Evaluating the Gaussian integral.

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1 Plain old Gaussian.

QM solutions appear to involve a lot of Gaussian integrals. Looking at one of the problems in [McMahon(2005)] I tried to recall how to evaluate the simplest of these. Google says the trick is squaring and polar substitution. Let's try this.

Solve

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx$$

$$\begin{aligned}
I^2 &= \int_{x=-\infty}^{\infty} e^{-\alpha x^2} dx \int_{y=-\infty}^{\infty} e^{-\alpha y^2} dy \\
&= \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-\alpha r^2} r dr d\theta \\
&= 2\pi \left. \frac{e^{-\alpha r^2}}{-2\alpha} \right|_{r=0}^{\infty} \\
&= \frac{\pi}{\alpha}
\end{aligned}$$

So we have

$$I = \sqrt{\frac{\pi}{\alpha}}$$

2 A couple higher order Gaussian's and normalization exersize.

In order to do the normalization exersize for

$$\psi = \left(A e^{-\frac{x^2}{a}} + B x e^{-\frac{x^2}{b}} \right) e^{-ict} \tag{1}$$

We want to calculate

$$\int \psi^* \psi = |A|^2 e^{-2\frac{x^2}{a}} + |B|^2 x^2 e^{-2\frac{x^2}{b}} + (A\bar{B} + B\bar{A}) x e^{-x^2\left(\frac{1}{a}+\frac{1}{b}\right)}$$

so we need the $n = 1, 2$ versions of the following Gaussians

$$I_n = \int x^n e^{-\alpha x^2} dx$$

The $n = 1$ case is directly integratable:

$$\begin{aligned}
I_1 &= \int x e^{-\alpha x^2} dx \\
&= \int \left(\frac{e^{-\alpha x^2}}{-2\alpha} \right)' dx \\
&= 0
\end{aligned}$$

(integration bounds are $\pm\infty$ so the exponential vanishes).
Next. I_2 follows with integration by parts

$$\begin{aligned}
I_2 &= \int x^2 e^{-\alpha x^2} dx \\
&= \int x \left(x e^{-\alpha x^2} \right) dx \\
&= \int x \left(\frac{e^{-\alpha x^2}}{-2\alpha} \right)' dx \\
&= -\int \frac{e^{-\alpha x^2}}{-2\alpha} dx \\
&= \frac{1}{2\alpha} \int e^{-\alpha x^2} dx \\
&= \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}}
\end{aligned}$$

So the normalization required for 1 is

$$1 = |A|^2 \sqrt{\frac{\pi a}{2}} + \frac{|B|^2}{2} \sqrt{\frac{\pi b}{2}} \frac{b}{2}$$

The values a, and b are presumably due to boundary conditions, and this then fixes $|A|$ in terms of $|B|$ or the other way around.

3 Generalized.

Let

$$I_n = \int_{-\infty}^{\infty} x^n e^{-\alpha x^2} dx \tag{2}$$

We've solved this for $I_0 = \sqrt{\pi/\alpha}$, $I_1 = 0$, and I_2. A quick calculation shows that $I_{2k+1} = 0$ too:

$$\begin{aligned}
I_n &= \int_{-\infty}^{\infty} x^n e^{-\alpha x^2} dx \\
&= \int_{0}^{\infty} x^n e^{-\alpha x^2} dx + \int_{-\infty}^{0} x^n e^{-\alpha x^2} dx \\
&= \int_{0}^{\infty} x^n e^{-\alpha x^2} dx + \int_{\infty}^{0} (-x)^n e^{-\alpha x^2} (-dx) \\
&= \int_{0}^{\infty} x^n e^{-\alpha x^2} dx + \int_{0}^{\infty} (-x)^n e^{-\alpha x^2} dx \\
&= \int_{0}^{\infty} (x^n + (-x)^n) e^{-\alpha x^2} dx
\end{aligned}$$

But if n is odd $(-x)^n = -x^n$, so this is zero.
Now, for n even, we can integrate by parts, as done for I_2.

$$\begin{aligned}
I_{2m} &= \int x^{2m} e^{-\alpha x^2} dx \\
&= \int x^{2m-1} \left(x e^{-\alpha x^2} \right) dx \\
&= \int x^{2m-1} \left(\frac{e^{-\alpha x^2}}{-2\alpha} \right)' dx \\
&= -\int (2m-1) x^{2m-2} \frac{e^{-\alpha x^2}}{-2\alpha} dx
\end{aligned}$$

This gives us a recurrance relationship for the even order terms

$$I_{2m} = \frac{2m-1}{2\alpha} I_{2m-2}. \tag{3}$$

Expanding this explicitly for the first few m shows the pattern

$$
\begin{aligned}
I_2 &= \tfrac{2-1}{2\alpha} I_0 &&= \tfrac{1}{2\alpha}\sqrt{\tfrac{\pi}{\alpha}} \\
I_4 &= \tfrac{4-1}{2\alpha}\tfrac{2-1}{2\alpha} I_0 &&= \tfrac{3.1}{(2\alpha)^2}\sqrt{\tfrac{\pi}{\alpha}} \\
I_6 &= \tfrac{6-1}{2\alpha}\tfrac{4-1}{2\alpha}\tfrac{2-1}{2\alpha} I_0 &&= \tfrac{5.3.1}{(2\alpha)^3}\sqrt{\tfrac{\pi}{\alpha}}
\end{aligned}
$$

Or

$$I_0 = \sqrt{\frac{\pi}{\alpha}} \tag{4}$$

$$I_{2m-1} = 0 \tag{5}$$

$$I_{2m} = \frac{(2m-1)(2m-3)\cdots(3)(1)}{(2\alpha)^m}\sqrt{\frac{\pi}{\alpha}} \tag{6}$$

References

[McMahon(2005)] D. McMahon. *Quantum Mechanics Demystified*. McGraw-Hill Professional, 2005.