Chapter 2 Quiz solutions for QM Demystified book.

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Contents

1 Motivation.

Work through the quiz problems from [McMahon(2005)]. I have treated these problems as only a guideline. Instead the questions have been used as a base to develop some comfort in the subject, exploring the math and physics past the scope of the quiz itself.

2 Problem 1. Separation of variables.

This problem was to show that separation of variables leads to an exponential energy/phase term. Let's try this, but do it instead for three dimensions and explore a bit.

We try a test solution of the form

$$\psi = X(x)Y(y)Z(z)T(t)$$

and substitute into

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\partial_t\psi$$

differentiating and dividing by ψ we have

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) + V = i\hbar\frac{T'}{T}$$

We set the right hand side equal to a constant E to be determined by boundary value conditions. According to the dimensionals of V this E constant can be seen to neccessarily be an energy of some sort. In terms of this energy, we have for the function T

$$i\hbar\frac{T'}{T} = E$$

With a solution of

$$T(t) = e^{-iEt/\hbar}$$

Now, the left hand side imposes some constraints on E, but these will be potential dependent. The simplest case, for the wave function of a free particle, is where $V = 0$.

In that case we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\frac{2mE}{\hbar^2}$$

The sum of each of the terms involved all identically equal a constant, which is perhaps reasonable to assume to be negative. If we do so and impose the usual sort of separation of variables constrait, requiring each of the X''/X, Y''/Y, and Z''/Z terms to separately equal some negative constant (to be fixed by boundary conditions), we can write

$$\begin{aligned}
\frac{X''}{X} &= -{k_1}^2 \\
\frac{Y''}{Y} &= -{k_2}^2 \\
\frac{Z''}{Z} &= -{k_3}^2
\end{aligned}$$

So we have for the complete equation a solution proportional to

$$\psi = XYZT = \exp(i(\mathbf{k} \cdot \mathbf{x} - Et/\hbar))$$

With the additonal boundary value constraint of

$$\mathbf{k}^2 = \frac{2mE}{\hbar^2}$$

2.1 Time independent equation.

Given that we can express the time variation of the wave function as an exponential, we can use this to calculate the time independent equation. Let

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x})e^{-iEt/\hbar}$$

Subst back into the equation gives

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi(\mathbf{x})e^{-iEt/\hbar} = E\psi(\mathbf{x})e^{-iEt/\hbar}$$

Or

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi(\mathbf{x}) = E\psi(\mathbf{x})$$

2.2 Constant potential.

Let's consider a slightly more general potential with V constant in some spatial interval.

By inspection it appears that we should perhaps have

$$\psi = \exp(i(\mathbf{k} \cdot \mathbf{x} - (E - V)t/\hbar))$$

Let's check if this is right. We want

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V - i\hbar\partial_t \right) \psi = 0$$

which means that we have

$$\begin{aligned}
0 &= \left(\frac{\hbar^2}{2m}\mathbf{k}^2 + V - i\hbar \frac{-i(E - V)}{\hbar} \right) \psi \\
&= \left(\frac{\hbar^2}{2m}\mathbf{k}^2 + V - (E - V) \right) \psi \\
&= \left(\frac{\hbar^2}{2m}\mathbf{k}^2 - (E - 2V) \right) \psi
\end{aligned}$$

so our wave number energy constraint is

$$\mathbf{k}^2 = \frac{2m(E - 2V)}{\hbar^2}$$

This looks a bit strange, but can be fixed up writing $E' = E - 2V$. Then our test solution takes the form

$$\begin{aligned}
\psi &= \exp(i(\mathbf{k} \cdot \mathbf{x} - (E' + V)t/\hbar)) \\
\mathbf{k}^2 &= \frac{2m(E' + V)}{\hbar^2}
\end{aligned}$$

If you guess wrong, the math forces you to fix the mistake! A final check to see if this is kosher

$$
\begin{aligned}
0 &= \left(\frac{\hbar^2}{2m} \mathbf{k}^2 + V - i\hbar \frac{-i(E+V)}{\hbar} \right) \psi \\
&= \left(\frac{\hbar^2}{2m} \mathbf{k}^2 + V - (E+V) \right) \psi \\
&= \left(\frac{\hbar^2}{2m} \mathbf{k}^2 - E \right) \psi
\end{aligned}
$$

Okay, we're cool now, and things even make sense. A reasonable interpretation is probably introduction of a constant potential raises the minimum ground state energy.

Based on just the math, we don't know that E is neccessarily positive, so in addition to the trigonometric solution above is it also reasonable to allow for possible hyperbolic solutions? If so then the following should also be allowed

$$
\begin{aligned}
\psi &= \exp(\mathbf{k} \cdot \mathbf{x} - i(E+V)t/\hbar) \\
\mathbf{k}^2 &= \frac{2mE}{\hbar^2}
\end{aligned}
$$

We need some physics to augment the math in order to determine what form of solution is actually valid. Some of that physics likely comes in the form of the boundary conditions and perhaps other constraints such as normalization.

2.3 General solution.

Using superposition we should be able to form a wave packet in integral form by allowing for any set of $\mathbf{k}$ vectors. Suppose we assemble a test solution by summing over possible wave numbers

$$
\psi = \int A(\mathbf{k}) \exp(\mathbf{k} \cdot \mathbf{x} - i(E+V)t/\hbar) dk_1 dk_2 dk_3
$$

For generality, allowing both trigonometric and hyperbolic solutions we can allow the coordinates of $\mathbf{k}$ to be real, imaginary, or zero.

Does this work? Let's take derivatives and see what constraints we require if it does.

$$
\begin{aligned}
0 &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V - i\hbar\partial_t\right)\psi \\
&= \int A(\mathbf{k})\left(-\frac{\hbar^2}{2m}\sum_{j=1}^{3}{k_j}^2 - E\right)\exp(\mathbf{k}\cdot\mathbf{x} - i(E+V)t/\hbar)dk_1dk_2dk_3
\end{aligned}
$$

Abusing notation somewhat for this complex "vector" $\mathbf{k}$ by writing $\mathbf{k}^2 = \sum{k_j}^2$, we have as a general solution for this constant potential wave equation

$$
\psi(\mathbf{x},t) = \int A(\mathbf{k})\exp\left(\mathbf{k}\cdot\mathbf{x} - i\left(-\frac{\hbar^2\mathbf{k}^2}{2m} + V\right)\frac{t}{\hbar}\right)dk_1dk_2dk_3
$$

So, for trig solutions (plane waves) we have $\mathbf{k}$ purely imaginary, our energy integration constant $E = -\hbar^2\mathbf{k}^2/2m$ takes a positive value, whereas for hyperbolic solutions, it is negative.

I don't know if it is physically reasonable to allow for hyperbolic solutions. If not, then we should just factor out an explicit i, and write

$$
\psi(\mathbf{x},t) = \int A(\mathbf{k})\exp\left(i\left(\mathbf{k}\cdot\mathbf{x} - \left(\frac{\hbar^2\mathbf{k}^2}{2m} + V\right)\frac{t}{\hbar}\right)\right)dk_1dk_2dk_3
$$

3 Problem 2. Probabilities for a polynomial wavefunction.

The wave function to work with is

$$
\psi = C\frac{1+ix}{1+ix^2}
$$

With probability density

$$
\psi\psi^* = C^2\frac{1+x^2}{1+x^4}
$$

3.1 normalize it.

We can do the normalization with a complex integral over an upper half plane semicircular contour. On the arc we the integral can be parameterized with

$$\begin{aligned}
z &= Re^{i\theta} \\
dz &= Rie^{i\theta} d\theta
\end{aligned}$$

$$\int \psi\psi^* dz = C^2 \int \frac{1 + R^2 e^{2i\theta}}{1 + R^4 e^{4i\theta}} Rie^{i\theta} d\theta$$

This is of order R^3/R^4 so will vanish at infinity.

For the remaining part of the integral we integrate on $[-\infty, \infty]$, but duck up and back around the poles $\sqrt{i}$, and $i\sqrt{i}$ in counterclockwise circles.

Now, the only problem is to remember how to do the integral around the poles. Suppose we have

$$I = \oint \frac{f(z)}{z - z_0} dz$$

for a function that is regular at z_0, and integrate in an infinitesimal loop around z_0. That contour is parameterized by

$$\begin{aligned}
dz &= rie^{i\theta} d\theta \\
z - z_0 &= re^{i\theta}
\end{aligned}$$

So this little contour integral has the value

$$I = \oint f(z) i d\theta$$

if the contour is made small enough that $f(z)$ doesn't vary, then that function takes the value $f(z_0)$, and we have for a clockwise contour the value

$$I = \oint \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Now for this probability density we can do a partial fractions split around the poles $\{\pm\sqrt{i}, \pm i\sqrt{i}\}$ of the form

$$\frac{1 + x^2}{1 + x^4} = \frac{A}{x - \sqrt{i}} + \frac{B}{x + \sqrt{i}} + \frac{C}{x - i\sqrt{i}} + \frac{D}{x + i\sqrt{i}}$$

but we don't really need do to this algebra. Instead for pole p since it is first order we can write

$$\psi(x)\psi^*(x) = C^2 \left(\frac{1+x^2}{1+x^4}(x-p) \right) \frac{1}{x-p}$$

The left hand factor here is then regular at the pole, and we can use L'Hopitals rule to evaluate what value this takes at the pole.

For our first quadrant pole we have

$$\begin{aligned}
\left(\frac{1+x^2}{1+x^4}(x-\sqrt{i}) \right)\Big|_{x=\sqrt{i}} &= \frac{1+i}{4i\sqrt{i}} \\
&= -i\sqrt{2}/4
\end{aligned}$$

and for the second quadrant pole we have

$$\begin{aligned}
\left(\frac{1+x^2}{1+x^4}(x-\sqrt{i}) \right)\Big|_{x=i\sqrt{i}} &= \frac{1+(i\sqrt{i})^2}{4(i\sqrt{i})^3} \\
&= \frac{1-i}{4\sqrt{i}} \\
&= \sqrt{2}\frac{(1-i)^2}{8} \\
&= -i\sqrt{2}/4
\end{aligned}$$

Combining all the contours we have for the integral now

$$\begin{aligned}
0 &= 0 + I + 2(-2\pi i)(-i\sqrt{2}/4)C^2 \\
&= 0 + I - \pi\sqrt{2}C^2
\end{aligned}$$

Therefore for the unit probability we have as desired

$$C = \frac{1}{\sqrt{\pi\sqrt{2}}}$$

3.2 definite integral of probablility.

Next part of the problem was to evaluate the probability of finding the particle in a specific region ($[0,1]$ specifically).

Here we need a definite integral, so none of the contour integration tricks will help. At least I remember how to do that now.

Let's actually do the partial fractions split

$$
\begin{aligned}
\frac{x^2+1}{x^4+1} &= \frac{x^2+1}{x^4-i^2} \\
&= \frac{1}{2}\left(\frac{1}{x^2-i}+\frac{1}{x^2+i}\right) \\
&= \frac{1}{4\sqrt{i}}\left(\frac{1}{x-\sqrt{i}}-\frac{1}{x+\sqrt{i}}\right)+\frac{1}{4\sqrt{-i}}\left(\frac{1}{x-\sqrt{-i}}-\frac{1}{x+\sqrt{-i}}\right)
\end{aligned}
$$

Antidifferentiation gives

$$
\frac{1}{4\sqrt{i}}\ln\left(\frac{x-\sqrt{i}}{x+\sqrt{i}}\right)+\frac{1}{4\sqrt{-i}}\ln\left(\frac{x-\sqrt{-i}}{x+\sqrt{-i}}\right)
$$

It should be possible to simplify this, or use it to verify the contour integral, or directly evaluate the integral for the $[0,1]$ range of the problem, but this particular form is proving somewhat intractable (or I've made mistakes). A lazier way is to invoke webmathematica, which gives

$$
\int \frac{1+x^2}{1+x^4}dx = \frac{-\tan^{-1}(1-\sqrt{2}x)+\tan^{-1}(1+\sqrt{2}x)}{\sqrt{2}}
$$

$$
\int C^2\frac{1+x^2}{1+x^4}dx = \frac{1}{2\pi}(-\tan^{-1}(1-\sqrt{2}x)+\tan^{-1}(1+\sqrt{2}x))
$$

However calculating this for $x=0$ gives zero and for $x=1$ gives 0.25, whereas the text gives ≈ 0.52. The text can't be correct since the density is symmetric, which would imply that the probabilty to find it in $[-1,1]$ is $1.04 > 1$.

4 Inverse first order wave function.

This problem was to normalize

$$
\psi = \frac{1}{x}e^{i\omega t} \quad x \in [1,2]
$$

and to calculate the probability to find the particle in $[1.5,2]$.

4.1 normalization.

$$\begin{aligned}\int |\phi|^2 dx &= \int \frac{1}{x^2} dx \\ &= \frac{1}{1} - \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

So our normalized wave function is

$$\psi = \frac{\sqrt{2}}{x} e^{i\omega t}$$

4.2 probablility in a range.

By inspection this probability is

$$P(a, b) = 2\left(\frac{1}{a} - \frac{1}{b}\right)$$

So for $[1.5, 2]$ we have

$$P = 2\left(\frac{2}{3} - \frac{1}{2}\right) = \frac{1}{3}$$

4.3 Probability current.

This wave function provides a super simple example to try a current calculation with.

$$J = \frac{\hbar}{2mi}\left(\psi^* \psi_x - \psi {\psi^*}_x\right)$$

The time factor will cancel out, leaving

$$\begin{aligned}J &= \frac{\hbar}{2mi}\left(\psi \psi_x - \psi \psi_x\right) \\ &= 0\end{aligned}$$

Okay, that's a too simple probability calculation exersize! Not interesting.

4.4 expectation values.

4.4.1 position

Okay, those were pretty easy integration exersizes. How about using these to verify the Hiesenberg uncertaintly principle. That should be easy enough with this simple wavefunction.

$$
\begin{aligned}
< x > &= \int_1^2 x \frac{2}{x^2} dx = 2(\ln(2) - \ln(1)) = 2\ln(2) \\
< x^2 > &= \int_1^2 x^2 \frac{2}{x^2} dx = 2 - 1 = 1
\end{aligned}
$$

Using the formula for standard deviation on page 50 we have

$$
\begin{aligned}
\Delta x &= \sqrt{< x^2 > - < x >^2} \\
&= \sqrt{1 - (2\ln(2))^2}
\end{aligned}
$$

which is a complex number?

Let's go back to the statistical definition of standard deviation from school and see if this makes sense.

$$
\begin{aligned}
\sigma^2 &= E(x - \bar{x})^2 \\
&= \frac{1}{N} \sum (x_i - \bar{x})^2 \\
&= \frac{1}{N} \sum \left({x_i}^2 - 2x_i\bar{x} + \bar{x}^2 \right) \\
&= \frac{1}{N} \sum {x_i}^2 - 2\bar{x}^2 + \frac{N}{N}\bar{x}^2 \\
&= \frac{1}{N} \sum {x_i}^2 - \bar{x}^2
\end{aligned}
$$

In the QM notation this is

$$
(\Delta x)^2 = < x^2 > - < x >^2
$$

which is strictly positive. Okay, so I must have a mistake above somewhere.

Let's look at the expectation value of x. Does $2\ln(2) = 1.386$ make sense? How about a discrete average that approximates it

$$
\frac{2}{3} \left(1\frac{1}{1^2} + 1.5\frac{1}{1.5^2} + 2\frac{1}{2^2} \right) = \frac{13}{9} = 1.444
$$

Okay, this makes sense, and it doesn't make sense in the discrete approximation that $< x^2 >$ could be lower, so that integration must be wrong. Take II

$$< x^2 > = \int_1^2 x^2 \frac{2}{x^2} dx = 2x|_1^2 = 4 - 2 = 2$$

Okay, that's better ... dumb mistake. Our std deviation is therefore

$$\Delta x = \sqrt{2 - 4(\ln(2))^2} \approx 2.80$$

4.4.2 momentum

Now, how about the momentum variance?

$$\begin{aligned}
< p > &= -i\hbar \int_1^2 2\frac{1}{x} \left(\frac{1}{x}\right)' dx \\
&= 2i\hbar \int_1^2 \frac{1}{x^3} dx \\
&= i\hbar \left(-\frac{1}{4} + \frac{1}{1}\right) dx \\
&= \frac{3}{4} i\hbar
\end{aligned}$$

$$\begin{aligned}
< p^2 > &= -\hbar^2 \int_1^2 2\frac{1}{x} \left(\frac{1}{x}\right)'' dx \\
&= -\hbar^2 \int_1^2 2\frac{1}{x} \left(-\frac{1}{x^2}\right)' dx \\
&= -4\hbar^2 \int_1^2 \frac{1}{x^4} dx \\
&= -\frac{4}{3}\hbar^2 \left(-\frac{1}{8} + 1\right) \\
&= -\frac{7}{6}\hbar^2
\end{aligned}$$

$$\begin{aligned}
\frac{< p^2 > - < p >^2}{\hbar^2} &= \frac{1}{2}\left(-\frac{7}{3} + \frac{9}{8}\right) \\
&= -\frac{29}{48}
\end{aligned}$$

This gives

$$\Delta x \Delta p = i\hbar \sqrt{\frac{29}{24}(1 - 2(\ln(2))^2)} \approx (0.217) i\hbar$$

We expect

$$\Delta x \Delta p > \hbar/2$$

and ended up with an imaginary value where the $\hbar$ factor is less than 0.5. Something's fishy here, and I don't think it's my algebra this time.

How about

$$\begin{aligned}
\frac{|< p^2 >| - |< p >|^2}{\hbar^2} &= \frac{1}{2}\left(\frac{7}{3} - \frac{9}{8}\right) \\
&= \frac{29}{48}
\end{aligned}$$

Not any different, except that the factor of i vanishes. Now the wikipedia uncertainly article presents this way differently. How to reconsile the ideas here?

5 Problem 4.

Unnormalized wavefunction is

$$\psi = \frac{1}{x^2 + 9}$$

5.1 Second order pole contour integral.

Now, in the problem, the normalization is given and only the position expectation and variance is asked for. This normalization factor is interesting to calculate however, since to do the contour integral for this one we have to deal with a double pole, and I'd also forgotten how to do those.

Suppose, again, that we have a regular function $f(z)$ in the neighbourhood of z_0. We want to calculate the double pole integral at that point.

$$I = \oint \frac{f(z)}{(z - z_0)^2} dz$$

Integration by parts looks like the way to go.

$$
\begin{aligned}
I &= \oint \frac{f(z)}{(z-z_0)^2} dz \\
&= \oint f(z) \left(\frac{-1}{z-z_0} \right)' dz \\
&= \oint \left(\left(-\frac{f(z)}{z-z_0} \right)' + \frac{f(z)'}{z-z_0} \right) dz
\end{aligned}
$$

Now, for a circular contour around z_0, we have $z = z_0 + Re^{i\theta} = z_0 + Re^{i(\theta+2\pi)}$, so

$$
\begin{aligned}
-\frac{f(z)}{z-z_0}\Big|_{z_0+Re^{i\phi}}^{z_0+Re^{i(\phi+2\pi)}} &= -\frac{f(z_0+Re^{i\phi} - f(z_0+Re^{i(\phi+2\pi)}}{e}^{-i\phi} \\
&= 0
\end{aligned}
$$

So the integral of the first term is zero, and we know how to deal with second provided the derivative is regular at the point of interest.

$$
\begin{aligned}
\oint \frac{f(z)}{(z-z_0)^2} dz &= \oint \frac{f(z)'}{z-z_0} dz \\
&= 2\pi i f(z)'\big|_{z=z_0}
\end{aligned}
$$

5.2 normalize.

Now, we are set to normalize the wave function. We have a pole at $\pm 3i$

$$
\begin{aligned}
I &= \int |\psi|^2 \\
&= \int_{z=-\infty}^{\infty} dz \frac{1}{(z-3i)^2} \frac{1}{(z+3i)^2}
\end{aligned}
$$

Picking a semicircular arc we have

$$
I + -2\pi i \left(\frac{1}{(z+3i)^2} \right)' \Bigg|_{z=3i} = 0
$$

The integral is therefore

$$
\begin{aligned}
I &= 2\pi i(-2)\frac{1}{(6i)^3} \\
&= \pi\frac{1}{54}
\end{aligned}
$$

and the normalized wave function is

$$
\psi = \sqrt{\frac{54}{\pi}}\frac{1}{x^2+9}
$$

as given in the problem.

5.3 expectation and variance values.

The expectation value for the position operator is just

$$
< x >= \frac{54}{\pi}\int \frac{x}{(x^2+9)^2}dx = 0
$$

Since it is an odd function. For the square we have

$$
\begin{aligned}
< x^2 > &= \frac{54}{\pi}\int \frac{x^2}{(x^2+9)^2}dx \\
&= \frac{54}{\pi}2\pi i\left(\frac{x^2}{(x+3i)^2}\right)'\Bigg|_{3i} \\
&= \cdots \quad \text{some algebra} \\
&= 9
\end{aligned}
$$

Now, how about the momentum? This one is odd too, and therefore zero

$$
< p >= -i\hbar\frac{54}{\pi}\int \frac{1}{(x^2+9)}\frac{-2x}{(x^2+9)^2}dx = 0
$$

Last we have the squared momentum operator expectation

$$
\begin{aligned}
< p^2 > &= -\hbar^2 \frac{54}{\pi} \int \frac{1}{(x^2+9)} \left(\frac{-2x}{(x^2+9)^2} \right)' dx \\
&= -\hbar^2 \frac{54}{\pi} \int \frac{1}{(x^2+9)} \left(\frac{-2}{(x^2+9)^2} - 2x \frac{-2(2)(x^2+9)(2x)}{(x^2+9)^4} \right) dx \\
&= -\hbar^2 \frac{54}{\pi} \int \frac{-2}{(x^2+9)^3} \left(1 - 8x^2 \frac{1}{x^2+9} \right) dx \\
&= -\hbar^2 \frac{54}{\pi} \int \frac{-2}{(x^2+9)^4} \left(9 - 7x^2 \right) dx \\
&= 2\hbar^2 \frac{54}{\pi} \int \frac{1}{(x-3i)^4} \frac{1}{(x+3i)^4} \left(9 - 7x^2 \right) dx
\end{aligned}
$$

Damn. Now we need a fourth order pole (third derivative) residue. This is getting messy. Backing up one step to put things in a nice form for cheating with mathematica we have

$$
< p^2 > = \hbar^2 \frac{108}{\pi} \int \frac{1}{(x^2+9)^4} \left(9 - 7x^2 \right) dx
$$

and the cheat gives us

$$
\int \frac{1}{(x^2+9)^4} \left(9 - 7x^2 \right) dx = \frac{-3x(-729 + 24x^2 + x^4) + (9+x^2)^3 \tan^{-1}(x/3)}{1944(9+x^2)^3}
$$

The first term will drop out for the infinite range, leaving

$$
2 \tan^{-1}(\infty/3) \frac{1}{1944} = \pi \frac{1}{1944}
$$

So, if all went well we have

$$
\begin{aligned}
< p^2 > &= \hbar^2 \frac{108}{\pi} \pi \frac{1}{1944} \\
&= \hbar^2 \frac{1}{18}
\end{aligned}
$$

This provides another numerical verification of the Heisenberg uncertainty relation

$$
\Delta p \Delta x = \hbar \frac{1}{3\sqrt{2}} 3 \approx 0.7\hbar > \hbar/2
$$

6 Problem 5.

Find position and momentum expectation values for

$$\psi = A(x^5 - ax^3)$$

... however, the problem doesn't define the range for the wave function. For a symmetric finite interval it is simple enough to show that these are zero, but the solution in the back of the text just wanted "ill posed" for an infinite range.

7 Problem 6. Is X operator Hermitian.

$$< x >^* = \left(\int \psi^* x \psi dx \right)^* = < x >$$

answer is therefore yes (with $< ix >$ being Skew-Hermitian).

Momentum operator follows the same way with integration by parts, but I've written that up recently in [Joot()] so won't repeat it here.

8 Problem 7. A current calculation.

$$\psi = (Ae^{ip\hbar x} + Be^{-ip\hbar x})e^{-ip^2 t/2m\hbar}$$

The probability density is just

$$|\psi|^2 = (Ae^{ip\hbar x} + Be^{-ip\hbar x})(A^* e^{-ip\hbar x} + B^* e^{ip\hbar x})$$

But this doesn't have to be expanded for the continuity calculation, since we only want the time derivative which is zero since this isn't a function of time

$$\frac{\partial |\psi|^2}{\partial t} = 0.$$

Now the current density is

$$J = \frac{\hbar}{2mi} \left(\psi^* \partial_x \psi - \psi \partial_x \psi^* \right)$$

the spatial derivatives leave the time phase term untouched so the conjugation takes those out, leaving

$$
\begin{aligned}
J &= \frac{\hbar}{2mi} ip\hbar \left((A^* e^{-ip\hbar x} + B^* e^{ip\hbar x})(A e^{ip\hbar x} - B e^{-ip\hbar x}) - (A e^{ip\hbar x} + B e^{-ip\hbar x})(-A^* e^{-ip\hbar x} + B^* e^{ip\hbar x}) \right) \\
&= \frac{\hbar}{2mi} ip\hbar \left(2|A|^2 - 2|B|^2 \right)
\end{aligned}
$$

So we have $\nabla \cdot J = 0$ since this is a constant, and therefore

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 + 0 = 0$$

9 Problem 8.

A square well problem in $[0, a]$ with

$$\psi = i\frac{\sqrt{3}}{2}\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) e^{-iE_1 t/\hbar} + \frac{1}{2}\sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right) e^{-iE_3 t/\hbar}$$

Using separation of variables for the wave equation in this case we have

$$-\frac{\hbar^2}{2m}\frac{X''}{X} = i\hbar\frac{T'}{T} = E$$

We have

$$T = e^{-iEt/\hbar}$$

and

$$X = A \sin\left(\frac{\sqrt{2mE}x}{\hbar}\right) = A \sin\left(\frac{k\pi x}{a}\right)$$

To normalizing X we need squared sine

$$\sin^2(u) = \frac{1}{-4}(e^{2iu} + e^{-2iu} - 2) = \frac{1}{-2}\cos(2u) + \frac{1}{2}$$

So we can integrate to find the normalization factor

$$\int_0^a X^2 = A^2 \frac{a}{2} = 1$$

for

$$
\begin{aligned}
T_k &= e^{-iE_k t/\hbar} \\
X_k &= \sqrt{\frac{2}{a}} \sin\left(\frac{k\pi x}{a}\right) \\
E_k &= \frac{1}{2m}\left(\frac{k\pi x}{a}\right)^2
\end{aligned}
$$

9.1 Is it normalized?

Yes, $3/4 + 1/4 = 1$.

9.2 What are the values of the energy?

The work for this is above:

$$
\begin{aligned}
E_1 &= \frac{1}{2m}\left(\frac{\pi x}{a}\right)^2 \\
E_3 &= \frac{1}{2m}\left(\frac{3\pi x}{a}\right)^2
\end{aligned}
$$

9.3 expectation of position

We can write the wave function in terms of basis functions for convience

$$
\begin{aligned}
\phi_m &= \sqrt{\frac{2}{a}} \sin\left(\frac{kx\pi}{a}\right) \\
\psi &= i\frac{\sqrt{3}}{2}\phi_1 e^{-iE_1 t/\hbar} + \frac{1}{2}\phi_3 e^{-iE_3 t/\hbar}
\end{aligned}
$$

or more generally

$$
\psi = \sum_m c_m(t)\phi_m
$$

In terms of this Fourier series our position expectation is

$$
\begin{aligned}
< x > &= \sum_{m,n} \int c_m^* (\phi_m)^* x c_n \phi_n \\
&= \sum_{m,n} c_m^* c_n \int x \phi_m^* \phi_n \\
&= \sum_{m,n} c_m^* c_n \frac{2a}{\pi^2} \int_0^a (a\pi x/a) \sin(m\pi x/a) \sin(n\pi x/a) a(\pi dx/a) \\
&= \sum_{m,n} c_m^* c_n \frac{2a}{\pi^2} \int_0^\pi u \sin(mu) \sin(nu) du
\end{aligned}
$$

For the integral for $m \neq n$ we have

$$
\begin{aligned}
\int u \sin(mu) \sin(nu) du = \frac{1}{2} &\left(\frac{\cos((m-n)x)}{(m-n)^2} - \frac{\cos((m+n)x)}{(m+n)^2} \right) \\
+ \frac{1}{2} &\left(\frac{x \sin((m-n)x)}{m-n} - \frac{x \sin((m+n)x)}{m+n} \right)
\end{aligned}
$$

The sine terms will drop out at zero and π, and the cosine terms will subtract out since they are the same at the boundaries.

$$
< x > = \sum_n |c_n|^2 \frac{2a}{\pi^2} \int_0^\pi u \sin^2(nu) du
$$

Now, for the integral we have

$$
\int u \sin^2(nu) du = \frac{u^2}{4} - \frac{\cos(2nu)}{8n^2} - \frac{u \sin(2nu)}{4n}
$$

again the sine terms are zero, the cosines subtract away and we have only the first term making a contribution at the upper bound. This gives

$$
\begin{aligned}
< x > &= \sum_n |c_n|^2 \frac{2a}{\pi^2} \frac{\pi^2}{4} \\
&= \frac{a}{2} \sum_n |c_n|^2 \\
&= \frac{a}{2}
\end{aligned}
$$

This is kind of cool. A likely interpretation is that for any wave function whatsoever for this particle in the box we have equal probability of finding the particle at any particular point. Because of this it makes sense that the average value for the location of the particle is exactly the average of the positions available in the box.

9.4 expectation of momentum

For the momentum expectation we want

$$
\begin{aligned}
< p > &= \sum_{m,n} c_n^* c_m - i\hbar \frac{2}{a} \int_0^a \sin(n\pi x/a)(m\pi/a)\cos(m\pi x/a) dx (\pi/a)(a/\pi) \\
&= \sum_{m,n} c_n^* c_m - i\hbar \frac{2m}{a} \int_0^\pi \sin(nu)\cos(mu) du
\end{aligned}
$$

Since the integral above is zero for all m, n, we have for any wave function for the particle in a box

$$< p >= 0$$

This trivially shows that the statement of the problem that

$$\frac{dm < x >}{dt} =< p >$$

is in fact true, but this doesn't say much since both sides are zero.

References

[Joot()] Peeter Joot. Notes on susskind's qm lectures. "`http://sites.google.com/site/peeterjoot/math2009/qm_susskind.pdf`".

[McMahon(2005)] D. McMahon. *Quantum Mechanics Demystified*. McGraw-Hill Professional, 2005.