms of its inverse transform

$$
\phi_2(x)) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{\phi}_2(u)e^{iux/\hbar}du
\tag{C.10}
$$

The product transform is now

$$
\begin{aligned}
\tilde{\Phi}_{12}(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi_1(x)\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{\phi}_2(u)e^{iux/\hbar}due^{-ipx/\hbar}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} du\tilde{\phi}_2(u)\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi_1(x)e^{-ix(p-u)/\hbar}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} du\tilde{\phi}_2(u)\tilde{\phi}_1(p-u) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dv\tilde{\phi}_1(v)\tilde{\phi}_2(p-v)
\end{aligned}
\tag{C.11}
$$

So we have product transform expressed by the convolution integral, but have an extra $1/\sqrt{2\pi\hbar}$ factor in this form

$$\phi_1(x)\phi_2(x) \Leftrightarrow \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dv \tilde{\phi}_1(v)\tilde{\phi}_2(p-v) \tag{C.12}$$

c.3.2 *Conjugation*

Next we need to see how the conjugate transforms. This is pretty straight forward

$$\begin{aligned}
\phi^*(x) &\Leftrightarrow \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi^*(x) e^{-ipx/\hbar dx} \\
&= \left(\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(x) e^{ipx/\hbar dx} \right)^*
\end{aligned} \tag{C.13}$$

So we have

$$\phi^*(x) \Leftrightarrow \left(\tilde{\phi}(-p)\right)^* \tag{C.14}$$

c.3.3 *Rayleigh's Energy Theorem*

Now, we should be set to prove the energy theorem. Let us start with the momentum domain integral and translate back to position basis

$$\begin{aligned}
\int_{-\infty}^{\infty} dp \tilde{\phi}(p)\tilde{\phi}^*(p) &= \int_{-\infty}^{\infty} dp \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \phi(x) e^{-ipx/\hbar} \tilde{\phi}^*(p) \\
&= \int_{-\infty}^{\infty} dx \phi(x) \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \tilde{\phi}^*(p) e^{-ipx/\hbar} \\
&= \int_{-\infty}^{\infty} dx \phi(x) \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \tilde{\phi}^*(-p) e^{ipx/\hbar} \\
&= \int_{-\infty}^{\infty} dx \phi(x) \mathcal{F}^{-1}(\tilde{\phi}^*(-p))
\end{aligned} \tag{C.15}$$

This is exactly our desired result

$$\int_{-\infty}^{\infty} dp \tilde{\phi}(p)\tilde{\phi}^*(p) = \int_{-\infty}^{\infty} dx \phi(x)\phi^*(x) \tag{C.16}$$

Hmm. Did not even need the convolution as the systems book did. Will have to look over how they did this more closely. Regardless, this method was nicely direct.

D

A CHEATSHEET FOR FOURIER TRANSFORM CONVENTIONS

D.1 A CHEATSHEET FOR DIFFERENT FOURIER INTEGRAL NOTATIONS

Damn. There are too many different notations for the Fourier transform. Examples are:

$$
\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) \exp\left(-2\pi i k x\right) dx \\
\tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp\left(-i k x\right) dx \\
\tilde{f}(p) &= \sqrt{\frac{1}{2\pi\hbar}} \int_{-\infty}^{\infty} f(x) \exp\left(\frac{-ipx}{\hbar}\right) dx
\end{aligned}
\tag{D.1}
$$

There are probably many more, with other variations such as using hats over things instead of twiddles, and so forth.

Unfortunately each of these have different numeric factors for the inverse transform. Having just been bitten by rogue factors of 2π after innocently switching notations, it seems worthwhile to express the Fourier transform with a general fudge factor in the exponential. Then it can be seen at a glance what constants are required in the inverse transform given anybody's particular choice of the transform definition.

Where to put all the factors can actually be seen from the QM formulation since one is free to treat $\hbar$ as an arbitrary constant, but let us do it from scratch in a mechanical fashion without having to think back to QM as a fundamental.

Suppose we define the Fourier transform as

$$
\begin{aligned}
\tilde{f}(s) &= \kappa \int_{-\infty}^{\infty} f(x) \exp\left(-i\alpha s x\right) dx \\
f(x) &= \kappa' \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i\alpha x s\right) ds
\end{aligned}
\tag{D.2}
$$

Now, what factor do we need in the inverse transform to make things work out right? With the Rigor Police on holiday, let us expand the inverse transform integral in terms of the original transform and see what these numeric factors must then be to make this work out.

Omitting temporarily the κ factors to be determined we have

$$\begin{aligned}
f(x) &\propto \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u) \exp\left(-i\alpha s u\right) du \right) \exp\left(i\alpha x s\right) ds \\
&= \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} \exp\left(i\alpha s(x-u)\right) ds \\
&= \int_{-\infty}^{\infty} f(u) du \lim_{R\to\infty} 2\pi \frac{1}{\pi\alpha(x-u)} \sin\left(\alpha R(x-u)\right) \\
&= \int_{-\infty}^{\infty} f(u) du 2\pi\delta\left(\alpha(x-u)\right) \\
&= \frac{1}{\alpha} \int_{-\infty}^{\infty} f(v/\alpha) dv 2\pi\delta\left(\alpha x - v\right) \\
&= \frac{2\pi}{\alpha} f((\alpha x)/\alpha) \\
&= \frac{2\pi}{\alpha} f(x)
\end{aligned} \tag{D.3}$$

Note that to get the result above, after switching order of integration, and assuming that we can take the principle value of the integrals, the usual ad-hoc sinc and exponential integral identification of the delta function was made

$$\begin{aligned}
\mathrm{PV} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(isx\right) ds &= \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \exp\left(isx\right) ds \\
&= \lim_{R\to\infty} \frac{\sin\left(Rx\right)}{\pi x} \\
&\equiv \delta\left(x\right)
\end{aligned} \tag{D.4}$$

The end result is that we will need to fix

$$\kappa\kappa' = \frac{\alpha}{2\pi} \tag{D.5}$$

to have the transform pair produce the desired result. Our transform pair is therefore

$$\tilde{f}(s) = \kappa \int_{-\infty}^{\infty} f(x) \exp\left(-i\alpha s x\right) dx \Leftrightarrow f(x) = \frac{\alpha}{2\pi\kappa} \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(i\alpha s x\right) ds \tag{D.6}$$

D.2 A SURVEY OF NOTATIONS

From eq. (D.6) we can express the required numeric factors that accompany all the various forward transforms conventions. Let us do a quick survey of the bookshelf, ignoring differences in the i's and j's, differences in the transform variables, and so forth.

From my old systems and signals course, with the book [18] we have, $\kappa = 1$, and $\alpha = 2\pi$

$$
\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) \exp\left(-2\pi i s x\right) dx \\
f(x) &= \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(2\pi i s x\right) ds
\end{aligned}
\tag{D.7}
$$

The mathematician's preference, and that of [2], and [5] appears to be the nicely symmetrical version, with $\kappa = 1/\sqrt{2\pi}$, and $\alpha = 1$

$$
\begin{aligned}
\tilde{f}(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp\left(-isx\right) dx \\
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(isx\right) ds
\end{aligned}
\tag{D.8}
$$

From the old circuits course using [21], and also in the excellent text [29], we have $\kappa = 1$, and $\alpha = 1$

$$
\begin{aligned}
\tilde{f}(s) &= \int_{-\infty}^{\infty} f(x) \exp\left(-isx\right) dx \\
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(s) \exp\left(isx\right) ds
\end{aligned}
\tag{D.9}
$$

and finally, the QM specific version from [33], with $\alpha = p/\hbar$, and $\kappa = 1/\sqrt{2\pi\hbar}$ we have

$$
\begin{aligned}
\tilde{f}(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{ipx}{\hbar}\right) dx \\
f(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{f}(p) \exp\left(\frac{ipx}{\hbar}\right) dp
\end{aligned}
\tag{D.10}
$$

E

PROJECTION WITH GENERALIZED DOT PRODUCT

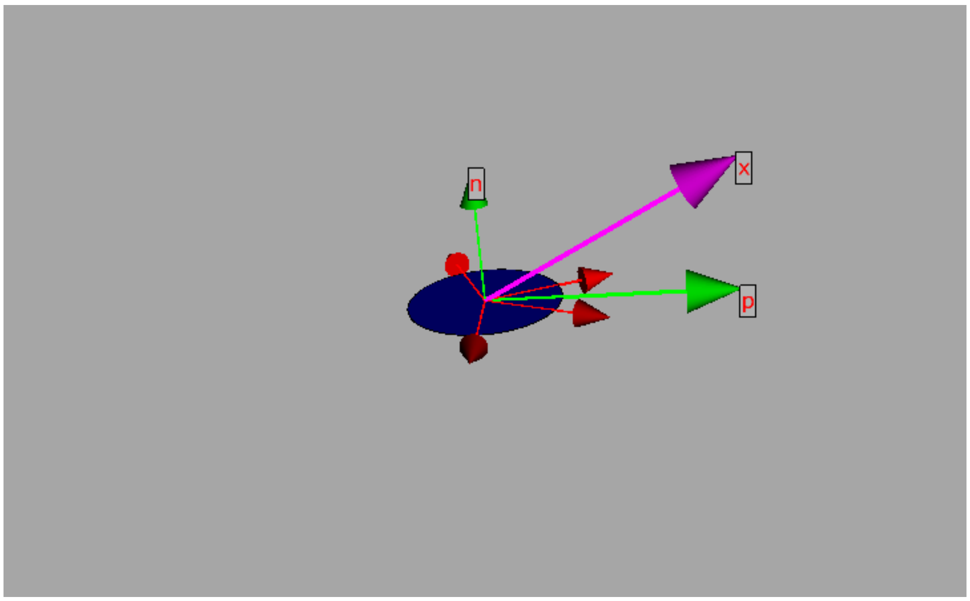

Figure E.1: Visualizing projection onto a subspace

We can geometrically visualize the projection problem as in fig. E.1. Here the subspace can be pictured as a plane containing a set of mutually perpendicular basis vectors, as if one has visually projected all the higher dimensional vectors onto a plane.

For a vector $\mathbf{x}$ that contains some part not in the space we want to find the component in the space $\mathbf{p}$, or characterize the projection operation that produces this vector, and also find the space of vectors that lie perpendicular to the space.

Expressed in terms of the Euclidean dot product this perpendicularity can be expressed explicitly as $U^{\mathrm{T}}\mathbf{n} = 0$. This is why we say that $\mathbf{n}$ is in the null space of U^{T}, $N(U^{\mathrm{T}})$ not the null space of U itself ($N(U)$). One perhaps could say this is in the null or perpendicular space of the set $\{u_i\}$, but the typical preference to use columns as vectors makes this not entirely unnatural.

In a complex vector space with $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^*\mathbf{v}$ transposition no longer expresses this null space concept, so the null space is the set of $\mathbf{n}$, such that $U^*\mathbf{n} = 0$, so one would say $\mathbf{n} \in N(U^*)$.

One can generalize this projection and nullity to more general dot products. Let us examine the projection matrix calculation with respect to a more arbitrary inner product. For an inner product that is conjugate linear in the first variable, and linear in second variable we can write:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* A \mathbf{v} \tag{E.1}$$

This is the most general complex bilinear form, and can thus represent any complex dot product.

The problem is the same as above. We want to repeat the projection derivation done with the Euclidean dot product, but be more careful with ordering of terms since we now using a non-commutative dot (inner) product.

We are looking for vectors $\mathbf{p} = \sum a_i \mathbf{u}_i$, and $\mathbf{e}$ such that

$$\mathbf{x} = \mathbf{p} + \mathbf{e} \tag{E.2}$$

If the inner product defines the projection operation we have for any $\mathbf{u}_i$

$$\begin{aligned}
0 &= \langle \mathbf{u}_i, \mathbf{e} \rangle \\
&= \langle \mathbf{u}_i, \mathbf{x} - \mathbf{p} \rangle \\
\implies & \\
\langle \mathbf{u}_i, \mathbf{x} \rangle &= \langle \mathbf{u}_i, \mathbf{p} \rangle \\
&= \langle \mathbf{u}_i, \sum_j a_j \mathbf{u}_j \rangle \\
&= \sum_j a_j \langle \mathbf{u}_i, \mathbf{u}_j \rangle
\end{aligned} \tag{E.3}$$

In matrix form, this is

$$\Big[\langle \mathbf{u}_i, \mathbf{x} \rangle\Big]_i = \Big[\langle \mathbf{u}_i, \mathbf{u}_j \rangle\Big]_{ij} [a_i]_i$$

Or

$$A = [a_i]_i = \frac{1}{\Big[\langle \mathbf{u}_i, \mathbf{u}_j \rangle\Big]_{ij}} \Big[\langle \mathbf{u}_i, \mathbf{x} \rangle\Big]_i$$

We can also write our projection in terms of A:

$$\mathbf{p} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix} A = UA$$

Thus the projection vector can be written:

$$\mathbf{p} = U \frac{1}{\left[\langle \mathbf{u}_i, \mathbf{u}_j \rangle\right]_{ij}} \left[\langle \mathbf{u}_i, \mathbf{x} \rangle\right]_i$$

In matrix form this is:

$$\mathrm{Proj}_U(\mathbf{x}) = \left(U \frac{1}{U^* A U} U^* A \right) \mathbf{x} \tag{E.4}$$

Writing $W^* = U^* A$, this is

$$\mathrm{Proj}_U(\mathbf{x}) = \left(U \frac{1}{W^* U} W^* A \right) \mathbf{x}$$

which is what the wikipedia article on projection calls an oblique projection. Q: Can any oblique projection be expressed using just an alternate dot product?

F

MATHEMATICA NOTEBOOKS

These Mathematica notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in

https://raw.github.com/peeterjoot/mathematica/master/.

The free Wolfram CDF player, is capable of read-only viewing these notebooks to some extent.

- Sep 15, 2011 gabook/matrixVectorPotentialsTrig.nb

  Some trig double angle reductions.

- Sep 15, 2011 gabook/pendulumDouble.nb

  Generate some double pendulum figures.

- May 4, 2014 gabook/sphericalSurfaceAndVolumeElements.nb

  In a 4D Euclidean space, this notebook calculates the spherical tangent space basis for a spherical parameterization of span e2,e3,e4 and their duals on that volume. The duals are calculated using the Geometric algebra methods, instead of matrix inversion. This notebook uses clifford.m

- May 4, 2014 gabook/sphericalSurfaceAndVolumeElementsMinkowski.nb

  In a 4D Minkowski space, this notebook calculates the spherical tangent space basis for a spherical parameterization of span e2,e3,e4 and their duals on that volume. The duals are calculated using the Geometric algebra methods, instead of matrix inversion. This notebook uses clifford.m

G

FURTHER READING

There is a wealth of information on the subject available online, but finding information at an appropriate level may be difficult. Not all resources use the same notation or nomenclature, and one can get lost in a sea of product operators. Some of the introductory material also assumes knowledge of various levels of physics. This is natural since the algebra can be utilized well to expresses many physics concepts. While natural, this can also be intimidating if one is unprepared, so mathematics that one could potentially understand may be presented in a fashion that is inaccessible.

G.1 GEOMETRIC ALGEBRA FOR COMPUTER SCIENCE BOOK

The book Geometric Algebra For Computer Science. by Dorst, Fontijne, and Mann has one of the best introductions to the subject that I have seen. It is also fairly inexpensive ($60 Canadian). Compared for example to Hestenes's "From Clifford Algebra to Geometric Calculus" which I have seen listed on amazon.com with a default price of $250, discounted to $150.

This book contains particularly good introductions to the dot and wedge products, both for vectors, and the generalizations. How these can be applied and what they can be used to model is covered excellently.

Compromises have been made in this book on the order to present information, and what level of detail to use and when. Many proofs are deferred or placed only in the appendix. For example, they introduce (define) a scalar product initially (denoted with an asterisk (*)), and define this using a determinant without motivation. This allows for development of a working knowledge of how to apply the subject.

Once an ability to apply has been developed they proceed with an axiomatic development. I would consider an axiomatic approach to the subject very important since there is a sea of identities associated with the algebra. Figuring out which ones are consequences of the others can be difficult, if one starts with definitions that are not fundamental. One can easily go in circles and wonder really are the basic rules (this was my first impression starting with the Hestenes book "New Foundations for Classical Mechanics". The book "Geometric Algebra for Physicists" has an excellent axiomatic development. It however notably makes a similar compromise first introducing the algebra with a dot plus wedge product formulation to develop some familiarity.

This book has three parts. The first is on the algebra, covering the generalized dot and wedge products, rotors, projections, join, linear transformations as outermorphisms, and all the rest of the basic material that one would expect. It does this excellently.

The second portion of this book is on the use of a 5D conformal model for 3D graphics (adding a point at infinity on top of the normal extra viewport dimension that traditional graphics applications use). I can not comment too much on this part of the book since I loaned it to a friend after reading the first and last parts of the book.

The last part of the book is on implementation, and makes for an interesting read. Details on their Gaigen implementation are discussed, as are performance and code size implications of their implementation.

The only thing negative I have to say about this book is the unfortunate introduction of an alternate notation for the generalized dot product (L and backwards L). This is distracting if one started, like I did, with the Hestenes, Cambridge, or Baylis papers or books, and their notation dominates the literature as far as I can tell. This does not take too long to adjust, since one mostly just has to mentally substitute dots for L's (although there are some subtle differences where this transposition does not necessarily work).

G.2 GAVIEWER

Performing the GAViewer tutorial exercises is a great way to build some intuition to go along with the math (putting the geometric back in the algebra).

There are specific GAViewer exercises that you can do independent of the book, and there is also an excellent interactive tutorial 2003 Game Developer Lecture available here:

Interactive GA tutorial. UvA GA Website: Tutorials

(they have hijacked GAViewer here to use as presentation software, but you can go through things at your own pace, and do things such as rotating viewpoints). Quite neat, and worth doing just to play with the graphical cross product manipulation even if you decide not to learn GA.

G.3 OTHER RESOURCES FROM DORST, FONTIJNE, AND MANN

There are other web resources available associated with this book that are quite good. The best of these is GAViewer, a graphical geometric calculator that was the product of some of the research that generated this book.

See , or his paper itself .

Some other links:

Geometric algebra (Clifford algebra)

This is a good tutorial , as it focuses on the geometrical rather than have any tie to physics (fun but more to know). The following looks like a slightly longer updated version:

GA: a practical tool for efficient geometric representation (Dorst)

G.4 LEARNING GA

Of the various GA primers and workbooks above, here are a couple specific documents that are noteworthy, and some direct links to a few things that can be found by browsing that were noteworthy. This is an interactive GA tutorial/presentation for a game programmers conference

that provides a really good intro and has a lot of examples that I found helpful to get an intuitive feel for all the various product operations and object types. Even if you weare not trying to learn GA, if you have done any traditional vector algebra/calculus, IMO its worthwhile to download this just to just to see the animation of how the old cross product varies with changes to the vectors. You have to download the GAViewer program (graphical vector calculator) to run the presentation. Once you do that you can use it for other calculation examples, such as those available in these examples of how to use GAViewer as a standalone tool.. Note that the book the drills are from use a different notation for dot product (with a slightly different meaning and uses an oriented L symbol dependent on the grades of the blades.

Jaap Suter's GA primer. His website, which is referenced in various GA papers no longer (at least obviously) has this primer on it any longer (Sept/2008).

Ian Bell's introduction to GA

This author has a wide range of GA information, but looking at it will probably give you a headache.

GA wikipedia

There are a number of comparisons here between GA identities and traditional vector identities, that may be helpful to get oriented.

- Maths - Clifford / Geometric Algebra - Martin Baker

A GA intro, a small part in the much larger Euclidean space website.

- As mentioned above there is a lot of learning GA content available in the Cambridge/Baylis/Hestenes/Dorst/... sites.

G.5 CAMBRIDGE

The Cambridge GA group has a number of Geometric Algebra publications, including the book

Geometric Algebra for Physicists

This book has an excellent introductory treatment of a number of basic GA concepts, a number of which are much easier to follow than similar content in Hestenes's "New Foundations for Classical Mechanics". When it comes to physics content in this book there are a lot of details left out, so it is not the best for learning the physics itself if you are new to the topic in question.

Much of the content of their book is actually available online in their publications above, but it is hard to beat coherent organization and a paper version that you can mark up.

Some other online learning content from the Cambridge group includes

Introduction to Geometric Algebra

This is an HTML version of the Imaginary numbers are not real paper.

A nice starting point is lect1.pdf from the Cambridge PartIII physics course on GA applications . Only at the very end of this first pdf is any real physics content. taught to what sounds like final year undergrad physics students. The first parts of this do not need much physics knowledge.

G.6 BAYLIS

Wiliam Baylis GA page

He uses a scalar plus vector multivector representation for relativity (APS, Algebra of Physical Space), and an associated conjugate length operation. You will find an intro relativity, GA workbook, and some papers on GA applied to physics here. Also based on his APS approach is the following wikibook:

Physics in the Language of Geometric Algebra. An Approach with the Algebra of Physical Space

G.7 HESTENES

Hestenes main page for GA is Geometric Calculus R & D Home Page

This includes a number of primers and introductions to the subject such as Geometric Algebra Primer. As described in the Introduction page for this primer, this is a workbook, and reading should not be passive.

Also available is his Oersted Lecture, which contains a good introduction.

If you do not have his "New Foundations of Classical Mechanics" book, you can find some of the dot-product/wedge-product reduction formulas in the following non-metric treatment of GA.

Also interesting is this Gauge Theory Gravity with Geometric Algebra paper. This has an introduction to STA (Space Time Algebra) as used in the Cambridge books. This also shows at a high level where one can go with a lot of these ideas (like the grad F = J formulation of Maxwell's equation, a multivector form that incorporates all of the traditional four vector Maxwell's equations). Nice teaser document if you intend to use GA for physics study, but hard to read even the consumable bits because they are buried in among a lot of other higher level math and physics.

Hestenes, Li and Rockwood in their paper New Algebraic Tools for Classical Geometry in G. Sommer (ed.) Geom. Computing with Clifford Algebras (Springer, 2001) treat outermorphisms and determinants in a separate subsection entitled "Outermorphism" of section 1.3 Linear Transformations:

This is a comprehensive doc. Content includes:

- GA intro boilerplate.
- Projection and Rejection.
- Meet and Join.
- Reciprocal vectors (dual frame).
- Vector differentiation.
- Linear transformations.
- Determinants and outermorphisms.
- Rotations.
- Simplexes and boundaries
- Dual quaternions.

G.8 ECKHARD M. S. HITZER (UNIVERSITY OF FUKUI)

From Eckhard's Geometric Algebra Topics.

Since these are all specific documents, and all at a fairly consumable level for a new learner, I have listed them here specifically:

- Axioms of geometric algebra
- The use of quadratic forms in geometric algebra
- The geometric product and derived products
- Determinants in geometric algebra
- Gram-Schmidt orthogonalization in geometric algebra
- What is an imaginary number?
- Simplical calculus:

G.9 ELECTRODYNAMICS

John Denker has a number of GA docs that all appear very readable. One such doc is:

Electromagnetism using Geometric Algebra versus Components

This is a nice little doc (there is also an HTML version, but it is very hard to read, and the first time I saw it I actually missed a lot of content).

The oft repeated introduction to GA is not in this doc, so you have to know the basics first. Denker takes the $\nabla F = J/c\epsilon_0$ equation and unpacks it in a brute force but understandable fashion, and shows that these are identical to the vector differential form of Maxwell's equations. A few other E&M constructs are shown in their GA form (covariant form of Lorentz force equation, Lagrangian density, Stress tensor, Poynting Vector. There are also many good comments on notation issues.

A cautionary note if you have read any of the Cambridge papers. This doc uses a -+++ metric instead of the +— used in those docs.

Some other Denker GA papers:

- Magnetic field of a straight wire.
- Clifford Intro.

  Very nice axiomatic introduction with excellent commentary. Also includes an STA intro.
- Complex numbers.
- Area and Volume.
- Rotations.

(have not read all these yet).

Richard E. Harke, An Introduction to the Mathematics of the Space-Time Algebra

This is a nice complete little doc ($\tilde{4}0$ pages), where many basic GA constructs are developed axiomatically with associated proofs. This includes some simplical calculus and outermorphism content, and eventually moves on to STA and Lorentz rotations.

G.10 MISC

- A blog like subscription service that carries abstracts for various papers on or using Geometric Algebra.

G.11 COLLECTIONS OF OTHER GA BOOKMARKS

- Geomerics. Graphics software for Games, Geometric Algebra references and description.

- Ramon González Calvet us GA links.
- R. W. Gray's GA links.
- Cambridge groups GA urls.

G.12 EXTERIOR ALGEBRA AND DIFFERENTIAL FORMS

- Grassmann Algebra Book

  Pdf files of a book draft entitled Grassmann Algebra: Exploring applications of extended vector algebra with Mathematica.

  This has some useful info. In particular, a great example of solving linear systems with the wedge product.

- The Cornell Library Historical Mathematics Monographs - hyde on grassman
- A Geometric Approach to Differential Forms by David Bachman

G.13 SOFTWARE

- Gaigen 2
- CLICAL for Clifford Algebra Calculations
- nklein software. Geoma.

Part XIII

CHRONOLOGY

H

CHRONOLOGICAL INDEX

QM formulation, with hbar's, of the Fourier transform pair, and Rayleigh Energy theorem, as seen in the book "Quantum Mechanics Demystified". Very non-rigorous treatment, good only for intuition. Also derive the Rayleigh Energy theorem used (but not proved) in this text.

- January 11, 2009 106 Schrödinger equation probability conservation

  Schrodinger probability density and current conservation equation, and comparison of four-vectorized current to Dirac Lagrangian.

  Calculating the rate of change of probability, and using Schrodinger's equation and its conjugate allows for the definition of a probability current, and an electromagnetic like probability-density/current-density conservation law.

  What I thought was interesting was that if you put this into a four vector form as a spacetime divergence (ie: the Lorentz gauge of electrodynamics), the resulting 'four-component' current vector needs only a $\gamma^0 \partial_0$ term to be added to it, for that current itself to be the Dirac Lagrangian (omitting the local-gauge term eA). So it looks like taking the spacetime divergence of the Dirac Lagrangian essentially gives you the probability/current conservation equation (except now this would also produce an extra timelike term not there in the original Schrodinger's equation.) There are some notational differences with the wikipedia form of the Dirac Lagrangian, but I believe all the basic content is there once those differences are accounted for. Very surprising to see the Dirac Lagrangian fall so naturally out of the Schrodinger (non-relativistic) equation.

  I also observe that the probability wave function is perhaps naturally expressed as a relativistic four vector (with a γ_0 term factored out). I still don't understand how Maxwell's equation and QM fit together, but with Maxwell's equation or Lagrangian expressible strictly in terms of four vectors (or the four-gradient and four-curl of such four vectors), there would be a logical cleanliness if one could also express the (relativistic) QM laws strictly in terms of four vectors. Definitely worth playing with.

- January 13, 2009 48 Polar velocity and acceleration

  Straight up column matrix vectors and complex number variants of radial motion derivatives.

- January 18, 2009 95 Time rate of change of the Poynting vector, and its conservation law

  These notes contain the conservation calculation itself, and verify the end result of Schwartz's tricky relativistic argument, that I have yet to understand, to put the conservation into a divergence form that is volume integrable.

  The derivation itself is not too hard. Reconciling all the different notations is actually the tricky bit. Schwartz does this in terms of the dual field tensors F and G, Doran/Lasenby have their GA $F\gamma_k F$ formulation, wikipedia had something different either of than those, and I'd seen in another paper that Jackson used something completely different. At the time I did not have Jackson to see how he did it.

Very interesting here is that we end up with what looks like the Lorentz force law by only looking at conservation requirements based on Maxwell's equation itself. Calling the Poynting vector a field momentum density by analogy (because it showed up in what appeared to be an Energy/momentum (density) four vector) is then seen to be very justifiable. Previously I'd seen that it took two Lagrangians for electrodynamics. One for the fields and one for the interaction term. But now it looks like the interaction term follows from the fields (in a hand waving, fuzzy, not yet fully understood way). Quite interesting, and worth more thought, but seeing how one gets the interaction term from the QM field equation should probably take precedence.

- January 19, 2009 113 Fourier Solutions to Heat and Wave equations

  Apply the series technique to solve for the general time evolution of a wave function for a free (no potential) particle constrained to a circle, and the transform method for a one dimensional linear (non-periodic) scenario.

- January 21, 2009 D A cheatsheet for Fourier transform conventions

- January 25, 2009 69 Electrodynamic wave equation solutions

  Carry the separation of variables to a reasonable point of completion, deriving a tidy relativistic solution for $F_{\mu\nu}$. After this try generalizing that a bit with some intuition that turned out to be busted. Left my dead ends as a marker pointing where not to go in the future.

- January 26, 2009 115 Fourier transform solutions to the wave equation

  Produces the $f(x,t) = g(x - vt)$ solution quite nicely! This works in a fashion for the 2 and 3D cases too, but there the Green's function doesn't reduce nicely to a delta function as in the 1D case.

- January 29, 2009 116 Fourier transform solutions to Maxwell's equation

  Work out a Green's function solution of sorts for the non-homogeneous Maxwell's equation.

- January 31, 2009 117 First order Fourier transform solution of Maxwell's equation

  Application of the Fourier transform to the spacetime split of the gradient term of Maxwell's equation allows for a complete solution of both the vacuum and current forced fields without requiring any computation with four vector potentials. Presuming I got all the math right, this is a beautiful application of both Fourier theory and the STA algebra. Note that the Rigor police are thoroughly away on vacation in this particular set of notes!

- February 1, 2009 118 4D Fourier transforms applied to Maxwell's equation

  Wow, using a spacetime Fourier transform for a Maxwell's solution is much simpler. This is a neat result.

- February 3, 2009 119 Fourier series Vacuum Maxwell's equations

  Go through Bohm's treatment that preps for the Rayleigh-Jeans result in his quantum book in a more natural way. I use complex exponentials, with the STA pseudoscalar for i, and use the much simpler STA Maxwell vacuum equation as the base.

- February 7, 2009 121 Lorentz Gauge Fourier Vacuum potential solutions

  Split from the first order treatment.

- February 8, 2009 120 Plane wave Fourier series solutions to the Maxwell vacuum equation

  My first attempt is getting confusing, especially after seeing after the fact that plane wave constraints on the solution are required for the solution to maintain a grade two form. Summarizes results from the first attempt in a more coherent, albeit denser, form.

- February 13, 2009 98 Lorentz force relation to the energy momentum tensor

  Express the energy momentum tensor in terms of the four vector Lorentz force. This builds on the previous observation that the $T(\gamma_0)$ is related to the work done against the Lorentz force.

- February 17, 2009 99 Energy momentum tensor relation to Lorentz force

- February 18, 2009 114 Poisson and retarded Potential Green's functions from Fourier kernels

  Work through the details of how to derive the Poisson integral kernel starting with the Fourier transform derived Green's function. Do the same thing with the wave equation, and produce the retarded and advanced form solutions. A few years in the works since seeing them in Feynman and wondering where they came from. Feb 25. Did a reduction of the 1D forced wave equation's Green function to a difference of unit step functions. Have to compute derivatives to see if this really works.

- February 26, 2009 39 Spherical and hyperspherical parametrization

  Volume calculations for 1-sphere (circle), 2-sphere (sphere), 3-sphere (hypersphere). Followup with a calculation of the differential volume element for the hypersphere (ie: Minkowski spaces of signature (+,-,-,-). Plan to use these results in an attempt to reduce the 4D hyperbolic Green's functions that we get from Fourier transforming Maxwell's equation.

- March 13, 2009 18 Levi-Civitica summation identity

  A summation identity given in Byron and Fuller, ch 1. Initial proof with a perl script, then note equivalence to bivector dot product.

- March 18, 2009 91 Lorentz force rotor formulation

  Time evolution of a particle in a field as a bivector differential equation, solving for the active Lorentz transformation on the rest frame worldline. Work it out at my own pace in both the GA and tensor formalism.

- April 18, 2009 73 Biot Savart Derivation

- April 28, 2009 37 Developing some intuition for Multivariable and Multivector Taylor Series

  Explicit expansion and Hessian matrix connection. Factor out the gradient from the direction derivative for a few different multivector spaces.

- May 23, 2009 89 Lorentz boost of Lorentz force equations

  My own attempt to walk through the Lorentz transformation of the pair of Lorentz force and power equations, as done in Bohm's 'The Special Theory of Relativity'. Bohm's text left out a number of details, as well as had a number of sign typos and some dropped terms. Try to get it right. Was able to do some of it, but part of the final "the reader can verify bits" have me stumped. How to do those last bits is not obvious to me, which is likely why Bohm left this out of this pseudo-layman book. This set of notes starts off with a large digression on how to express and translate from the GA hyperbolic exponential Lorentz boost formulation to the "classical" coordinate and vector representations used in the Bohm text and other places. My initial reason for writing that up for myself all in one place was that I intended to try the Lorentz force boost procedure of the Bohm text completely in GA form, but I also have not gotten to attempting that. My goal was to finish the details of the "old-fashioned" way first, but the algebra for that way is so messy I don't see how to do it.

- May 28, 2009 66 Macroscopic Maxwell's equation

  Got my "new" second hand 2nd ed. of Jackson's Classical Electrodynamics in the mail, and got distracted reading the introduction. Turns out that a trivector "current" term (with basis vectors in the Dirac vector space) to supplement the four-vector current completely summarizes the mess of $B, D, H, E, M, P, J, \rho$ variables nicely in a fashion very similar to the $\nabla F = J$ variation of Maxwell's equation for the microscopic case.

- June 1, 2009 63 Poincare transformations

  A paper used a specific antisymmetric object for linearized Poincare transformations. Try to figure this out. Turns out to be a representation of the bivector that encodes the plane of rotation or spacetime boost plane.

- June 21, 2009 79 Wave equation form of Maxwell's equations

  Fill in missing details from Jackson, and find the wave equation from Maxwell's equations with and without Geometric Algebra

- June 27, 2009 62 Relativistic Doppler formula

  Deriving the Doppler shift result with a Lorentz boost is much simpler than the time dilation argument in wikipedia.

- July 2, 2009 80 Space time algebra solutions of the Maxwell equation for discrete frequencies

  Exploring vacuum Maxwell solutions using Geometric Algebra formalism. Motivate with Fourier transform techniques, and examine the result and constraints required for solution.

- July 27, 2009 110 Bivector form of quantum angular momentum operator

  Exploring a wedge product formulation of the angular momentum operator in Cartesian and spherical polar representations. Lots of good stuff here!

- July 30, 2009 81 Transverse electric and magnetic fields

  Coupling between transverse and propagation direction components of wave guide solutions is examined using Geometric Algebra.

- Aug 6, 2009 82 Comparing phasor and geometric transverse solutions to the Maxwell equation

  Attempting to use the pseudoscalar as the imaginary in a wave equation phasor expression leads to specific results. Examine these and contrast to scalar imaginary phasors.

- Aug 10, 2009 83 Covariant Maxwell equation in media

  Formulate the Maxwell equation in media (from Jackson) without an explicit spacetime split.

- Aug 14, 2009 92 (INCOMPLETE) Geometry of Maxwell radiation solutions

  After having some trouble with pseudoscalar phasor representations of the wave equation, step back and examine the geometry that these require. Find that the use of $I\hat{\mathbf{z}}$ for the imaginary means that only transverse solutions can be encoded.

- Aug 16, 2009 111 Graphical representation of Spherical Harmonics for $l = 1$

  Observations that the first set of spherical harmonic associated Legendre eigenfunctions have a natural representation as projections from rotated spherical polar rotation points.

- Aug 31, 2009 34 Generator of rotations in arbitrary dimensions.

  Similar to the exponential translation operator, the exponential operator that generates rotations is derived. Geometric Algebra is used (with an attempt to make this somewhat understandable without a lot of GA background). Explicit coordinate expansion is also covered, as well as a comparison to how the same derivation technique could be done with matrix only methods. The results obtained apply to Euclidean and other metrics and also to all dimensions, both 2D and greater or equal to 3D (unlike the cross product form).

- Sept 6, 2009 112 Bivector grades of the squared angular momentum operator.

  The squared angular momentum operator can potentially have scalar, bivector, and (four) pseudoscalar components (depending on the dimension of the space). Here just the bivector grades of that product are calculated. With this the complete factorization of the Laplacian can be obtained.

- Sept 13, 2009 93 Relativistic classical proton electron interaction.

  An attempt to setup (but not yet solve) the equations for relativistically correct proton electron interaction.

- Sept 20, 2009 35 Spherical Polar unit vectors in exponential form.

  An exponential representation of spherical polar unit vectors. This was observed when considering the bivector form of the angular momentum operator, and is reiterated here independent of any quantum mechanical context.

- Sept 24, 2009 84 Electromagnetic Gauge invariance.

  Show the gauge invariance of the Lorentz force equations. Start with the four vector representation since these transformation relations are simpler there and then show the invariance in the explicit space and time representation.

- Dec 1, 2009 41 Polar form for the gradient and Laplacian.

  Explore a chain rule derivation of the polar form of the Laplacian, and the validity of my old First year Professor's statements about divergence of the gradient being the only way to express the general Laplacian. His insistence that the grad dot grad not being generally valid is reconciled here with reality, and the key is that the action on the unit vectors also has to be considered.

Part XIV

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Part XV

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