Lorentz Gauge Fourier Vacuum potential solutions.

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Contents

1 Motivation.

In [Joot()] a first order Fourier solution of the Vacuum Maxwell equation was performed. Here a comparative potential solution is obtained.

1.1 Notation.

The 3D Fourier series notation developed for this treatment can be found in the original notes [Joot()]. Also included there is a table of notation, much of which is also used here.

2 Second order treatment with potentials.

2.1 With the Lorentz gauge.

Now, it appears that Bohm's use of potentials allows a nice comparison with the harmonic oscillator. Let's also try a Fourier solution of the potential equations. Again, use STA instead of the traditional vector equations, writing $A = (\phi + \mathbf{a})\gamma_0$, and employing the Lorentz gauge $\nabla \cdot A = 0$ we have for $F = \nabla \wedge A$ in cgs units

FIXME: Add $\mathbf{a}$, and ψ to notatational table below with definitions in terms of $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{H}}$ (or the other way around).

$$\nabla^2 A = 4\pi J$$

Again with a spacetime split of the gradient

$$\nabla = \gamma^0(\partial_0 + \boldsymbol{\nabla}) = (\partial_0 - \boldsymbol{\nabla})\gamma_0$$

our four Laplacian can be written

$$\begin{aligned}
(\partial_0 - \boldsymbol{\nabla})\gamma_0\gamma^0(\partial_0 + \boldsymbol{\nabla}) &= (\partial_0 - \boldsymbol{\nabla})(\partial_0 + \boldsymbol{\nabla}) \\
&= \partial_{00} - \boldsymbol{\nabla}^2
\end{aligned}$$

Our vacuum field equation for the potential is thus

$$\partial_{tt} A = c^2 \boldsymbol{\nabla}^2 A \tag{1}$$

Now, as before assume a Fourier solution and see what follows. That is

$$A(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{A}_{\mathbf{k}}(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{2}$$

Applied to each component this gives us

$$\begin{aligned}
\hat{A}''_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} &= c^2 \hat{A}_{\mathbf{k}}(t) \sum_m \frac{\partial^2}{(\partial x^m)^2} e^{-2\pi i \sum_j k_j x^j/\lambda_j} \\
&= c^2 \hat{A}_{\mathbf{k}}(t) \sum_m (-2\pi i k_m/\lambda_m)^2 e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&= -c^2 \mathbf{k}^2 \hat{A}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}$$

So we are left with another big ass set of simplest equations to solve

$$\hat{A}''_{\mathbf{k}} = -c^2 \mathbf{k}^2 \hat{A}_{\mathbf{k}}$$

Note that again the origin point $\mathbf{k} = (0, 0, 0)$ is a special case. Also of note this time is that $\hat{A}_{\mathbf{k}}$ has vector and trivector parts, unlike $\hat{F}_{\mathbf{k}}$ which being derived from dual and non-dual components of a bivector was still a bivector.

It appears that solutions can be found with either left or right handed vector valued integration constants

$$\begin{aligned}
\hat{A}_{\mathbf{k}}(t) &= \exp(\pm ic\mathbf{k}t) C_{\mathbf{k}} \\
&= D_{\mathbf{k}} \exp(\pm ic\mathbf{k}t)
\end{aligned}$$

Since these are equal at $t = 0$, it appears to imply that these commute with the complex exponentials as was the case for the bivector field.

For the $\mathbf{k} = 0$ special case we have solutions

$$\hat{A}_{\mathbf{k}}(t) = D_0 t + C_0$$

It doesn't seem unreasonable to require $D_0 = 0$. Otherwise this time dependent DC Fourier component will blow up at large and small values, while periodic solutions are sought.

Putting things back together we have

$$A(\mathbf{x}, t) = \sum_{\mathbf{k}} \exp(\pm ic\mathbf{k}t) C_{\mathbf{k}} \exp(-i\mathbf{k} \cdot \mathbf{x})$$

Here again for $t = 0$, our integration constants are found to be determined completely by the initial conditions

$$A(\mathbf{x}, 0) = \sum_{\mathbf{k}} C_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{3}$$

So we can write

$$C_{\mathbf{k}} = \frac{1}{V} \int A(\mathbf{x}', 0) e^{i\mathbf{k}\cdot\mathbf{x}'} d^3 x'$$

In integral form this is

$$A(\mathbf{x}, t) = \int \sum_{\mathbf{k}} \exp(\pm i\mathbf{k}ct) A(\mathbf{x}', 0) \exp(i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')) \tag{4}$$

This, somewhat suprisingly, is strikingly similar to what we had for the bivector field. That was:

$$F(\mathbf{x}, t) = \int G(\mathbf{x} - \mathbf{x}', t) F(\mathbf{x}', 0) d^3 x' \tag{5}$$

$$G(\mathbf{x}, t) = \frac{1}{V} \sum_{\mathbf{k}} \exp(i\mathbf{k}ct) \exp(-i\mathbf{k} \cdot \mathbf{x}) \tag{6}$$

We cannot however commute the time phase term to construct a one sided Green's function for this potential solution (or perhaps we can but if so shown or attempted to show that this is possible). We also have a plus or minus variation in the phase term due to the second order nature of the harmonic oscillator equations for our Fourier coefficients.

2.2 Comparing the first and second order solutions

A consequense of working in the Lorentz gauge ($\nabla \cdot A = 0$) is that our field solution should be a gradient

$$
\begin{aligned}
F &= \nabla \wedge A \\
&= \nabla A
\end{aligned}
$$

FIXME: expand this out using 4 to compare to the first order solution.

References

[Joot()] Peeter Joot. Fourier series vacuum maxwell's equations. "http://sites.google.com/site/peeterjoot/math2009/fourier_series_maxwell.pdf".