# Four vector velocity addition notes. 

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## 1 Motivation.

Reconcile four vector transformed velocity coordinates with non-covariant form. Specifically, equations (10) and (191) in [Pauli(1981)] look considerably different on the surface, but must have the same content.

Equations (10) were also derived in a bit more detail than in Pauli's book in [Joot()] and are

$$
\begin{align*}
u_{x} & =\frac{u_{x}^{\prime}+v}{1+v u_{x}^{\prime} / c^{2}}  \tag{1}\\
u_{y} & =\frac{u_{y}^{\prime}}{\gamma\left(1+v u_{x}^{\prime} / c^{2}\right)}  \tag{2}\\
u_{z} & =\frac{u_{z}^{\prime}}{\gamma\left(1+v u_{x}^{\prime} / c^{2}\right)}  \tag{3}\\
\gamma^{-1} & =\sqrt{1-v^{2} / c^{2}} \tag{4}
\end{align*}
$$

whereas equations (191) are given as

$$
\begin{align*}
& u^{1^{\prime}}=\gamma\left(u^{1}+i(v / c) u^{4}\right)  \tag{5}\\
& u^{2^{\prime}}=u^{2}  \tag{6}\\
& u^{3^{\prime}}=u^{3}  \tag{7}\\
& u^{4^{\prime}}=\gamma\left(u^{4}-i(v / c) u^{1}\right) \tag{8}
\end{align*}
$$

## 2 Derive the transformed velocity equations.

Pauli uses a $(+,+,+,-)$ metric, with $c t=x^{4}=-x_{4}$. For much of his SR treatment he also uses the Minkowski representation $x^{4}=x_{4}=i c t$. In the first representation we have

$$
\begin{aligned}
-c^{2} & =\frac{d x^{\mu}}{d \tau} \frac{d x_{\mu}}{d \tau} \\
& =\frac{d x^{k}}{d \tau} \frac{d x_{k}}{d \tau}+\frac{d x^{4}}{d \tau} \frac{d x_{4}}{d \tau} \\
& =\left(\frac{d t}{d \tau}\right)^{2}\left(\sum_{k=1}^{3}\left(\frac{d x^{k}}{d t}\right)^{2}-\left(\frac{d x^{4}}{d t}\right)^{2}\right) \\
& =\left(\frac{d t}{d \tau}\right)^{2}\left(\mathbf{u}^{2}-c^{2}\right)
\end{aligned}
$$

Shuffling and taking roots produces a $\gamma$ factor by virtue of the invariant

$$
\begin{aligned}
& \frac{d t}{d \tau} \\
& \quad=\frac{1}{\sqrt{1-\mathbf{u}^{2} / c^{2}}}
\end{aligned}
$$

This is enough to write the proper velocity in terms of a space time split

$$
\begin{aligned}
\dot{X} & =\left(\frac{d x^{\mu}}{d \tau}\right) \\
& =\frac{1}{\sqrt{1-\mathbf{u}^{2} / c^{2}}}(\mathbf{u}, c)
\end{aligned}
$$

As a four vector this can be Lorentz boosted. For an x-axis boost we have

$$
\begin{aligned}
{\left[\begin{array}{l}
u^{1} \\
u^{2} \\
u^{3} \\
u^{4}
\end{array}\right]^{\prime} } & =\left[\begin{array}{cccc}
\gamma & 0 & 0 & -\gamma \beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\gamma \beta & 0 & 0 & \gamma
\end{array}\right]\left[\begin{array}{l}
u^{1} \\
u^{2} \\
u^{3} \\
u^{4}
\end{array}\right] \\
\gamma & =\frac{1}{\sqrt{1-\beta^{2}}}
\end{aligned}
$$

Expanding this we have

$$
\begin{align*}
& u^{1^{\prime}}=\gamma\left(u^{1}-\beta u^{4}\right)  \tag{9}\\
& u^{2^{\prime}}=u^{2}  \tag{10}\\
& u^{3^{\prime}}=u^{3}  \tag{11}\\
& u^{4^{\prime}}=\gamma\left(u^{4}-\beta u^{1}\right) \tag{12}
\end{align*}
$$

In the imaginary representation the Lorentz transform takes the form

$$
\left[\begin{array}{l}
u^{1} \\
u^{2} \\
u^{3} \\
u^{4}
\end{array}\right]^{\prime}=\left[\begin{array}{cccc}
\gamma & 0 & 0 & i \gamma \beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-i \gamma \beta & 0 & 0 & \gamma
\end{array}\right]\left[\begin{array}{l}
u^{1} \\
u^{2} \\
u^{3} \\
u^{4}
\end{array}\right]
$$

Let's verify that this produces the same result by expansion

$$
\begin{aligned}
& u^{1^{\prime}}=\gamma\left(u^{1}+i \beta u^{4}\right) \\
& u^{2^{\prime}}=u^{2} \\
& u^{3^{\prime}}=u^{3} \\
& u^{4^{\prime}}=\gamma\left(u^{4}-\beta i u^{1}\right)
\end{aligned}
$$

with $u^{4} \rightarrow i u^{4}$ to switch to a real representation this is

$$
\begin{aligned}
& u^{1^{\prime}}=\gamma\left(u^{1}-\beta u^{4}\right) \\
& u^{2^{\prime}}=u^{2} \\
& u^{3^{\prime}}=u^{3} \\
& u^{4^{\prime}}=\gamma\left(u^{4}-\beta u^{1}\right)
\end{aligned}
$$

Good. This matches equations 9 . Now, we want to put these in an explicit space time representation to compare against 1. Since those are in real form, work with the real representation instead of the imaginary Minkowski representation for such a comparison.

### 2.1 WRONG: Non-covariant representation of the transformed velocity.

Expanding out the proper time derivatives (assuming that $d x^{\prime} / d t^{\prime}=v$ is a correct interpretation of the math), we have

$$
\begin{aligned}
& \frac{1}{\sqrt{1-v^{2} / c^{2}}} \frac{d x^{\prime 1}}{d t^{\prime}}=\frac{1}{\sqrt{1-v^{2} / c^{2}}} \frac{1}{\sqrt{1-\mathbf{u}^{2} / c^{2}}}\left(\frac{d x^{1}}{d t}-\beta c\right) \\
& \frac{1}{\sqrt{1-v^{2} / c^{2}}} \frac{d x^{\prime 2}}{d t^{\prime}}=\frac{1}{\sqrt{1-\mathbf{u}^{2} / c^{2}}} \frac{d x^{2}}{d t} \\
& \frac{1}{\sqrt{1-v^{2} / c^{2}}} \frac{d x^{\prime 3}}{d t^{\prime}}=\frac{1}{\sqrt{1-\mathbf{u}^{2} / c^{2}}} \frac{d x^{3}}{d t} \\
& \frac{1}{\sqrt{1-v^{2} / c^{2}}} \frac{d x^{\prime 4}}{d t^{\prime}}=\frac{1}{\sqrt{1-v^{2} / c^{2}}} \frac{1}{\sqrt{1-\mathbf{u}^{2} / c^{2}}}\left(c-\beta \frac{d x^{1}}{d t}\right)
\end{aligned}
$$

Hmm. That doesn't appear to match.

## 3 Try again from scratch.

### 3.1 Boost a stationary particle.

Instead of starting with a proper velocity with a spatial component, let's cut the complexity and consider the simplest case, a particle at rest. The worldline (in two dimensions) for a particle in its rest frame is

$$
X=(0, c t)
$$

The proper velocity for this particle is

$$
u=\frac{d X}{d \tau}=\left(0, c \frac{d t}{d \tau}\right)
$$

But since this is a particle in its rest frame $d t / d \tau=1$, this proper velocity is

$$
u=(0, c)
$$

Observe that the norm of this vector (still using the time negative metric signature) is

$$
u \cdot u=0^{2}-c^{2}=-c^{2}
$$

Now, what happens when we apply a Lorentz boost to this?

$$
u^{\prime}=\left[\begin{array}{cc}
\gamma & -\gamma \beta \\
-\gamma \beta & \gamma
\end{array}\right]\left[\begin{array}{l}
0 \\
c
\end{array}\right]
$$

This is

$$
u^{\prime}=\gamma\left[\begin{array}{c}
-\beta  \tag{13}\\
1
\end{array}\right] c
$$

What's the norm of this vector. It should be unchanged, so let's verify.

$$
\begin{aligned}
u^{\prime} \cdot u^{\prime} & =\gamma^{2}\left((-\beta)^{2}-1^{2}\right) c^{2} \\
& =-\gamma^{2}\left(1-\beta^{2}\right) c^{2} \\
& =-c^{2}
\end{aligned}
$$

Good, still have the expected $-c^{2}$ value. For this boosted vector, what is $d t^{\prime} / d \tau^{\prime}$ ? Note that in general for the components of $u^{\prime}$ we have

$$
\frac{d x^{\prime \mu}}{d \tau^{\prime}}=\frac{d x^{\prime \mu}}{d t^{\prime}} \frac{d t^{\prime}}{d \tau^{\prime}}
$$

and in particular we have $u^{\prime 4}=c d t^{\prime} / d \tau$

$$
\begin{aligned}
u^{\prime 4} & =\frac{d x^{\prime 4}}{d t^{\prime}} \frac{d t^{\prime}}{d \tau^{\prime}} \\
& =c \frac{d t^{\prime}}{d \tau^{\prime}}
\end{aligned}
$$

Comparing to 13 we have

$$
\begin{aligned}
u^{\prime 4} & =\gamma c \\
& =c \frac{d t^{\prime}}{d \tau^{\prime}}
\end{aligned}
$$

and therefore can write

$$
\frac{d t^{\prime}}{d \tau^{\prime}}=\gamma
$$

Similarly the spatial velocity of the particle in the boosted frame is

$$
\begin{aligned}
u^{\prime 1} & =\frac{d x^{\prime 1}}{d t^{\prime}} \frac{d t^{\prime}}{d \tau^{\prime}} \\
& =u_{x}^{\prime} \frac{d t^{\prime}}{d \tau^{\prime}} \\
& =-\gamma v
\end{aligned}
$$

So we have

$$
u_{x}^{\prime}=-v
$$

This seems to make sense. We move the frame along the positive x -axis, so a particle at rest at the origin of the stationary frame has a velocity $v$ in the opposite direction from the viewpoint of something at rest in the moving frame.

### 3.2 Apply a second boost transformation.

Okay, treating the almost too simple case in detail was helpful to see where to go next. Now that we have a view of a particle at rest from a moving frame, let's apply another boost so we have a second frame moving with relative velocity $\beta^{\prime}$ with respect to the moving frame. Our transformation is

$$
L^{\prime}=\left[\begin{array}{cc}
\gamma^{\prime} & -\gamma^{\prime} \beta^{\prime} \\
-\gamma^{\prime} \beta^{\prime} & \gamma^{\prime}
\end{array}\right]
$$

this second transformation takes the original proper velocity to

$$
u^{\prime \prime}=\gamma \gamma^{\prime}\left[\begin{array}{cc}
1 & -\beta^{\prime} \\
-\beta^{\prime} & 1
\end{array}\right]\left[\begin{array}{c}
-\beta \\
1
\end{array}\right] c
$$

This is

$$
u^{\prime \prime}=\gamma \gamma^{\prime}\left[\begin{array}{c}
-\left(\beta+\beta^{\prime}\right)  \tag{14}\\
1+\beta \beta^{\prime}
\end{array}\right] c
$$

Let's verify that we still have our invariant norm.

$$
\begin{aligned}
u^{\prime \prime} \cdot u^{\prime \prime} & =\gamma^{2} \gamma^{\prime 2}\left(\left(\beta+\beta^{\prime}\right)^{2}-\left(1+\beta \beta^{\prime}\right)^{2}\right) c^{2} \\
& =\gamma^{2} \gamma^{\prime 2}\left(\beta^{2}+\beta^{\prime 2}+2 \beta \beta^{\prime}-1-2 \beta \beta^{\prime}-\beta^{2} \beta^{\prime 2}\right) c^{2} \\
& =\gamma^{2} \gamma^{\prime 2}\left(\beta^{2}\left(1-\beta^{\prime 2}\right)-\left(1-\beta^{\prime 2}\right)\right) c^{2} \\
& =-\gamma^{2} \gamma^{\prime 2}\left(1-\beta^{\prime 2}\right)\left(1-\beta^{2}\right) c^{2} \\
& =-c^{2}
\end{aligned}
$$

Now, we have $u^{\prime \prime 4}=c d t^{\prime \prime} / d \tau^{\prime \prime}$ as before, so from equation 14 the new compound $\gamma$ factor can be picked off

$$
\frac{d t^{\prime \prime}}{d \tau^{\prime \prime}}=\gamma \gamma^{\prime}\left(1+\beta \beta^{\prime}\right)
$$

Using this and chain rule again we have the spatial velocity in the second moving frame for the particle at rest in the original frame. This is

$$
\begin{aligned}
u_{x}^{\prime \prime} & =\frac{\frac{d x^{\prime \prime}}{d t^{\prime \prime}}}{\frac{d t^{\prime \prime}}{d \tau^{\prime \prime}}} \\
& =\frac{-\gamma \gamma^{\prime}\left(\beta+\beta^{\prime}\right) c}{\gamma \gamma^{\prime}\left(1+\beta \beta^{\prime}\right)} \\
& =\frac{-\left(\beta+\beta^{\prime}\right) c}{1+\beta \beta^{\prime}} \\
& =\frac{-\left(v+v^{\prime}\right)}{1+v v^{\prime} / c^{2}}
\end{aligned}
$$

Okay, good. From consideration of proper velocities and their transformations we have something that is of the form of Pauli's equation 10 (here equation 1p, which is the standard form for colinear relativistic velocity addition.

There is a difference though, namely that Pauli's equation 10 expresses the reverse transformation. Shuffling equation 1 to solve for $u_{x}^{\prime}$, we have

$$
u_{x}\left(1+v u_{x}^{\prime}\right)=u_{x}^{\prime}+v
$$

which gives

$$
u_{x}^{\prime}=\frac{u_{x}+(-v)}{1+(-v) u_{x}}
$$

An algebraic inversion of the equation has exactly the same form, but with the velocity negated in sign.

Now with $u_{x}=-v^{\prime}$ we have an identification between this twice boosted frame observing the particle at rest in the original frame.

### 3.3 Perpendicular directions.

Now, the only thing left to understand is the spatial representation of the boosted velocity for the perpendicular to the boost direction components.

To do so, let's treat a more general case for the proper velocity of a particle as seen in some observers "rest frame". Given the particle worldline

$$
X=\left(x^{\mu}\right)
$$

The proper velocity is

$$
\frac{d X}{d \tau}=\left(\frac{x^{k}}{d t}, c\right) \frac{d t}{d \tau}
$$

Writing

$$
\begin{aligned}
u_{x} & =\frac{x^{1}}{d t} \\
u_{y} & =\frac{x^{2}}{d t} \\
u_{z} & =\frac{x^{3}}{d t} \\
\gamma_{0} & =\frac{d t}{d \tau}
\end{aligned}
$$

Application of a boost produces

$$
\begin{aligned}
u^{\prime} & =\left[\begin{array}{cccc}
\gamma & 0 & 0 & -\gamma \beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\gamma \beta & 0 & 0 & \gamma
\end{array}\right]\left[\begin{array}{c}
u_{x} \\
u_{y} \\
u_{z} \\
c
\end{array}\right] \gamma_{0} \\
& =\left[\begin{array}{c}
\gamma_{0} \gamma\left(u_{x}-\beta c\right) \\
\gamma_{0} u_{y} \\
\gamma_{0} u_{z} \\
\gamma_{0} \gamma\left(-\beta u_{x}+c\right)
\end{array}\right]
\end{aligned}
$$

In particular we have

$$
\frac{d x^{\prime}}{d \tau^{\prime}}=\gamma_{0} \gamma\left(1-\beta u_{x} / c\right)
$$

So can write

$$
\begin{aligned}
u_{x}^{\prime} & =\frac{\gamma_{0} \gamma\left(u_{x}-\beta c\right)}{\gamma_{0} \gamma\left(1-\beta u_{x} / c\right)} \\
u_{y}^{\prime} & =\frac{\gamma_{0} u_{y}}{\gamma_{0} \gamma\left(1-\beta u_{x} / c\right)} \\
u_{z}^{\prime} & =\frac{\gamma_{0} u_{z}}{\gamma_{0} \gamma\left(1-\beta u_{x} / c\right)}
\end{aligned}
$$

Reversing signs in $\beta$ to invert and canceling common factors this is

$$
\begin{aligned}
& u_{x}=\frac{u_{x}^{\prime}+v}{1+v u_{x}^{\prime} / c^{2}} \\
& u_{y}=\frac{u_{y}^{\prime}}{\gamma\left(1+v u_{x}^{\prime} / c^{2}\right)} \\
& u_{z}=\frac{u_{z}^{\prime}}{\gamma\left(1+v u_{x}^{\prime} / c^{2}\right)}
\end{aligned}
$$

A final substitution of $\gamma^{-1}=\sqrt{1-v^{2} / c^{2}}$ and we have equation 1 as desired. Pauli says this step is easy, and that's true enough once the simpler cases are first understood.

## References

[Joot()] Peeter Joot. Some notes on pauli relativity velocity addition. 'http: //sites.google.com/site/peeterjoot/math/velocity_addition.pdf'.
[Pauli(1981)] W. Pauli. Theory of Relativity. Dover Publications, 1981.

