Comparison of two covariant Lorentz force Lagrangians

Peeter Joot   peeter.joot@gmail.com

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Contents

1 Motivation

In [Poisson(1999)], the covariant Lorentz force Lagrangian is given by

$$\mathcal{L} = \int A_\alpha j^\alpha d^4 x - m \int d\tau \tag{1}$$

which is not quadratic in proper time as seen previously in [Joot(a)] , and [Joot(b)]

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} m v^2 + q A \cdot (v/c) \qquad (2) \\ &= \frac{1}{2m} \left(m v + \frac{q}{c} A \right)^2 - \frac{q^2}{2mc^2} A^2 \qquad (3)\end{aligned}$$

These two forms are identical, but the second is expressed explicitly in terms of the conjugate momentum, and calls out the explicit kinetic vs potential terms in the Lagrangian nicely. Note that both forms assume $\gamma_0^2 = 1$, unlike 1, which must assume a time negative line element.

2 Lagrangian with Quadradic Velocity

For review purposes lets once again compute the equations of motion with an evaluation of the Euler-Lagrange equations. With hindsight this can also be done more compactly than in previous notes.

We carry out the evaluation of the Euler-Lagrange equations in vector form

$$\begin{aligned}
0 &= \nabla \mathcal{L} - \frac{d}{d\tau} \nabla_v \mathcal{L} \\
&= \left(\nabla - \frac{d}{d\tau} \nabla_v \right) \left(\frac{1}{2} m v^2 + q A \cdot (v/c) \right) \\
&= q \nabla (A \cdot (v/c)) - \frac{d}{d\tau} \nabla_v \left(\frac{1}{2} m v^2 + q A \cdot (v/c) \right)
\end{aligned}$$

The middle term here is the easiest and we essentially want the gradient of a vector square.

$$\begin{aligned}
\nabla x^2 &= \gamma^\mu \partial_\mu x^\alpha x_\alpha \\
&= 2 \gamma^\mu x_\mu
\end{aligned}$$

This is

$$\nabla x^2 = 2x \tag{4}$$

The same argument would work for $\nabla_v v^2 = \gamma^\mu \partial (\dot{x}^\alpha \dot{x}_\alpha)/\partial \dot{x}^\mu$, but is messier to write and read.

Next we need the gradient of the $A \cdot v$ dot product, where $v = \gamma_\mu \dot{x}^\mu$ is essentially a constant. We have

$$\begin{aligned}
\nabla (A \cdot v) &= \langle \nabla (A \cdot v) \rangle_1 \\
&= \frac{1}{2} \langle \dot{\nabla} (\dot{A} v + v \dot{A}) \rangle_1 \\
&= \frac{1}{2} \left((\nabla \cdot A) v + (\nabla \wedge A) \cdot v + (v \cdot \nabla) A - \underbrace{(v \wedge \nabla) \cdot A}_{v(\nabla \cdot A) - \nabla (A \cdot v)} \right)
\end{aligned}$$

Cancelling $v(\nabla \cdot A)$ terms, and rearranging we have

$$\nabla (A \cdot v) = (\nabla \wedge A) \cdot v + (v \cdot \nabla) A \tag{5}$$

Finally we want

$$\begin{aligned}
\nabla_v (A \cdot v) &= \gamma^\mu \frac{\partial A_\alpha v^\alpha}{\partial \dot{x}^\mu} \\
&= \gamma^\mu A_\mu
\end{aligned}$$

Which is just

$$\nabla_v (A \cdot v) = A \tag{6}$$

Putting these all together we have

$$0 = q \left((\nabla \wedge A) \cdot v/c + (v/c \cdot \nabla) A \right) - \frac{d}{d\tau} (mv + qA/c)$$

The only thing left is the proper time derivative of A, which by chain rule is

$$\begin{aligned}
\frac{dA}{d\tau} &= \frac{\partial A}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tau} \\
&= v^\mu \partial_\mu A \\
&= (v \cdot \nabla) A
\end{aligned}$$

So our $(v \cdot \nabla) A$ terms cancel and with $F = \nabla \wedge A$ we have our covariant Lorentz force law

$$\frac{d(mv)}{d\tau} = qF \cdot v/c \tag{7}$$

3 Lagrangian with Absolute Velocity

Now, with

$$d\tau = \sqrt{\frac{dx}{d\lambda}} d\lambda$$

it appears from 1 that we can form a different Lagrangian

$$\mathcal{L} = \alpha m |v| + qA \cdot v/c \tag{8}$$

where α is a constant to be determined. Most of the work of evaluating the variational derivative has been done, but we need $\nabla_v |v|$, omitting dots this is

$$
\begin{aligned}
\nabla |x| &= \gamma^\mu \partial_\mu \sqrt{x^\alpha x_\alpha} \\
&= \gamma^\mu \frac{1}{2\sqrt{x^2}} \partial_\mu (x^\alpha x_\alpha) \\
&= \gamma^\mu \frac{1}{\sqrt{x^2}} x_\mu \\
&= \frac{x}{|x|}
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
\nabla_v |v| &= \frac{v}{|v|} \\
&= \frac{v}{c}
\end{aligned}
$$

which gives us

$$
\alpha \frac{d(mv/c)}{d\tau} = qF \cdot v/c \tag{9}
$$

This fixes the constant $\alpha = c$, and we now have a new form for the Lagrangian

$$
\mathcal{L} = m|v|c + qA \cdot v/c \tag{10}
$$

Observe that only after varying the Lagrangian can one make use of the $|v| = c$ equality.

References

[Joot(a)] Peeter Joot. Revisit Lorentz force from Lagrangian. "`http://sites.google.com/site/peeterjoot/geometric-algebra/lorentz_force.pdf`", a.

[Joot(b)] Peeter Joot. Lorentz force Lagrangian with conjugate momentum. "`http://sites.google.com/site/peeterjoot/math2009/lorentz_force_p_qA.pdf`", b.

[Poisson(1999)] E. Poisson. An introduction to the Lorentz-Dirac equation. *Arxiv preprint gr-qc/9912045*, 1999.