Lorentz invariance of energy momentum four vector.

Peeter Joot peeter.joot@gmail.com

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Contents

1 Motivation.

A blurb on Lorentz invariance eventually removed from other notes. Probably want to merge this with my treatment of application of the chain rule to the wave equation as a method of finding the Lorentz boost matrix.

2 Prerequiste concepts. Wave equation, and Lorentz invariance.

An unforced mechanical wave described by a function $\psi(t, \mathbf{x})$, propagating undamped and unforced with velocity v is described by the familiar equation

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} - \boldsymbol{\nabla}^2 \psi = 0 \tag{1}$$

For the purposes of this discussion, a relativistic wave is described by (1) with two additional conditions. The first is that the wave speed is $v = c$, the speed of light. The second condition required for the label relativistic is a restriction on the allowed coordinate transformations. These are the linear transformations of space time coordinates $(t, x, y, z) \rightarrow (t', x', y', z')$ for which the wave equation retains precisely this form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}-\frac{\partial^2\psi}{\partial x'^2}-\frac{\partial^2\psi}{\partial y'^2}-\frac{\partial^2\psi}{\partial z'^2} \tag{2}$$

Such transformations, the Lorentz transformations, are those that introduce no cross term such as $\partial^2\psi/\partial x'\partial y'$, and do not change the wave velocity. One can show that spatial rotations such as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{3}$$

Or Lorentz boosts such as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{4}$$

Or any composition of such transformations meet this requirement.

In the linear transformations above the space and time coordinates were merged into a single vector representation, the particle worldline vector, often written with shorthand such as

$$X \equiv (ct, \mathbf{x}) \tag{5}$$

For such a vector, a Lorentz length can be defined

$$X^2 \equiv c^2t^2 - \mathbf{x}^2 \equiv c^2t^2 - \mathbf{x}\cdot\mathbf{x} \tag{6}$$

Without specific discussion of the wave equation, a more usual but equivalent definition of Lorentz transformations, are those that leave this Lorentz length unchanged, as in

$$c^2t^2 - \mathbf{x}^2 = c^2t'^2 - \mathbf{x}'^2 \tag{7}$$

If we introduce a vector space time derivative operator

$$\nabla \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \tag{8}$$

The Lorentz invariant length of this vector operator is in fact our wave equation operator

$$\Box \equiv \nabla^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} \tag{9}$$

It is clear that the original requirement for wave equation invariance (2) is also contained within this definition of Lorentz invariant length.

Unit vectors with respect to Lorentz length are neccessarily Lorentz invariant. Considering for example the time rate of change of a particle worldline we have

$$\left(\frac{d}{dt}(ct, \mathbf{x})\right)^2 = c^2 - \mathbf{v}^2 \tag{10}$$

which implies that the Lorentz length of

$$\frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}(1, \mathbf{v}/c) \tag{11}$$

is just one. For the purposes of this Schrödinger equation discussion a scaling of this four vector so that it has dimensions of energy is required

$$\frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}(mc^2, m\mathbf{v}c) \tag{12}$$

Algebraically, the Lorentz length of this four vector can easily be confirmed to be $(mc^2)^2$. This quantity we will identify as an energy-momentum four vector as follows

$$P = (E/c, \mathbf{p}) \tag{13}$$

With energy defined as

$$E \equiv \frac{mc^2}{\sqrt{1 - \mathbf{v}^2/c^2}} \tag{14}$$

and spatial momentum defined as

$$\mathbf{p} \equiv \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}} \tag{15}$$

It is beyond the scope of these notes to provide a good justification for this identification. [1]

From (13) that Lorentz length of the energy momentum four vector is

$$P^2 = E^2/c^2 - \mathbf{p}^2 = m^2c^2 \tag{16}$$

[1] This is a dodge, and having to make a statement like this shows that it is beyond the scope of the author's understanding to coherently justify this identification. In the spirit of my engineering education I can at least work with it.

References