# Stokes theorem applied to vector and bivector fields.

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## 1. Vector Stokes Theorem

I found my self forgetting stokes theorem once again. Redo this for the simplest case of a parallelogram area element.

What I recall is that we have on one side the curl dotted into the plane of the surface area element

$$\int (\nabla \wedge A) \cdot d^2 x \tag{1}$$

and on the other side a loop integral

$$\oint A \cdot dx \tag{2}$$

Comparing the two we should end up with the same form and thus determine the form of the grade two Stokes equation (i.e. for curl of a vector).

### *1.1. Bivector product part.*

$$\begin{aligned}
(\nabla \wedge A) \cdot d^2 x &= (\nabla \wedge A) \cdot \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \right) d\alpha d\beta \\
&= \partial_\mu A_\nu \frac{\partial x^\sigma}{\partial \alpha} \frac{\partial x^\epsilon}{\partial \beta} (\gamma^\mu \wedge \gamma^\nu) \cdot (\gamma_\sigma \wedge \gamma_\epsilon) d\alpha d\beta \\
&= \partial_\mu A_\nu \frac{\partial x^\sigma}{\partial \alpha} \frac{\partial x^\epsilon}{\partial \beta} ({\delta^\mu}_\epsilon {\delta^\nu}_\sigma - {\delta^\mu}_\sigma {\delta^\nu}_\epsilon) d\alpha d\beta \\
&= \partial_\mu A_\nu \left( \frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} - \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \right) d\alpha d\beta
\end{aligned}$$

So we have

$$(\nabla \wedge A) \cdot d^2x = -\partial_\mu A_\nu \frac{\partial(x^\mu, x^\nu)}{\partial(\alpha, \beta)} d\alpha d\beta \tag{3}$$

### *1.2. **Loop integral part.***

Integrating around a parallelogram spacetime area element with sides $d\alpha \partial x/\partial\alpha$ and $d\beta \partial x/\partial\beta$, as depicted in figure (1), we have

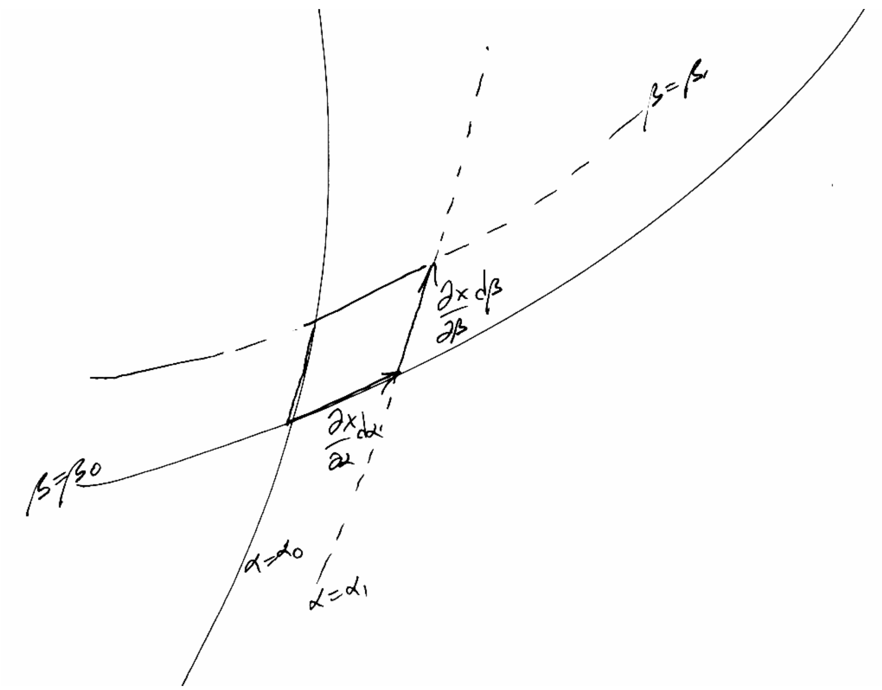


Figure 1: Surface area element

$$\begin{aligned}
\oint A \cdot dx &= \int A|_{\beta=\beta_0} \cdot \frac{\partial x}{\partial \alpha} d\alpha + A|_{\alpha=\alpha_1} \cdot \frac{\partial x}{\partial \beta} d\beta + A|_{\beta=\beta_1} \cdot \left( -\frac{\partial x}{\partial \alpha} d\alpha \right) + A|_{\alpha=\alpha_0} \cdot \left( -\frac{\partial x}{\partial \beta} d\beta \right) \\
&= \int \left( A|_{\alpha=\alpha_1} - A|_{\alpha=\alpha_0} \right) \cdot \frac{\partial x}{\partial \beta} d\beta - \left( A|_{\beta=\beta_1} - A|_{\beta=\beta_0} \right) \cdot \frac{\partial x}{\partial \alpha} d\alpha \\
&= \int \frac{\partial A}{\partial \alpha} \cdot \frac{\partial x}{\partial \beta} d\alpha d\beta - \frac{\partial A}{\partial \beta} \cdot \frac{\partial x}{\partial \alpha} d\beta d\alpha
\end{aligned}$$

Expanding the derivatives in terms of coordinates we have

$$\begin{aligned}
\frac{\partial A}{\partial \sigma} &= \frac{\partial A_\mu}{\partial \sigma} \gamma^\mu \\
&= \frac{\partial A_\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial \sigma} \gamma^\mu \\
&= \partial_\nu A_\mu \frac{\partial x^\nu}{\partial \sigma} \gamma^\mu
\end{aligned}$$

and

$$\frac{\partial x}{\partial \sigma} = \frac{\partial x^\nu}{\partial \sigma}\gamma_\nu$$

Assembling we have

$$\oint A \cdot dx = \int \partial_\nu A_\mu \left( \frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} - \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\mu}{\partial \alpha} \right) d\alpha d\beta$$

In terms of the Jacobian used in (3) we have

$$\oint A \cdot dx = \int \partial_\mu A_\nu \frac{\partial(x^\mu, x^\nu)}{\partial(\alpha, \beta)} d\alpha d\beta$$

Comparing the two we have only a sign difference so the conclusion is that Stokes for a vector field (considering only a flat parallelogram area element) is

$$\int (\nabla \wedge A) \cdot d^2 x = \oint A \cdot dx \tag{4}$$

Observe that there's an implied orientation of the area element on the LHS, required to match up with the orientation of the RHS integral.

## 2. **Bivector Stokes Theorem**

A parallelepiped volume element is depicted in figure (2). Three parameters $\alpha$, $\beta$, $\sigma$ generate a set of differential vector displacements spanning the three dimensional subspace

Writing the displacements

$$\begin{aligned}
dx_\alpha &= \frac{\partial x}{\partial \alpha} d\alpha \\
dx_\beta &= \frac{\partial x}{\partial \beta} d\beta \\
dx_\sigma &= \frac{\partial x}{\partial \sigma} d\sigma
\end{aligned}$$

We have for the front, right and top face area elements

$$\begin{aligned}
dA_F &= dx_\alpha \wedge dx_\beta \\
dA_R &= dx_\beta \wedge dx_\sigma \\
dA_T &= dx_\sigma \wedge dx_\alpha
\end{aligned}$$

These are the surfaces of constant parametrization, respectively, $\sigma = \sigma_1$, $\alpha = \alpha_1$, and $\beta = \beta_1$. For a bivector, the flux through the surface is therefore

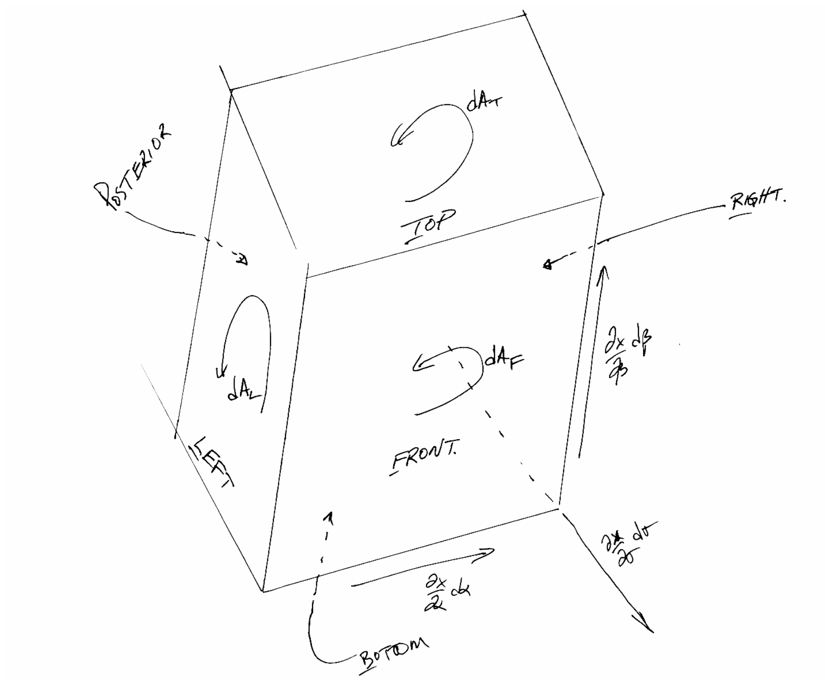


Figure 2: Differential volume element

$$
\begin{aligned}
\int B \cdot dA &= (B_{\sigma_1} \cdot dA_F - B_{\sigma_0} \cdot dA_P) + (B_{\alpha_1} \cdot dA_R - B_{\alpha_0} \cdot dA_L) + (B_{\beta_1} \cdot dA_T - B_{\beta_0} \cdot dA_B) \\
&= d\sigma \frac{\partial B}{\partial \sigma} \cdot (dx_\alpha \wedge dx_\beta) + d\alpha \frac{\partial B}{\partial \alpha} \cdot (dx_\beta \wedge dx_\sigma) + d\beta \frac{\partial B}{\partial \beta} \cdot (dx_\sigma \wedge dx_\alpha)
\end{aligned}
$$

Written out in full this is a bit of a mess

$$
\int B \cdot dA = d\alpha d\beta d\sigma \partial_\mu B \cdot \left( \left( -\frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \alpha} + \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} + \frac{\partial x^\mu}{\partial \beta} \frac{\partial x^\nu}{\partial \sigma} \frac{\partial x^\epsilon}{\partial \alpha} \right) (\gamma_\nu \wedge \gamma_\epsilon) \right) \tag{5}
$$

It should equal, at least up to a sign, $\int (\nabla \wedge B) \cdot d^3x$. Expanding the latter is probably easier than regrouping the mess, and doing so we have

$$
\begin{aligned}
(\nabla \wedge B) \cdot d^3 x &= d\alpha d\beta d\sigma (\gamma^\mu \wedge \partial_\mu B) \cdot \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \\
&= d\alpha d\beta d\sigma \frac{1}{2} (\gamma^\mu \partial_\mu B + \partial_\mu B \gamma^\mu) \cdot \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \\
&= d\alpha d\beta d\sigma \frac{1}{2} \left\langle (\gamma^\mu \partial_\mu B + \partial_\mu B \gamma^\mu) \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \right\rangle \\
&= d\alpha d\beta d\sigma \frac{1}{2} \partial_\mu B \cdot \left\langle \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \gamma^\mu + \gamma^\mu \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \right\rangle_2 \\
&= d\alpha d\beta d\sigma \partial_\mu B \cdot \left( \left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \cdot \gamma^\mu \right)
\end{aligned}
$$

Expanding just that trivector-vector dot product

$$
\begin{aligned}
\left( \frac{\partial x}{\partial \alpha} \wedge \frac{\partial x}{\partial \beta} \wedge \frac{\partial x}{\partial \sigma} \right) \cdot \gamma^\mu &= \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} (\gamma_\lambda \wedge \gamma_\nu \wedge \gamma_\epsilon) \cdot \gamma^\mu \\
&= \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} (\gamma_\lambda \wedge \gamma_\nu {\delta_\epsilon}^\mu - \gamma_\lambda \wedge \gamma_\epsilon {\delta_\nu}^\mu + \gamma_\nu \wedge \gamma_\epsilon {\delta_\lambda}^\mu)
\end{aligned}
$$

So we have

$$
\begin{aligned}
(\nabla \wedge B) \cdot d^3 x &= d\alpha d\beta d\sigma \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \partial_\mu B \cdot (\gamma_\lambda \wedge \gamma_\nu {\delta_\epsilon}^\mu - \gamma_\lambda \wedge \gamma_\epsilon {\delta_\nu}^\mu + \gamma_\nu \wedge \gamma_\epsilon {\delta_\lambda}^\mu) \\
&= d\alpha d\beta d\sigma \partial_\mu B \cdot \left( \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\mu}{\partial \sigma} \gamma_\lambda \wedge \gamma_\nu + \frac{\partial x^\lambda}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \gamma_\epsilon \wedge \gamma_\lambda + \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \gamma_\nu \wedge \gamma_\epsilon \right) \\
&= d\alpha d\beta d\sigma \partial_\mu B \cdot \left( \left( \frac{\partial x^\nu}{\partial \alpha} \frac{\partial x^\epsilon}{\partial \beta} \frac{\partial x^\mu}{\partial \sigma} + \frac{\partial x^\epsilon}{\partial \alpha} \frac{\partial x^\mu}{\partial \beta} \frac{\partial x^\nu}{\partial \sigma} + \frac{\partial x^\mu}{\partial \alpha} \frac{\partial x^\nu}{\partial \beta} \frac{\partial x^\epsilon}{\partial \sigma} \right) \gamma_\nu \wedge \gamma_\epsilon \right)
\end{aligned}
$$

Noting that an $\epsilon$, $\nu$ interchange in the first term inverts the sign, we have an exact match with (5), thus fixing the sign for the bivector form of Stokes theorem for the orientation picked in this diagram

$$
\int (\nabla \wedge B) \cdot d^3 x = \int B \cdot d^2 x \tag{6}
$$

Like the vector case, there is a requirement to be very specific about the meaning given to the oriented surfaces, and the corresponding oriented volume element (which could be a volume subspace of a greater than three dimensional space).