Tensor derivation of Maxwell equation (non-dual part) from Lagrangian.

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Contents

1 Motivation.

Looking through my notes for a purely tensor derivation of Maxwell's equation, and not finding one. Have done this on paper a number of times, but writing it up once for reference to refer to for signs will be useful.

2 Lagrangian.

Notes containing derivations of Maxwell's equation

$$\nabla F = J/\epsilon_0 c \tag{1}$$

From the Lagrangian

$$\mathcal{L} = -\frac{\epsilon_0}{2}(\nabla \wedge A)^2 + \frac{J}{c} \cdot A \tag{2}$$

can be found in [Joot(a)], and the earlier [Joot(b)].

We will work from the scalar part of this Lagrangian, expressed strictly in tensor form

$$\mathcal{L} = \frac{\epsilon_0}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{c} J_\mu \cdot A^\mu \tag{3}$$

3 Calculation.

3.1 Preparation.

In preparation, an expansion of the Faraday tensor in terms of potentials is desirable

$$\begin{aligned}
F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu \\
&= 2(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)
\end{aligned}$$

So we have

$$\mathcal{L} = \frac{\epsilon_0}{2} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{c} J_\mu \cdot A^\mu$$

3.2 Derivatives.

We want to compute

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = \sum \partial_\beta \frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} \tag{4}$$

Starting with the LHS we have

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} = \frac{1}{c} J^\alpha$$

and for the RHS

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= \frac{\epsilon_0}{2} \frac{\partial}{\partial(\partial_\beta A_\alpha)} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \frac{\epsilon_0}{2} \left(F^{\beta\alpha} + \partial^\mu A^\nu \frac{\partial}{\partial(\partial_\beta A_\alpha)} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right) \\
&= \frac{\epsilon_0}{2} \left(F^{\beta\alpha} + \partial^\beta A^\alpha - \partial^\alpha A^\beta \right) \\
&= \epsilon_0 F^{\beta\alpha}
\end{aligned}$$

Taking the β derivatives and combining the results for the LHS and RHS this is

$$\partial_\beta F^{\beta\alpha} = \frac{1}{\epsilon_0 c} J^\alpha \tag{5}$$

3.3 Compare to STA form.

To verify that no sign errors have been made during the index manipulations above, this result should also match the STA Maxwell equation of 1, the vector part of which is

$$\nabla \cdot F = J/\epsilon_0 c$$

Dotting the LHS with γ^α we have

$$\begin{aligned}
(\nabla \cdot F) \cdot \gamma^\alpha &= ((\gamma^\mu \partial_\mu) \cdot (\frac{1}{2} F^{\beta\sigma}(\gamma^\beta \wedge \gamma_\sigma))) \cdot \gamma^\alpha \\
&= \frac{1}{2} \partial_\mu F^{\beta\sigma}(\gamma^\mu \cdot (\gamma_\beta \wedge \gamma_\sigma)) \cdot \gamma^\alpha \\
&= \frac{1}{2} \left(\partial_\beta F^{\beta\sigma} \gamma_\sigma - \partial_\sigma F^{\beta\sigma} \gamma_\beta \right) \cdot \gamma^\alpha \\
&= \frac{1}{2} \left(\partial_\beta F^{\beta\alpha} - \partial_\sigma F^{\alpha\sigma} \right) \\
&= \partial_\beta F^{\beta\alpha}
\end{aligned}$$

This gives us

$$\partial_\beta F^{\beta\alpha} = J^\alpha / \epsilon_0 c \tag{6}$$

In agreement with 5.

References

[Joot(a)] Peeter Joot. Derivation of euler-lagrange field equations. "http://sites.google.com/site/peeterjoot/geometric-algebra/field_lagrangian.pdf", a.

[Joot(b)] Peeter Joot. Complex valued lagrangian derivation of sta maxwell's equation. "http://sites.google.com/site/peeterjoot/geometric-algebra/lagrangian_field_density.pdf", b.