Two particle center of mass Laplacian change of variables.

Originally appeared at:
http://sites.google.com/site/peeterjoot/math2009/twoParticleCMLaplacian.pdf

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Nov 30, 2009 $RCSfile : twoParticleCMLaplacian.tex, v$ *Last Revision* : 1.3
Date : 2009/12/0102 : 35 : 29

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Exercise 15.2 in [1] is to do a center of mass change of variables for the two particle Hamiltonian

$$H = -\frac{\hbar^2}{2m_1}{\boldsymbol{\nabla}_1}^2 - \frac{\hbar^2}{2m_2}{\boldsymbol{\nabla}_2}^2 + V(\mathbf{r}_1 - \mathbf{r}_2). \tag{1}$$

Before trying this, I was surprised that this would result in a diagonal form for the transformed Hamiltonian, so it is well worth doing the problem to see why this is the case. He uses

$$\boldsymbol{\xi} = \mathbf{r}_1 - \mathbf{r}_2 \tag{2}$$

$$\boldsymbol{\eta} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2). \tag{3}$$

Lets use coordinates ${x_k}^{(1)}$ for $\mathbf{r}_1$, and ${x_k}^{(2)}$ for $\mathbf{r}_2$. Expanding the first order partial operator for $\partial/\partial {x_1}^{(1)}$ by chain rule in terms of $\boldsymbol{\eta}$, and $\boldsymbol{\xi}$ coordinates we have

$$\begin{aligned}
\frac{\partial}{\partial x_1^{(1)}} &= \frac{\partial \eta_k}{\partial x_1^{(1)}}\frac{\partial}{\partial \eta_k} + \frac{\partial \xi_k}{\partial x_1^{(1)}}\frac{\partial}{\partial \xi_k} \\
&= \frac{m_1}{M}\frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \xi_1}.
\end{aligned}$$

We also have

$$\begin{aligned}
\frac{\partial}{\partial x_1^{(2)}} &= \frac{\partial \eta_k}{\partial x_1^{(2)}}\frac{\partial}{\partial \eta_k} + \frac{\partial \xi_k}{\partial x_1^{(2)}}\frac{\partial}{\partial \xi_k} \\
&= \frac{m_2}{M}\frac{\partial}{\partial \eta_1} - \frac{\partial}{\partial \xi_1}.
\end{aligned}$$

The second partials for these x coordinates are not a diagonal quadratic second partial operator, but are instead

$$\frac{\partial}{\partial x_1^{(1)}}\frac{\partial}{\partial x_1^{(1)}} = \frac{(m_1)^2}{M^2}\frac{\partial^2}{\partial \eta_1 \partial \eta_1} + \frac{\partial^2}{\partial \xi_1 \partial \xi_1} + 2\frac{m_1}{M}\frac{\partial^2}{\partial \xi_1 \partial \eta_1} \tag{4}$$

$$\frac{\partial}{\partial x_1^{(2)}}\frac{\partial}{\partial x_1^{(2)}} = \frac{(m_2)^2}{M^2}\frac{\partial^2}{\partial \eta_1 \partial \eta_1} + \frac{\partial^2}{\partial \xi_1 \partial \xi_1} - 2\frac{m_2}{M}\frac{\partial^2}{\partial \xi_1 \partial \eta_1}. \tag{5}$$

The desired result follows directly, since the mixed partial terms conveniently cancel when we sum $(1/m_1)\partial/\partial x_1^{(1)}\partial/\partial x_1^{(1)} + (1/m_2)\partial/\partial x_1^{(2)}\partial/\partial x_1^{(2)}$. This leaves us with

$$H = \frac{-\hbar^2}{2} \sum_{k=1}^{3} \left(\frac{1}{M} \frac{\partial^2}{\partial \eta_k \partial \eta_k} + \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \frac{\partial^2}{\partial \xi_k \partial \xi_k} \right) + V(\boldsymbol{\xi}), \tag{6}$$

With the shorthand of the text

$$\boldsymbol{\nabla}_\eta = \sum_k \frac{\partial^2}{\partial \eta_k \partial \eta_k} \tag{7}$$

$$\boldsymbol{\nabla}_\xi = \sum_k \frac{\partial^2}{\partial \xi_k \partial \xi_k}, \tag{8}$$

this is the result to be proven.

References

[1] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989.